Triple (and quadruple) soft-gluon radiation in QCD hard scattering

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and arXiv 2022.xxxx for soft $\mathrm{g q \bar{q}}$ radiation

Loops and Legs Ettal 29.04.2022

Outline

- Review of tree-level soft current for 1 or 2 soft gluons

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- Tree-level current for 3 soft gluons
- Squared currents for 1 or 2 or 3 soft gluons
  - Colour structure with dipoles and quadrupoles
  - Kinematical coefficients in strong energy ordering
  - Collinear singularities

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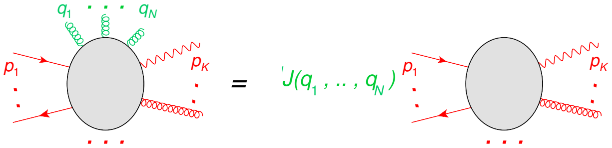
- Review of tree-level soft current for 1 or 2 soft gluons
- Tree-level current for 3 soft gluons
- Squared currents for 1 or 2 or 3 soft gluons
  - Colour structure with dipoles and quadrupoles
  - Kinematical coefficients in strong energy ordering
  - Collinear singularities
- Squared currents with 3 hard partons
  - c-number factorization
  - Casimir scaling (violation)
  - Multi-eikonal formula in strong energy ordering
- Squared currents with 2 hard partons
  - Same analysis as for 3 hard partons
  - Extension to 4 soft gluons: colour monster, quartic Casimir,. . .

Motivations

- High-precision LHC data $\leftrightarrow$ high precision in theoretical predictions
  - to test our present understanding of the Standard Model
  - to discover (probably tiny) signals of new physics
- Explicit knowledge of soft/collinear factorization of scattering amplitudes necessary in resummed calculations at N^3LL
- Calculation of large logarithmic terms can be used to obtain approximated fixed-order results
- Soft/collinear factorization provides the theoretical basis of parton shower algorithms for Monte Carlo event generators

- Behaviour of scattering amplitude $\mathcal{M}(\{p_k\},\{q_i\})$ when some external gluons $q_i = \xi \bar{q}_i$ become soft ($\xi \to 0$)



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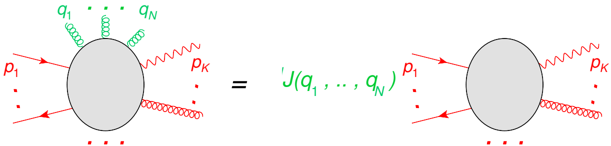
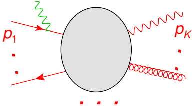


- Leading $1/\xi$: only gluon insertions on the external hard legs

QED: $$J(q) = J^{\mu}(q)\varepsilon_{\mu}(q) = \sum_{k=1}^{K} \frac{e_k p_k^{\mu}}{p_k \cdot q}\varepsilon_{\mu}(q)$$



QCD: $$J^{a}(q) = J^{a,\mu}(q)\varepsilon_{\mu}(q) = \sum_{k=1}^{K} \frac{g T_k^a p_k^{\mu}}{p_k \cdot q}\varepsilon_{\mu}(q)$$

$$\begin{cases} (T_q^a)_{bc} = t_{bc}^a \\ (T_{\bar{q}}^a)_{bc} = -t_{cb}^a \\ (T_g^a)_{bc} = \mathrm{i} f^{abc} \end{cases}$$

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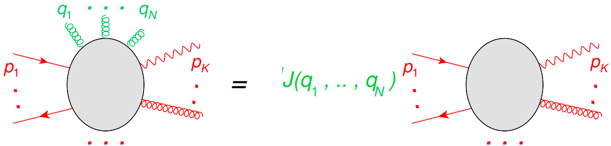

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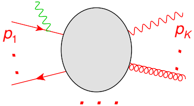

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$$\sum_{k=1}^{K} e_k = 0\,, \qquad \sum_{k=1}^{K} T_k^a \overset{\text{cs}}{=} 0 \quad \Longrightarrow \quad q_{\mu} J^{a,\mu} \overset{\text{cs}}{=} 0$$

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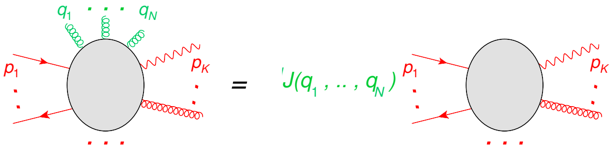

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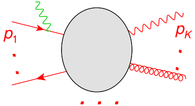

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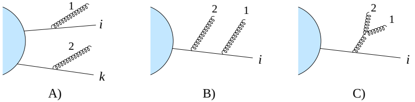
GRAV: $$J(q) = J^{\mu\nu}(q)\varepsilon_{\mu\nu}(q) = \sum_{k=1}^{K} \frac{\kappa p_k^{\nu} p_k^{\mu}}{p_k \cdot q} \varepsilon_{\mu\nu}(q)$$

2 soft gluon emission



[Catani, Grazzini '00] "Independent" + max. non-abelian correlation

$$J^{a_1 a_2}_{\mu_1 \mu_2}(q_1, q_2) = J^{a_1}_{\mu_1}(q_1) * J^{a_2}_{\mu_2}(q_2) + \Gamma^{a_1 a_2}_{\mu_1 \mu_2}(q_1, q_2) \qquad A*B \equiv \frac{1}{2}(AB+BA)$$

$$\Gamma^{a_1 a_2}_{\mu_1 \mu_2}(q_1, q_2) = \mathrm{i} f^{a_1 a_2 \, b} \sum_{k \in \text{hard}} T^b_k \, \gamma^{\mu_1 \mu_2}_k(q_1, q_2)$$

$$\gamma^{\mu_1 \mu_2}_k(q_1, q_2) = \frac{1}{p_k \cdot (q_1 + q_2)} \left\{ \frac{p_k^{\mu_1} p_k^{\mu_2}}{2 \, p_k \cdot q_1} + \frac{1}{q_1 \cdot q_2} \left(p_k^{\mu_1} q_1^{\mu_2} + \frac{1}{2} g^{\mu_1 \mu_2} p_k \cdot q_2 \right) \right\} - \left(1 \leftrightarrow 2 \right)$$

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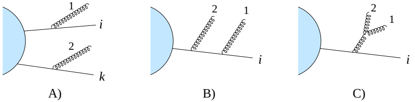

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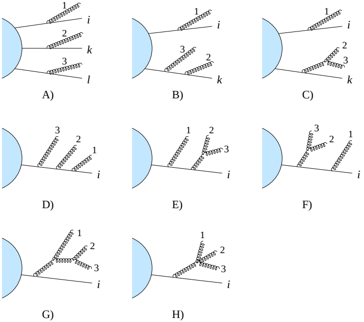
- conservation of current: $q_1^{\mu_1} J^{a_1 a_2}_{\mu_1 \mu_2}(q_1, q_2)\, \varepsilon^{\mu_2}(q_2) \overset{\text{cs}}{=} 0$
- Abelian case ($f^{abc} = 0$): only independent emission
- QCD: $J(1) * J(2) \implies$ colour-correlations with hard partons (furthermore currents do not commute)

3 soft gluon emission



- Eikonal vertices on hard lines
- Exact vertices elsewhere
- Exact propagators of soft gluons (light-cone / covariant) gauges

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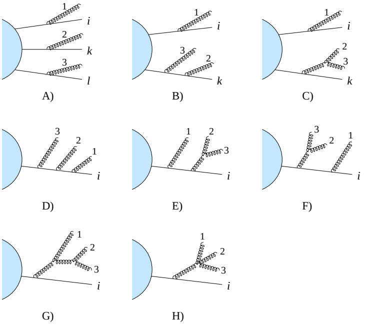

- Eikonal vertices on hard lines
- Exact vertices elsewhere
- Exact propagators of soft gluons (light-cone / covariant) gauges
- Current is conserved on colour-singlet states
- Irreducible term has colour structure $T^b \sum_s f^{a_1 a_2 s} f^{s a_3 b}$

$$J(1,2,3) = J(1) * J(2) * J(3) + \left[\sum_{\mathrm{cyc.123}} J(1) * \Gamma(2,3) \right] + \Gamma(1,2,3)$$

$$\Gamma(1,2,3) = \sum_{k \in \mathrm{hard}} T_k^b \sum_{\mathrm{cyc.123}} f^{a_1 a_2,\, a_3 b}\, \gamma_k^{\mu_1 \mu_2 \mu_3}(q_1, q_2; q_3)$$

Kinematical coefficient of maximally non-abelian term

$$
\begin{aligned}
\gamma_k(1,2;3) = \frac{1}{p_k\cdot q_{123}} \Bigg\{ & \frac{1}{12}\,\frac{p_k^{\mu_1} p_k^{\mu_2} p_k^{\mu_3}\; p_k\cdot(3q_3 - q_{12})}{p_k\cdot q_2\;\, p_k\cdot q_3\;\, p_k\cdot q_{12}} \\
& + \frac{p_k^{\mu_3}\; p_k\cdot(q_3-q_{12})}{p_k\cdot q_3\;\, p_k\cdot q_{12}\;\, q_{12}^2}\left(\frac{1}{2} g^{\mu_1\mu_2}\, p_k\cdot q_1 + p_k^{\mu_2}\, q_2^{\mu_1}\right) \\
& + \frac{1}{q_{123}^2\;\, q_{12}^2}\Big[q_{12}^2\, p_k^{\mu_1}\, g^{\mu_2\mu_3} + 2q_2^{\mu_1} g^{\mu_2\mu_3}\, p_k\cdot(q_3-q_{12}) \\
& + 4\, q_3^{\mu_1}\, q_1^{\mu_2}\, p_k^{\mu_3} + 4\, q_2^{\mu_1}\, p_k^{\mu_2}\, q_{12}^{\mu_3} \\
& + g^{\mu_1\mu_2}\left(q_{23}^2\, p_k^{\mu_3} + q_1^{\mu_3}\, p_k\cdot(q_{13}-3q_2)\right)\Big] \Bigg\} - \Big(1 \leftrightarrow 2\Big)
\end{aligned}
$$

Notation: $q_{ij} = q_i + q_j$, $\qquad q_{123} = q_1 + q_2 + q_3$

Soft factors for colour-ordered subamplitudes

Colour-ordered subamplitudes decomposition *[Berends, Giele '89]*

$$M(1,\cdots,n) = \sum_{\mathrm{perm}(1,\cdots,n-1)} \mathrm{tr}(t^{a_1}\cdots t^{a_n})\; C(1,\cdots,n)$$

In the soft limit $k_2,\ldots k_m \to 0$

$$C(1,\underline{2},\cdots,\underline{m},m+1,\cdots,n) = s_{1,\underline{2},\cdots,\underline{m},m+1}\, C(1,m+1,\cdots,n)$$

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Soft factors in terms of kinematical coefficients

$$s_{i\underline{1}k} = j_k(1) - j_i(1) \qquad \left(j_k^\mu(1) \equiv \frac{p_k^\mu}{p_k\cdot q_1}\right)$$

$$s_{i\underline{12}k} = \gamma_i(1,2) + \tfrac{1}{2}[j_i(1)j_i(2) - j_i(1)j_k(2)] + \binom{1\leftrightarrow 2}{i\leftrightarrow k}$$

$$\begin{aligned} s_{i\underline{123}k} = {} & \gamma_i(1,2;3) + \gamma_i(3,2;1) - \tfrac{1}{2}\left[\gamma_i(1,2)\,j_i(3) + j_i(1)\,\gamma_i(2,3)\right] \\ & + \gamma_i(1,2)\,j_k(3) + \tfrac{1}{2}\,j_i(1)\,j_i(2)\,j_k(3) - \tfrac{1}{6}\,j_i(1)\,j_i(2)\,j_i(3) - \binom{1\leftrightarrow 3}{i\leftrightarrow k} \end{aligned}$$

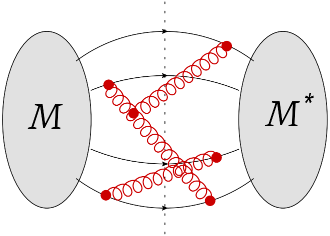
Square of soft current

Dirac's notation in colour $\otimes$ helicity space

$$\mathcal{M}^{c_1 c_2 \dots}_{\lambda_1 \lambda_2 \dots} =: \langle c_1, c_2 \dots | \otimes \langle \lambda_1, \lambda_2 \dots | \mathcal{M} \rangle$$

$$|\mathcal{M}(p_k, q_i)\rangle \simeq \boldsymbol{J}(q_i) |\mathcal{M}(p_k)\rangle$$

$$|\mathcal{M}(p_k, q_i)|^2 \simeq \langle \mathcal{M}(p_k)| \, |\boldsymbol{J}(q_i)|^2 \, |\mathcal{M}(p_k)\rangle$$



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$$|\mathcal{M}(p_k, q_i)|^2 \simeq \langle \mathcal{M}(p_k)| \, |\boldsymbol{J}(q_i)|^2 \, |\mathcal{M}(p_k)\rangle$$

Current conservation on colour-singlets $q_\ell^{\mu_\ell} J^{..a_\ell..}_{..\mu_\ell..}(..q_\ell..) \stackrel{\text{cs}}{=} 0$ $\Downarrow$

$$|\boldsymbol{J}(q_1 \cdots q_N)|^2 \stackrel{\text{cs}}{=} \left[\prod_{\ell=1}^{N} -g^{\mu_\ell \nu_\ell} \right] J^{a_1 \ldots a_N}_{\mu_1 \ldots \mu_N}(q_1 \cdots q_N)^\dagger J^{a_1 \ldots a_N}_{\nu_1 \ldots \nu_N}(q_1 \cdots q_N)$$

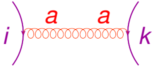
- Explicitly gauge invariant
- Still a colour operator that depends on the colour charges of the hard partons in $\mathcal{M}(p_k)$
- Important simplifications for 2 or 3 hard partons

Square of soft currents for 1 or 2 gluon emission

- Single gluon emission

$$|\boldsymbol{J}(q)|^2 \overset{\text{cs}}{=} -\sum_{i,k\in hard} \boldsymbol{T}_i \cdot \boldsymbol{T}_k\, \mathcal{S}_{ik}(q) =: W(q)$$

$$\mathcal{S}_{ik}(q) = \frac{p_i \cdot p_k}{p_i \cdot q \; p_k \cdot q}$$



$$\left(\boldsymbol{T}_i \cdot \boldsymbol{T}_k \equiv \sum_a T_i^a T_k^a \right)$$

colour dipoles

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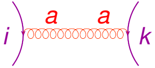

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colour dipoles

$$\mathcal{S}_{ik}(q) = \frac{p_i \cdot p_k}{p_i \cdot q \; p_k \cdot q}$$

- Double gluon emission
  involves just colour dipoles in irreducible correlation

$$|\boldsymbol{J}(q_1, q_2)|^2 \stackrel{\text{cs}}{=} W(q_1) * W(q_2) + W(q_1, q_2)$$

$$W(q_1, q_2) = -C_A \sum_{i,k \in hard} \boldsymbol{T}_i \cdot \boldsymbol{T}_k \, \mathcal{S}_{ik}(q_1, q_2)$$

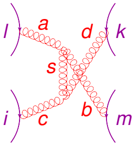
Square of soft current for 3 gluon emission

$$|\boldsymbol{J}(q_1,q_2,q_3)|^2 \stackrel{\text{cs}}{=} W(q_1)*W(q_2)*W(q_3) + \Big[\sum_{\text{cyc.}123} W(q_1)*W(q_2,q_3)\Big] + W(q_1,q_2,q_3)$$

Each W is gauge-invariant

The irreducible correlation involves dipoles and **quadrupoles**

$$W(q_1,q_2,q_3) = -C_A^2 \sum_{i,k} \boldsymbol{T}_i \cdot \boldsymbol{T}_k \, \mathcal{S}_{ik}(q_1,q_2,q_3) + \sum_{iklm} Q_{iklm} \, \mathcal{S}_{iklm}(q_1,q_2,q_3)$$



Irred. quadrupoles $Q_{iklm} \equiv \frac{1}{2} f^{ab,cd} \left(T_l^a \{ T_i^c, T_k^d \} T_m^b \; + \text{ h.c.} \right)$

Square of soft current for 3 gluon emission

- In strong energy ordering $E_1 \ll E_2 \ll E_3$ and massless hard partons

$$
\begin{aligned}
S_{ik}^{\text{seo}} = & \frac{2(p_i \cdot p_k)^3}{3(p_i \cdot q_1)(p_k \cdot q_1)(p_i \cdot q_2)(p_k \cdot q_2)(p_i \cdot q_3)(p_k \cdot q_3)} \\
& - \frac{2(p_i \cdot p_k)^2}{(q_1 \cdot q_2)(p_i \cdot q_1)(p_k \cdot q_2)(p_i \cdot q_3)(p_k \cdot q_3)} \\
& + \frac{2p_i \cdot p_k}{(q_1 \cdot q_3)(q_2 \cdot q_3)(p_i \cdot q_1)(p_k \cdot q_2)} + \text{perms.}\,\{1,2,3\}
\end{aligned}
$$

symmetric (!) under exchange of soft gluons momenta q_1, q_2, q_3

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\end{aligned}
$$

  symmetric (!) under exchange of soft gluons momenta q_1, q_2, q_3
- Quadrupole kinematical coefficient

$$
\begin{aligned}
S_{iklm}^{\mathrm{seo}} = & \frac{p_i \cdot p_k}{(p_i \cdot q_3)(p_k \cdot q_3)(p_l \cdot q_1)(p_m \cdot q_2)} \left\{ \frac{(p_m \cdot p_l)(p_l \cdot p_i)}{3(p_l \cdot q_2)(p_i \cdot q_1)} \right. \\
& \left. - \frac{p_m \cdot p_i}{3 p_i \cdot q_2} \left[\frac{p_l \cdot p_i}{p_i \cdot q_1} + \frac{p_l \cdot p_m}{p_m \cdot q_1} + \frac{2 p_l \cdot p_k}{p_k \cdot q_1} \right] + \frac{p_l \cdot q_3}{q_3 \cdot q_1} \left[\frac{2 p_m \cdot p_i}{p_i \cdot q_2} - \frac{p_m \cdot q_3}{q_3 \cdot q_2} \right] \right\} + (1 \leftrightarrow 2)
\end{aligned}
$$

  symmetric (!) in the exchange $q_1 \leftrightarrow q_2$.

Collinear singularities of squared currents

- $|\mathcal{M}|^2$ singular (not integrable) when momenta of two or more of external *massless* legs become collinear

Collinear singularities of squared currents

- $|\mathcal{M}|^2$ singular (not integrable) when momenta of two or more of external *massless* legs become collinear
- 1 gluon emission: singular when $q \simeq z\, p_B$

$$|\boldsymbol{J}(q)|^2 = -\sum_{i\neq k} \boldsymbol{T}_i \cdot \boldsymbol{T}_k \frac{p_i \cdot p_k}{p_i \cdot q \; p_k \cdot q} \simeq -\sum_{i\neq k} \boldsymbol{T}_i \cdot \boldsymbol{T}_k \, \frac{\delta_{iB} + \delta_{kB}}{z\, p_B \cdot q}$$

$$= \frac{2}{z\, p_B \cdot q} \boldsymbol{T}_B \cdot \Big(-\sum_{k\neq B} \boldsymbol{T}_k \Big) \overset{\text{cs}}{=} \frac{1}{p_B \cdot q} \, \frac{2C_B}{z} \quad \begin{cases} C_F & (B = \mathrm{q}) \\ C_A & (B = \mathrm{g}) \end{cases}$$

- absence of colour correlations $\Rightarrow$ colour coherence

Collinear singularities of squared currents

recall $|\boldsymbol{J}(1,2,3)|^2 \overset{\text{cs}}{=} W(1) * W(2) * W(3) + \left[\sum_{\text{cyc.123}} W(1) * W(2,3) \right] + W(1,2,3)$

Expansion in irreducible correlations reduces collinear singularities of W's

- $W(2,3)$
  - c_1 double-collinear limit of the 2 soft gluons (exact $P^{\mu\nu}_{g_1 g_2}$)
  - c_2 triple-collinear limit of the 2 soft gluons and a hard parton
- $W(1,2,3)_{\text{dipole}}$
  - c_3 double-collinear limit of 2 soft gluons (exact)
  - c_4 triple-collinear limit of the 3 soft gluons (exact $P^{\mu\nu}_{g_1 g_2 g_3}$)
  - c_5 quadruple-collinear limit of the 3 soft gluons and a hard parton $\Longrightarrow$ soft limit of $P^{ss'}_{g_1 g_2 g_3 C}$ (new!)
- $W(1,2,3)_{\text{quadrupole}}$ has no collinear singularity!

3 hard partons

Consider amplitudes with 3 hard partons A, B, C plus soft gluons (and colourless objs)

- $\{A, B, C\} = \{\mathrm{g}, \mathrm{q}, \bar{\mathrm{q}}\}$ or $\{A, B, C\} = \{\mathrm{g}, \mathrm{g}, \mathrm{g}\}$ (flavour cons.)

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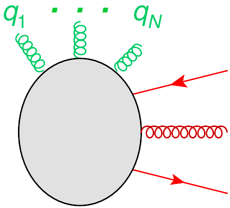

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c-number

$$|\boldsymbol{J}(q_1 \cdots q_N)|^2|\mathrm{g}\mathrm{q}\bar{\mathrm{q}}\rangle = |\mathrm{g}\mathrm{q}\bar{\mathrm{q}}\rangle\, |\boldsymbol{J}(q_1 \cdots q_N)|^2_{\mathrm{g}\mathrm{q}\bar{\mathrm{q}}}$$

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$$|\mathcal{M}(p_k, q_1 \cdots q_N)|^2 \simeq |\mathcal{M}(p_k)|^2\ |\boldsymbol{J}(q_1 \cdots q_N)|^2_{\mathrm{g}\mathrm{q}\bar{\mathrm{q}}}$$

- c-number soft-gluon factorization formula is valid at arbitrary loop orders

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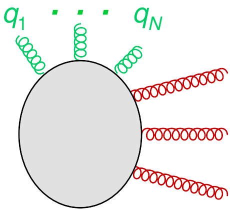
$$\langle abc|(\mathrm{ggg})_f\rangle = \mathrm{i}f^{abc} \qquad \langle abc|(\mathrm{ggg})_d\rangle = d^{abc}$$



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- $|\boldsymbol{J}(q_1 \cdots q_N)|^2_f \neq |\boldsymbol{J}(q_1 \cdots q_N)|^2_d$ for $N \geq 3$
  Quadrupole colour correlations remove degeneracy

3 hard partons: Radiation at tree level of 1 or 2 gluons

Dipoles $\propto$ Casimirs: $\boldsymbol{T}_A \cdot \boldsymbol{T}_B \, |ABC\rangle = \frac{1}{2}(C_C - C_A - C_B) \, |ABC\rangle$

Rewrite $W(q_1 \cdots q_N) \overset{\text{cs}}{=} -\frac{1}{2} C_A^{n-1} \sum_{i,k} \boldsymbol{T}_i \cdot \boldsymbol{T}_k \, w_{ik}(q_1 \cdots q_N)$

$$w_{AB} = \mathcal{S}_{AB} + \mathcal{S}_{BA} - \mathcal{S}_{AA} - \mathcal{S}_{BB}$$

$$w_{ABC} = w_{AB} + w_{AC} - w_{BC}$$

- $|\boldsymbol{J}(q)|^2_{ABC} = C_B \, w_{BC}(q) + C_A \, w_{ABC}(q)$

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$$w_{ABC} = w_{AB} + w_{AC} - w_{BC}$$

- $|\boldsymbol{J}(q)|^2_{ABC} = C_B \, w_{BC}(q) + C_A \, w_{ABC}(q)$
- $|\boldsymbol{J}(q_1, q_2)|^2_{ABC} = C_B^2 \, w_{BC}(q_1) \, w_{BC}(q_2) +$ $C_B C_A \, [w_{BC}(q_1, q_2) + w_{BC}(q_1) \, w_{ABC}(q_2) + w_{BC}(q_2) \, w_{ABC}(q_1)]$ $+ C_A^2 \, [w_{ABC}(q_1, q_2) + w_{ABC}(q_1) \, w_{ABC}(q_2)]$
- dependence on colour state of hard-partons entirely expressed through the Casimir coefficients C_A e C_B
- Casimir scaling:

ABC :	$\mathrm{gq\bar{q}}$	ggg
C_B :	C_F	C_A

3 hard partons: Radiation at tree level of 3 gluons

- Dipole term: Casimir scaling

$$
\begin{aligned}
|\boldsymbol{J}(q_1, q_2, q_3)|^{2,\mathrm{dip}}_{ABC} &= C_B^3 \, w_{BC}(q_1) w_{BC}(q_2) \, w_{BC}(q_3) \\
&+ C_B^2 C_A \, \left[w_{BC}(q_1) \, w_{BC}(q_2, q_3) + \cdots \right] \\
&+ C_B C_A^2 \, \left[w_{BC}(q_1, q_2, q_3) + \cdots \right] \\
&+ C_A^3 \left[w_{ABC}(q_1, q_2, q_3) + w_{ABC}(q_1) \, w_{ABC}(q_2) \, w_{ABC}(q_3) + \cdots \right]
\end{aligned}
$$

3 hard partons: Radiation at tree level of 3 gluons

- Dipole term: Casimir scaling

$$|\boldsymbol{J}(q_1,q_2,q_3)|^{2,\mathrm{dip}}_{ABC} = C_B^3\, w_{BC}(q_1) w_{BC}(q_2)\, w_{BC}(q_3)$$
$$+ C_B^2 C_A\, [w_{BC}(q_1)\, w_{BC}(q_2,q_3) + \cdots]$$
$$+ C_B C_A^2\, [w_{BC}(q_1,q_2,q_3) + \cdots]$$
$$+ C_A^3 \left[w_{ABC}(q_1,q_2,q_3) + w_{ABC}(q_1)\, w_{ABC}(q_2)\, w_{ABC}(q_3) + \cdots\right]$$

- Quadrupole term: (sizeable) Casimir scaling violation $\mathcal{O}\left(N_c^{-2}\right)$

$$|\boldsymbol{J}(q_1,q_2,q_3)|^{2,\mathrm{quad}}_{ABC} = \lambda_B N_c\, w^{\mathrm{quad}}_{ABC}(q_1,q_2,q_3)$$
$$|\mathrm{g q \bar{q}}\rangle : \lambda_F = 1/2\,, \qquad |f\rangle : \lambda_A = 3\,, \qquad |d\rangle : \lambda = 0$$

$$w^{\mathrm{quad}}_{ABC} \equiv [\mathcal{S}_{ABAB} - \mathcal{S}_{ABBA} + \mathcal{S}_{ABCA} - \mathcal{S}_{ABAC} + \mathcal{S}_{BAAC} - \mathcal{S}_{BACA}] + \mathrm{perm}\{A,B,C\}$$

collinear safe w.r.t angular integration over soft-gluon momenta

3 hard *gluons*: radiation at tree level of N gluons

- Strong energy ordering: $E_1 \ll E_2 \ll \cdots \ll E_N$

$$|\mathcal{M}(p_k, q_i)|^2 \simeq |\mathcal{M}(p_k)|^2 \, |\boldsymbol{J}(q_i)|^{2,\text{seo}}_{\text{ggg}_f} \left\{1 + \mathcal{O}\left(N_c^{-2}\right)\right\} \quad \text{(new)}$$

$$|\boldsymbol{J}(q_i)|^{2,\text{seo}}_{\text{ggg}} = C_A^N \; p_A \cdot p_B \;\; p_B \cdot p_C \;\; p_C \cdot p_A \; F_{\text{eik}}(p_A, p_B, p_C, q_1, \cdots, q_N)$$

$$F_{\text{eik}}(k_1, \cdots, k_M) \equiv \left[(k_1 \cdot k_2)(k_2 \cdot k_3) \ldots (k_{M-1} \cdot k_M)(k_M \cdot k_1)\right]^{-1} + \text{ineq. perm}\{k_1, \cdots, k_M\}$$

in terms of multi-eikonal F_{eik} *[Bassetto Ciafaloni Marchesini 83]*

- The 2 eigenvalues of squared current become degenerate, up to colour suppressed terms

2 hard partons

- $\{B, C\} = \{\mathrm{q}, \bar{\mathrm{q}}\}$ or $\{B, C\} = \{\mathrm{g}, \mathrm{g}\}$ (flavour cons.)
- 1-dimensional colour spaces $\Longrightarrow$ c-number factorization
- Non-abelian effects are in $SU(N_c)$ colour coefficients

$$
\begin{aligned}
|\boldsymbol{J}(q_1, q_2, q_3)|^2_{BC} = {} & C_B^3\, w_{BC}(q_1)\, w_{BC}(q_2)\, w_{BC}(q_3) \\
& + C_B^2\, C_A [w_{BC}(q_1)\, w_{BC}(q_2, q_3) + \text{cyc.perm.}(123)] \\
& + C_B\, C_A^2\, w_{BC}(q_1, q_2, q_3)
\end{aligned}
$$

- Casimir scaling valid up to 3 soft gluons

2 hard partons

- Strong energy ordering with 2 hard gluons: check of *[BCM]* [link a F_{eik}]

$$|\boldsymbol{J}(q_1 \cdots q_N)|^{2,\text{seo}}_{BC} = 2\, C_A^N \, (p_B \cdot p_C)^2 F_{eik}(p_B, p_C, q_1 \cdots q_N) + \mathcal{O}\left(N_c^{N-2}\right)$$

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- If one soft gluon is much harder than the others ($E_N \gg E_1,\cdots,E_{N-1}$)

$$|J(q_1\cdots q_N)|^2_{BC} \simeq |J(q_N)|^2_{BC}\;|J(q_1\cdots q_{N-1})|^2_{ABC}$$

$p_A \equiv q_N$

4 soft gluon sq.current constrained by 3 soft gluon sq.current

2 hard partons, 4 soft gluons

Expansion in irreducible correlations

$$
\begin{aligned}
|\boldsymbol{J}(1,2,3,4)|^2 \overset{\text{cs}}{=} & \; W(q_1) * W(q_2) * W(q_3) * W(q_4) \\
& + [W(q_1,q_2) * W(q_3) * W(q_4) + \text{ineq. perms. } \{1,2,3,4\}] \\
& + [W(q_1,q_2) * W(q_3,q_4) + \text{ ineq. perms. } \{1,2,3,4\}] \\
& + [W(q_1) * W(q_2,q_3,q_4) + \; (1 \leftrightarrow 2) + (1 \leftrightarrow 3) + (1 \leftrightarrow 4)] \\
& + W(q_1,q_2,q_3,q_4)
\end{aligned}
$$

Action on a colour-singlet $|BC\rangle$ state when $E_4 \gg E_{1,2,3}$

$$|\boldsymbol{J}(1,2,3,4)|^2_{BC} = C_B\, w_{BC}(q_4) \left[|\boldsymbol{J}(1,2,3)|^{2,\mathrm{dip}}_{4BC} + \lambda_B N_c\, w^{\mathrm{quad}}_{4BC}(q_1,q_2,q_3) \right]$$

New irreducible correlation: leading + subleading colour

$$W(q_1 \cdots q_4)|_{BC} = C_B \left[C_A^3\, w^{(L)}_{BC}(q_1 \cdots q_4) + \lambda_B N_c\, w^{(S)}_{BC}(q_1 \cdots q_4) \right]$$

2 hard partons, 4 soft gluons, energy ordering

$$W(q_1 \cdots q_4)|_{BC} = C_B \left[C_A^3 \, w_{BC}^{(L)}(q_1 \cdots q_4) + \lambda_B N_c \, w_{BC}^{(S)}(q_1 \cdots q_4) \right]$$

- Exact colour structure for massive hard particles and arbitrary soft energies; Kinematical coeffs. known in energy ordering
- Last term is related to quartic Casimir

$$C_B \, \lambda_B \, N_c = 2 \, \frac{d_{AB}^{(4)}}{D_B} - \frac{1}{12} \, C_B \, C_A^3$$

D_B dimension of B representation

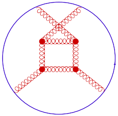
- Generalized Casimir scaling: $C_F \to C_A$, $\quad d_{AF}^{(4)} \to d_{AA}^{(4)}$
- First correction to multi-eikonal BCM formula $\quad \{B, C\} = \{\mathrm{g}, \mathrm{g}\}$

$$\begin{aligned} |\boldsymbol{J}(1, \cdots, 4)|_{BC}^{2,\mathrm{seo}} &= 2 \, C_A^4 (p_B \cdot p_C)^2 \, F_{eik}^{(6)}(p_B, p_C, q_1, q_2, q_3, q_4) \\ &+ 3 \, N_c^2 \, w_{BC}^{(S,\mathrm{seo})}(q_1, q_2, q_3, q_4) \end{aligned}$$

2 hard partons, 4 soft gluons, strong ordering

$$W(q_1 \cdots q_4)|_{BC} = C_B \left[C_A^3 \, w_{BC}^{(L)}(q_1 \cdots q_4) + \lambda_B N_c \, w_{BC}^{(S)}(q_1 \cdots q_4) \right]$$

- *[Dok Kho Mue Tro 91]* examined 4 soft gluon radiation from 2 massless hard partons in strong energy ordering ...
- ... and found a contribution $\sim C_B N_c$ from colour monster



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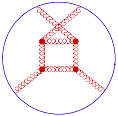

- Our results are fully consistent with the colour monster ...
- ...related to quartic Casimir $C_B \, \lambda_B \, N_c = 2 \, \frac{d_{AB}^{(4)}}{D_B} - \frac{1}{12} \, C_B \, C_A^3$
- $w_{BC}^{(S)}$ has collinear singularities $\Rightarrow \alpha_S^4 \, d_{AB}^{(4)} \, / \, \epsilon$ in coll.ev.kernels contribute to $\Gamma_{\rm cusp}^{(4)}$ term that violates Casimir scaling

Conclusions

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- With two hard partons: extended analysis to 4 soft gluons
  - Presented the full colour structure; kin.coeffs in energy ordering
  - Found col-monster contrib. $(\sim N_c^{-2})$ related to quartic Casimir
  - Generalization of Casimir scaling ($C_F \to C_A$, $d_{AF}^{(4)} \to d_{AA}^{(4)}$)
  - Colour-monster term has collinear singularities and contributes to the cusp anomalous dimension at α_S^4
  - Computed first correction $\mathcal{O}\left(1/N_c^2\right)$ to multi-eikonal BCM