STRONGLY-ORDERED INFRARED LIMITS FROM FACTORISATION

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Outline

- Introduction
- Infrared Counterterms
- Strong Ordering
- Outlook

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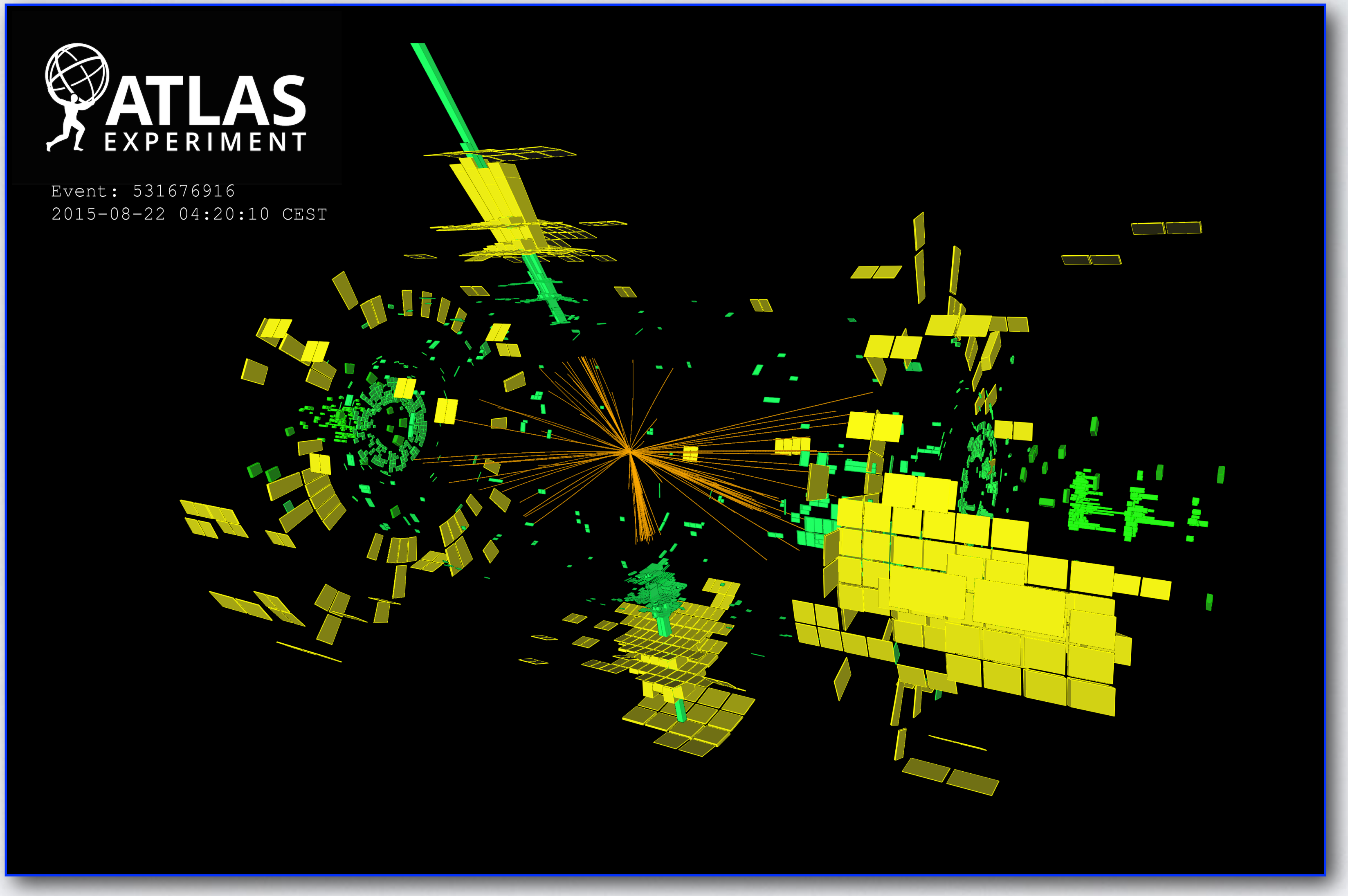
In collaboration with
Calum Milloy
Chiara Signorile-Signorile
Paolo Torrielli
Sandro Uccirati

INTRODUCTION



NLO Subtraction

The computation of a generic IRC-safe observable at NLO requires the combination

$$\frac{d\sigma_{\rm NLO}}{dX} = \lim_{d\to4}\left\{\int d\Phi_n\, V_n\, \delta_n(X) + \int d\Phi_{n+1}\, R_{n+1}\, \delta_{n+1}(X)\right\},$$

The necessary numerical integrations require finite ingredients in d=4. Define counterterms

$$K_{n+1}^{(\mathbf{1})} = \mathbf{L}^{(\mathbf{1})}\, R_{n+1}\,.$$

$$I_n^{(\mathbf{1})} \equiv \int d\Phi_{\mathrm{r},\,1}^{n+1}\, K_{n+1}^{(\mathbf{1})}\,,$$

Add and subtract the same quantity to the observable: each contribution is now finite.

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Search for the simplest fully local integrand K_{n+1} with the correct singular limits.

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Defining $\mathbf{L}^{(1)}$ with sectors

Minimize complexity: split phase space in sectors with sector function $\mathcal{W}_{ij}$ in order to have at most one soft (i) and one collinear (ij) singularity in each sector (FKS).

- Sector functions must form a partition of unity.
- In order not to appear in analytic integrations, sector functions must obey sum rules. Denoting with $\mathbf{S}_i$ the soft limit for parton i and $\mathbf{C}_{ij}$ the collinear limit for the ij pair,

$$\mathbf{S}_i \sum_{k\neq i} \mathcal{W}_{ik} = 1\,, \qquad \mathbf{C}_{ij} \sum_{ab\,\in\,\mathrm{perm}(ij)} \mathcal{W}_{ab} = 1\,, \qquad \longleftarrow \qquad \text{sum rules}$$

- Sector functions are defined in terms of Lorentz invariants before choosing an explicit parametrisation of phase space. A possible choice is

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum\limits_{k,\,l\neq k} \sigma_{kl}}\,, \qquad \text{with} \qquad \sigma_{ij} = \frac{1}{e_i\, w_{ij}}\,, \qquad e_i = \frac{s_{qi}}{s}\,, \qquad w_{ij} = \frac{s\, s_{ij}}{s_{qi}\, s_{qj}}\,.$$

- With the help of sector functions, one can now define a candidate counterterm

$$\mathbf{L}^{(1)} R_{n+1} = \sum_i \sum_{j\neq i} \Big(\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i \mathbf{C}_{ij}\Big) R_{n+1} \mathcal{W}_{ij}\,.$$

See (LM et al. 1809.09570)
and Sandro Uccirati's talk
for NNLO generalisation

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Phase-space mappings at NLO

In order to factorise a Born matrix element B_n with n on-shell particles conserving momentum, we need a mapping from the (n+1)-particle to the Born phase spaces. We use (CS)

$$\bar{k}_i^{(abc)} = k_i, \quad \text{if } i \neq a, b, c,$$

$$\bar{k}_b^{(abc)} = k_a + k_b - \frac{s_{ab}}{s_{ac} + s_{bc}}\, k_c\,, \qquad \bar{k}_c^{(abc)} = \frac{s_{abc}}{s_{ac} + s_{bc}}\, k_c\,,$$

We can now redefine soft and collinear limits to include the re-parametrisation. Explicitly

$$\begin{aligned}
\overline{\mathbf{S}}_i\, R(\{k\}) &= -\mathcal{N}_1 \sum_{l,\, m} \delta_{f_i g}\, \frac{s_{lm}}{s_{il}\, s_{im}}\, B_{lm}\Big(\{\bar{k}\}^{(ilm)}\Big)\,, \\
\overline{\mathbf{C}}_{ij}\, R(k) &= \frac{\mathcal{N}_1}{s_{ij}} \left[P_{ij}\, B\Big(\{\bar{k}\}^{(ijr)}\Big) + Q_{ij}^{\mu\nu}\, B_{\mu\nu}\Big(\{\bar{k}\}^{(ijr)}\Big)\right], \\
\overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, R(\{k\}) &= 2\mathcal{N}_1\, C_{f_j}\, \delta_{f_i g}\, \frac{s_{jr}}{s_{ij}\, s_{ir}}\, B\Big(\{\bar{k}\}^{(ijr)}\Big)\,,
\end{aligned}$$

Note that we have assigned parametrisation triplets differently in different terms. Then

$$\overline{K} = \sum_{i,j\neq i} \overline{K}_{ij}\,, \qquad \overline{K}_{ij} \equiv \left(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\right) R\, \mathcal{W}_{ij}\,,$$

Far from trivial beyond NLO!
Systematics needed.
(Del Duca and Lionetti 1910.01024)

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$$

NNLO Subtraction

The pattern of cancellations is more intricate at higher orders

$$\frac{d\sigma_{\mathrm{NNLO}}}{dX} = \lim_{d\to 4}\left\{\int d\Phi_n\, VV_n\, \delta_n(X) + \int d\Phi_{n+1}\, RV_{n+1}\, \delta_{n+1}(X) + \int d\Phi_{n+2}\, RR_{n+2}\, \delta_{n+2}(X)\right\},$$

More counterterm functions need to be defined

$$K^{(\mathbf{1})}_{n+2} = \mathbf{L}^{(1)}\, RR_{n+2}\,, \qquad K^{(\mathbf{2})}_{n+2} = \mathbf{L}^{(2)}\, RR_{n+2}\,, \qquad K^{(\mathbf{12})}_{n+2} = \mathbf{L}^{(1)}\,\mathbf{L}^{(2)}\, RR_{n+2}\,, \qquad K^{(\mathbf{RV})}_{n+1} = \mathbf{L}^{(1)}\, RV_{n+1}\,.$$

$$I^{(\mathbf{1})}_{n+1} = \int d\Phi^{n+2}_{\mathrm{r},1}\, K^{(\mathbf{1})}_{n+2}\,, \quad I^{(\mathbf{12})}_{n+1} = \int d\Phi^{n+2}_{\mathrm{r},1}\, K^{(\mathbf{12})}_{n+2}\,, \quad I^{(\mathbf{2})}_{n} = \int d\Phi^{n+2}_{\mathrm{r},2}\, K^{(\mathbf{2})}_{n+2}\,, \quad I^{(\mathbf{RV})}_{n} = \int d\Phi^{n+1}_{\mathrm{r},1}\, K^{(\mathbf{RV})}_{n+1}\,.$$

A finite expression for the observable in d=4 must combine several ingredients

$$\begin{aligned}\frac{d\sigma_{\mathrm{NNLO}}}{dX} &= \int d\Phi_n\Big[VV_n + \underline{I^{(\mathbf{2})}_n} + \underline{I^{(\mathbf{RV})}_n}\Big]\,\delta_n(X)\\ &+ \int d\Phi_{n+1}\Big[\Big(RV_{n+1} + \underline{I^{(\mathbf{1})}_{n+1}}\Big)\,\delta_{n+1}(X) - \Big(\underline{K^{(\mathbf{RV})}_{n+1}} + \underline{I^{(\mathbf{12})}_{n+1}}\Big)\,\delta_n(X)\Big]\\ &+ \int d\Phi_{n+2}\Big[RR_{n+2}\,\delta_{n+2}(X) - \underline{K^{(\mathbf{1})}_{n+2}}\,\delta_{n+1}(X) - \Big(\underline{K^{(\mathbf{2})}_{n+2}} - \underline{K^{(\mathbf{12})}_{n+2}}\Big)\,\delta_n(X)\Big].\end{aligned}$$

N^{3}LO Subtraction

A systematic generalisation to higher orders is possible. At three loops one finds

$$
\begin{aligned}
\frac{d\sigma_{\rm N3LO}}{dX} = & \int d\Phi_n \left[VVV_n + I_n^{(\mathbf{3})} + I_n^{(\mathbf{RVV})} + I_n^{(\mathbf{RRV},\,\mathbf{2})} \right] \delta_n(X) \\
& + \int d\Phi_{n+1} \Bigg[\Big(RVV_{n+1} + I_{n+1}^{(\mathbf{2})} + I_{n+1}^{(\mathbf{RRV},\,\mathbf{1})} \Big)\, \delta_{n+1}(X) \\
& \qquad\qquad - \Big(K_{n+1}^{(\mathbf{RVV})} + I_{n+1}^{(\mathbf{23})} + I_{n+1}^{(\mathbf{RRV},\,\mathbf{12})} \Big)\, \delta_n(X) \Bigg] \\
& + \int d\Phi_{n+2} \Bigg\{ \Big(RRV_{n+2} + I_{n+2}^{(\mathbf{1})} \Big)\, \delta_{n+2}(X) - \Big(K_{n+2}^{(\mathbf{RRV,1})} + I_{n+2}^{(\mathbf{12})} \Big)\, \delta_{n+1}(X) \\
& \qquad\qquad - \Bigg[\Big(K_{n+2}^{(\mathbf{RRV},\,\mathbf{2})} + I_{n+2}^{(\mathbf{13})} \Big) - \Big(K_{n+2}^{(\mathbf{RRV,12})} + I_{n+2}^{(\mathbf{123})} \Big) \Bigg]\, \delta_n(X) \Bigg\} \\
& + \int d\Phi_{n+3} \Bigg[RRR_{n+3}\, \delta_{n+3}(X) - K_{n+3}^{(\mathbf{1})}\, \delta_{n+2}(X) - \Big(K_{n+3}^{(\mathbf{2})} - K_{n+3}^{(\mathbf{12})} \Big)\, \delta_{n+1}(X) \\
& \qquad\qquad - \Big(K_{n+3}^{(\mathbf{3})} - K_{n+3}^{(\mathbf{13})} - K_{n+3}^{(\mathbf{23})} + K_{n+3}^{(\mathbf{123})} \Big)\, \delta_n(X) \Bigg] ,
\end{aligned}
$$

A general formula for N^{k}LO subtraction is available, involving $p = 2^{(k+1)} - 2 - k$ counterterms.

See Sandro Uccirati's talk

LASST status

- So far the formalism is developed for massless partons.
- At NLO we have a full-fledged subtraction formalism, and simple integrals.
- NLO numerical implementation is under way.
- At NNLO Local Analytic Subtraction has been achieved for final state radiation.
  - A complete set of NNLO sector functions with the desired sum rules is available.
  - Flexible phase space mappings for single and double unresolved limits exist.
  - Phase space mappings have been checked not to misalign nested limits.
  - All integrals for final state radiation are done analytically, without IBP techniques.
- The numerical implementation at NNLO is the natural next step.
- Generalisation to initial state radiation requires work but no new concepts.
- More `interesting' integrals may arise with massive partons.

INFRARED COUNTERTERMS

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Soft cross sections: pictorial

Consider first the (academic) case of purely soft final state divergences.

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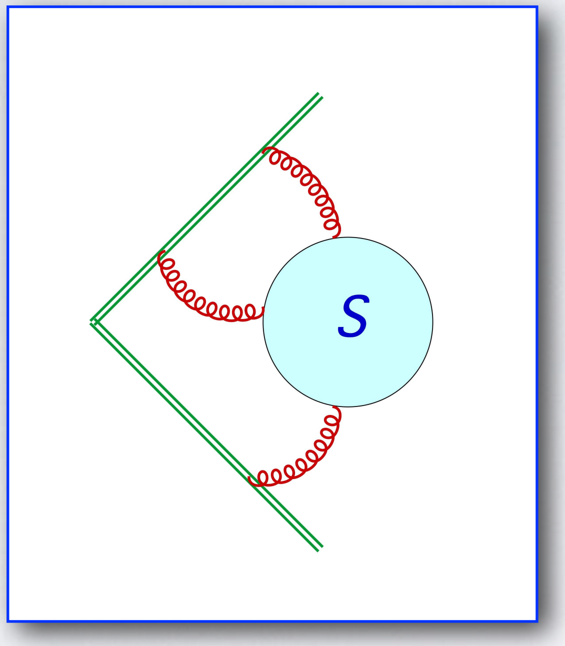
At amplitude level poles factorise and exponentiate.



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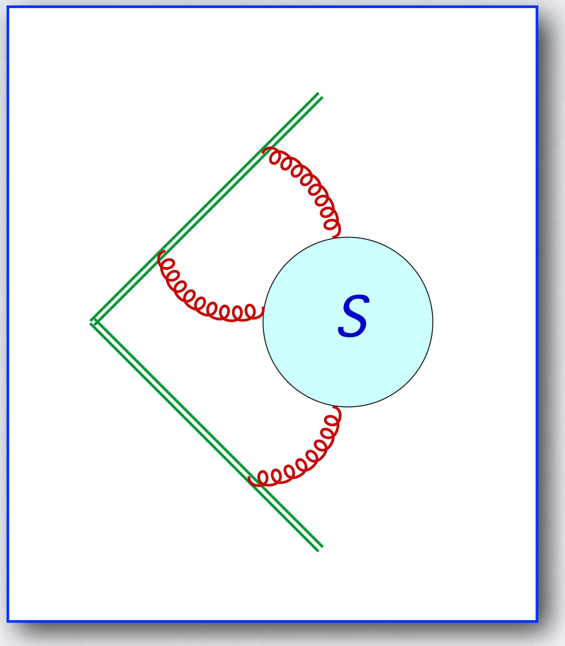
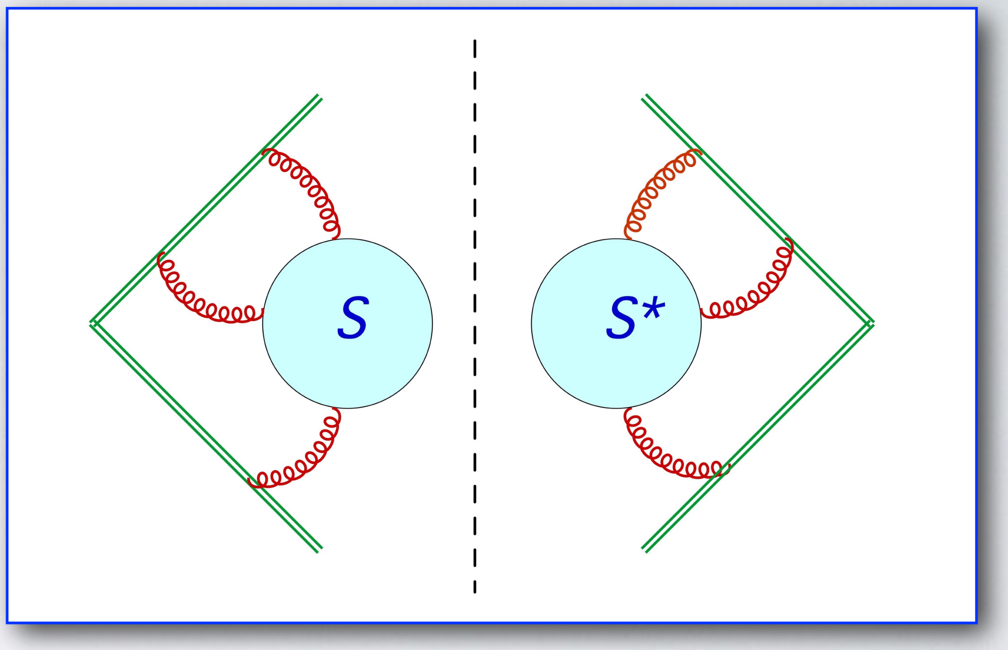


We need to build cross-section level quantities.



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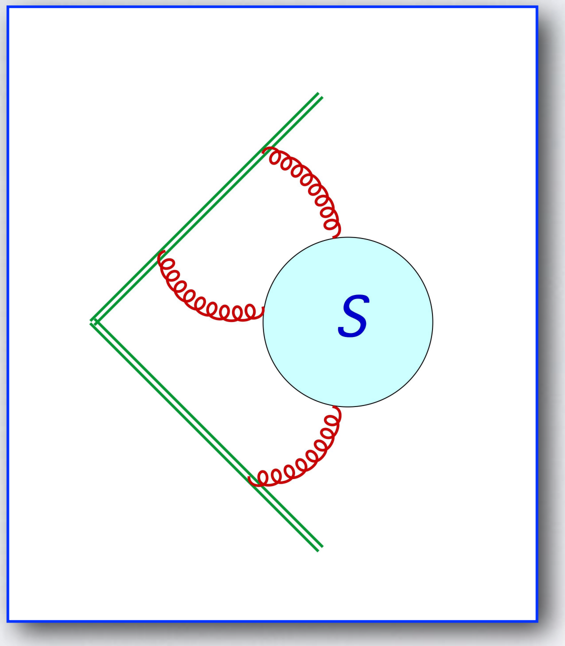

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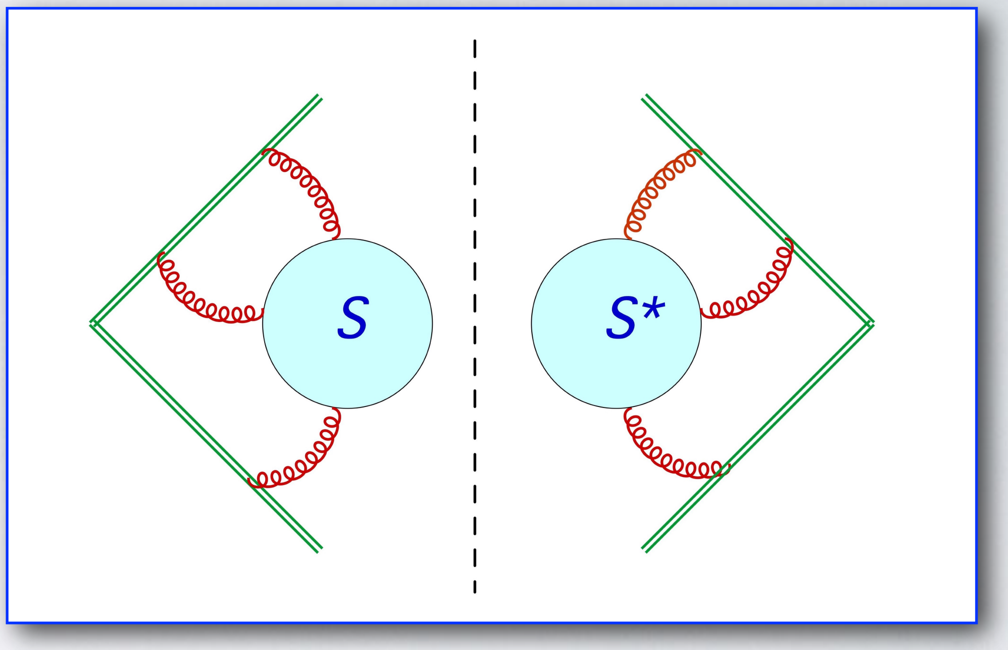
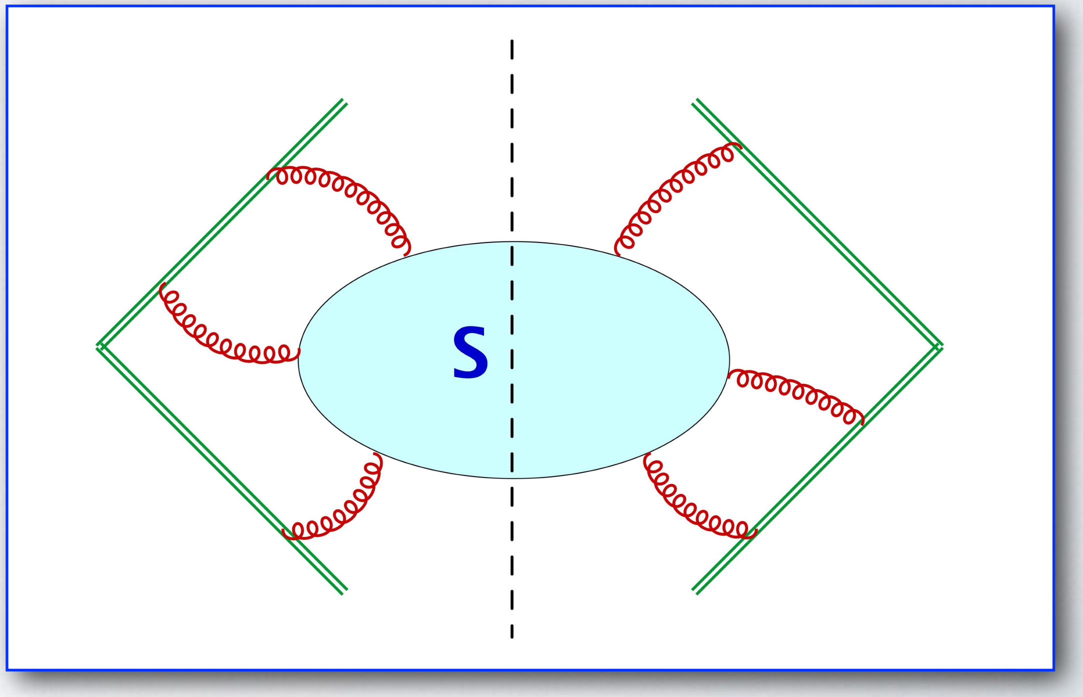


- Inclusive eikonal cross sections are finite.
- They are building blocks for threshold and Q_T resummations.
- They are defined by gauge-invariant operator matrix elements.
- Fixing the quantum numbers of particles crossing the cut one obtains local soft counterterms.



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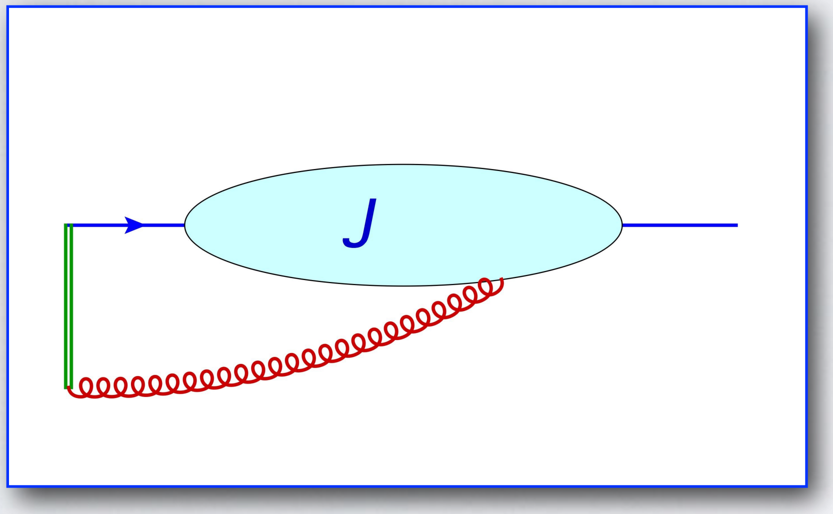

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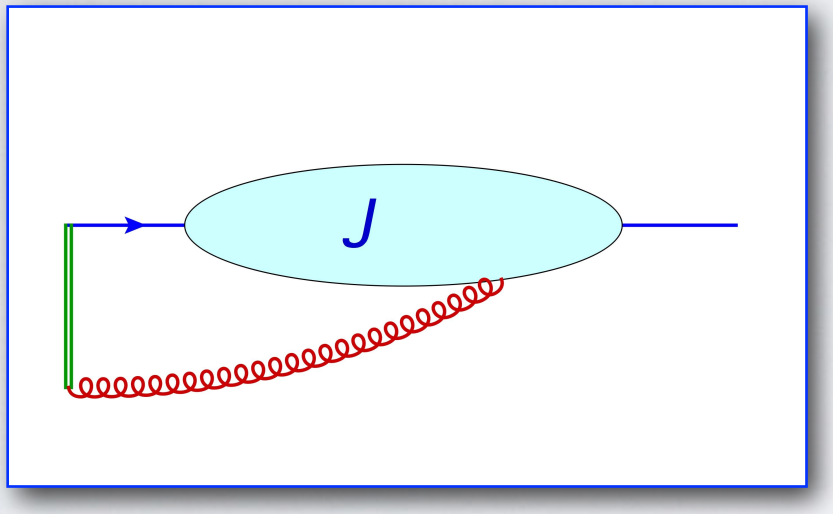
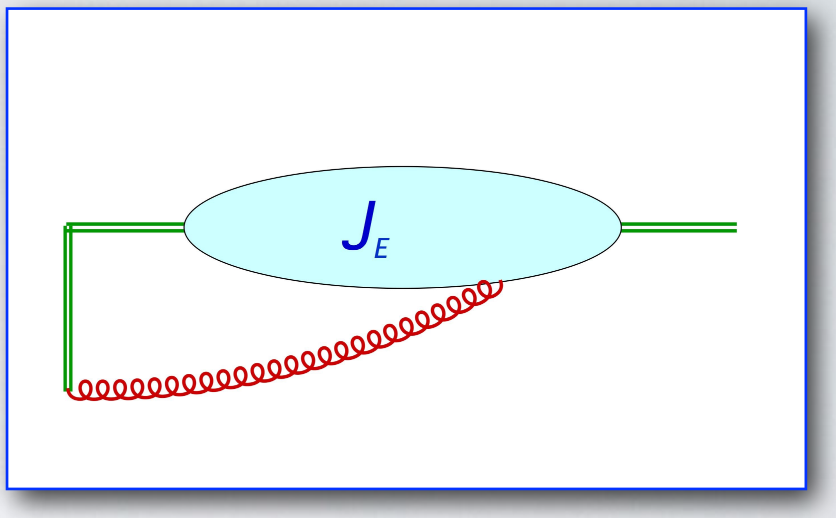


Soft-collinear poles can be subtracted



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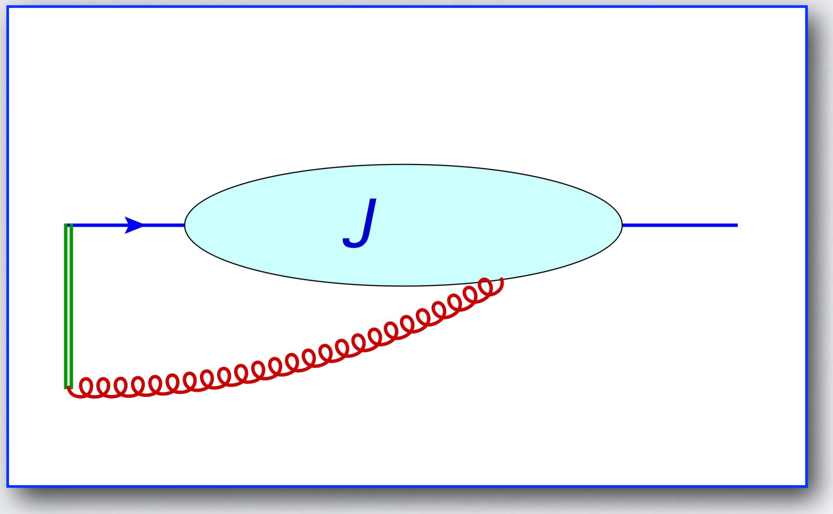

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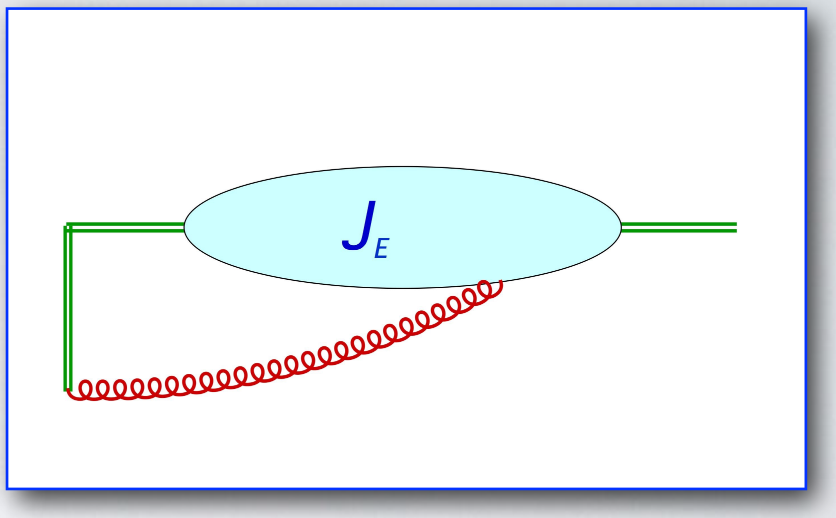
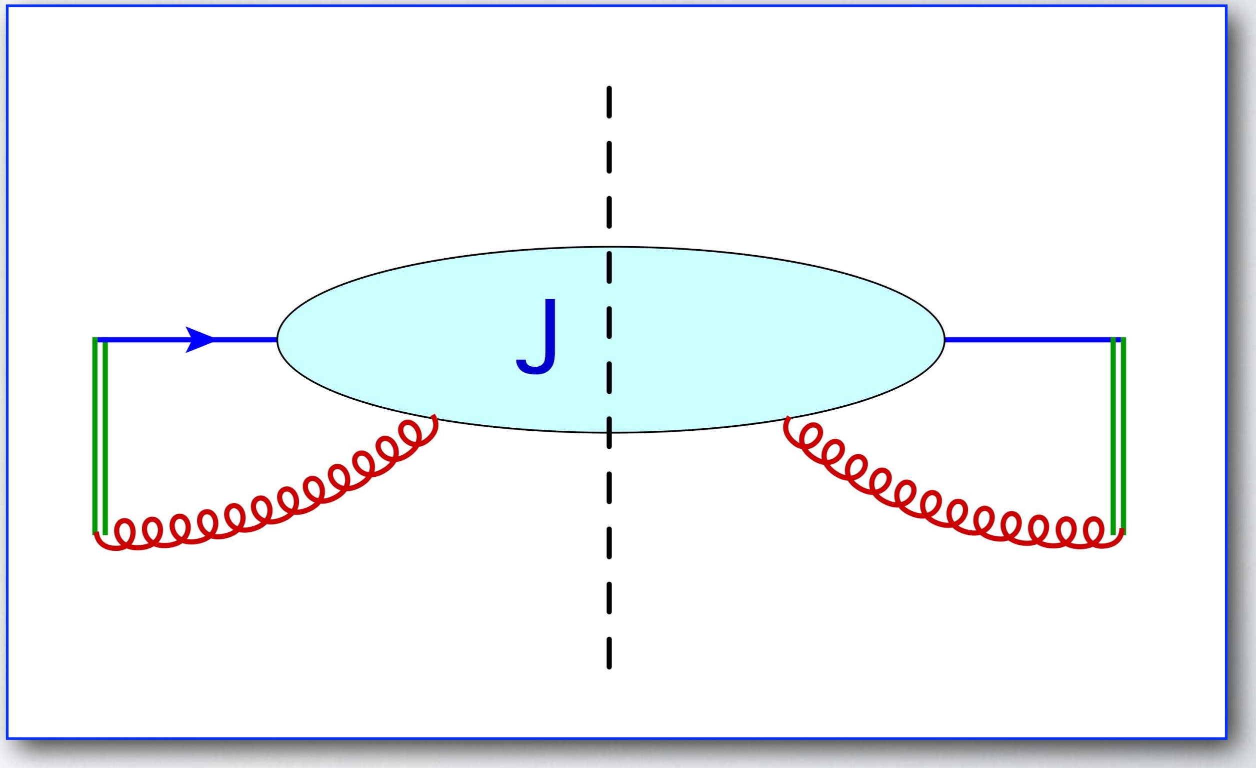


- Inclusive `jet cross sections’ are finite.
- They are building blocks for threshold and Q_T resummations.
- They are defined by gauge-invariant operator matrix elements.
- Fixing the quantum numbers of particles crossing the cut one obtains local collinear counterterms.
- Eikonal jet cross sections subtract the soft-collinear double counting.



See also Feige, Schwartz 2014

Soft counterterms: all orders

Introduce eikonal form factors for the emission of m soft partons from n hard ones.

$$\begin{aligned}\mathcal{S}_{n,\,m}\left(k_1,\ldots,k_m;\beta_i\right) &\equiv \langle k_1,\lambda_1;\ldots;k_m,\lambda_m|\prod_{i=1}^{n}\Phi_{\beta_i}(\infty,0)\ |0\rangle \\ &\equiv \epsilon^{*\,(\lambda_1)}_{\mu_1}(k_1)\ldots\epsilon^{*\,(\lambda_m)}_{\mu_m}(k_m)\,J_{\mathcal{S}}^{\mu_1\ldots\mu_m}\left(k_1,\ldots,k_m;\beta_i\right) \\ &\equiv \sum_{p=0}^{\infty}\mathcal{S}^{(p)}_{n,\,m}\left(k_1,\ldots,k_m;\beta_i\right)\end{aligned}$$

These matrix elements define soft gluon multiple emission currents. They are gauge invariant and they contain loop corrections to all orders.

Existing finite order calculations and all-order arguments are consistent with the factorisation

$$\mathcal{A}_{n,\,m}\left(k_1,\ldots,k_m;p_i\right) = \mathcal{S}_{n,\,m}\left(k_1,\ldots,k_m;\beta_i\right)\mathcal{H}_n(p_i) + \mathcal{R}_{n,\,m}\left(k_1,\ldots,k_m;p_i\right)$$

with corrections that are finite in dimensional regularisation, and integrable in the soft gluon phase space. It is a working assumption: a formal all-order proof is still lacking.

Soft counterterms: all orders

The factorisation is reflected at cross-section level, for fixed final state quantum numbers.

$$\sum_{\lambda_i} \left|\mathcal{A}_{n,m}\left(k_1,\ldots,k_m;p_i\right)\right|^2 \simeq \mathcal{H}_n^\dagger(p_i)\, S_{n,m}\left(k_1,\ldots,k_m;\beta_i\right)\, \mathcal{H}_n(p_i)$$

The *cross-section level* "radiative soft functions" are Wilson-line squared matrix elements

$$\begin{aligned} S_{n,m}\Big(\{k_m\},\{\beta_i\}\Big) &\equiv \sum_{p=0}^{\infty} S^{(p)}_{n,m}\Big(\{k_m\},\{\beta_i\}\Big) \\ &\equiv \sum_{\{\lambda_i\}} \langle 0|\, \overline{T}\left[\prod_{i=1}^{n}\Phi_{\beta_i}(0,\infty)\right] |k_1,\lambda_1;\ldots;k_m,\lambda_m\rangle \langle k_1,\lambda_1;\ldots;k_m,\lambda_m|\, T\left[\prod_{i=1}^{n}\Phi_{\beta_i}(\infty,0)\right] |0\rangle\,, \end{aligned}$$

These functions provide a complete list of local soft subtraction counterterms, to all orders. Indeed, summing over particle numbers and integrating over the soft phase space one finds

$$\sum_{m=0}^{\infty}\int d\Phi_m\, S_{n,m}\Big(\{k_m\};\{\beta_i\}\Big) = \langle 0|\, \overline{T}\left[\prod_{i=1}^{n}\Phi_{\beta_i}(0,\infty)\right] T\left[\prod_{i=1}^{n}\Phi_{\beta_i}(\infty,0)\right] |0\rangle\,.$$

"Completeness relation"

This is a finite fully inclusive soft cross section, order by order in perturbation theory.

Collinear counterterms: all orders

For collinear poles, introduce jet matrix elements for the emission of m partons. For quarks

$$\overline{u}_s(p)\,\mathcal{J}_{q,\,m}\,(k_1,\ldots,k_m;p,n)\;\equiv\;\langle p,s;k_1,\lambda_1;\ldots;k_m,\lambda_m|\,\overline{\psi}(0)\,\Phi_n(0,\infty)\,|0\rangle$$

At *cross-section level*, "radiative jet functions" can be defined as Fourier transforms of squared matrix elements, to account for the non-trivial momentum flow. We propose

$$\begin{aligned} J_{q,\,m}(\{k_m\};l,p,n)\;&\equiv\;\sum_{p=0}^{\infty}J^{(p)}_{q,\,m}(\{k_m\};l,p,n)\\ \equiv\int d^dx\,\mathrm{e}^{\mathrm{i}l\cdot x}\sum_{\{\lambda_m\}}&\langle 0|\,\overline{T}\Big[\Phi_n(\infty,x)\,\psi(x)\Big]\,|p,s;\{k_m,\lambda_m\}\rangle\,\langle p,s;\{k_m,\lambda_m\}|\,T\Big[\overline{\psi}(0)\,\Phi_n(0,\infty)\Big]\,|0\rangle\;, \end{aligned}$$

These functions provide a complete list of local collinear counterterms, to all orders. Summing over particle numbers and integrating over the collinear phase space one finds

$$\sum_{m=0}^{\infty}\int d\Phi_{m+1}\,J_{q,\,m}(\{k_m\};l,p,n)\;=\;\mathrm{Disc}\Bigg[\int d^dx\,\mathrm{e}^{\mathrm{i}l\cdot x}\;\langle 0|\,T\Big[\Phi_n(\infty,x)\psi(x)\overline{\psi}(0)\Phi_n(0,\infty)\Big]\,|0\rangle\Bigg]\,.$$

A "two-point function", finite order by order in perturbation theory. Note however

- The collinear limit must still be taken (as $l^2 \to 0$), unlike the case of radiative soft functions.
- $n^2 \neq 0$ avoids spurious collinear poles, but is cumbersome → use SCET-like anti-collinear n^μ.

NLO subtraction

The outlines of a subtraction procedure emerge. Begin by expanding the virtual matrix element

$$\mathcal{A}_n(p_i) = \Big[\mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(1)}(p_i) + \sum_{i=1}^{n}\Big(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i)\Big)\,\mathcal{S}_n^{(0)}(\beta_i)\,\mathcal{H}_n^{(0)}(p_i)\Big]\Big(1 + \mathcal{O}\left(\alpha_s^2\right)\Big)$$

From the master formula, get the virtual poles of the cross section in terms of virtual kernels

$$V_n \equiv 2\,\mathbf{Re}\Big[\mathcal{A}_n^{(0)*}\mathcal{A}_n^{(1)}\Big] \simeq \mathcal{H}_n^{(0)\,\dagger}(p_i)S_{n,\,0}^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \sum_i\Big(J_{i,\,0}^{(1)}(p_i) - J_{\mathrm{E},\,i,\,0}^{(1)}(\beta_i)\Big)\Big|\mathcal{A}_n^{(0)}(p_i)\Big|^2$$

Go through the list of proposed soft and collinear counterterms to collect the relevant ones

$$S_{n,0}^{(1)}(\beta_i) + \int d\Phi_1\, S_{n,\,1}^{(0)}(k,\beta_i) = \mathrm{finite}$$

$$J_{i,\,0}^{(1)}(l,p,n) + \int d\Phi_1\, J_{i,\,1}^{(0)}(k;l,p,n) = \mathrm{finite}$$

Construct the appropriate local functions.

$$K_{n+1}^{\mathrm{NLO,\,s}} = \mathcal{H}_n^{(0)\,\dagger}(p_i)\,S_{n,\,1}^{(0)}(k,\beta_i)\,\mathcal{H}_n^{(0)}(p_i)$$

$$K_{n+1}^{\mathrm{NLO,\,c}} = \sum_{i=1}^{n} J_{i,\,1}^{(0)}(k_i;l,p_i,n_i)\,\Big|\mathcal{A}_n^{(0)}\left(p_1,\ldots,p_{i-1},l,p_{i+1},\ldots,p_n\right)\Big|^2$$

with a similar expression for the anti-subtraction of the soft-collinear region in terms of $J_{\mathcal{E}}$.

A “top-down” approach

$$\mathcal{A}_n(p_i) = \prod_{i=1}^{n} \left[\frac{\mathcal{J}_i(p_i, n_i)}{\mathcal{J}_{E,i}(\beta_i, n_i)}\right] \mathcal{S}_n(\beta_j)\, \mathcal{H}_n(p_i)$$

NLO subtraction

The outlines of a subtraction procedure emerge. Begin by expanding the virtual matrix element

$$\mathcal{A}_n(p_i) = \Big[\mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(1)}(p_i) + \sum_{i=1}^{n} \Big(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i)\Big)\, \mathcal{S}_n^{(0)}(\beta_i)\, \mathcal{H}_n^{(0)}(p_i)\Big] \Big(1 + \mathcal{O}\left(\alpha_s^2\right)\Big)$$

From the master formula, get the virtual poles of the cross section in terms of virtual kernels

$$V_n \equiv 2\,\mathbf{Re}\Big[\mathcal{A}_n^{(0)*}\mathcal{A}_n^{(1)}\Big] \simeq \mathcal{H}_n^{(0)\,\dagger}(p_i) S_{n,\,0}^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \sum_i \Big(J_{i,\,0}^{(1)}(p_i) - J_{\mathrm{E},\,i,\,0}^{(1)}(\beta_i)\Big) \Big|\mathcal{A}_n^{(0)}(p_i)\Big|^2$$

Go through the list of proposed soft and collinear counterterms to collect the relevant ones

$$S_{n,0}^{(1)}(\beta_i) + \int d\Phi_1\, S_{n,\,1}^{(0)}(k, \beta_i) = \text{finite}$$

$$J_{i,\,0}^{(1)}(l,p,n) + \int d\Phi_1\, J_{i,\,1}^{(0)}(k; l, p, n) = \text{finite}$$

Construct the appropriate local functions.

$$K_{n+1}^{\mathrm{NLO,\,s}} = \mathcal{H}_n^{(0)\,\dagger}(p_i)\, S_{n,\,1}^{(0)}(k, \beta_i)\, \mathcal{H}_n^{(0)}(p_i)$$

$$K_{n+1}^{\mathrm{NLO,\,c}} = \sum_{i=1}^{n} J_{i,\,1}^{(0)}(k_i; l, p_i, n_i) \left|\mathcal{A}_n^{(0)}\left(p_1, \ldots, p_{i-1}, l, p_{i+1}, \ldots, p_n\right)\right|^2$$

with a similar expression for the anti-subtraction of the soft-collinear region in terms of $J_{\mathcal{E}}$.

STRONG ORDERING

Soft refactorisation: tree level

The tree-level double soft-gluon current simplifies considerably in the strong-ordering limit

$$\left[J^{(0),\,\mathrm{s.o.}}_{\mathrm{CG}}\right]^{a_1 a_2}_{\mu_1 \mu_2}(k_1, k_2; \beta_i) \, = \, \left(J^{(0)\,a_2}_{\mu_2}(k_2)\,\delta^{a_1 a} + \mathrm{i} g_s\, f^{a_1 a_2 a}\, \frac{k_{1,\,\mu_2}}{k_1 \cdot k_2} \right) J^{(0)}_{\mu_1,\,a}(k_1)\,,$$

$$J^{(0)\,a}_{\mu}(k) \, = \, g_s \sum_{i=1}^{n} \frac{\beta_{i,\,\mu}}{\beta_i \cdot k}\, T_i^a$$

One may define a strongly-ordered soft form factor by contracting with physical polarisations

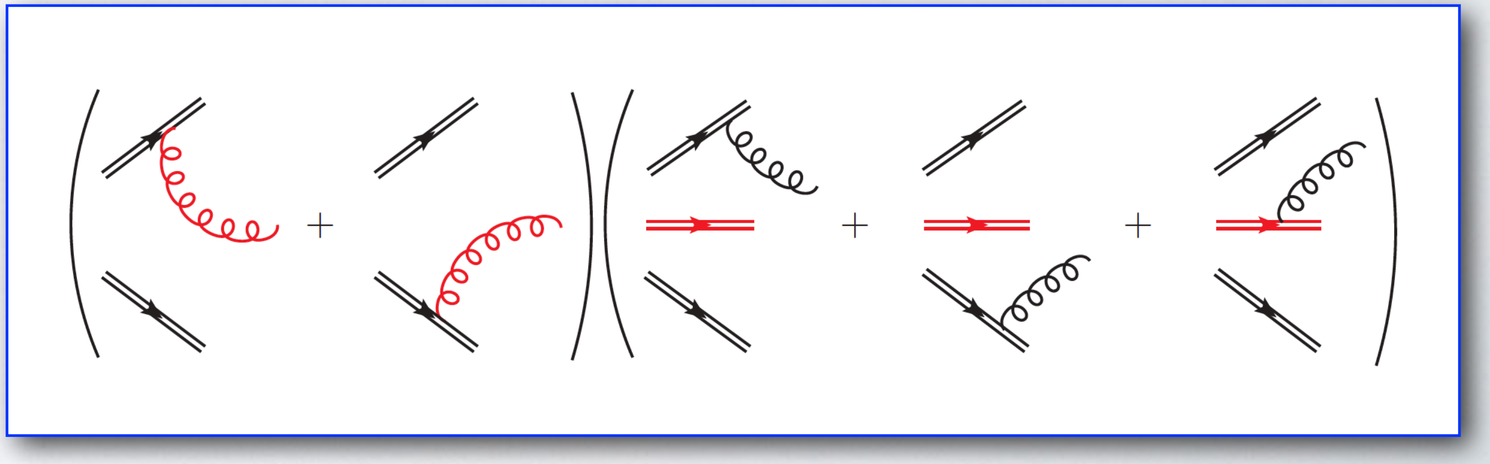
$$\left[\mathcal{S}^{(0)}_{n;\,1,\,1}\right]^{a_1 a_2}(k_1, k_2; \beta_i) = \epsilon^{*\,\mu_1}(k_1)\,\epsilon^{*\,\mu_2}(k_2)\, \left[J^{(0),\,\mathrm{s.o.}}_{\mathrm{CG}}\right]^{a_1 a_2}_{\mu_1 \mu_2}(k_1, k_2; \beta_i)\,.$$

The form factor is given by an interesting "re-factorisation" of the double-radiative soft function

$$\begin{aligned} \left[\mathcal{S}^{(0)}_{n;\,1,\,1}\right]^{a_1 a_2}_{\{d_i e_i\}}(k_1, k_2; \beta_i) \quad &\equiv \quad \langle k_2, a_2|\; \Phi_{\beta_{k_1}}^{\;a_1 b}(0,\infty) \prod_{i=1}^{n} \Phi_{\beta_i,\,d_i}^{\quad\;\; c_i}(0,\infty)\; |0\rangle \\ &\qquad \times\, \langle k_1, b|\, \prod_{i=1}^{n} \Phi_{\beta_i,\,c_i e_i}(0,\infty)\, |0\rangle \Big|_{\mathrm{tree}} \\ &= \quad \left[\mathcal{S}^{(0)}_{n+1,1}\right]^{a_2,\,a_1 b}_{\{d_i c_i\}}(k_2; \beta_{k_1}, \beta_i)\; \left[\mathcal{S}^{(0)}_{n,1}\right]_{b,\,\{c_i e_i\}}(k_1; \beta_i)\,, \end{aligned}$$

Notice the non-trivial colour structure: the product is ordered.

The original system of n Wilson lines radiates the harder gluon, which then "Wilsonises". The augmented system of (n+1) Wilson lines radiates the softer gluon



Soft refactorisation: tree level

This framework generalises to arbitrary patterns of strong ordering for multiple soft radiation at tree level. For example for strongly-ordered triple radiation one can define

$$
\begin{aligned}
\left[\mathcal{S}^{(0)}_{n;1,1,1}\right]^{a_1 a_2 a_3}_{\{f_i e_i\}}(k_1, k_2, k_3; \beta_i) \; & \equiv \; \left[\mathcal{S}^{(0)}_{n+2,1}\right]^{a_3}_{\{f_i d_i\},\, a_1 b_1,\, a_2 b_2} \left[\mathcal{S}^{(0)}_{n+1,1}\right]^{b_2}_{\{d_i c_i\},\, b_1 g_1} \left[\mathcal{S}^{(0)}_{n,1}\right]^{g_1}_{\{c_i e_i\}} \\
& = \; \langle k_3, a_3 |\, \Phi^{a_1 b_1}_{\beta_{k_1}}(0,\infty)\, \Phi^{a_2 b_2}_{\beta_{k_2}}(0,\infty) \prod_{i=1}^{n} \Phi^{f_i d_i}_{\beta_i}(0,\infty) | 0 \rangle \\
& \quad \times \langle k_2, b_2 |\, \Phi^{b_1 g_1}_{\beta_{k_1}}(0,\infty) \prod_{i=1}^{n} \Phi^{d_i c_i}_{\beta_i}(0,\infty) | 0 \rangle \\
& \quad \times \langle k_1, g_1 | \prod_{i=1}^{n} \Phi^{c_i e_i}_{\beta_i}(0,\infty) | 0 \rangle \Big|_{\rm tree} \, ,
\end{aligned}
$$

Computing the form factors, one reproduces the strongly-ordered limit of (Catani et al. 2019).

$$
\begin{aligned}
\left[\mathcal{S}^{(0)}_{n;1,1,1}\right]^{a_1 a_2 a_3} \; = \; & \epsilon^*_{\mu_3}(k_3)\, \epsilon^*_{\mu_2}(k_2)\, \epsilon^*_{\mu_1}(k_1) \\
& \times \left[J^{\mu_3}_{a_3}(k_3)\, \delta^{a_1 b_1}\, \delta^{a_2 b_2} + {\rm i} g_s\, f^{a_1 a_3 b_1}\, \delta^{a_2 b_2} \frac{k_1^{\mu_3}}{k_1 \cdot k_3} + {\rm i} g_s\, f^{a_2 a_3 b_2}\, \delta^{a_1 b_1} \frac{k_2^{\mu_3}}{k_2 \cdot k_3} \right] \\
& \times \left[J^{\mu_2}_{b_2}(k_2)\, \delta^{b_1 c_1} + {\rm i} g_s\, f^{b_1 b_2 c_1} \frac{k_1^{\mu_2}}{k_1 \cdot k_2} \right] J^{\mu_1}_{c_1}(k_1) \, ,
\end{aligned}
$$

See Dimitri Colferai's talk

- Generalising to strongly-ordered soft radiation of m gluons is natural (and tested for m=3).
- Similar definitions hold for soft form factors for multiple ordered subsets of several gluons.
- Preliminary evidence suggests that similar soft re-factorisations may hold to higher orders.

Strongly-ordered soft counterterms

The top-down approach suggests an expression for the soft real-virtual counterterm

$$K_{n+1}^{(\mathbf{RV}),\,\mathrm{s}} \,=\, \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,1}^{(1)}\, \mathcal{H}_n^{(0)} \,+\, \text{finite}$$

Collinear poles?

The refactorisation of strongly-ordered soft radiation suggests an expression for the soft K$^{(12)}$

$$\begin{aligned} K_{n+2}^{(\mathbf{12}),\,\mathrm{s}} \;&=\; \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,1,1}^{(0)}\, \mathcal{H}_n^{(0)} \\ &=\; \mathcal{H}_n^{(0)\,\dagger} \left[\mathcal{S}_{n,\,1}^{b,\,(0)}(\beta_i;k_1)\right]^\dagger \left[\mathcal{S}_{n+1,\,1}^{a_2,\,a_1 b\,(0)}(\beta_i,\beta_{k_1};k_2)\right]^\dagger \mathcal{S}_{n+1,\,1}^{a_2,\,a_1 c,(0)}(\beta_i,\beta_{k_1};k_2)\, \mathcal{S}_{n,\,1}^{c,\,(0)}(\beta_i;k_1)\, \mathcal{H}_n^{(0)} \\ &\equiv\; \mathcal{H}_n^{(0)\,\dagger} \left[\mathcal{S}_{n,\,1}^{b}(\beta_i;k_1)\right]^\dagger S_{n+1,\,1}^{bc,(0)}(\beta_i,\beta_{k_1};k_2)\, \mathcal{S}_{n,\,1}^{c}(\beta_i;k_1)\, \mathcal{H}_n^{(0)} \end{aligned}$$

One can now use the finiteness of inclusive soft cross sections to cancel soft poles arising from the phase-space integration of K$^{(12)}$, using

$$S_{n+1,\,0}^{bc,(1)}(\beta_i,\beta_{k_1}) \,+\, \int d\Phi_1(k_2)\, S_{n+1,\,1}^{bc,(0)}(\beta_i,\beta_{k_1};k_2) \,=\, \text{finite}$$

"Completeness relation"

This gives a new expression for the real-virtual soft counterterm

$$K_{n+1}^{(\mathbf{RV}),\,\mathrm{s}} \,=\, \mathcal{H}_n^{(0)\,\dagger} \left[\mathcal{S}_{n,\,1}^{b,\,(0)}(\beta_i;k_1)\right]^\dagger S_{n+1,\,0}^{bc,(1)}(\beta_i,\beta_{k_1})\, \mathcal{S}_{n,\,1}^{c,\,(0)}(\beta_i;k_1)\, \mathcal{H}_n^{(0)} \,+\, \text{finite}$$

A "bottom-up" approach

The two definitions have identical soft poles, which was checked with a non-trivial calculation.

A top-down approach

This result is better understood by taking more seriously the idea of refactorisation

- The radiative soft function is not a pure counterterm: it has IR poles and finite contributions.
- It can be considered as an amplitude in the presence of sources: virtual IR poles will factorise.

Applying the standard soft-jet-hard factorisation for scattering amplitudes we write

$$\mathcal{A}_n(p_i) = \prod_{i=1}^{n}\left[\frac{\mathcal{J}_i(p_i,n_i)}{\mathcal{J}_{E,i}(\beta_i,n_i)}\right]\mathcal{S}_n(\beta_j)\,\mathcal{H}_n(p_i) \quad\to\quad \mathcal{S}_{n,1}(k;\beta_i) = \frac{\mathcal{J}_g(k,n)}{\mathcal{J}_{E,g}(\beta_k,n)}\,\mathcal{S}_{n+1,0}(\beta_k,\beta_i)\,\mathcal{S}^{\rm fin}_{n,1}(k;\beta_i)$$

Expanding to one-loop order, the terms containing IR poles are

$$\mathcal{S}^{(1)}_{n,1}(k;\beta_i) = \underline{\mathcal{S}^{(1)}_{n+1,0}(\beta_k,\beta_i)\,\mathcal{S}^{(0)}_{n,1}(k;\beta_i)} + \underline{\Big(\mathcal{J}^{(1)}_g(k,n) - \mathcal{J}^{(1)}_{E,g}(\beta_k,n)\Big)}\;\mathcal{S}^{(0)}_{n,1}(k;\beta_i)$$

We recognise (upon squaring) the soft contribution to $K^{(RV)}$, plus hard collinear corrections.

This can be explicitly checked against the general expression for the soft limit of RV

$$S^{(1)}_{n,1}(k;\beta_i) = \mathbf{S}_k RV - \frac{\alpha_{\rm S}^2\mu^{2\epsilon}}{S_\epsilon}\sum_{i>j}^{n}\frac{\beta_i\cdot\beta_j}{\beta_i\cdot k\,\beta_j\cdot k}\,\mathbf{T}_i\cdot\mathbf{T}_j\left[\sum_{m=1}^{n}\frac{\gamma_m^{(1)}}{\epsilon}+\frac{b_0}{2\epsilon}\right]$$

To match the two calculations, one must subtract the hard-collinear poles of the virtual part.

See Oscar Braun-White's and
Prasanna Kumar Dhani's talks

Collinear refactorisation

The top-down approach suggests an expression for the collinear real-virtual counterterm

$$K_{n+1}^{(\mathbf{RV}),\,\mathrm{c},\,i} \;=\; \mathcal{H}_n^{(0)\,\dagger}\, J_{i,\,1}^{(1)}\, \mathcal{H}_n^{(0)} \;+\; \text{finite}$$

Soft poles?

In the bottom-up approach one starts with strongly-ordered collinear kernels, for example

$$\lim_{\theta_{12} \ll \theta_{13} \ll 1} RR_{n+2} \;=\; \frac{\mathcal{N}^2}{s_{12}\, s_{[12]3}}\, P_{gq}^{\alpha\beta}\big(z_{[12]}, q_\perp\big)\, d_{\alpha\mu}\big(k_{[12]}, n\big)\, P_{q\bar{q}}^{\mu\nu}\left(\frac{z_1}{z_{[12]}}, k_\perp\right) d_{\nu\beta}\big(k_{[12]}, n\big)\,,$$

$q \to q\, q\, \bar{q}$

This can be directly translated in the language of jet functions. At cross-section level

$$J_{q,1,1}^{(0)}\left(k_1, k_2; k_3, n\right)\Big|_{gg,\,\mathrm{ab.}} \;=\; J_{q,1}^{(0)}\left(k_1; k_{[23]}, n\right)\, J_{q,1}^{(0)}\left(k_2; k_3, n\right)\,,$$

$q \to q\, g\, g$, abelian

One can now use the finiteness of inclusive collinear cross sections to cancel collinear poles arising from the phase-space integration of $K^{(12)}$, using

$$K_{n+2}^{(\mathbf{12}),\,\mathrm{c},\,q} \;=\; \mathcal{H}_n^{(0)\,\dagger}\, J_{q,1,1}^{(0)}\, \mathcal{H}_n^{(0)}\,; \qquad J_{q,0}^{(1)}\big(k_{[23]}, n\big) + \int d\Phi_1(k_3)\, J_{q,1}^{(0)}\big(k_2; k_3, n\big) \;=\; \text{finite}$$

"Completeness
relation"

This gives a new expression for the real-virtual collinear counterterm

$$K_{n+1}^{(\mathbf{RV}),\,\mathrm{c},\,q} \;=\; \mathcal{H}_n^{(0)\,\dagger}\, J_{q,1}^{(0)}\big(k_1; k_{[23]}, n\big)\, J_{q,0}^{(1)}\big(k_{[23]}, n\big)\, \mathcal{H}_n^{(0)} \;+\; \text{finite}$$

Bottom-up
approach

The two definitions have identical collinear poles, which again calls for an explanation.

Preliminary

Top-down collinear

Once again, the result is better understood by means of a refactorisation of the radiative jet

- The radiative jet function has both UV and IR poles, as well as phase-space singularities.
- As before, it is an amplitude in the presence of sources: virtual IR poles will factorise.

Applying the standard soft-jet-hard factorisation for amplitudes we write

$$\mathcal{J}_{f,1}(k;p,n) \, = \left[\frac{\mathcal{J}(k,n_k)}{\mathcal{J}_E(\beta_k,n_k)} \, \frac{\mathcal{J}(p,n_p)}{\mathcal{J}_E(\beta_p,n_p)} \right] \mathcal{S}_3(\beta_k,\beta_p,\beta_n) \, \mathcal{J}^{\text{fin}}_{f,1}(k,p,n)$$

Expanding to one-loop order, the terms containing IR poles are

$$\mathcal{J}^{(1)}_{f,1}(k;p,n) \, = \left[\mathcal{J}^{(1)}(k,n_k) - \mathcal{J}^{(1)}_E(\beta_k,n_k) + \mathcal{J}^{(1)}(p,n_p) - \mathcal{J}^{(1)}_E(\beta_p,n_p) + \mathcal{S}^{(1)}_3(\beta_k,\beta_p,\beta_n) \right] \mathcal{J}^{\text{fin},\,0}_{f,1}(k,p,n)$$

One reconstructs (upon squaring) the collinear contribution to $K^{(RV)}$, plus soft corrections.

- For gg radiation, hard collinear terms are identical and phase space provides a factor 1/2.
- The three-point soft function does not affect collinear factorisation: it simplifies to a singlet quantity when the collinear limit is taken.

Detailed checks against collinear limits of RR_{n+2} require implementing phase-space mappings.

OUTLOOK

Outlook

- Infrared subtraction beyond NLO requires understanding all strongly-ordered IR limits.
- Factorisation provides definitions for local soft and collinear counterterms to all orders.
- Soft and collinear kernels are expressed by matrix elements of fields and Wilson lines.
- In strongly ordered limits the kernels refactorise into lower-order matrix elements.
- Known strongly ordered IR limits at NNLO and N3LO are reproduced by factorisation.
- "Completeness relations" link strongly-ordered kernels and real-virtual counterterms.
- Upon implementing phase-space mappings, the cancellation of RV poles can be checked.
- The refactorisation approach to strong-ordering generalises smoothly to higher orders.
- The architecture of infrared subtraction is becoming clear to all orders.

THANK YOU