Renormalization of non-singlet quark OMEs for off-forward hard scattering

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Loops and Legs in Quantum Field Theory 2022

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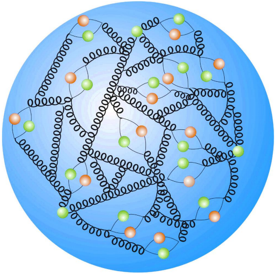
Introduction

- Important question: How do hadronic properties emerge from the properties of the constituent partons?
- Experimentally: Perform scattering experiments, like DIS or DVCS, that can resolve the inner hadron structure
- Theoretically: Study matrix elements of composite operators
- Related to distributions like PDFs (forward kinematics) and GPDs (off-forward kinematics)



Composite operators and distributions

Distributions are universal quantities related to hadronic matrix elements of composite operators

$$\langle p^+(p_1)|\mathcal{O}(p_3)|p^+(p_2)\rangle$$

During this talk: Focus on leading-twist flavor-non-singlet quark operators

$$\mathcal{O} = \mathcal{S}\overline{\psi}\lambda^{\alpha}\Gamma D_{\nu_1}\dots D_{\nu_N}\psi$$

with $D_\mu = \partial_\mu - ig_s A_\mu$ the covariant derivative.

Different operators/physics depending on Dirac structure Γ

- Wilson operators:

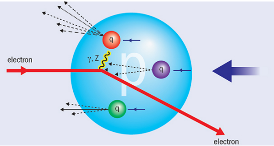
$$\mathcal{O}^W_{\mu_1\dots\mu_N} = \overline{\psi}\gamma_{\mu_1}D_{\mu_2}\dots D_{\mu_N}\psi$$

- Transversity operators:

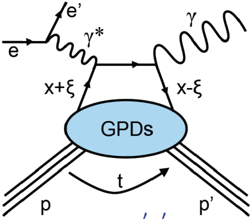
$$\mathcal{O}^T_{\mu\nu_1\dots\nu_N} = \overline{\psi}\sigma_{\mu\nu_1}D_{\nu_2}\dots D_{\nu_N}\psi$$

Matrix elements of Wilson operators

- Parton distributions (PDF): Longitudinal momentum/polarization carried by partons within fast-moving hadrons, accessible e.g. in inclusive DIS $e\,p \to e\,X$



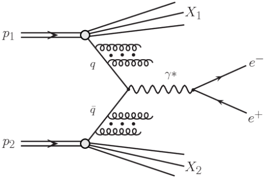
- Generalized parton distributions (GPDs): Transverse distributions of partons + contributions partonic orbital angular momentum to total hadronic spin, accessible e.g. in DVCS $e\,p \to e\,p\,\gamma$



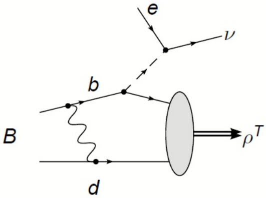
$ep \to e'p'\gamma$

Matrix elements of transversity operators

- Transversity distributions: Difference in probabilities of finding a parton in a transversly polarized nucleon polarized parallel to the nucleon spin and an oppositely polarized one, accessible e.g. in polarized Drell-Yan



- Transverse distribution amplitudes, accessible e.g. in ρ-meson production



Matrix elements of quark operators

As the matrix elements for the distributions are defined with respect to hadronic states, they cannot be calculated in perturbation theory

$\Rightarrow$ Direct extraction from experimental data (see e.g. [Brock et al., 1995]) or using lattice QCD (see e.g. [Alexandrou et al., 2020], [Ji et al., 2021], [Wang et al., 2021] for PDFs/GPDs and [Alexandrou et al., 2022a], [Scapellato et al., 2022], [Alexandrou et al., 2022b] for transversity distributions)

What about the scale-dependence of the distributions?

Scale-dependence of distributions

Because of the close connection between distributions and composite operators, their scale-dependence is determined by the operator anomalous dimensions, which can be calculated perturbatively in QCD

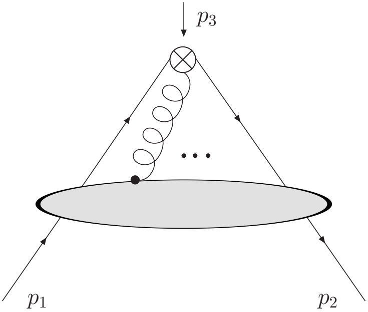
$$\frac{\mathrm{d}[\mathcal{O}]}{\mathrm{d}\ln\mu^2} = \gamma[\mathcal{O}], \quad \gamma \equiv a_s\gamma^{(0)} + a_s^2\gamma^{(1)} + \dots \text{ and } a_s \equiv \alpha_s/(4\pi)$$

Hence it is important to understand the renormalization properties of the leading-twist operators in (off-)forward kinematics

In practice, the anomalous dimensions are extracted by renormalizing partonic matrix elements of the operators

$$\langle\psi(p_1)|\mathcal{O}(p_3)|\overline{\psi}(p_2)\rangle$$

Operator anomalous dimensions



In practice consider $\mathcal{O}_N^W \equiv \overline{\psi}\not{\Delta}(\Delta \cdot D)^{N-1}\psi$ for the Wilson operators and $\mathcal{O}_N^T \equiv \overline{\psi}\frac{1}{2}[\gamma_\mu, \Delta](\Delta \cdot D)^{N-1}\psi$ for the transversity ones ($\Delta^2 = 0$)

The diagrams can be calculated automatically using e.g. ***Forcer*** [Ruijl et al., 2020] in ***Form*** [Vermaseren, 2000],[Kuipers et al., 2013] or ***Package-X*** [Patel, 2015] in ***Mathematica*** (use dimensional regularization and keep gauge dependence)

Operator renormalization

In forward kinematics, the operators renormalize multiplicatively

$$\mathcal{O}_{N+1} = Z_{N,N}[\mathcal{O}_{N+1}].$$

The anomalous dimensions are extracted from the Z-factors as

$$\gamma_{N,N} = -\frac{1}{Z_{N,N}} \frac{\mathrm{d} Z_{N,N}}{\mathrm{d} \ln \mu^2}.$$

The anomalous dimensions are related to the splitting functions by a Mellin transformation

$$\gamma_{NS}(N) \equiv \gamma_{N-1,N-1} = -\int_0^1 \mathrm{d}x \, x^{N-1} P_{NS}(x)$$

which determine the scale dependence of the forward distributions through the DGLAP equation [Gribov and Lipatov, 1972], [Altarelli and Parisi, 1977], [Dokshitzer, 1977]

$$\frac{\mathrm{d} f_{NS}(x, \mu^2)}{\mathrm{d} \ln \mu^2} = \int_x^1 \frac{dy}{y} P_{NS}(y) f_{NS}\left(\frac{x}{y}, \mu^2\right).$$

Operator renormalization

The Wilson forward anomalous dimensions are known completely up to the 3-loop level, and in certain limits up to the 5-loop level [Gross and Wilczek, 1973], [Floratos et al., 1977], [Gracey, 1994], [Moch et al., 2004], [Velizhanin, 2012a], [Ruijl et al., 2016], [Moch et al., 2017], [Herzog et al., 2019], [Velizhanin, 2020], [Blümlein et al., 2021]

The transversity forward anomalous dimensions are known to 3-loop order [Shifman and Vysotsky, 1981], [Baldracchini et al., 1981],[Artru and Mekhfi, 1990], [Hayashigaki et al., 1997], [Kumano and Miyama, 1997], [Vogelsang, 1998], [Blumlein, 2001], [Gracey, 2003], [Ablinger et al., 2011], [Velizhanin, 2012b], [Blümlein et al., 2021]

In the leading-n_f limit ($n_f \to \infty$) the all-order expression of the anomalous dimensions is also known ([Gracey, 1994] for Wilson and [Gracey, 2003] for transversity)

Operator renormalization

In off-forward kinematics, life is more difficult, since the operators will mix with total derivative operators under renormalization.

$$\mathcal{O}_{N+1} = Z_{N,N}[\mathcal{O}_{N+1}] + Z_{N,N-1}[\partial\mathcal{O}_N] + \cdots + Z_{N,0}[\partial^N\mathcal{O}_1]$$

For the full spin-(N+1) sector we then get the mixing matrix:

$$\begin{pmatrix} \mathcal{O}_{N+1} \\ \partial\mathcal{O}_N \\ \vdots \\ \partial^N\mathcal{O}_1 \end{pmatrix} = \begin{pmatrix} Z_{N,N} & Z_{N,N-1} & \dots & Z_{N,0} \\ 0 & Z_{N-1,N-1} & \dots & Z_{N-1,0} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & Z_{0,0} \end{pmatrix} \begin{pmatrix} [\mathcal{O}_{N+1}] \\ [\partial\mathcal{O}_N] \\ \vdots \\ [\partial^N\mathcal{O}_1] \end{pmatrix}$$

For a detailed study, one has to choose a basis for such operators. Several options exist in the literature, we will focus on two.

The Gegenbauer basis

In this basis the operators are expanded in terms of Gegenbauer polynomials

$$\mathcal{O}^{\mathcal{G}}_{N,k} = (\Delta\cdot\partial)^k \overline{\psi}(x)(\Delta\cdot\Gamma)C_N^{3/2}\left(\frac{\overleftarrow{D}\cdot\Delta - \Delta\cdot\overrightarrow{D}}{\overleftarrow{\partial}\cdot\Delta + \Delta\cdot\overrightarrow{\partial}}\right)\psi(x)$$

with [Olver et al., 2010]

$$C_N^{\nu}(z) = \frac{\Gamma(\nu+1/2)}{\Gamma(2\nu)}\sum_{l=0}^{N}(-1)^l\binom{N}{l}\frac{(N+l+2)!}{(l+1)!}\left(\frac{1}{2}-\frac{z}{2}\right)^l.$$

- This choice of operator basis is particularly useful when there is conformal symmetry, see [Efremov and Radyushkin, 1980a], [Belitsky and Müller, 1999], [Braun et al., 2017]
- The anomalous dimensions of the local Wilson operators in this basis are known up to the 3-loop level [Braun et al., 2017]. Evolution kernels for transversity are known to 2-loop order [Belitsky et al., 2000],[Mikhailov and Vladimirov, 2009]

The total derivative basis

In this approach we identify operators by counting powers of derivatives

$$\mathcal{O}^{\mathcal{D}}_{p,q,r} = (\Delta \cdot \partial)^{p} \Big\{ (\Delta \cdot D)^{q} \overline{\psi} (\Delta \cdot \Gamma)(\Delta \cdot D)^{r} \psi \Big\}$$

We now make the assumption of working in the chiral limit. This has 2 nice consequences:

- Total derivatives act as

$$\mathcal{O}^{\mathcal{D}}_{p,q,r} \ = \ \mathcal{O}^{\mathcal{D}}_{p-1,q+1,r} + \mathcal{O}^{\mathcal{D}}_{p-1,q,r+1},$$

- Left- and right-derivative operators renormalize with the same renormalization constants

$$\mathcal{O}^{\mathcal{D}}_{k,0,N} = \sum_{j=0}^{N} Z^{\mathcal{D}}_{N,N-j} \, [\mathcal{O}^{\mathcal{D}}_{k+j,0,N-j}] \, ,$$

$$\mathcal{O}^{\mathcal{D}}_{k,N,0} = \sum_{j=0}^{N} Z^{\mathcal{D}}_{N,N-j} \, [\mathcal{O}^{\mathcal{D}}_{k+j,N-j,0}]$$

The total derivative basis

- Since

$$\gamma_{N,k}^{\mathcal{D}} = -(Z_{N,j}^{\mathcal{D}})^{-1} \frac{\mathrm{d}\, Z_{j,k}^{\mathcal{D}}}{\mathrm{d} \ln \mu^2},$$

  the last property implies that left- and right-derivative operators have the same anomalous dimensions
- This choice of operator basis is useful in connecting continuum quantities to lattice ones for non-perturbative studies, see e.g. [Göckeler et al., 2005] and [Gracey, 2009]
- In this basis, the Wilson anomalous dimensions for low-N operators are known up to the 3-loop level (see [Gracey, 2009] for analytical results and [Kniehl and Veretin, 2020] for a numerical extension of these). For the transversity operators, the 1-loop anomalous dimensions are known [Shifman and Vysotsky, 1981], [Baldracchini et al., 1981], [Artru and Mekhfi, 1990], [Blumlein, 2001]

Relating the total derivative and Gegenbauer bases

We have derived a relation between the operators in the Gegenbauer basis and the operators in the total derivative basis

$$\mathcal{O}^{\mathcal{G}}_{N,k} = \frac{1}{2N!} \sum_{l=0}^{N} (-1)^l \binom{N}{l} \frac{(N+l+2)!}{(l+1)!} \mathcal{O}^{\mathcal{D}}_{k-l,0,l} \,.$$

Writing the corresponding relation for the renormalized operators, this in turn relates the anomalous dimensions in both bases to each other

$$\sum_{j=0}^{N} (-1)^j \frac{(j+2)!}{j!} \gamma^{\mathcal{G}}_{N,j} = \frac{1}{N!} \sum_{j=0}^{N} (-1)^j \binom{N}{j} \frac{(N+j+2)!}{(j+1)!} \sum_{l=0}^{j} \gamma^{\mathcal{D}}_{j,l}.$$

Constraints on the anomalous dimensions

Using $\mathcal{O}^{\mathcal{D}}_{p,q,r} = \mathcal{O}^{\mathcal{D}}_{p-1,q+1,r} + \mathcal{O}^{\mathcal{D}}_{p-1,q,r+1}$ we can now derive a relation between bare operators

$$\mathcal{O}^{\mathcal{D}}_{0,N,0} - (-1)^N \sum_{j=0}^{N} (-1)^j \binom{N}{j} \mathcal{O}^{\mathcal{D}}_{j,0,N-j} = 0.$$

For renormalized operators

$$\mathcal{O}^{\mathcal{D}}_{k,0,N} = \sum_{j=0}^{N} Z^{\mathcal{D}}_{N,N-j} \, [\mathcal{O}^{\mathcal{D}}_{k+j,0,N-j}] \,,$$

$$\mathcal{O}^{\mathcal{D}}_{k,N,0} = \sum_{j=0}^{N} Z^{\mathcal{D}}_{N,N-j} \, [\mathcal{O}^{\mathcal{D}}_{k+j,N-j,0}]$$

this leads to a relation between the renormalization factors, and hence the operator anomalous dimensions, which has to be valid to all orders in a_s.

Constraints on the anomalous dimensions

$$\gamma^{\mathcal{D}}_{N,k} = \binom{N}{k} \sum_{j=0}^{N-k} (-1)^j \binom{N-k}{j} \gamma_{j+k,\,j+k} + \sum_{j=k}^{N} (-1)^k \binom{j}{k} \sum_{l=j+1}^{N} (-1)^l \binom{N}{l} \gamma^{\mathcal{D}}_{l,j}$$

e.g.

$$\gamma^{\mathcal{D}}_{2,1} = 2(\gamma_{1,1} - \gamma_{2,2}) - \gamma^{\mathcal{D}}_{2,1}.$$

- ✓ Order-independent consistency check
- ✓ Can be used to construct the full mixing matrix from the knowledge of the forward anomalous dimensions $\gamma_{N,N}$ + some boundary condition to ensure uniqueness of the solution ($\gamma^{\mathcal{D}}_{N,0}$, from Feynman diagrams)

4-step algorithm for constructing the ADM

1. Determine all-N expression for $\gamma^{\mathcal{D}}_{N,0}$ from Feynman diagrams
2. Calculate

$$\binom{N}{k}\sum_{j=0}^{N-k}(-1)^j\binom{N-k}{j}\gamma_{j+k,\,j+k}$$

and construct an Ansatz for the off-diagonal piece

3. Calculate

$$\sum_{j=k}^{N}(-1)^k\binom{j}{k}\sum_{l=j+1}^{N}(-1)^l\binom{N}{l}\gamma^{\mathcal{D}}_{l,j}$$

4. Substitute into the consistency relation $\Rightarrow$ System of equations + boundary condition for $k=0$

Non-trivial sums can be performed using algorithms of symbolic summation, nicely implemented e.g. in the ***Mathematica*** package ***Sigma*** [Schneider, 2004].

Results and discussion: Total derivative basis

- Wilson anomalous dimensions up to 5-loop order in the leading-n_f approximation. See [Moch and Van Thurenhout, 2021] for the explicit expressions.

$$
\begin{aligned}
\gamma_{N,k}^{\mathcal{D},(4)} = \; & \frac{16}{81} n_f^4 C_F \Bigg\{ \frac{1}{12}\Big(S_1(N)-S_1(k)\Big)^4 \Big(\frac{1}{N+2}-\frac{1}{N-k}\Big) \\
& +\frac{1}{3}\Big(S_1(N)-S_1(k)\Big)^3\Big(\frac{5}{3}\frac{1}{N-k}+\frac{2}{N+1}-\frac{11}{3}\frac{1}{N+2}+\frac{1}{(N+2)^2}\Big) \\
& +\frac{1}{2}\Big(S_1(N)-S_1(k)\Big)^2\Big(S_2(N)-S_2(k)\Big)\Big(\frac{1}{N+2}-\frac{1}{N-k}\Big) \\
& +\Big(S_1(N)-S_1(k)\Big)^2\Big(\frac{1}{3}\frac{1}{N-k}-\frac{13}{3}\frac{1}{N+1}+\frac{2}{(N+1)^2}+\frac{4}{N+2} \\
& -\frac{11}{3}\frac{1}{(N+2)^2}+\frac{1}{(N+2)^3}\Big) \\
& +\Big(S_1(N)-S_1(k)\Big)\Big(S_2(N)-S_2(k)\Big)\Big(\frac{5}{3}\frac{1}{N-k}+\frac{2}{N+1}-\frac{11}{3}\frac{1}{N+2}+\frac{1}{(N+2)^2}\Big) \\
& +\frac{2}{3}\Big(S_1(N)-S_1(k)\Big)\Big(S_3(N)-S_3(k)\Big)\Big(\frac{1}{N+2}-\frac{1}{N-k}\Big) \\
& +\Big(S_1(N)-S_1(k)\Big)\Big(\frac{2}{3}\frac{1}{N-k}+\frac{2}{N+1}-\frac{26}{3}\frac{1}{(N+1)^2}+\frac{4}{(N+1)^3}-\frac{8}{3}\frac{1}{N+2} \\
& +\frac{8}{(N+2)^2}-\frac{22}{3}\frac{1}{(N+2)^3}+\frac{2}{(N+2)^4}\Big)+\frac{1}{4}\Big(S_2(N)-S_2(k)\Big)^2\Big(\frac{1}{N+2}-\frac{1}{N-k}\Big) \\
& +\Big(S_2(N)-S_2(k)\Big)\Big(\frac{1}{3}\frac{1}{N-k}-\frac{13}{3}\frac{1}{N+1}+\frac{2}{(N+1)^2}+\frac{4}{N+2} \\
& -\frac{11}{3}\frac{1}{(N+2)^2}+\frac{1}{(N+2)^3}\Big)+\frac{2}{3}\Big(S_3(N)-S_3(k)\Big)\Big(\frac{5}{3}\frac{1}{N-k}+\frac{2}{N+1} \\
& -\frac{11}{3}\frac{1}{N+2}+\frac{1}{(N+2)^2}\Big)+\frac{1}{2}\Big(S_4(N)-S_4(k)\Big)\Big(\frac{1}{N+2}-\frac{1}{N-k}\Big)+\frac{2}{3}\frac{1}{N-k} \\
& -\frac{2}{3}\frac{1}{N+1}+\frac{2}{(N+1)^2}-\frac{26}{3}\frac{1}{(N+1)^3}+\frac{4}{(N+1)^4}-\frac{8}{3}\frac{1}{(N+2)^2}+\frac{8}{(N+2)^3} \\
& -\frac{22}{3}\frac{1}{(N+2)^4}+\frac{2}{(N+2)^5}\Bigg\}.
\end{aligned}
$$

Results and discussion: Total derivative basis

- Transversity anomalous dimensions up to 4-loop order in the leading-n_f approximation. See [Van Thurenhout, 2022] for the explicit expressions.

$$
\begin{aligned}
\gamma_{N,k}^{\mathcal{D},T,(3)} = &\frac{16}{27} n_f^3 C_F \Bigg\{ \frac{1}{(N+1)^4} - \frac{5}{3}\frac{1}{(N+1)^3} + [S_1(N) - S_1(k)] \Bigg(\frac{1}{(N+1)^3} - \frac{5}{3}\frac{1}{(N+1)^2} \\
&- \frac{4}{3}\frac{1}{N+1} + \frac{1}{N+2} + \frac{1}{3}\frac{1}{N-k} \Bigg) \\
&+ \Big([S_1(N) - S_1(k)]^2 + S_2(N) - S_2(k) \Big) \Bigg(\frac{1}{2}\frac{1}{(N+1)^2} - \frac{5}{6}\frac{1}{N+1} + \frac{5}{6}\frac{1}{N-k} \Bigg) \\
&+ \Bigg(\frac{1}{N+1} - \frac{1}{N-k} \Bigg) \Bigg(\frac{1}{6}[S_1(N) - S_1(k)]^3 + \frac{1}{2}[S_2(N) - S_2(k)][S_1(N) - S_1(k)] \\
&+ \frac{1}{3}[S_3(N) - S_3(k)] \Bigg) - \frac{4}{3}\frac{1}{(N+1)^2} + \frac{4}{3}\frac{1}{N+1} + \frac{1}{(N+2)^2} - \frac{5}{3}\frac{1}{N+2} \\
&+ \frac{1}{3}\frac{1}{N-k} \Bigg\}
\end{aligned}
$$

Results and discussion: Gegenbauer basis

- By using a basis transformation between the Gegenbauer and total derivative bases, we have an independent check of previous calculations, e.g. [Braun et al., 2017].
- We can also extend these previous calculations to the 4-loop level in the leading-n_f approximation.
- See [Moch and Van Thurenhout, 2021],[Van Thurenhout, 2022] for more details.

Conclusions and outlook

- New method for calculating off-diagonal elements of the mixing matrix of quark operators including total derivative ones, based purely on renormalization structure in chiral limit
- Independent check of previous calculations, both in the derivative and in Gegenbauer bases
- New results, e.g. 5-loop Wilson and 4-loop transversity anomalous dimensions in leading-n_f limit
- Possible to go beyond the leading-n_f limit, see [Moch and Van Thurenhout, 2021] for more details
- Nice advantage of our method: Well-suited for automation with computer algebra programs
- To do: Generalize method to (polarized) flavor singlet operators in QCD

Thank you for your attention!

Backup and references

1. Feynman diagrams and $\gamma_{N,0}$
2. ERBL evolution
3. ADM in the Geyer basis
4. Five-loop ADM in the leading-n_f limit
5. Calculating sums
6. Feynman rules for operator insertions
7. Some comments on *Forcer*
8. References

Feynman diagrams and $\gamma_{N,0}$

The renormalization pattern for the right-derivative operators is

$$\mathcal{O}_{N+1} = Z_\psi \left(Z_{N,N}[\mathcal{O}_{N+1}] + Z_{N,N-1}[\partial \mathcal{O}_N] + \cdots + Z_{N,0}[\partial^N \mathcal{O}_1] \right)$$

Here, Z_ψ represents the quark wave function renormalization factor, necessary to take into account the self-energy corrections for the off-shell quarks.

From this, it is clear that there is a direct relation between the bare OMEs and the sum of anomalous dimensions

$$\mathcal{O}_{N+1}/\epsilon \sim \sum_{i=0}^{N} \gamma_{N,i}^{\mathcal{D}} \equiv \mathcal{B}(N+1).$$

E.g. at 1-loop we simply have

$$\frac{\mathcal{O}_{N+1}^{(0)}}{\epsilon} = \gamma_\psi^{(0)} + \sum_{i=0}^{N} \gamma_{N,i}^{\mathcal{D},(0)}.$$

Feynman diagrams and $\gamma_{N,0}$

Furthermore, from the general consistency relation for $k = 0$ we can derive a recursion-type relation for the last column

$$\gamma^{\mathcal{D}}_{N,0} = (-)^N \left[\sum_{i=0}^{N} \gamma^{\mathcal{D}}_{N,i} - \sum_{j=1}^{N-1} (-)^j \binom{N}{j} \gamma^{\mathcal{D}}_{j,0} \right]$$

Hence, it follows that there is a direct relation between the bare OMEs and the last column of the mixing matrix

$$\gamma^{\mathcal{D}}_{N,0} = \sum_{i=0}^{N} (-1)^i \binom{N}{i} \mathcal{B}(i+1)$$

ERBL evolution

In momentum x-space, the scale-dependence of distribution amplitudes is determined by the ERBL evolution equations

[Efremov and Radyushkin, 1980a],[Efremov and Radyushkin, 1980b],[Lepage and Brodsky, 1979], [Lepage and Brodsky, 1980], which can be regarded to be the generalization of DGLAP to exclusive processes

$$\frac{\mathrm{d}\phi(x,\mu^2)}{\mathrm{d}\ln\mu^2} = \int_0^1 \mathrm{d}y\, V(x,y)\phi(y,\mu^2).$$

Here $\phi(x,\mu^2)$ is the distribution amplitude and $V(x,y)$ is the evolution kernel, which, similarly to the DGLAP case, is related to a double Mellin transform of the anomalous dimensions.

ADM in the Geyer basis

Another basis for local Wilson operators exists, namely

$$\mathcal{O}^{\mathcal{B}}_{N,k} \equiv \overline{\psi}\gamma(\overrightarrow{D} + \overleftarrow{D})^{N-k}(\overleftarrow{D} - \overrightarrow{D})^{k}\psi,$$

see e.g. [Geyer, 1982] and [Blümlein et al., 1999]. The contraction with an arbitrary light-like vector is understood, i.e.

$$\gamma \equiv \Delta^{\mu}\,\gamma_{\mu},$$
$$D \equiv \Delta^{\mu}\,D_{\mu}$$

and $\Delta^2 = 0$. These operators, and correspondingly their anomalous dimensions, can be related to those in the derivative basis. For this the relation

$$\mathcal{O}^{\mathcal{D}}_{0,N-k,k} = (-1)^{k}\sum_{j=0}^{k}\binom{k}{j}\mathcal{O}^{\mathcal{D}}_{j,N-j,0}$$

for the operators in the derivative basis is useful.

ADM in the Geyer basis

For the bare operators we then find

$$\mathcal{O}^{\mathcal{B}}_{N,k} = \sum_{i=0}^{N-k}\sum_{j=0}^{k}(-1)^i \binom{N-k}{i}\binom{k}{j}\sum_{l=0}^{i+j}(-1)^l\binom{i+j}{l}\mathcal{O}^{\mathcal{D}}_{l,N-l,0}$$

which leads to

$$[\mathcal{O}^{\mathcal{B}}_{N,k}] = \sum_{i=0}^{N-k}\sum_{j=0}^{k}(-1)^i \binom{N-k}{i}\binom{k}{j}\sum_{l=0}^{i+j}(-1)^l\binom{i+j}{l}\sum_{m=0}^{N-l}(-1)^m\,\gamma^{\mathcal{D}}_{N-l,m} \times \sum_{n=0}^{m}(-1)^n\binom{m}{n}[\mathcal{O}^{\mathcal{D}}_{N-m+n,0,m-n}]$$

for the renormalized ones.

ADM in the Geyer basis

The evolution of the local operators in this basis can be summarized as

$$\mu^2 \frac{d}{d\mu^2} \mathcal{O}^{\mathcal{B}}_{N,N} = \sum_{k=0}^{N} \frac{1 \pm (-1)^k}{2} \gamma^{\mathcal{B}}_{N,k} [\mathcal{O}^{\mathcal{B}}_{k,N}].$$

The relation between the anomalous dimensions in the Geyer and derivative bases then becomes

$$\gamma^{(0)}_{N,N} + \sum_{j=0}^{N-1} \frac{1 \pm (-1)^j}{2} \gamma^{\mathcal{B},(0)}_{N,j} = \pm(-1)^N \sum_{l=0}^{N} 2^l (-1)^l \binom{N}{l} \gamma^{\mathcal{D},(0)}_{l,0}$$

at the 1-loop level with the upper (lower) sign for even (odd) N. This provides an additional consistency check at this order.

Five-loop ADM in the leading-n_f limit

- By analyzing the leading-n_f anomalous dimensions up to 4 loops, it becomes clear that certain patterns start to emerge.
- The majority of terms in the L-loop anomalous dimensions can be deduced from the expression of the $(L-1)$-loop ones.
- What is left then is a small number of unknown terms, which can be fixed by using our consistency relation.
- This is how we determined the expression for the 5-loop Wilson anomalous dimensions. Note that this in principle can be extended to arbitrary orders.
- This method is also used to determine the leading-n_f mixing matrices for the transversity operators.

Calculating sums

Using our constructive algorithm, we have to evaluate single and double sums involving harmonic sums and denominators.

This can be done by using the principles of symbolic summation, see e.g. [Graham et al., 1989] or [Kauers and Paule, 2011] for extensive overviews.

Particularly helpful for these purposes is the Mathematica package *Sigma* [Schneider, 2004], which uses the creative telescoping algorithm.

Calculating sums

Telescoping: Suppose we have a summation $\sum_{k=a}^{N} f(k)$
$\rightarrow$ Find function $g(N)$ such that the summand can be written as

$$f(k) = \Delta g(k) \equiv g(k+1) - g(k)$$

$$\Rightarrow \sum_{k=a}^{N} f(k) = \sum_{k=a}^{N} g(k+1) - \sum_{k=a}^{N} g(k)$$
$$= g(N+1) - g(a)$$

$\sim$ discrete version of symbolic integration:

$$f(x) = Dg(x) \equiv \frac{d}{dx} g(x) \Rightarrow \int_a^b f(x)dx = g(b) - g(a)$$

Calculating sums

Creative telescoping [Zeilberger, 1991]: Suppose we have the summation $\sum_{k=a}^{b} f(n,k) \equiv S(n)$
$\Rightarrow$ Find functions $c_0(n), ..., c_d(n)$ and $g(n,k)$ such that

$$g(n,k+1) - g(n,k) = c_0(n) f(n,k) + ... + c_d(n) f(n+d,k)$$

Now apply summation on both sides of the equation

$$\Rightarrow g(n,b+1) - g(n,a) = c_0(n) \sum_{k=a}^{b} f(n,k) + ... + c_d(n) \sum_{k=a}^{b} f(n+d,k)$$

$\Rightarrow$ Inhomogeneous recurrence for original sum

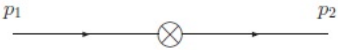
$$q(n) = c_0(n) S(n) + ... + c_d(n) S(n+d)$$

Calculating sums

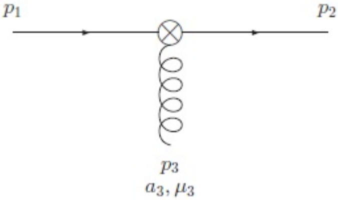
- In this way, *Sigma* generates and solves recurrence for given summation problem
- Solution consists of solution set for homogeneous recurrence + particular solution
- For final closed expression of summation: Determine linear combination of solutions that has same initial values as the given sum

Feynman rules for operator insertions

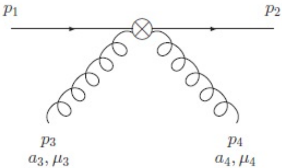
Contract OMEs with $\Delta_{\mu_1}...\Delta_{\mu_N}$, $\Delta^2 = 0$, see e.g. [Moch et al., 2017]



$$\not\Delta\,(\Delta\cdot p_2)^{N-1}$$



$$-g t^{a_3}\not\Delta\Delta^{\mu_3}\sum_{j_1=0}^{N-2}(p_2\cdot\Delta)^{N-2-j_1}(q\cdot\Delta-p_1\cdot\Delta)^{j_1}$$



$$g^2\not\Delta\Delta^{\mu_3}\Delta^{\mu_4}\sum_{j_1=0}^{N-3}\sum_{j_2=0}^{j_1}(p_2\cdot\Delta)^{N-3-j_1}(q\cdot\Delta-p_1\cdot\Delta)^{j_2}$$
$$\Big(t^{a_3}t^{a_4}(q\cdot\Delta-p_1\cdot\Delta-p_3\cdot\Delta)^{j_1-j_2}$$
$$+t^{a_4}t^{a_3}(q\cdot\Delta-p_1\cdot\Delta-p_4\cdot\Delta)^{j_1-j_2}\Big)$$

Some comments on *FORCER*

- *FORM* [Vermaseren, 2000], [Kuipers et al., 2013] program for the reduction of four-loop massless propagator-type integrals to master integrals
- Parametric IBP reductions
- Often possible to avoid explicit IBP reductions by reducing topologies to simpler ones (1-loop integrals, triangle rule, ...) $\rightarrow$ Automatized!
- Less diagrams for which actual IBP reductions are necessary, special rules for these

More details can be found in the original paper [Ruijl et al., 2020].

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