The 3-loop anomalous dimensions from off-shell operator matrix elements

LL022, Ettal,D

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DESY

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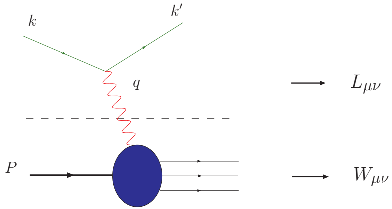
Outline

Based on:

Nucl.Phys.B 922 (2017) 1-40, Nucl.Phys.B 948 (2019) 114753, Nucl.Phys.B 971 (2021) 115542, JHEP 01 (2022) 193, 2202.03216 (Nucl. Phys. B in print) and in preparation

Theory of deep inelastic scattering



- Kinematic invariants:

$$Q^2 = -q^2, \qquad x = \frac{Q^2}{2P.q}$$

- The cross section factorizes into leptonic and hadronic tensor:

$$\frac{d^2\sigma}{dQ^2 dx} \sim L_{\mu\nu} W^{\mu\nu}$$

- The hadronic tensor can be expressed through structure functions:

$$W_{\mu\nu} = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, | \left[J_\mu^{\rm em}(\xi), J_\nu^{\rm em}(\xi) \right] | P \rangle$$
$$= \frac{1}{2x}\left(g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2}\right) F_L(x, Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x, Q^2)$$
$$+ i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho S^\sigma}{q \cdot P} g_1(x, Q^2) + i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho \left(q \cdot P S^\sigma - q \cdot S P^\sigma\right)}{(q \cdot P)^2} g_2(x, Q^2)$$

- F_L, F_2, g_1 and g_2 contain contributions from both, charm and bottom quarks.

Introduction

Light and Heavy Flavor Contributions are both important.

- They form a significant contribution to all structure functions particularly at small x and high Q^2.
- Concise 3-loop corrections are needed to determine $\alpha_s(M_Z)$, m_c and perhaps m_b.
- The accuracy of measurements at the LHC reaches a level of precision requiring 3-loop VFNS matching.

NNLO: [S. Alekhin, J. Blümlein, S. Moch and R. Placakyte (Phys. Rev. D96 (2017))]

$$\alpha_s(M_Z^2) = 0.1147 \pm 0.0008$$

$$m_c(m_c) = 1.252 \pm 0.018(exp)\ {}^{+0.03}_{-0.02}\ (scale),\ {}^{+0.00}_{-0.07}\ (thy)\text{GeV} \qquad (\overline{\text{MS}}\text{-scheme})$$

Yet approximate NNLO treatment for F_2 (non-negligible error) [H. Kawamura et al. (Nucl. Phys. B864 (2012))]

NS & PS corrections are exact [J. Ablinger et al. (Nucl. Phys. B886 & B890 (2014))]

EIC: many more high precision data ahead for various detailed unpolarized and polarized precision measurements.

One important ingredient to the description of the scaling violations are the anomalous dimensions.

The status of the QCD anomalous dimensions

Unpolarized anomalous dimensions

1 Loop:

- Gross, Wilczek 1974a,b
- Georgi, Politzer 1974

2 Loop:

- Floratos, D. Ross, Sachrajda 1977, 1978
- Gonzalex-Arroyo, Lopez, Yndurain 1979, Gonzalex-Arroyo and Lopez 1980
- Curci, Furmanski and Petronzio 1980, Furmanski and Petronzio 1980
- Floratos, Kounnas and Lacaze 1981
- Hamberg and van Neerven 1992 [all former flaws clarified.]
- Ellis and Vogelsang 1996; Matiounine, Smith and van Neerven 1998; Moch and Vermaseren 1999; Blümlein, Marquard, Schönwald and Schneider 2022

3 Loop:

- Moch, Vermaseren, Vogt 2004a,b
- Ablinger et al. 2010, 2014abc, 2017
- Anastasiou et al. 2015, Duhr, Dulat and Mistlberger, 2020 (implicit)
- Blümlein, Marquard, Schönwald and Schneider 2021

The status of the QCD anomalous dimensions

During the 1990ies until 2004 moments of the 3-loop anomalous dimensions were calculated up to $N = 16$ in the non-singlet case in 2004, which has been quite an effort at that time.

Polarized anomalous dimensions

1 Loop:

- Sasaki 1975
- Ahmed and Ross 1975

2 Loop:

- Mertig and van Neerven 1995
- Vogelsang 1995
- Matiounine, Smith and van Neerven 1998

3 Loop:

- Moch, Vermaseren, Vogt 2014, 2015
- Behring et al. 2019
- Blümlein, Marquard, Schönwald and Schneider 2021

The status of the QCD anomalous dimensions

Transversity

1 Loop:

- Artru and Mekhfi 1990
- Ioffe and Khojamirian 1995
- Blümlein 2001 [and many other papers]

2 Loop:

- Hayashigaki, Kanazawa and Koike 1995
- Kumano and Miyama 1997
- Vogelsang 1998
- Blümlein, Klein, Tödtli 2009 (including 3-loop TF moments)

3 Loop:

- Velizhanin 2012 (diophantine guess)
- Blümlein, Marquard, Schönwald and Schneider 2021

The status of the QCD anomalous dimensions

Leading N_f series [partial results]

- Gracey 1994, 1996, 2000, 2003, 2006 a,b
- Gracey and Bennett 1998 a,b

Leading small x series

- Fadin, Kuraev and Lipatov 1975, Balitsky and Lipatov 1978
- Jaroszewicz 1982
- Kirschner and Lipatov 1983
- Catani and Hautmann, 1994
- Bartels, Ermolaev and Ryskin, 1995, 1996
- Blümlein and Vogt 1995, 1996
- Cougoulic, Kovchegov et al.: 2204.11898, large N_c, confirm known $ln^a(x)$ structure for ΔP_{gg}

Next-to-leading order small x series

- Fadin and Lipatov 1998

4–loop moments

- Baikov and Chetyrkin 2006, Baikov, Chetyrkin and Kühn 2015
- Velizhanin 2014
- Ruijl et al. 2016, Davies et al., Moch et al., 2017, 2021

5–loop moments

- Herzog et al. 2019

The evolution equations

The different anomalous dimensions

- 3 non–singlet anomalous dimensions (starting at 1-, 2-, and 3-loop) $\gamma_{qq}^{\rm NS,\pm}, \gamma_{qq}^{\rm NS,s}$
- 4 singlet anomalous dimensions $\gamma_{qq}^{\rm PS}, \gamma_{qg}, \gamma_{gq}, \gamma_{gg}$ [$\gamma_{qq}^{\rm PS}$ contributes from 2-loops.]
- both in the unpolarized and polarized case: $\gamma_{ij} \rightarrow \Delta\gamma_{ij}$
- + expansion coefficients for the so called alien operators, including new anomalous dimensions.

Evolution equations

$$
\begin{aligned}
\Sigma(N) &= \sum_{k=1}^{N_F} q_k(N) + \bar{q}_k(N) \\
\gamma_{ij} &= \sum_{k=0}^{\infty} a_s^{k+1} \gamma_{ij} \\
\frac{dq^{\rm NS,\pm,(s)}(N,a_s)}{d\ln(\mu^2)} &= \sum_{k=0}^{\infty} a_s^{k+1} \gamma_{qq}^{(k),\rm NS,\pm(s)}(N) \cdot q^{\rm NS,\pm(s)}(N,a_s) \\
\frac{d}{d\ln(\mu^2)} \begin{pmatrix} \Sigma(N) \\ G(N) \end{pmatrix} &= \sum_{k=0}^{\infty} a_s^{k+1} \begin{pmatrix} \gamma_{qq}^{(k)} & \gamma_{qg}^{(k)} \\ \gamma_{gq}^{(k)} & \gamma_{gg}^{(k)} \end{pmatrix} \cdot \begin{pmatrix} \Sigma(N) \\ G(N) \end{pmatrix}
\end{aligned}
$$

Computation Methods at Three Loops

Forward Compton Amplitude:

- Moch, Vermaseren and Vogt 2004a,b, 2014; based on `Mincer` Gorishnii, Larin, Surguladze, 1989.
- straightforward in the quarkonic case; gauge invariant framework
- necessitates scalar currents to obtain the gluonic anomalous dimensions in the unpolarized case
- polarized case: gravitational currents in the case of the gluonic anomalous

Massive On-Shell Operator Matrix Elements:

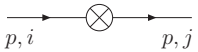
- Blümlein at al. 2014 a,b,c, 2015, 2017, 2019
- allows to calculate all contributions $\propto T_F$
- gauge invariant framework

Massless Off-Shell Operator Matrix Elements:

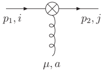
- Blümlein, Marquard, Schönwald and Schneider 2021 a,b, 2022 a [2 loop complete]
- allows to calculate all anomalous dimensions
- gauge variant framework
- no further problems in the non–singlet and polarized singlet case
- mixing with new (alien) operator matrix elements in the unpolarized case Falcioni, Herzog 2022
- the previous 2-loop studies by Matiounine, van Neerven and Smith 1998 contain some errors

Calculation of the 3-loop operator matrix elements

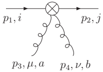
The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules + 1 more five-gluon graph. $p^2 < 0$ implies breaking of gauge invariance.



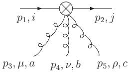
$$\delta^{ij}\Delta\!\!\!/\gamma_\pm(\Delta\cdot p)^{N-1}\,, \quad N \geq 1$$



$$g t^a_{ji}\Delta^\mu\Delta\!\!\!/\gamma_\pm\sum_{j=0}^{N-2}(\Delta\cdot p_1)^j(\Delta\cdot p_2)^{N-j-2}\,, \quad N\geq 2$$



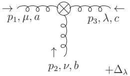
$$g^2\Delta^\mu\Delta^\nu\Delta\!\!\!/\gamma_\pm\sum_{j=0}^{N-3}\sum_{l=j+1}^{N-2}(\Delta p_2)^j(\Delta p_1)^{N-l-2}\Big[(t^at^b)_{ji}(\Delta p_1+\Delta p_4)^{l-j-1}+(t^bt^a)_{ji}(\Delta p_1+\Delta p_3)^{l-j-1}\Big]\,,$$
$$N\geq 3$$



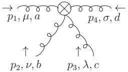
$$\begin{aligned}
&g^3\Delta_\mu\Delta_\nu\Delta_\rho\Delta\!\!\!/\gamma_\pm\sum_{j=0}^{N-4}\sum_{l=j+1}^{N-3}\sum_{m=l+1}^{N-2}(\Delta.p_2)^j(\Delta.p_1)^{N-m-2}\\
&\Big[\ (t^at^bt^c)_{ji}(\Delta.p_4+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_5+\Delta.p_1)^{m-l-1}\\
&+(t^at^ct^b)_{ji}(\Delta.p_4+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_4+\Delta.p_1)^{m-l-1}\\
&+(t^bt^at^c)_{ji}(\Delta.p_3+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_5+\Delta.p_1)^{m-l-1}\\
&+(t^bt^ct^a)_{ji}(\Delta.p_3+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_3+\Delta.p_1)^{m-l-1}\\
&+(t^ct^at^b)_{ji}(\Delta.p_3+\Delta.p_4+\Delta.p_1)^{l-j-1}(\Delta.p_4+\Delta.p_1)^{m-l-1}\\
&+(t^ct^bt^a)_{ji}(\Delta.p_3+\Delta.p_4+\Delta.p_1)^{l-j-1}(\Delta.p_3+\Delta.p_1)^{m-l-1}\Big]\,,\\
&N\geq 4
\end{aligned}$$

$$\gamma_+ = 1\,, \quad \gamma_- = \gamma_5\,.$$

$$\frac{1+(-1)^N}{2}\delta^{ab}(\Delta\cdot p)^{N-2}\Big[g_{\mu\nu}(\Delta\cdot p)^2-(\Delta_\mu p_\nu+\Delta_\nu p_\mu)\Delta\cdot p+p^2\Delta_\mu\Delta_\nu\Big]\,, \quad N\geq 2$$



$$\begin{aligned}
&-ig\frac{1+(-1)^N}{2}f^{abc}\Bigg(\\
&\Big[(\Delta_\nu g_{\lambda\mu}-\Delta_\lambda g_{\mu\nu})\Delta\cdot p_1+\Delta_\mu(p_{1,\nu}\Delta_\lambda-p_{1,\lambda}\Delta_\nu)\Big](\Delta\cdot p_1)^{N-2}\\
&+\Delta_\lambda\Big[\Delta\cdot p_1p_{2,\mu}\Delta_\nu+\Delta\cdot p_2p_{1,\nu}\Delta_\mu-\Delta\cdot p_1\Delta\cdot p_2g_{\mu\nu}-p_1\cdot p_2\Delta_\mu\Delta_\nu\Big]\\
&\times\sum_{j=0}^{N-3}(-\Delta\cdot p_1)^j(\Delta\cdot p_2)^{N-3-j}\\
&+\left\{\begin{smallmatrix}p_1\to p_2\to p_3\to p_1\\ \mu\to\nu\to\lambda\to\mu\end{smallmatrix}\right\}+\left\{\begin{smallmatrix}p_1\to p_3\to p_2\to p_1\\ \mu\to\lambda\to\nu\to\mu\end{smallmatrix}\right\}\Bigg)\,, \quad N\geq 2
\end{aligned}$$



$$\begin{aligned}
&g^2\frac{1+(-1)^N}{2}\Big(f^{abe}f^{cde}O_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4)\\
&+f^{ace}f^{bde}O_{\mu\lambda\nu\sigma}(p_1,p_3,p_2,p_4)+f^{ade}f^{bce}O_{\mu\sigma\nu\lambda}(p_1,p_4,p_2,p_3)\Big)\,,\\
&O_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4)=\Delta_\nu\Delta_\lambda\Big\{-g_{\mu\sigma}(\Delta\cdot p_3+\Delta\cdot p_4)^{N-2}\\
&+[p_{4,\mu}\Delta_\sigma-\Delta\cdot p_4g_{\mu\sigma}]\sum_{i=0}^{N-3}(\Delta\cdot p_3+\Delta\cdot p_4)^i(\Delta\cdot p_4)^{N-3-i}\\
&-[p_{1,\sigma}\Delta_\mu-\Delta\cdot p_1g_{\mu\sigma}]\sum_{i=0}^{N-3}(-\Delta\cdot p_1)^i(\Delta\cdot p_3+\Delta\cdot p_4)^{N-3-i}\\
&+[\Delta\cdot p_1\Delta\cdot p_4g_{\mu\sigma}+p_1\cdot p_4\Delta_\mu\Delta_\sigma-\Delta\cdot p_4p_{1,\sigma}\Delta_\mu-\Delta\cdot p_1p_{4,\mu}\Delta_\sigma]\\
&\times\sum_{i=0}^{N-4}\sum_{j=0}^{i}(-\Delta\cdot p_1)^{N-4-i}(\Delta\cdot p_3+\Delta\cdot p_4)^{i-j}(\Delta\cdot p_4)^j\Big\}\\
&-\left\{\begin{smallmatrix}p_1\leftrightarrow p_2\\ \mu\leftrightarrow\nu\end{smallmatrix}\right\}-\left\{\begin{smallmatrix}p_3\leftrightarrow p_4\\ \lambda\leftrightarrow\sigma\end{smallmatrix}\right\}+\left\{\begin{smallmatrix}p_1\leftrightarrow p_2,\ p_3\leftrightarrow p_4\\ \mu\leftrightarrow\nu,\ \lambda\leftrightarrow\sigma\end{smallmatrix}\right\}\,, \quad N\geq 2
\end{aligned}$$

The unpolarized massless OMEs

Vector case:

$$O^{\rm NS}_{q,r;\mu_1\ldots\mu_N} = i^{N-1}\mathbf{S}\left[\overline{\psi}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\frac{\lambda_r}{2}\psi\right] - \text{trace terms},$$

$$O^{\rm NS,5}_{q,r;\mu_1\ldots\mu_N} = i^{N-1}\mathbf{S}\left[\overline{\psi}\gamma_5\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\frac{\lambda_r}{2}\psi\right] - \text{trace terms}$$

Transversity case:

$$O^{\rm NS,tr}_{q,r;\mu\mu_1\ldots\mu_N} = i^{N-1}\mathbf{S}\left[\overline{\psi}\sigma_{\mu\mu_1}D_{\mu_2}\ldots D_{\mu_N}\frac{\lambda_r}{2}\psi\right] - \text{trace terms}\ ,$$

where $\sigma_{\mu\nu} = (i/2)[\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu]$.

$$\Delta\hat{A}_{ij} = \langle q(p), j|O_i^{\rm S,(5)}|q(p), j\rangle.$$

The unpolarized massless OMEs

The operator matrix elements have the representation

$$\hat{A}_{qq}^{\rm NS} = \left[\not{\Delta}\hat{A}_{qq}^{\rm NS,phys} + \not{p}\frac{\Delta.p}{p^2}\hat{A}_{qq}^{\rm NS,EOM}\right](\Delta.p)^{N-1}.$$

with $\Delta.\Delta = 0$.

$$\hat{A}_{qq}^{\rm NS,phys} = \frac{1}{4(\Delta.p)^N}\,\mathrm{tr}\left[\left(\not{p} - \frac{p^2}{\Delta.p}\not{\Delta}\right)\hat{A}_{qq}^{\rm NS}\right],$$

$$\hat{A}_{qq}^{\rm NS,EOM} = \frac{1}{4(\Delta.p)^N}\,\mathrm{tr}\left[\not{\Delta}\hat{A}_{qq}^{\rm NS}\right].$$

$$\hat{A}_{qq}^{\rm NS,5,phys} = \frac{1}{4(\Delta.p)^N}\,\mathrm{tr}\left[\left(\not{p} - \frac{p^2}{\Delta.p}\not{\Delta}\right)\gamma_5\hat{A}_{qq}^{\rm NS,5}\right],$$

$$\hat{A}_{qq}^{\rm NS,5,EOM} = \frac{1}{4(\Delta.p)^N}\,\mathrm{tr}\left[\not{\Delta}\gamma_5\hat{A}_{qq}^{\rm NS,5}\right].$$

The polarized massless OMEs

$$O^{\rm S,5}_{q;\mu_1\ldots\mu_N} = i^{N-1}{\bf S}\left[\overline{\psi}\gamma_5\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\psi\right] - \text{ trace terms},$$

$$O^{\rm S,5}_{g;\mu_1\ldots\mu_N} = 2i^{N-2}{\bf S}\,{\bf tr}\left[\frac{1}{2}\varepsilon^{\mu_1\alpha\beta\gamma}F^a_{\beta\gamma}D_{\mu_2}\ldots D_{\mu_{N-1}}F^{\mu_N}_{\alpha,a}\right] - \text{ trace terms}$$

$$\Delta\hat{A}^{\rm PS}_{qq} = \left[\gamma_5\not{\Delta}\Delta\hat{A}^{\rm PS,phys}_{qq} + \gamma_5\not{p}\frac{\Delta.p}{p^2}\Delta\hat{A}^{\rm PS,EOM}_{qq}\right](\Delta.p)^{N-1}$$

$$\Delta\hat{A}_{qg} = \varepsilon_{\mu\nu\alpha\beta}\Delta^\alpha p^\beta\frac{1}{\Delta.p}\Delta\hat{A}^{\rm phys}_{qg}(\Delta.p)^{N-1}$$

$$\Delta\hat{A}_{gq} = \left[\gamma_5\not{\Delta}\Delta\hat{A}^{\rm phys}_{gq} + \gamma_5\not{p}\frac{\Delta.p}{p^2}\hat{A}^{\rm EOM}_{gq}\right](\Delta.p)^{N-1}$$

$$\Delta\hat{A}_{gg,\mu\nu} = \varepsilon_{\mu\nu\alpha\beta}\Delta^\alpha p^\beta\frac{1}{\Delta.p}\Delta\hat{A}^{\rm phys}_{gq}(\Delta.p)^{N-1},$$

The polarized massless OMEs

$$\Delta\hat{A}_{iq}^{\rm phys} = -\frac{1}{4(D-2)(D-3)}\varepsilon_{\mu\nu\rho\sigma}p^\rho\Delta^\sigma \mathrm{tr}\left[\not p\gamma^\mu\gamma^\nu\Delta\hat{A}_{iq}\right](\Delta.p)^{-N-1}$$
$$-\frac{p^2}{4(D-2)(D-3)}(\Delta.p)^{-N-2}\varepsilon_{\mu\nu\rho\sigma}p^\rho\Delta^\sigma \mathrm{tr}\left[\not\Delta\gamma^\mu\gamma^\nu\Delta\hat{A}_{iq}\right],$$
$$\Delta\hat{A}_{ig}^{\rm phys} = \frac{1}{(D-2)(D-3)}\varepsilon_{\mu\nu\rho\sigma}\Delta^\rho p^\sigma(\Delta.p)^{-N-1}\Delta\hat{A}_{ig}^{\mu\nu},$$

We use the Larin scheme Larin 1993, being a consistent scheme, to describe γ^5 in $D = 4 + \varepsilon$-dimensions:

$$\gamma^5 = \frac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma,$$
$$\not\Delta\gamma^5 = \frac{i}{6}\varepsilon_{\mu\nu\rho\sigma}\Delta^\mu\gamma^\nu\gamma^\rho\gamma^\sigma.$$

The Levi-Civita symbols are now contracted in D dimensions,

$$\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\lambda\tau\gamma} = -\mathrm{Det}[g^\beta_\omega], \quad \beta = \alpha,\lambda,\tau,\gamma; \;\; \omega = \mu,\nu,\rho,\sigma.$$

OME structure: Renormalization

- coupling constant renormalization
- gauge parameter renormalization
- renormalization of the local operator
- The OMEs remain gauge–dependent (off-shellness).

The partly renormalized polarized singlet OMEs, $\tilde{A}_{ij}^{\rm phys}$, read

$$\tilde{A}_{ij}^{\rm phys} = 1 + a\left[\frac{a_{ij}^{(1,-1)}}{\varepsilon} + a_{ij}^{(1,0)} + a_{ij}^{(1,1)}\varepsilon\right] + a^2\left[\frac{a_{ij}^{(2,-2)}}{\varepsilon^2} + \frac{a_{ij}^{(2,-1)}}{\varepsilon} + a_{ij}^{(2,0)}\right] + a^3\left[\frac{a_{ij}^{(3,-3)}}{\varepsilon^3} + \frac{a_{ij}^{(3,-2)}}{\varepsilon^2} + \frac{a_{ij}^{(3,-1)}}{\varepsilon}\right].$$

$$Z_{ij}^{\rm S} = \delta_{ij} + a\frac{\gamma_{ij}^{(0)}}{\varepsilon} + a^2\left[\frac{1}{\varepsilon^2}\left(\frac{1}{2}\gamma_{il}^{(0)}\gamma_{lj}^{(0)} + \beta_0\gamma_{ij}^{(0)}\right) + \frac{1}{2\varepsilon}\gamma_{ij}^{(1)}\right] + a^3\left[\frac{1}{\varepsilon^3}\left(\frac{1}{6}\gamma_{il}^{(0)}\gamma_{lk}^{(0)}\gamma_{kj}^{(0)} + \beta_0\gamma_{il}^{(0)}\gamma_{lj}^{(0)} + \frac{4}{3}\beta_0^2\gamma_{ij}^{(0)}\right)\right.$$
$$\left. + \frac{1}{\varepsilon^2}\left(\frac{1}{6}\gamma_{il}^{(1)}\gamma_{lj}^{(0)} + \frac{1}{3}\gamma_{il}^{(0)}\gamma_{lj}^{(1)} + \frac{2}{3}\beta_0\gamma_{ij}^{(1)} + \frac{2}{3}\beta_1\gamma_{ij}^{(0)}\right) + \frac{1}{3\varepsilon}\gamma_{ij}^{(2)}\right].$$

$$A_{ij}^{\rm phys} = (Z_{ik}^S)^{-1}\tilde{A}_{kj}^{\rm phys}.$$

Technical aspects of the computation

The program chain:

- The 3-loop calculation is by now fully automated.
- Diagram generation: `QGRAF`, Nogueira, 1991.
- Lorentz algebra: `FORM`, Vermaseren and Tentyukov, 2000, 2010.
- Color algebra: `Color`, van Ritbergen, Schellekens, Vermaseren 1999.
- IBP reductions: `Crusher`, Marquard & Seidel.
- Initial values, Chetyrkin, Tkachov, 1981, Tkachov, 1981, Gehrmann et al. 2010; Lee, Smirnov, Smirnov 2010, 2011.
- Method of arbitrary high moments to generated large sets of Mellin moments, here 3000: `SolveCoupledSystems`, Blümlein and Schneider, 2017.
- Determination of the recurrences for the OMEs: `Guess` in `Sage`, Blümlein, Kauers, Schneider, 2009; Kauers et al. 2009, 2015.
- Solution of the recurrences: `Sigma`, Schneider, 2007–
- Reduction of the solution `HarmoncisSums`: Ablinger 2010– using shuffle-algebras Blümlein, 2003 and structural relations [i.e. double integer and differential relations] Blümlein, 2009.

Mass production of moments is no problem since 2017 anymore. Various physical problems have already been computed by us based on 8000 moments.

Technical aspects of the computation

Expression of the results in Mellin N and z space

Harmonic Sums:

Vermaseren 1998, Blümlein and Kurth, 1998

$$S_{b,\vec{a}}(N) = \sum_{k=1}^{N} \frac{(\mathrm{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \quad S_\emptyset = 1, \quad b, a_i \in \mathbb{Z}\backslash\{0\}, N \in \mathbb{N}\backslash\{0\}.$$

Their Mellin inversion to the splitting functions $P_{qq}(z)$

$$\gamma_{qq}(N) = -\int_0^1 dz z^{N-1} P_{qq}(z)$$

Harmonic Polylogarithms:

Remiddi and Vermaseren, 1999

$$\mathrm{H}_{b,\vec{a}}(z) = \int_0^z dx f_b(x) \mathrm{H}_{\vec{a}}(x), \quad \mathrm{H}_\emptyset = 1, \quad b, a_i \in \{-1, 0, 1\},$$

with the alphabet of letters

$$\mathfrak{A}_{\mathrm{H}} = \left\{ f_0(z) = \frac{1}{z}, \quad f_{-1}(z) = \frac{1}{1+z}, \quad f_1(z) = \frac{1}{1-z} \right\}.$$

Technical aspects of the computation

- Consider the example of the unpolarized 3-loop singlet case
- The quoted quantities are all less in the different non–singlet and polarized singlet cases
- Number of 1PI diagrams: 3324
- Number of moments needed:
  - $\gamma_{qq}^{(2),\mathrm{PS}}$: 588
  - $\gamma_{qg}^{(2)}$: 1248
  - $\gamma_{gq}^{(2)}$: 1274
  - $\gamma_{gg}^{(2)}$: 1860
- largest difference equation: d = 373, o = 16. [Far larger difference equations can be solved by `Sigma`.]
- overall computation time: $\sim$ 3 weeks.

All 3-loop anomalous dimensions can be represented by 10 basic harmonic sums only after using structural relations (in the complex plane):

$$\left\{S_1, S_{2,1}, S_{-2,1}, S_{-3,1}, S_{-4,1}, S_{2,1,1}, S_{-2,1,1}, S_{2,1,-2}, S_{-3,1,1}, S_{-2,1,1,1}\right\}.$$

The number of moments obtained by the method of arbitrary high moments is several orders of magnitude higher than the one which can be produced using e.g. `Mincer`.

Technical aspects of the computation

The Larin and M-scheme in the polarized singlet case Matiounine, van Neerven and Smith, 1998

$$
\begin{aligned}
\Delta\gamma_{ij}^{(0),\mathrm{M}} &= \Delta\gamma_{ij}^{(0),\mathrm{L}} \\
\Delta\gamma_{qq}^{(1),\mathrm{NS},\mathrm{M}} &= \Delta\gamma_{qq}^{(1),\mathrm{NS},\mathrm{L}} + 2\beta_0 z_{qq}^{(1)}, \\
\Delta\gamma_{qq}^{(1),\mathrm{PS},\mathrm{M}} &= \Delta\gamma_{qq}^{(1),\mathrm{PS},\mathrm{L}}, \\
\Delta\gamma_{qg}^{(1),\mathrm{M}} &= \Delta\gamma_{qg}^{(1),\mathrm{L}} + \Delta\gamma_{qg}^{(0)} z_{qq}^{(1)}, \\
\Delta\gamma_{gq}^{(1),\mathrm{M}} &= \Delta\gamma_{gq}^{(1),\mathrm{L}} - \Delta\gamma_{gq}^{(0)} z_{qq}^{(1)}, \\
\Delta\gamma_{gg}^{(1),\mathrm{M}} &= \Delta\gamma_{gg}^{(1),\mathrm{L}}. \\
\Delta\gamma_{qq}^{(2),\mathrm{NS},\mathrm{M}} &= \Delta\gamma_{qq}^{(2),\mathrm{NS},\mathrm{L}} - 2\beta_0 \left(\left(z_{qq}^{(1)}\right)^2 - 2 z_{qq}^{(2),\mathrm{NS}}\right) + 2\beta_1 z_{qq}^{(1)}, \\
\Delta\gamma_{qq}^{(2),\mathrm{PS},\mathrm{M}} &= \Delta\gamma_{qq}^{(2),\mathrm{PS},\mathrm{L}} + 4\beta_0 z_{qq}^{(2),\mathrm{PS}}, \\
\Delta\gamma_{qg}^{(2),\mathrm{M}} &= \Delta\gamma_{qg}^{(2),\mathrm{L}} + \Delta\gamma_{qg}^{(1),\mathrm{M}} z_{qq}^{(1)} + \Delta\gamma_{qg}^{(0)} \left(z_{qq}^{(2)} - \left(z_{qq}^{(1)}\right)^2\right), \\
\Delta\gamma_{gq}^{(2),\mathrm{M}} &= \Delta\gamma_{gq}^{(2),\mathrm{L}} - \Delta\gamma_{gq}^{(1),\mathrm{M}} z_{qq}^{(1)} - \Delta\gamma_{gq}^{(0)} z_{qq}^{(2)}, \\
\Delta\gamma_{gg}^{(2),\mathrm{M}} &= \Delta\gamma_{gg}^{(2),\mathrm{L}},
\end{aligned}
$$

Technical aspects of the computation

$$
\begin{aligned}
z_{qq}^{(1)} &= -\frac{8C_F}{N(N+1)}, \\
z_{qq}^{(2),\mathrm{NS}} &= C_F T_F N_F \frac{16\big(-3-N+5N^2\big)}{9N^2(1+N)^2} + C_A C_F \left\{ -\frac{4R_1}{9N^3(1+N)^3} - \frac{16}{N(1+N)} S_{-2} \right\} \\
&\quad + C_F^2 \left\{ \frac{8\big(2+5N+8N^2+N^3+2N^4\big)}{N^3(1+N)^3} + \frac{16(1+2N)}{N^2(1+N)^2} S_1 \right. \\
&\quad \left. + \frac{16}{N(1+N)} S_2 + \frac{32}{N(1+N)} S_{-2} \right\}, \\
z_{qq}^{(2),\mathrm{PS}} &= 8 C_F T_F N_F \frac{(N+2)(1+N-N^2)}{N^3(N+1)^3}, \\
z_{qq}^{(2)} &= z_{qq}^{(2),\mathrm{NS}} + z_{qq}^{(2),\mathrm{PS}}
\end{aligned}
$$

The simplest 3-loop anomalous dimensions: $\gamma_{qq}^{\mathrm{NS},(\mathrm{s})}$ and $\Delta\gamma_{qq}^{\mathrm{NS},(\mathrm{s})}$

- Moch, Vermaseren, Vogt, 2004a; Blümlein, Marquard, Schönwald and Schneider 2021a
- $\gamma_{qq}^{\mathrm{NS},(\mathrm{s})}$ stems from considering the odd moments of the unpolarized pure singlet OME.

$$\gamma_{\mathrm{NS}}^{(2),\mathrm{s}} = 4N_F \frac{d_{abc}d^{abc}}{N_C} \Bigg[\frac{2P_{60}}{(-1+N)N^5(1+N)^5(2+N)} - \frac{P_{61}}{(-1+N)N^4(1+N)^4(2+N)} S_1 + \left(-\frac{2P_{62}}{(-1+N)N^3(1+N)^3(2+N)} - \frac{4\left(2+N+N^2\right)^2}{(-1+N)N^2(1+N)^2(2+N)} S_1 \right) S_{-2} - \frac{\left(2+N+N^2\right)}{N^2(1+N)^2} \left[S_3 - 2S_{-3} + 4S_{-2,1}\right] \Bigg],$$

The simplest 3-loop anomalous dimensions: $\gamma_{qq}^{\mathrm{NS},(\mathrm{s})}$ and $\Delta\gamma_{qq}^{\mathrm{NS},(\mathrm{s})}$

- $\Delta\gamma_{qq}^{\mathrm{NS},(\mathrm{s})}$ rules the evolution of the polarized electro-weak interference structure function $g_5^-(x, Q^2)$, cf. Blümlein and Kochelev, 1996.
- It has only even moments (although being polarized).
- It is best calculated using the forward Compton amplitude for a vector and and axial-vector current.
- Moch, Vermaseren, Vogt, 2015; Blümlein, Marquard, Schönwald and Schneider 2021b

$$
\begin{aligned}
\Delta\gamma_{\mathrm{NS}}^{(2),\mathrm{s}} \quad = \quad & -16\frac{1+(-1)^N}{2} N_F \frac{d^{abc}d_{abc}}{N_c}\Bigg[\frac{1}{N^2} + \frac{2Q_1}{N^4(1+N)^4}S_1 + \frac{2(2+3N+3N^2)}{N^2(1+N)^2} \\
& \times[S_3 - 2S_{-3} + 4S_{-2,1}] + \left(\frac{4(2+4N+4N^2+N^3+N^4)}{N^3(1+N)^3} + \frac{8(-1+N)(2+N)}{N^2(1+N)^2}S_1\right) \\
& \times S_{-2}\Bigg]
\end{aligned}
$$

The non-singlet anomalous dimension $\gamma_{qq}^{+,\mathrm{NS}}$

$$\gamma_{\rm NS}^{(2),+} = \frac{1}{2}\left[1+(-1)^N\right]$$
$$\times\Big\{ C_F^2 \Big\{ C_A \Big[\frac{72P_3}{N^2(1+N)^2}\zeta_3 + \frac{32P_{15}}{9N^2(1+N)^2}S_{-2,1} - \frac{16P_{17}}{9N^2(1+N)^2}S_3 + \frac{P_{33}}{18N^4(1+N)^4}$$
$$+\Big(-\frac{16P_{29}}{9N^4(1+N)^4} - \frac{4288}{9}S_2 + \frac{64(-12+31N+31N^2)}{3N(1+N)}S_3 + 320S_4 - 1024S_{3,1}$$
$$+\frac{64(-84+31N+31N^2)}{3N(1+N)}S_{-2,1} + 3712S_{-2,2} + 3840S_{-3,1} - 7168S_{-2,1,1}\Big)S_1 + \Big(256S_3$$
$$+1792S_{-2,1}\Big)S_1^2 + \Big(\frac{4P_{19}}{9N^2(1+N)^2} - 832S_3 - 5248S_{-2,1}\Big)S_2 + \frac{352}{3}S_2^2$$
$$+\frac{16(-30+151N+151N^2)}{3N(1+N)}S_4 + \Big(-\frac{16P_{22}}{9N^2(1+N)^3} + (-\frac{64P_9}{9N^2(1+N)^2} - 256S_2\Big)S_1$$
$$+\frac{32(12+31N+31N^2)S_2}{3N(1+N)} + 64S_3 + 5376S_{2,1} - 384S_{-2,1} + 576\zeta_3\Big)S_{-2}$$
$$+\Big(-\frac{32(8+3N+3N^2)}{N(1+N)} + 512S_1\Big)S_{-2}^2 + \Big(\frac{32(108+31N+31N^2)}{3N(1+N)}S_1 - \frac{16P_{16}}{9N^2(1+N)^2}$$
$$-1152S_1^2 + 2624S_2 + 960S_{-2}\Big)S_{-3} + \Big(\frac{16(138+35N+35N^2)}{3N(1+N)} - 1472S_1\Big)S_{-4}$$
$$+2304S_{-5} + 768S_{2,3} + 2688S_{2,-3} - \frac{64(-24+29N+29N^2)}{3N(1+N)}S_{3,1} - 768S_{4,1}$$
$$+\frac{32(-174+31N+31N^2)}{3N(1+N)}S_{-2,2} - 3648S_{-2,3} - \frac{1920}{N(1+N)}S_{-3,1} + 1728S_{-4,1}$$
$$-5376S_{2,1,-2} + 1536S_{3,1,1} - \frac{128(-84+31N+31N^2)}{3N(1+N)}S_{-2,1,1} - 1536S_{-2,1,-2}$$
$$-5376S_{-2,2,1} - 5376S_{-3,1,1} + 10752S_{-2,1,1,1}\Big] + T_F N_F \Big[-\frac{16P_5}{9N^2(1+N)^2}S_2 + \frac{4P_{34}}{9N^4(1+N)^4}$$
$$+\Big(-\frac{8P_{13}}{9N^2(1+N)^2} + \frac{1280}{9}S_2 - \frac{512}{3}S_3 - \frac{512}{3}S_{-2,1} + 128\zeta_3\Big)S_1 - \frac{128}{3}S_2^2$$
$$+\frac{64(12+29N+29N^2)}{9N(1+N)}S_3 - \frac{512}{3}S_4 + \Big(-\frac{128(-3+10N+16N^2)}{9N^2(1+N)^2} + \frac{2560}{9}S_1 - \frac{256}{3}S_2\Big)$$
$$\times S_{-2} + \Big(\frac{128(3+10N+10N^2)}{9N(1+N)} - \frac{256}{3}S_1\Big)S_{-3} - \frac{256}{3}S_{-4} - \frac{256(-3+10N+10N^2)}{9N(1+N)}S_{-2,1}$$
$$+\frac{256}{3}S_{3,1} - \frac{256}{3}S_{-2,2} + \frac{1024}{3}S_{-2,1,1} - \frac{32(2+3N+3N^2)}{N(1+N)}\zeta_3\Big]\Big\}$$
$$+C_F\Big\{T_F^2N_F^2\Big[\frac{8P_{28}}{27N^3(1+N)^3} - \frac{128}{27}S_1 - \frac{640}{27}S_2 + \frac{128}{9}S_3\Big] + C_A^2\Big[-\frac{24P_3}{N^2(1+N)^2}\zeta_3$$
$$\frac{32P_{11}}{9N^2(1+N)^2}S_{-2,1} + \frac{8P_{18}}{9N^2(1+N)^2}S_3 + \frac{P_{32}}{54N^3(1+N)^3} + \Big(\frac{4P_{35}}{3N^4(1+N)^4}$$
$$-\frac{16(-8+11N+11N^2)}{N(1+N)}S_3 - 256S_4 + 512S_{3,1} - \frac{64(-24+11N+11N^2)}{3N(1+N)}S_{-2,1}$$
$$-1024S_{-2,2} - 1024S_{-3,1} + 2048S_{-2,1,1}\Big)S_1 + \Big(-128S_3 - 512S_{-2,1}\Big)S_1^2 + \Big(-\frac{8344}{27}$$
$$+384S_3 + 1536S_{-2,1}\Big)S_2 - \frac{16(-24+55N+55N^2)}{3N(1+N)}S_4 + 64S_5 + \Big(\frac{32P_{10}}{9N^2(1+N)^2}S_1$$
$$+\frac{16P_{27}}{9N^3(1+N)^3} - \frac{352}{3}S_2 - 64S_3 - 1536S_{2,1} + 128S_{-2,1} - 192\zeta_3\Big)S_{-2}$$
$$+\Big(\frac{48(2+N+N^2)}{N(1+N)} - 192S_1\Big)S_{-2}^2 + \Big(\frac{16P_{12}}{9N^2(1+N)^2} - \frac{32(24+11N+11N^2)}{3N(1+N)}S_1$$
$$+256S_1^2 - 768S_2 - 320S_{-2}\Big)S_{-3} + \Big(-\frac{16(30+13N+13N^2)}{3N(1+N)} + 320S_1\Big)S_{-4}$$
$$-704S_{-5} - 384S_{2,3} - 768S_{2,-3} + \frac{64(-12+11N+11N^2)}{3N(1+N)}S_{3,1} + 384S_{4,1}$$
$$-\frac{32(-48+11N+11N^2)}{3N(1+N)}S_{-2,2} + 1088S_{-2,3} + \frac{512}{N(1+N)}S_{-3,1} - 448S_{-4,1}$$
$$+1536S_{2,1,-2} - 768S_{3,1,1} + \frac{128(-24+11N+11N^2)}{3N(1+N)}S_{-2,1,1} + 512S_{-2,1,-2} + 1536[S_{-2,2,1}$$
$$+S_{-3,1,1}] - 3072S_{-2,1,1,1}\Big] + C_A T_F N_F\Big[-\frac{8P_{31}}{27N^3(1+N)^3} + \Big(-\frac{16P_{30}}{27N^3(1+N)^3} + 64S_3$$
$$+\frac{256}{3}S_{-2,1} - 128\zeta_3\Big)S_1 + \frac{5344}{27}S_2 - \frac{32(3+14N+14N^2)}{3N(1+N)}S_3 + \frac{320}{3}S_4 + \Big(-\frac{1280}{9}S_1$$
$$+\frac{64(-3+10N+16N^2)}{9N^2(1+N)^2} + \frac{128}{3}S_2\Big)S_{-2} + \Big(-\frac{64(3+10N+10N^2)}{9N(1+N)} + \frac{128}{3}S_1\Big)S_{-3}$$
$$+\frac{128}{3}S_{-4} - \frac{256}{3}S_{3,1} + \frac{128(-3+10N+10N^2)}{9N(1+N)}S_{-2,1} + \frac{128}{3}S_{-2,2} - \frac{512}{3}S_{-2,1,1}$$
$$+\frac{32(2+3N+3N^2)}{N(1+N)}\zeta_3\Big]\Big\} + C_F^3\Big\{-\frac{48P_3}{N^2(1+N)^2}\zeta_3 + \frac{8P_4}{N^2(1+N)^2}S_3 + \frac{P_{36}}{N^5(1+N)^5}$$
$$+\Big(\frac{8P_{26}}{N^4(1+N)^4} - \frac{128(1+2N)}{N^2(1+N)^2}S_2 + 128S_2^2 - 384S_3 + 128S_4 + 512S_{3,1} - 3328S_{-2,2}$$
$$-\frac{384(-4+N+N^2)}{N(1+N)}S_{-2,1} - 3584S_{-3,1} + 6144S_{-2,1,1}\Big)S_1 + \Big(-\frac{64(1+3N+3N^2)}{N^3(1+N)^3}$$
$$-1536S_{-2,1}\Big)S_1^2 + \Big(\frac{4P_{25}}{N^3(1+N)^3} + 512S_3 + 4352S_{-2,1}\Big)S_2 - \frac{32(2+3N+3N^2)}{N(1+N)}S_2^2$$
$$\frac{32(2+15N+15N^2)}{N(1+N)}S_4 + (\frac{32P_{24}}{N^3(1+N)^3} + \Big(-\frac{128(5+7N+3N^2)}{N^2(1+N)^2} + 512S_2\Big)S_1$$
$$-\frac{64(4+3N+3N^2)}{N(1+N)}S_2 + 128S_3 - 4608S_{2,1} + 256S_{-2,1} - 384\zeta_3\Big)S_{-2} + \Big(\frac{128}{N(1+N)}$$
$$-256S_1\Big)S_{-2}^2 + \Big(\frac{32(8+5N+9N^2)}{N^2(1+N)^2} - \frac{64(20+3N+3N^2)}{N(1+N)}S_1 + 1280S_1^2 - 2176S_2$$
$$-640S_{-2}\Big)S_{-3} + \Big(-\frac{32(26+3N+3N^2)}{N(1+N)} + 1664S_1\Big)S_{-4} - 1792S_{-5} - 384S_{2,3}$$
$$-2304S_{2,-3} + \frac{128(-2+3N+3N^2)}{N(1+N)}S_{3,1} + 384S_{4,1} - \frac{64(4-N+3N^2)}{N^2(1+N)^2}S_{-2,1}$$
$$-\frac{64(-26+3N+3N^2)}{N(1+N)}S_{-2,2} + 2944S_{-2,3} + \frac{1792}{N(1+N)}S_{-3,1} - 1664S_{-4,1}$$
$$+4608S_{2,1,-2} - 768S_{3,1,1} + \frac{768(-4+N+N^2)}{N(1+N)}S_{-2,1,1} + 1024S_{-2,1,-2}$$
$$+4608[S_{-2,2,1} + S_{-3,1,1}] - 9216S_{-2,1,1,1}\Big\}\Big\},$$

The non-singlet anomalous dimension $\gamma_{qq}^{-,\mathrm{NS}}$

$$\gamma_{\rm NS}^{(2),-} = \frac{1}{2}\left[1-(-1)^N\right]$$

$$\times\Biggl\{C_F^2\Biggl\{C_A\Biggl[\frac{16\left(-126+6N+427N^2+770N^3+385N^4\right)}{9N^2(1+N)^2}[1-S_3]+\frac{72\zeta_3 P_{37}}{N^2(1+N)^2}$$

$$+\frac{32P_{43}}{9N^2(1+N)^2}S_{-2,1}-\frac{16P_{17}}{9N^2(1+N)^2}+\frac{P_{57}}{18N^5(1+N)^5}+\Biggl(-\frac{16P_{50}}{9N^4(1+N)^4}-\frac{4288}{9}S_2$$

$$+\frac{64\left(-12+31N+31N^2\right)}{3N(1+N)}S_3+320S_4-1024S_{3,1}+\frac{64\left(-84+31N+31N^2\right)}{3N(1+N)}S_{-2,1}$$

$$+3712S_{-2,2}+3840S_{-3,1}-7168S_{-2,1,1}\Biggr)S_1+\Biggl(256S_3+1792S_{-2,1}\Biggr)S_1^2+\Biggl(\frac{4P_{53}}{9N^3(1+N)^3}$$

$$-832S_3-5248S_{-2,1}\Biggr)S_2+\frac{352}{3}S_2^2+\frac{16\left(-30+151N+151N^2\right)}{3N(1+N)}S_4+\Biggl(-\frac{16P_{45}}{9N^3(1+N)^2}$$

$$+\Biggl(-\frac{64P_{40}}{9N^2(1+N)^2}-256S_2\Biggr)S_1+\frac{32\left(12+31N+31N^2\right)}{3N(1+N)}S_2+64S_3+5376S_{2,1}-384S_{-2,1}$$

$$+576\zeta_3\Biggr)S_{-2}+\Biggl(-\frac{32\left(8+3N+3N^2\right)}{N(1+N)}+512S_1\Biggr)S_{-2}^2+\Biggl(\frac{32\left(108+31N+31N^2\right)}{3N(1+N)}S_1$$

$$-\frac{16P_{44}}{9N^2(1+N)^2}-1152S_1^2+2624S_2+960S_{-2}\Biggr)S_{-3}+\Biggl(\frac{16\left(138+35N+35N^2\right)}{3N(1+N)}-1472S_1\Biggr)$$

$$\times S_{-4}+2304S_{-5}+768S_{2,3}+2688S_{2,-3}-\frac{64\left(-24+29N+29N^2\right)}{3N(1+N)}S_{3,1}-768S_{4,1}$$

$$+\frac{32\left(-174+31N+31N^2\right)}{3N(1+N)}S_{-2,2}-3648S_{-2,3}-\frac{1920S_{-3,1}}{N(1+N)}+1728S_{-4,1}-5376S_{2,1,-2}$$

$$+1536S_{3,1,1}-\frac{128\left(-84+31N+31N^2\right)}{3N(1+N)}S_{-2,1,1}-1536S_{-2,1,-2}-5376S_{-2,2,1}-5376S_{-3,1,1}$$

$$+10752S_{-2,1,1,1}\Biggr]+T_FN_F\Biggl[-\frac{16P_5}{9N^2(1+N)^2}S_2+\frac{4P_{54}}{9N^4(1+N)^4}+\Biggl(-\frac{8P_{51}}{9N^3(1+N)^3}+\frac{1280}{9}S_2$$

$$-\frac{512}{3}S_3-\frac{512}{3}S_{-2,1}+128\zeta_3\Biggr)S_1-\frac{128}{3}S_2^2+\frac{64\left(12+29N+29N^2\right)}{9N(1+N)}S_3-\frac{512}{3}S_4$$

$$+\Biggl(-\frac{128\left(-3+10N+16N^2\right)}{9N^2(1+N)^2}+\frac{2560}{9}S_1-\frac{256}{3}S_2\Biggr)S_{-2}+\Biggl(\frac{128\left(3+10N+10N^2\right)}{9N(1+N)}$$

$$-\frac{256}{3}S_1\Biggr)S_{-3}-\frac{256}{3}S_{-4}+\frac{256}{3}S_{3,1}-\frac{256\left(-3+10N+10N^2\right)}{9N(1+N)}S_{-2,1}-\frac{256}{3}S_{-2,2}$$

$$+\frac{1024}{3}S_{-2,1,1}-\frac{32\left(2+3N+3N^2\right)}{N(1+N)}\zeta_3\Biggr]\Biggr\}+C_F\Biggl\{T_F^2N_F^2\Biggl[\frac{8P_{28}}{27N^3(1+N)^3}-\frac{128}{27}S_1$$

$$-\frac{640}{27}S_2+\frac{128}{9}S_3\Biggr]+C_A^2\Biggl[-\frac{24P_{37}}{N^2(1+N)^2}\zeta_3-\frac{32P_{41}}{9N^2(1+N)^2}S_{-2,1}+\frac{8P_{18}}{9N^2(1+N)^2}S_3$$

$$+\frac{P_{59}}{54N^5(1+N)^5}+\Biggl(\frac{4P_{55}}{3N^4(1+N)^4}-\frac{16\left(-8+11N+11N^2\right)}{N(1+N)}S_3-256S_4+512S_{3,1}$$

$$-\frac{64\left(-24+11N+11N^2\right)}{3N(1+N)}S_{-2,1}-1024S_{-2,2}-1024S_{-3,1}+2048S_{-2,1,1}\Biggr)S_1$$

$$+\Biggl(-128S_3-512S_{-2,1}\Biggr)S_1^2+\Biggl(-\frac{8344}{27}+384S_3+1536S_{-2,1}\Biggr)S_2+64S_5$$

$$-\frac{16\left(-24+55N+55N^2\right)}{3N(1+N)}S_4+\Biggl(\frac{32P_{39}}{9N^2(1+N)^2}S_1+\frac{16P_{49}}{9N^3(1+N)^3}-\frac{352}{3}S_2-64S_3$$

$$-1536S_{2,1}+128S_{-2,1}-192\zeta_3\Biggr)S_{-2}+\Biggl(\frac{48\left(2+N+N^2\right)}{N(1+N)}-192S_1\Biggr)S_{-2}^2+\Biggl(\frac{16P_{42}}{9N^2(1+N)^2}$$

$$-\frac{32\left(24+11N+11N^2\right)}{3N(1+N)}S_1+256S_1^2-768S_2-320S_{-2}\Biggr)S_{-3}+\Biggl(-\frac{16\left(30+13N+13N^2\right)}{3N(1+N)}$$

$$+320S_1\Biggr)S_{-4}-704S_{-5}-384S_{2,3}-768S_{2,-3}+\frac{64\left(-12+11N+11N^2\right)}{3N(1+N)}S_{3,1}$$

$$+384S_{4,1}-\frac{32\left(-48+11N+11N^2\right)S_{-2,2}}{3N(1+N)}+1088S_{-2,3}+\frac{512S_{-3,1}}{N(1+N)}-448S_{-4,1}$$

$$+1536S_{2,1,-2}-768S_{3,1,1}+\frac{128\left(-24+11N+11N^2\right)}{3N(1+N)}S_{-2,1,1}+512S_{-2,1,-2}+1536[S_{-2,2,1}$$

$$+S_{-3,1,1}]-3072S_{-2,1,1,1}\Biggr]+C_AT_FN_F\Biggl[-\frac{8P_{56}}{27N^4(1+N)^4}+\Biggl(-\frac{16P_{52}}{27N^3(1+N)^3}+64S_3$$

$$+\frac{256}{3}S_{-2,1}-128\zeta_3\Biggr)S_1+\frac{5344}{27}S_2-\frac{32\left(3+14N+14N^2\right)}{3N(1+N)}S_3+\frac{320}{3}S_4$$

$$+\Biggl(\frac{64\left(-3+10N+16N^2\right)}{9N^2(1+N)^2}-\frac{1280}{9}S_1+\frac{128}{3}S_2\Biggr)S_{-2}+\Biggl(-\frac{64\left(3+10N+10N^2\right)}{9N(1+N)}$$

$$+\frac{128}{3}S_1\Biggr)S_{-3}+\frac{128}{3}S_{-4}-\frac{256}{3}S_{3,1}+\frac{128\left(-3+10N+10N^2\right)}{9N(1+N)}S_{-2,1}+\frac{128}{3}S_{-2,2}$$

$$-\frac{512}{3}S_{-2,1,1}+\frac{32\left(2+3N+3N^2\right)}{N(1+N)}\zeta_3\Biggr]\Biggr\}+C_F^3\Biggl\{-\frac{48\zeta_3P_{37}}{N^2(1+N)^2}+\frac{8P_{38}}{N^2(1+N)^2}S_3$$

$$+\frac{P_{58}}{N^5(1+N)^5}+\Biggl(\frac{8P_{48}}{N^4(1+N)^4}-\frac{128(1+2N)}{N^2(1+N)^2}S_2+128S_2^2-384S_3+128S_4+512S_{3,1}$$

$$-\frac{384\left(-4+N+N^2\right)}{N(1+N)}S_{-2,1}-3328S_{-2,2}-3584S_{-3,1}+6144S_{-2,1,1}\Biggr)S_1$$

$$+\Biggl(-\frac{64\left(1+3N+3N^2\right)}{N^3(1+N)^3}-1536S_{-2,1}\Biggr)S_1^2+\Biggl(\frac{4P_{47}}{N^3(1+N)^3}+512S_3+4352S_{-2,1}\Biggr)S_2$$

$$-\frac{32\left(2+3N+3N^2\right)}{N(1+N)}S_2^2-\frac{32\left(2+15N+15N^2\right)}{N(1+N)}S_4+\Biggl(\frac{32P_{46}}{N^3(1+N)^3}+512S_2$$

$$+\Biggl(\frac{128\left(1-N+3N^2\right)}{N^2(1+N)^2}+512S_2\Biggr)S_1-\frac{64\left(4+3N+3N^2\right)}{N(1+N)}S_2-512S_2+128S_3-4608S_{2,1}$$

$$+256S_{-2,1}-384\zeta_3\Biggr)S_{-2}+\Biggl(\frac{128}{N(1+N)}-256S_1\Biggr)S_{-2}^2+\Biggl(\frac{32\left(20+17N+21N^2\right)}{N^2(1+N)^2}+1280S_1^2$$

$$-2176S_2-\frac{64\left(20+3N+3N^2\right)}{N(1+N)}S_1-640S_{-2}\Biggr)S_{-3}+\Biggl(-\frac{32\left(26+3N+3N^2\right)}{N(1+N)}+1664S_1\Biggr)S_{-4}$$

$$-1792S_{-5}-384S_{2,3}-2304S_{2,-3}+\frac{128\left(-2+3N+3N^2\right)}{N(1+N)}S_{3,1}+384S_{4,1}-1664S_{-4,1}$$

$$+4608S_{2,1,-2}-\frac{64\left(16+11N+15N^2\right)}{N^2(1+N)^2}S_{-2,1}-\frac{64\left(-26+3N+3N^2\right)}{N(1+N)}S_{-2,2}+2944S_{-2,3}$$

$$+\frac{1792}{N(1+N)}S_{-3,1}-768S_{3,1,1}+\frac{768\left(-4+N+N^2\right)}{N(1+N)}S_{-2,1,1}+1024S_{-2,1,-2}+4608[S_{-2,2,1}$$

$$+S_{-3,1,1}]-9216S_{-2,1,1,1}\Biggr\}\Biggr\}.$$

The non-singlet anomalous dimension for transversity $\gamma_{qq}^{(2),+,\mathrm{NS},\mathrm{tr}}$

$$
\begin{aligned}
\gamma_{\rm NS}^{(2),{\rm tr},+} &= \frac{1}{2}[1+(-1)^N] \\
&\times \Biggl\{ C_F \Biggl\{ T_F^2 N_F^2 \Biggl[\frac{8(-8+17N+17N^2)}{9N(1+N)} - \frac{128}{27}S_1 - \frac{640}{27}S_2 + \frac{128}{9}S_3 \Biggr] \\
&+ C_A T_F N_F \Biggl[-\frac{16(-22+45N+45N^2)}{9N(1+N)} + \Biggl(-\frac{16(9+209N+209N^2)}{27N(1+N)} + 64S_3 \\
&+ \frac{256}{3}S_{-2,1} - 128\zeta_3 \Biggr) S_1 + \frac{5344}{27}S_2 - \frac{448}{3}S_3 + \frac{320}{3}S_4 + \Biggl(-\frac{1280}{9}S_1 + \frac{128}{3}S_2 \Biggr) S_{-2} \\
&+ \Biggl(-\frac{640}{9} + \frac{128}{3}S_1 \Biggr) S_{-3} + \frac{128}{3}S_{-4} - \frac{256}{3}S_{3,1} + \frac{1280}{9}S_{-2,1} + \frac{128}{3}S_{-2,2} - \frac{512}{3}S_{-2,1,1} \\
&+ 96\zeta_3 \Biggr] + C_A^2 \Biggl[\frac{-968+1657N+1657N^2}{18N(1+N)} + \Biggl(\frac{4P_{14}}{3(-1+N)N(1+N)(2+N)} - 176S_3 \\
&-256S_4 + 512S_{3,1} - \frac{704}{3}S_{-2,1} - 1024S_{-2,2} - 1024S_{-3,1} + 2048S_{-2,1,1} \Biggr) S_1 + \Biggl(-128S_3 \\
&-512S_{-2,1} \Biggr) S_1^2 + \Biggl(-\frac{8344}{27} + 384S_3 + 1536S_{-2,1} \Biggr) S_2 + \frac{3112}{9}S_3 - \frac{880}{3}S_4 + 64S_5 \\
&+ \Biggl(\frac{16P_2}{(-1+N)N(1+N)(2+N)} + \frac{32(-241+134N+134N^2)S_1}{9(-1+N)(2+N)} - \frac{352}{3}S_2 - 64S_3 \\
&-1536S_{2,1} + 128S_{-2,1} - 192\zeta_3 \Biggr) S_{-2} + \Biggl(48 - 192S_1 \Biggr) S_{-2}^2 + \Biggl(256S_1^2 - 768S_2 - 320S_{-2} \\
&+ \frac{32(-107+67N+67N^2)}{9(-1+N)(2+N)} - \frac{352}{3}S_1 \Biggr) S_{-3} + \Biggl(-\frac{208}{3} + 320S_1 \Biggr) S_{-4} - 704S_{-5} - 384S_{2,3} \\
&-768S_{2,-3} + \frac{704}{3}S_{3,1} + 384S_{4,1} - \frac{64(-107+67N+67N^2)S_{-2,1}}{9(-1+N)(2+N)} - \frac{352}{3}S_{-2,2} \\
&+1088S_{-2,3} - 448S_{-4,1} + 1536[S_{2,1,-2} + S_{-2,2,1} + S_{-3,1,1}] - 768S_{3,1,1} \\
&+ \frac{1408}{3}S_{-2,1,1} + 512S_{-2,1,-2} - 3072S_{-2,1,1,1} - \frac{24(-6+5N+5N^2)\zeta_3}{(-1+N)(2+N)} \Biggr] \Biggr\} \\
&+ C_F^2 \Biggl\{ T_F N_F \Biggl[92 + \Biggl(-\frac{8(-8+55N+55N^2)}{3N(1+N)} + \frac{1280}{9}S_2 - \frac{512}{3}S_3 - \frac{512}{3}S_{-2,1} + 128\zeta_3 \Biggr) \\
&\times S_1 - \frac{80}{3}S_2 - \frac{128}{3}S_2^2 + \frac{1856}{9}S_3 - \frac{512}{3}S_4 + \Biggl(\frac{2560}{9}S_1 - \frac{256}{3}S_2 \Biggr) S_{-2} + \Biggl(\frac{1280}{9} \\
&- \frac{256}{3}S_1 \Biggr) S_{-3} - \frac{256}{3}S_{-4} + \frac{256}{3}S_{3,1} - \frac{2560}{9}S_{-2,1} - \frac{256}{3}S_{-2,2} + \frac{1024}{3}S_{-2,1,1} - 96\zeta_3 \Biggr] \\
&+ C_A \Biggl[-\frac{151}{2} + \Biggl(-\frac{8(-206+211N+211N^2)}{3(-1+N)N(1+N)(2+N)} - \frac{4288}{9}S_2 + \frac{1984}{3}S_3 + 320S_4 - 1024S_{3,1} \\
&+ \frac{1984}{3}S_{-2,1} + 3712S_{-2,2} + 3840S_{-3,1} - 7168S_{-2,1,1} \Biggr) S_1 + \Biggl(256S_3 + 1792S_{-2,1} \Biggr) S_1^2 \\
&+ \Biggl(\frac{604}{3} - 832S_3 - 5248S_{-2,1} \Biggr) S_2 + \frac{352}{3}S_2^2 - \frac{6160}{9}S_3 + \frac{2416}{3}S_4 \\
&+ \Biggl(-\frac{48P_2}{(-1+N)N(1+N)(2+N)} + \Biggl(-\frac{64P_7}{9(-1+N)N(1+N)(2+N)} - 256S_2 \Biggr) S_1 \\
&+ \frac{992}{3}S_2 + 64S_3 + 5376S_{2,1} - 384S_{-2,1} + 576\zeta_3 \Biggr) S_{-2} + \Biggl(-96 + 512S_1 \Biggr) S_{-2}^2 \\
&+ \Biggl(-\frac{32(-187+134N+134N^2)}{9(-1+N)(2+N)} + \frac{992}{3}S_1 - 1152S_1^2 + 2624S_2 + 960S_{-2} \Biggr) S_{-3} \\
&+ \Biggl(\frac{560}{3} - 1472S_1 \Biggr) S_{-4} + 2304S_{-5} + 768S_{2,3} + 2688S_{2,-3} - \frac{1856}{3}S_{3,1} - 768S_{4,1} \\
&+ \frac{64(-187+134N+134N^2)S_{-2,1}}{9(-1+N)(2+N)} + \frac{992}{3}S_{-2,2} - 3648S_{-2,3} + 1728S_{-4,1} \\
&-5376[S_{2,1,-2} + S_{-2,2,1} + S_{-3,1,1}] + 1536[S_{3,1,1} - S_{-2,1,-2}] - \frac{3968}{3}S_{-2,1,1} \\
&+10752S_{-2,1,1,1} + \frac{72(-6+5N+5N^2)\zeta_3}{(-1+N)(2+N)} \Biggr] \Biggr\} \\
&+ C_F^3 \Biggl\{ -29 + \Biggl(\frac{384(-1+N+N^2)}{(-1+N)N(1+N)(2+N)} + 128S_2^2 - 384S_3 + 128S_4 + 512S_{3,1} \\
&-384S_{-2,1} - 3328S_{-2,2} - 3584S_{-3,1} + 6144S_{-2,1,1} \Biggr) S_1 - 256S_{-2}^2 S_1 + \Biggl(12 + 512S_3 \\
&+4352S_{-2,1} \Biggr) S_2 - 96S_2^2 + 104S_3 - 480S_4 + \Biggl(\frac{32P_2}{(-1+N)N(1+N)(2+N)} \\
&+ \Biggl(\frac{384}{(-1+N)N(1+N)(2+N)} + 512S_2 \Biggr) S_1 - 192S_2 + 128S_3 - 4608S_{2,1} + 256S_{-2,1} \\
&-384\zeta_3 \Biggr) S_{-2} + \Biggl(\frac{192}{(-1+N)(2+N)} - 192S_1 + 1280S_1^2 - 2176S_2 - 640S_{-2} \Biggr) S_{-3} \\
&+ \Biggl(-96 + 1664S_1 \Biggr) S_{-4} - 1792S_{-5} + 384[-S_{2,3} + S_{3,1} + S_{4,1}] - 2304S_{2,-3} \\
&- \frac{384S_{-2,1}}{(-1+N)(2+N)} - 1536S_1^2 S_{-2,1} - 192S_{-2,2} + 2944S_{-2,3} - 1664S_{-4,1} + 4608S_{2,1,-2} \\
&-768S_{3,1,1} + 768S_{-2,1,1} + 1024S_{-2,1,-2} + 4608[S_{-2,2,1} + S_{-3,1,1}] - 9216S_{-2,1,1,1} \\
&- \frac{48(-6+5N+5N^2)\zeta_3}{(-1+N)(2+N)} \Biggr\} \Biggr\},
\end{aligned}
$$

The non-singlet anomalous dimension for transversity $\gamma_{qq}^{(2),-,\mathrm{NS,tr}}$

$$\gamma_{\rm NS}^{(2),{\rm tr},-} = \frac{1}{2}\,[1-(-1)^N]\,\Big\{C_F\Big\{T_F^2N_F^2\Big[\frac{8(-8+17N+17N^2)}{9N(1+N)} - \frac{128}{27}S_1 - \frac{640}{27}S_2 + \frac{128}{9}S_3\Big]$$

$$+C_AT_FN_F\Big[-\frac{16(-8+49N+90N^2+45N^3)}{9N(1+N)^2} + \Big(-\frac{16(-27+209N+209N^2)}{27N(1+N)}$$

$$+64S_3 + \frac{256}{3}S_{-2,1} - 128\zeta_3\Big)S_1 + \frac{5344}{27}S_2 - \frac{448}{3}S_3 + \frac{320}{3}S_4 + \Big(-\frac{1280}{9}S_1 + \frac{128}{3}S_2\Big)S_{-2}$$

$$+\Big(-\frac{640}{9} + \frac{128}{3}S_1\Big)S_{-3} + \frac{128}{3}S_{-4} - \frac{256}{3}S_{3,1} + \frac{1280}{9}S_{-2,1} + \frac{128}{3}S_{-2,2} - \frac{512}{3}S_{-2,1,1}$$

$$+96\zeta_3\Big] + C_A^2\Big[\frac{P_{23}}{18(-1+N)N(1+N)^2(2+N)} + \Big(\frac{4(12+245N+245N^2)}{3N(1+N)} - 176S_3$$

$$-256S_4 + 512S_{3,1} - \frac{704}{3}S_{-2,1} - 1024S_{-2,2} - 1024S_{-3,1} + 2048S_{-2,1,1}\Big)S_1 + \Big(-128S_3$$

$$-512S_{-2,1}\Big)S_1^2 + \Big(-\frac{8344}{27} + 384S_3 + 1536S_{-2,1}\Big)S_2 + \frac{3112}{9}S_3 - \frac{880}{3}S_4 + 64S_5$$

$$+\Big(\frac{16(-5+3N+3N^2)}{(-1+N)(2+N)} + \frac{32(81+134N+134N^2)}{9N(1+N)}S_1 - \frac{352}{3}S_2 - 64S_3 - 1536S_{2,1}$$

$$+128S_{-2,1} - 192\zeta_3\Big)S_{-2} + \Big(48 - 192S_1\Big)S_{-2}^2 + \Big(\frac{32(81+67N+67N^2)}{9N(1+N)} - \frac{352}{3}S_1$$

$$+256S_1^2 - 768S_2 - 320S_{-2}\Big)S_{-3} + \Big(-\frac{208}{3} + 320S_1\Big)S_{-4} - 704S_{-5} - 384S_{2,3} - 768S_{2,-3}$$

$$+\frac{704}{3}S_{3,1} + 384S_{4,1} - \frac{64(81+67N+67N^2)S_{-2,1}}{9N(1+N)} - \frac{352}{3}S_{-2,2} + 1088S_{-2,3} - 448S_{-4,1}$$

$$+1536S_{2,1,-2} - 768S_{3,1,1} + \frac{1408}{3}S_{-2,1,1} + 512S_{-2,1,-2} + 1536[S_{-2,2,1} + S_{-3,1,1}]$$

$$-3072S_{-2,1,1,1} - \frac{24(12+5N+5N^2)\zeta_3}{N(1+N)}\Big]\Big\}$$

$$+C_F^2\Big\{T_FN_F\Big[\frac{4(112+415N+414N^2+207N^3)}{9N(1+N)^2} + \Big(-\frac{8(8+55N+55N^2)}{3N(1+N)} + \frac{1280}{9}S_2$$

$$-\frac{512}{3}S_3 - \frac{512}{3}S_{-2,1} + 128\zeta_3\Big)S_1 - \frac{80}{3}S_2 - \frac{128}{3}S_2^2 + \frac{1856}{9}S_3 - \frac{512}{3}S_4 + \Big(\frac{2560}{9}S_1$$

$$-\frac{256}{3}S_2\Big)S_{-2} + \Big(\frac{1280}{9} - \frac{256}{3}S_1\Big)S_{-3} - \frac{256}{3}S_{-4} + \frac{256}{3}S_{3,1} - \frac{2560}{9}S_{-2,1} - \frac{256}{3}S_{-2,2}$$

$$+\frac{1024}{3}S_{-2,1,1} - 96\zeta_3\Big] + C_A\Big[\frac{P_{20}}{18(-1+N)N(1+N)^2(2+N)}$$

$$+\Big(-\frac{8P_6}{3(-1+N)N^2(1+N)^2(2+N)} - \frac{4288}{9}S_2 + \frac{1984}{3}S_3 + 320S_4 - 1024S_{3,1}$$

$$+\frac{1984}{3}S_{-2,1} + 3712S_{-2,2} + 3840S_{-3,1} - 7168S_{-2,1,1}\Big)S_1 + \Big(256S_3 + 1792S_{-2,1}\Big)S_1^2$$

$$+\Big(\frac{4(-24+151N+151N^2)}{3N(1+N)} - 832S_3 - 5248S_{-2,1}\Big)S_2 + \frac{352}{3}S_2^2 - \frac{6160}{9}S_3 + \frac{2416}{3}S_4$$

$$+\Big(-\frac{48(-5+3N+3N^2)}{(-1+N)(2+N)} + \Big(-\frac{64P_8}{9(-1+N)N(1+N)(2+N)} - 256S_2\Big)S_1 + \frac{992}{3}S_2$$

$$+64S_3 + 5376S_{2,1} - 384S_{-2,1} + 576\zeta_3\Big)S_{-2} + \Big(-96 + 512S_1\Big)S_{-2}^2$$

$$+\Big(-\frac{32(243+134N+134N^2)}{9N(1+N)} + \frac{992}{3}S_1 - 1152S_1^2 + 2624S_2 + 960S_{-2}\Big)S_{-3} + \Big(\frac{560}{3}$$

$$-1472S_1\Big)S_{-4} + 2304S_{-5} + 768S_{2,3} + 2688S_{2,-3} - \frac{1856}{3}S_{3,1} - 768S_{4,1}$$

$$+\frac{64(243+134N+134N^2)}{9N(1+N)}S_{-2,1} + \frac{992}{3}S_{-2,2} - 3648S_{-2,3} + 1728S_{-4,1} - 5376S_{2,1,-2}$$

$$-1536[S_{-2,1,-2} - S_{3,1,1}] - 5376[S_{-2,2,1} + S_{-3,1,1}] - \frac{3968}{3}S_{-2,1,1} + 10752S_{-2,1,1,1}$$

$$+\frac{72(12+5N+5N^2)\zeta_3}{N(1+N)}\Big]\Big\} + C_F^3\Big\{\frac{P_{21}}{(-1+N)N(1+N)^2(2+N)}$$

$$+\Big(-\frac{32P_1}{(-1+N)N^2(1+N)^2(2+N)} + 128S_2^2 - 384[S_3 + S_{-2,1}] + 128S_4 + 512S_{3,1}$$

$$-3328S_{-2,2} - 3584S_{-3,1} + 6144S_{-2,1,1}\Big)S_1 - 256S_{-2}^2S_1 + \Big(\frac{4(16+3N+3N^2)}{N(1+N)}$$

$$+512S_3 + 4352S_{-2,1}\Big)S_2 - 96S_2^2 + 104S_3 - 480S_4 + \Big(\frac{32(-5+3N+3N^2)}{(-1+N)(2+N)}$$

$$+\Big(\frac{128(-9+4N+4N^2)}{(-1+N)N(1+N)(2+N)} + 512S_2\Big)S_1 - 192S_2 + 128S_3 - 4608S_{2,1}$$

$$+256S_{-2,1} - 384\zeta_3\Big)S_{-2} + \Big(\frac{576}{N(1+N)} - 192S_1 + 1280S_1^2 - 2176S_2 - 640S_{-2}\Big)S_{-3}$$

$$+\Big(-96 + 1664S_1\Big)S_{-4} - 1792S_{-5} - 384[S_{2,3} - S_{3,1} - S_{4,1}] - 2304S_{2,-3} - \frac{1152}{N(1+N)}S_{-2,1}$$

$$-1536S_1^2S_{-2,1} - 192S_{-2,2} + 2944S_{-2,3} - 1664S_{-4,1} + 4608S_{2,1,-2} - 768[S_{3,1,1} - S_{-2,1,1}]$$

$$+1024S_{-2,1,-2} + 4608[S_{-2,2,1} + S_{-3,1,1} - S_{-2,1,1,1}] - \frac{48(12+5N+5N^2)\zeta_3}{N(1+N)}\Big\}\Big\}.$$

The polarized singlet anomalous dimension $\Delta\gamma_{qq}^{(2),\mathrm{PS}}$

$$
\begin{aligned}
\Delta\gamma_{qq}^{(2),\mathrm{PS}} = \; & C_F \Biggl[T_F^2 N_F^2 \Biggl[-\frac{64(2+N)P_{30}}{27N^4(1+N)^4} + \frac{64(2+N)\bigl(6+10N-3N^2+11N^3\bigr)}{9N^3(1+N)^3} S_1 \\
& -\frac{32(N-1)(2+N)}{3N^2(1+N)^2}[S_1^2+S_2] \Biggr] + C_A T_F N_F \Biggl[\frac{8P_9}{3N^3(1+N)^3} S_1^2 + \frac{8P_{10}}{3N^3(1+N)^3} S_2 \\
& + \frac{16P_{61}}{27N^5(1+N)^5} + \Biggl(-\frac{16P_{51}}{9N^4(1+N)^4} + \frac{32(N-1)(2+N)}{N^2(1+N)^2} S_2 \Biggr) S_1 \\
& -\frac{32(N-1)(2+N)}{3N^2(1+N)^2} S_1^3 + \frac{16\bigl(-58+23N+23N^2\bigr)}{3N^2(1+N)^2} S_3 + \Biggl(-\frac{32P_1}{N^3(1+N)^3} \\
& + \frac{64(N-1)(2+N)}{N^2(1+N)^2} S_1 \Biggr) S_{-2} + \frac{32\bigl(-10+7N+7N^2\bigr)}{N^2(1+N)^2} S_{-3} - \frac{64(N-1)(2+N)}{N^2(1+N)^2} S_{2,1} \\
& -\frac{64\bigl(-2+3N+3N^2\bigr)}{N^2(1+N)^2} S_{-2,1} - \frac{192(N-1)(2+N)}{N^2(1+N)^2} \zeta_3 \Biggr] \Biggr] \\
& + C_F^2 T_F N_F \Biggl[-\frac{16(2+N)P_{54}}{N^5(1+N)^5} + \Biggl(\frac{16(2+N)P_{27}}{N^4(1+N)^4} - \frac{32(N-1)(2+N)}{N^2(1+N)^2} S_2 \Biggr) S_1 \\
& -\frac{8(N-1)(2+N)\bigl(2+3N+3N^2\bigr)}{N^3(1+N)^3} S_1^2 + \frac{32(N-1)(2+N)}{3N^2(1+N)^2} S_1^3 \\
& -\frac{8(2+N)\bigl(14+23N+11N^3\bigr)}{N^3(1+N)^3} S_2 - \frac{224(N-1)(2+N)}{3N^2(1+N)^2} S_3 \\
& + \frac{64(N-1)(2+N)}{N^2(1+N)^2} S_{2,1} + \frac{192(N-1)(2+N)}{N^2(1+N)^2} \zeta_3 \Biggr]
\end{aligned}
$$

The polynomials in the non-singlet cases are different from those in the polarized singlet case.

The polarized singlet anomalous dimension $\Delta\gamma_{qg}^{(2)}$

$$
\begin{aligned}
\Delta\gamma_{qg}^{(2)} = {} & C_F T_F^2 N_F^2 \Bigg[\frac{4P_{64}}{27N^5(1+N)^5} + \Bigg(-\frac{32\big(-24+4N+47N^2\big)}{27N^2(1+N)} - \frac{32(N-1)}{3N(1+N)}S_2\Bigg)S_1 \\
& + \frac{32(N-1)(3+10N)}{9N^2(1+N)}S_1^2 - \frac{32(N-1)}{9N(1+N)}S_1^3 + \frac{32(5N-1)}{3N^2(1+N)}S_2 + \frac{320(N-1)}{9N(1+N)}S_3\Bigg] \\
& + C_A C_F T_F N_F \Bigg[\frac{8P_{29}}{3N^3(1+N)^3}S_2 + \frac{P_{65}}{27N^5(1+N)^5(2+N)} + \Bigg(-\frac{384(N-1)}{N(1+N)}S_{2,1} \\
& + \frac{16P_{60}}{27N^4(1+N)^4(2+N)} + \frac{16\big(75+14N+18N^2+N^3\big)S_2}{3N^2(1+N)^2} + \frac{640(N-1)}{3N(1+N)}S_3 \\
& - \frac{192(N-1)}{N(1+N)}\zeta_3\Bigg)S_1 + \Bigg(-\frac{8P_{25}}{9N^3(1+N)^3} + \frac{160(N-1)}{N(1+N)}S_2\Bigg)S_1^2 \\
& + \frac{16\big(3-31N-18N^2+10N^3\big)}{9N^2(1+N)^2}S_1^3 + \frac{32(N-1)}{3N(1+N)}S_1^4 - \frac{64(N-1)}{N(1+N)}S_2^2 \\
& - \frac{16(N-1)\big(240-17N+19N^2\big)}{9N^2(1+N)^2}S_3 + \Bigg(\frac{128(N-1)\big(-4-N+N^2\big)}{N^2(1+N)^2(2+N)}S_1 \\
& - \frac{32P_{24}}{N^3(1+N)^3(2+N)} + \frac{192(N-1)}{N(1+N)}S_1^2\Bigg)S_{-2} + \frac{96(N-1)}{N(1+N)}S_{-2}^2 \\
& + \frac{32(N-1)(2+N)(-1+3N)}{N^2(1+N)^2}S_{-3} + \frac{96(N-1)\big(4+N+N^2\big)}{N^2(1+N)^2}S_{2,1} \\
& + \frac{160(N-1)}{N(1+N)}S_{-4} + \frac{64(N-1)}{N(1+N)}S_{3,1} - \frac{128(N-1)^2}{N^2(1+N)^2}S_{-2,1} + \frac{64(N-1)}{N(1+N)}S_{-2,2} \\
& + \frac{192(N-1)}{N(1+N)}S_{2,1,1} - \frac{256(N-1)}{N(1+N)}S_{-2,1,1} - \frac{192(N-1)\big(-5+3N+3N^2\big)}{N^2(1+N)^2}\zeta_3\Bigg] \\
& + C_A T_F^2 N_F^2 \Bigg[\frac{16P_{55}}{27N^4(1+N)^4} + \Bigg(\frac{64\big(23+50N+10N^2+19N^3\big)}{27N(1+N)^3} - \frac{32(N-1)}{3N(1+N)} \\
& \times S_2\Bigg)S_1 - \frac{64\big(-2+5N^2\big)}{9N(1+N)^2}S_1^2 + \frac{32(N-1)}{9N(1+N)}S_1^3 - \frac{64\big(-2+6N+5N^2\big)}{9N(1+N)^2}S_2 \\
& + \frac{64(N-1)}{9N(1+N)}S_3 - \frac{128(-2+5N)}{9N(1+N)}S_{-2} + \frac{128(N-1)}{3N(1+N)}[S_{-3}+S_{2,1}]\Bigg] \\
& + C_A^2 N_F T_F \Bigg[\frac{16P_{34}}{9N^3(1+N)^3}S_2 - \frac{8P_{70}}{27N^5(1+N)^5(2+N)} + \Bigg(-\frac{8P_{56}}{27N^4(1+N)^4} \\
& + \frac{8\big(-72+181N-48N^2+11N^3\big)}{3N^2(1+N)^2}S_2 - \frac{704(N-1)}{3N(1+N)}S_3 + \frac{128(N-1)}{N(1+N)}S_{2,1} \\
& + \frac{512(N-1)}{N(1+N)}S_{-2,1} + \frac{192(N-1)}{N(1+N)}\zeta_3\Bigg)S_1 + \Bigg(\frac{16P_{31}}{9N^3(1+N)^3} - \frac{160(N-1)}{N(1+N)}S_2\Bigg)S_1^2 \\
& - \frac{8\big(-24-59N+11N^3\big)}{9N^2(1+N)^2}S_1^3 - \frac{16(N-1)}{3N(1+N)}S_1^4 - \frac{16(N-1)}{N(1+N)}S_2^2 \\
& - \frac{16\big(345-428N+11N^3\big)}{9N^2(1+N)^2}S_3 - \frac{32(N-1)}{N(1+N)}S_4 + \Bigg(\frac{32P_{49}}{9N^3(1+N)^3(2+N)} \\
& - \frac{64(-5+N)(-1+2N)}{N^2(1+N)^2}S_1 - \frac{192(N-1)}{N(1+N)}S_1^2 - \frac{128(N-1)}{N(1+N)}S_2\Bigg)S_{-2} \\
& - \frac{96(N-1)}{N(1+N)}S_{-2}^2 + \Bigg(-\frac{32\big(69-92N+11N^3\big)}{3N^2(1+N)^2} - \frac{512(N-1)}{N(1+N)}S_1\Bigg)S_{-3} \\
& - \frac{352(N-1)}{N(1+N)}S_{-4} - \frac{32(N-1)\big(24+11N+11N^2\big)}{3N^2(1+N)^2}S_{2,1} - \frac{128(N-1)}{N(1+N)}S_{3,1} \\
& - \frac{64(-7+11N)}{N^2(1+N)^2}S_{-2,1} + \frac{448(-1+N)}{N(1+N)}S_{-2,2} + \frac{512(-1+N)}{N(1+N)}S_{-3,1} \\
& - \frac{768(-1+N)}{N(1+N)}S_{-2,1,1} + \frac{96(-1+N)\big(-8+3N+3N^2\big)}{N^2(1+N)^2}\zeta_3\Bigg] \\
& + C_F^2 T_F N_F \Bigg[-\frac{8P_{26}}{N^3(1+N)^3}S_1^2 + \frac{8P_{28}}{N^3(1+N)^3}S_2 + \frac{P_{63}}{N^4(1+N)^5(2+N)} \\
& + \Bigg(-\frac{8P_{53}}{N^4(1+N)^4} - \frac{8\big(-6+7N+28N^2+3N^3\big)}{N^2(1+N)^2}S_2 - \frac{704(N-1)}{3N(1+N)}S_3 \\
& + \frac{256(N-1)}{N(1+N)}S_{2,1}\Bigg)S_1 - \frac{8(N-1)\big(-10-9N+3N^2\big)}{3N^2(1+N)^2}S_1^3 - \frac{16(N-1)}{3N(1+N)}S_1^4 \\
& - \frac{48(N-1)}{N(1+N)}S_2^2 - \frac{16(N-1)\big(-22+27N+3N^2\big)}{3N^2(1+N)^2}S_3 - \frac{160(N-1)}{N(1+N)}S_4 \\
& + \Bigg(\frac{64P_{21}}{N^2(1+N)^3(2+N)} - \frac{256(N-1)}{N(1+N)^2}S_1 - \frac{128(N-1)}{N(1+N)}S_2\Bigg)S_{-2} - \frac{64(N-1)}{N(1+N)}S_{-2}^2 \\
& + \Bigg(-\frac{128(N-1)^2}{N^2(1+N)^2} - \frac{256(N-1)}{N(1+N)}S_1\Bigg)S_{-3} - \frac{320(N-1)}{N(1+N)}S_{-4} - \frac{128(N-1)}{N^2(1+N)^2}S_{2,1} \\
& + \frac{64(N-1)}{N(1+N)}S_{3,1} + \frac{256(N-1)}{N(1+N)^2}S_{-2,1} + \frac{128(N-1)}{N(1+N)}S_{-2,2} + \frac{256(N-1)}{N(1+N)}S_{-3,1} \\
& - \frac{192(N-1)}{N(1+N)}S_{2,1,1} + \frac{96(N-1)\big(-2+3N+3N^2\big)}{N^2(1+N)^2}\zeta_3\Bigg]
\end{aligned}
$$

The polarized singlet anomalous dimension $\Delta\gamma_{gq}^{(2)}$

$$
\begin{aligned}
\Delta\gamma_{gq}^{(2)} = {} & C_F^2 \Biggl[C_A \Biggl[\frac{4P_{39}}{9N^3(1+N)^3} S_2 + \frac{P_{73}}{54(N-1)N^5(1+N)^5} + \Biggl(-\frac{4P_{58}}{27N^4(1+N)^4} \\
& - \frac{8\bigl(30+203N+177N^2+49N^3\bigr)}{3N^2(1+N)^2} S_2 - \frac{640(2+N)}{3N(1+N)} S_3 + \frac{64(2+N)}{N(1+N)} S_{2,1} \\
& + \frac{128(2+N)}{N(1+N)} S_{-2,1} + \frac{288(2+N)}{N(1+N)} \zeta_3 \Biggr) S_1 + \Biggl(\frac{4P_{36}}{9N^3(1+N)^3} - \frac{16(2+N)}{N(1+N)} S_2 \Biggr) S_1^2 \\
& - \frac{8\bigl(6+85N+132N^2+50N^3\bigr)}{9N^2(1+N)^2} S_1^3 + \frac{8(2+N)\bigl(102+65N+29N^2\bigr)}{9N^2(1+N)^2} S_3 \\
& + \frac{16(2+N)}{3N(1+N)} S_1^4 - \frac{32(2+N)}{N(1+N)} S_4 + \Biggl(-\frac{16P_{44}}{(N-1)N^3(1+N)^3} + \frac{64(7+3N)}{N(1+N)} S_1 \\
& - \frac{96(2+N)}{N(1+N)} S_1^2 \Biggr) S_{-2} + \Biggl(\frac{32(2+N)(4+3N)}{N(1+N)^2} - \frac{64(2+N)}{N(1+N)} S_1 \Biggr) S_{-3} \\
& + \frac{80(2+N)}{N(1+N)} [S_{-2}^2 + S_{-4}] + \frac{16(2+N)\bigl(-6+11N+11N^2\bigr)}{3N^2(1+N)^2} S_{2,1} + \frac{224(2+N)}{N(1+N)} S_{3,1} \\
& - \frac{32(2+N)(5+3N)}{N(1+N)^2} S_{-2,1} + \frac{32(2+N)}{N(1+N)} S_{-2,2} - \frac{96(2+N)}{N(1+N)} S_{2,1,1} \\
& - \frac{128(2+N)}{N(1+N)} S_{-2,1,1} - \frac{432(2+N)\zeta_3}{N(1+N)} \Biggr] + T_F N_F \Biggl[\frac{2P_{71}}{27(N-1)N^5(1+N)^5} \\
& + \Biggl(\frac{32(2+N)P_{14}}{27N^3(1+N)^3} + \frac{208(2+N)}{3N(1+N)} S_2 \Biggr) S_1 - \frac{16(2+N)\bigl(-3+16N+37N^2\bigr)}{9N^2(1+N)^2} S_1^2 \\
& + \frac{80(2+N)}{9N(1+N)} S_1^3 - \frac{16(2+N)\bigl(9+46N+67N^2\bigr)}{9N^2(1+N)^2} S_2 + \frac{256(2+N)}{9N(1+N)} S_3 \\
& + \frac{256}{(N-1)N^2(1+N)^2} S_{-2} - \frac{64(2+N)}{3N(1+N)} S_{2,1} - \frac{128(2+N)}{N(1+N)} \zeta_3 \Biggr] \Biggr] \\
& + C_F \Biggl[T_F^2 N_F^2 \Biggl[\frac{64(2+N)\bigl(3+7N+N^2\bigr)}{9N(1+N)^3} + \frac{64(2+N)(2+5N)}{9N(1+N)^2} S_1 \\
& - \frac{32(2+N)}{3N(1+N)} [S_1^2 + S_2] \Biggr] + C_A T_F N_F \Biggl[\frac{8P_{57}}{27(N-1)N^3(1+N)^4} \\
& + \Biggl(-\frac{16P_{37}}{27N^3(1+N)^3} + \frac{80(2+N)S_2}{3N(1+N)} \Biggr) S_1 + \frac{16\bigl(18+116N+129N^2+43N^3\bigr)}{9N^2(1+N)^2} S_1^2 \\
& - \frac{80(2+N)}{9N(1+N)} S_1^3 + \frac{16\bigl(-2+16N+9N^2+N^3\bigr)}{3N^2(1+N)^2} S_2 + \frac{512(2+N)}{9N(1+N)} S_3 \\
& + \Biggl(-\frac{64P_7}{3(-1+N)N^2(1+N)^2} + \frac{256(2+N)S_1}{3N(1+N)} \Biggr) S_{-2} + \frac{128(2+N)}{3N(1+N)} [S_{-3} - S_{-2,1}] \\
& + \frac{128(2+N)}{N(1+N)} \zeta_3 \Biggr] + C_A^2 \Biggl[\frac{2P_{35}}{3N^3(1+N)^3} S_2 - \frac{4P_{72}}{27(N-1)N^5(1+N)^5} \\
& + \Biggl(\frac{4P_{62}}{27(N-1)N^4(1+N)^4} - \frac{4\bigl(120+158N+141N^2+55N^3\bigr)}{3N^2(1+N)^2} S_2 + \frac{128(2+N)}{3N(1+N)} S_3 \\
& + \frac{128(2+N)}{N(1+N)} S_{-2,1} - \frac{96(2+N)}{N(1+N)} \zeta_3 \Biggr) S_1 + \Biggl(-\frac{2P_{38}}{9N^3(1+N)^3} + \frac{48(2+N)}{N(1+N)} S_2 \Biggr) S_1^2 \\
& + \frac{4\bigl(24+158N+165N^2+55N^3\bigr)}{9N^2(1+N)^2} S_1^3 - \frac{8(2+N)}{3N(1+N)} S_1^4 - \frac{40(2+N)}{N(1+N)} S_2^2 \\
& - \frac{8\bigl(-186+295N+528N^2+176N^3\bigr)}{9N^2(1+N)^2} S_3 - \frac{16(2+N)}{N(1+N)} S_4 \\
& + \Biggl(-\frac{32P_{12}}{3(N-1)N^2(1+N)^2} S_1 + \frac{16P_{47}}{3(N-1)N^3(1+N)^3} + \frac{96(2+N)}{N(1+N)} S_1^2 \\
& - \frac{64(2+N)}{N(1+N)} S_2 \Biggr) S_{-2} - \frac{16(2+N)}{N(1+N)} S_{-2}^2 + \Biggl(-\frac{16\bigl(-126-13N+66N^2+22N^3\bigr)}{3N^2(1+N)^2} \\
& - \frac{192(2+N)}{N(1+N)} S_1 \Biggr) S_{-3} - \frac{176(2+N)}{N(1+N)} S_{-4} + \frac{32\bigl(-30+13N+33N^2+11N^3\bigr)}{3N^2(1+N)^2} S_{-2,1} \\
& - \frac{64(2+N)}{N(1+N)} S_{3,1} + \frac{224(2+N)}{N(1+N)} S_{-2,2} + \frac{256(2+N)}{N(1+N)} S_{-3,1} - \frac{384(2+N)}{N(1+N)} S_{-2,1,1} \\
& + \frac{144(2+N)}{N(1+N)} \zeta_3 \Biggr] \Biggr] + C_F^3 \Biggl[-\frac{2(2+N)P_{19}}{N^3(1+N)^3} S_2 + \frac{P_{66}}{2(N-1)N^5(1+N)^5} \\
& + \Biggl(-\frac{4(2+N)P_{43}}{N^4(1+N)^4} + \frac{4(2+N)\bigl(-2+19N+39N^2\bigr)}{N^2(1+N)^2} S_2 + \frac{128(2+N)}{3N(1+N)} S_3 \\
& - \frac{64(2+N)}{N(1+N)} S_{2,1} - \frac{192(2+N)}{N(1+N)} \zeta_3 \Biggr) S_1 + \Biggl(\frac{2(2+N)P_3}{N^3(1+N)^3} - \frac{32(2+N)}{N(1+N)} S_2 \Biggr) S_1^2 \\
& + \frac{4(2+N)\bigl(-2+3N+15N^2\bigr)}{3N^2(1+N)^2} S_1^3 - \frac{8(2+N)}{3N(1+N)} S_1^4 - \frac{24(2+N)}{N(1+N)} S_2^2 \\
& + \frac{64(2+N)\bigl(-1+3N^2\bigr)}{3N^2(1+N)^2} S_3 - \frac{48(2+N)}{N(1+N)} S_4 + \Biggl(\frac{32P_{22}}{(N-1)N^2(1+N)^3} \\
& - \frac{128(2+N)}{N(1+N)^2} S_1 - \frac{64(2+N)}{N(1+N)} S_2 \Biggr) S_{-2} - \frac{16(2+N)\bigl(-2+3N+3N^2\bigr)}{N^2(1+N)^2} S_{2,1} \\
& - \frac{96(2+N)}{N(1+N)} S_{-2}^2 + \Biggl(-\frac{64(N-1)(2+N)}{N^2(1+N)^2} - \frac{128(2+N)}{N(1+N)} S_1 \Biggr) S_{-3} \\
& - \frac{160(2+N)}{N(1+N)} [S_{-4} + S_{3,1}] + \frac{128(2+N)}{N(1+N)^2} [S_{-2,1} + (1+N)S_{-3,1}] \\
& + \frac{64(2+N)}{N(1+N)} S_{-2,2} + \frac{96(2+N)}{N(1+N)} S_{2,1,1} + \frac{288(2+N)\zeta_3}{N(1+N)} \Biggr]
\end{aligned}
$$

The polarized singlet anomalous dimension $\Delta\gamma_{gg}^{(2)}$

$$
\begin{aligned}
\Delta\gamma_{gg}^{(2)} = {} & C_A T_F^2 N_F^2 \Biggl[-\frac{16P_8}{27N^2(1+N)^2}S_1 - \frac{4P_{48}}{27N^3(1+N)^3}\Biggr] + C_F \Biggl[T_F^2 N_F^2 \Biggl[-\frac{8P_{59}}{27N^4(1+N)^4} \\
& + \frac{64(N-1)(2+N)\bigl(-6-8N+N^2\bigr)}{9N^3(1+N)^3}S_1 + \frac{32(N-1)(2+N)}{3N^2(1+N)^2}S_1^2 \\
& - \frac{32(N-1)(2+N)}{N^2(1+N)^2}S_2\Biggr] + C_A T_F N_F \Biggl[\frac{8P_6}{N^3(1+N)^3}S_2 - \frac{8P_9}{3N^3(1+N)^3}S_1^2 \\
& + \frac{2P_{77}}{27(N-1)N^5(1+N)^5(2+N)} + \Biggl(-\frac{8P_{67}}{9(-1+N)N^4(1+N)^4(2+N)} \\
& - \frac{32(N-1)(2+N)}{N^2(1+N)^2}S_2 + 128\zeta_3\Biggr)S_1 + \frac{32(N-1)(2+N)}{3N^2(1+N)^2}S_1^3 - \frac{32\bigl(34+N+N^2\bigr)}{3N^2(1+N)^2} \\
& \times S_3 + \Biggl(\frac{128P_2}{(N-1)N^2(1+N)^2(2+N)}S_1 - \frac{32P_{23}}{(N-1)N^2(1+N)^3(2+N)}\Biggr)S_{-2} \\
& - \frac{192\bigl(4-N-N^2\bigr)}{N^2(1+N)^2}S_{-3} + \frac{64(N-1)(2+N)}{N^2(1+N)^2}S_{2,1} - \frac{128\bigl(-8+N+N^2\bigr)}{N^2(1+N)^2}S_{-2,1} \\
& - \frac{64(-3+N)(4+N)}{N^2(1+N)^2}\zeta_3\Biggr]\Biggr] + C_A^3 \Biggl[\frac{64P_{16}}{9N^2(1+N)^2}S_{-2,1} - \frac{32P_{18}}{9N^2(1+N)^2}S_3 \\
& + \frac{P_{74}}{27(N-1)N^5(1+N)^5(2+N)} + \Biggl(\frac{4P_{69}}{9(N-1)N^4(1+N)^4(2+N)} \\
& - \frac{64P_{17}}{9N^2(1+N)^2}S_2 + 128S_2^2 + \frac{16\bigl(-96+11N+11N^2\bigr)}{3N(1+N)}S_3 + 192S_4 \\
& + \frac{1024}{N(1+N)}S_{-2,1} - 640S_{-2,2} - 768S_{-3,1} + 1024S_{-2,1,1}\Biggr)S_1 \\
& + \Biggl(-\frac{256\bigl(1+3N+3N^2\bigr)}{N^3(1+N)^3} + 128S_3 - 256S_{-2,1}\Biggr)S_1^2 + \Biggl(-\frac{16P_{41}}{9N^3(1+N)^3} \\
& + 64S_3 + 640S_{-2,1}\Biggr)S_2 - \frac{256}{N(1+N)}S_2^2 - \frac{384}{N(1+N)}S_4 + 64S_5 \\
& + \Biggl(\frac{32P_{52}}{9(N-1)N^3(1+N)^3(2+N)} + \Bigl(-\frac{64P_{32}}{9(-1+N)N(1+N)^2(2+N)} + 256S_2\Biggr) \\
& \times S_1 - \frac{512}{N(1+N)}S_2 + 128S_3 - 768S_{2,1}\Biggr)S_{-2} + \Biggl(-\frac{16\bigl(24+11N+11N^2\bigr)}{3N(1+N)} \\
& + 64S_1\Biggr)S_{-2}^2 + \Biggl(-\frac{32P_{15}}{9N^2(1+N)^2} - \frac{1536}{N(1+N)}S_1 + 384S_1^2 - 320S_2\Biggr)S_{-3} \\
& + \Biggl(-\frac{1024}{N(1+N)} + 512S_1\Biggr)S_{-4} - 192S_{-5} - 384S_{2,-3} + \frac{1280}{N(1+N)}S_{-2,2} \\
& + 384S_{-2,3} + \frac{1536}{N(1+N)}S_{-3,1} - 384S_{-4,1} + 768S_{2,1,-2} - \frac{2048}{N(1+N)}S_{-2,1,1} \\
& + 768[S_{-2,2,1} + S_{-3,1,1}] - 1536S_{-2,1,1,1}\Biggr] \\
& + C_F^2 T_F N_F \Biggl[-\frac{4P_{75}}{(N-1)N^5(1+N)^5(2+N)} + \Biggl(\frac{32(N-1)(2+N)S_2}{N^2(1+N)^2} \\
& - \frac{16P_{42}}{N^4(1+N)^4}\Biggr)S_1 + \frac{8(N-1)(2+N)\bigl(2+3N+3N^2\bigr)}{N^3(1+N)^3}S_1^2 - \frac{32(N-1)(2+N)}{3N^2(1+N)^2} \\
& \times S_1^3 - \frac{8(2+N)\bigl(2-11N-16N^2+9N^3\bigr)}{N^3(1+N)^3}S_2 + \frac{32\bigl(10+7N+7N^2\bigr)}{3N^2(1+N)^2}S_3 \\
& + \Biggl(-\frac{64\bigl(10+N+N^2\bigr)}{(N-1)N(1+N)(2+N)} + \frac{512}{N^2(1+N)^2}S_1\Biggr)S_{-2} + \frac{256}{N^2(1+N)^2}S_{-3} \\
& - \frac{64(N-1)(2+N)}{N^2(1+N)^2}S_{2,1} - \frac{512}{N^2(1+N)^2}S_{-2,1} + \frac{192\bigl(-2-N-N^2\bigr)}{N^2(1+N)^2}\zeta_3\Biggr] \\
& + C_A^2 T_F N_F \Biggl[\frac{32P_4}{9N^2(1+N)^2}S_2 + \frac{32P_{11}}{9N^2(1+N)^2}S_{-3} - \frac{64P_{11}}{9N^2(1+N)^2}S_{-2,1} \\
& + \frac{16P_{13}}{9N^2(1+N)^2}S_3 + \frac{2P_{76}}{27(N-1)N^5(1+N)^5(2+N)} + \Biggl(\frac{1280}{9}S_2 - \frac{64}{3}S_3 \\
& - \frac{8P_{68}}{27(-1+N)N^4(1+N)^4(2+N)} - 128\zeta_3\Biggr)S_1 + \frac{64}{3}S_{-2}^2 \\
& + \Biggl(\frac{64P_{45}}{9(N-1)N^2(1+N)^2(2+N)}S_1 - \frac{32P_{50}}{9(N-1)N^3(1+N)^3(2+N)}\Biggr)S_{-2} \\
& + \frac{128\bigl(-3+2N+2N^2\bigr)}{N^2(1+N)^2}\zeta_3\Biggr]
\end{aligned}
$$

Comparison to the literature

- Our results agree in the 3-loop non–singlet vector cases with Moch, Vermaseren, Vogt, 2004a, 2015.
- In the transversity case the formerly guessed expressions in Velizhanin, 2012 agree with the results of our direct calculation.
- We agree with all our earlier results obtained from massive OMEs at $O(T_F)$ (2014-2019).
- in the polarized singlet case we agree with Moch, Vermaseren, Vogt, 2014 in the M-scheme.
- Our results agree with (some of) the predictions by Gracey & Gracey et al. for the leading N_F terms.
- We also have obtained already the EOM, NGI, WI OMEs to 2-loops at higher order in the dimensional parameter ε.

Small *x* predictions: non–singlet and polarized singlet cases

- The small *x* approaches based on infrared evolution equations (IEE) or pomeron exchange never specify the factorization scheme of the derived evolution kernels/anomalous dimensions. This complicates comparisons.
- We agree with the LO non-singlet prediction for $\gamma_{qq}^{(2),\mathrm{NS},\pm}$ prediction by Kirschner and Lipatov, 1983, after correcting errors, cf. Blümlein and Vogt, 1995.
- There were no small *x* predictions for $(\Delta)\gamma_{qq}^{(2),\mathrm{NS},(\mathrm{s})}$ and the transversity anomalous dimensions.
- The LO polarized singlet small *x* predictions by Bartels, Ermolaev and Ryskin, 1996 are not in the M-scheme from 3-loop order, which our comparison shows, while there is agreement at 1- and 2-loop order.
- Agreement had been reached by Moch, Vermaseren, Vogt, 2014, however, by considering a physical scheme, defined by their 2x2 Matrix of QCD and gravitational Wilson coefficients.
- Here it essential to prove that the small *x* Wilson coefficients are of less singular order in the IEE approach, which has been investigated by Kiyo, Kodaira and Tochimura, 1996 in the non-singlet case.

Conclusions and Outlook

- We have calculated all flavor non-singlet and the polarized singlet three–loop anomalous dimensions by using the method of massless off–shell operator matrix elements. Here the method of arbitrary high Mellin moments is of instrumental importance. The current problem requires less than 2000 moments.
- The unpolarized and polarized transversity anomalous dimensions have been calculated for the first time ab initio.
- We agree with the results of earlier calculations using either the forward Compton amplitude or massive on-shell OMEs and the available predictions in for the large N_F terms.
- The comparison to pomeron-dynamics based equations, like the infrared evolution equations, in the small x range are more subtle to perform because the calculation scheme is not specified explicitly. Yet we agree with the leading series predictions given there too in a certain physical scheme.
- The two–loop unpolarized case is completely understood, correcting errors in the literature.
- The calculation of the three-loop unpolarized singlet anomalous dimensions by using the same method is forthcoming.
- The method appears to be useful to calculate 4-loop quantities.