Negative energy effects in processes involving bound particles

Outline:

- Negative energy states: part of a complete basis
- Example where important: bound muon decay
- Qualitative picture: Klein paradox vs. bound state wave fnc
- Dirac vs. Klein-Gordon
- Remaining qualitative questions

Loops and Legs 2022 Ettal

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Complete basis: states with both signs of energy

Example: Dirac equation for a free particle

$$i\partial_t \psi = (\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m)\, \psi$$

We solve it by substituting

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \exp\left(i\boldsymbol{p} \cdot \boldsymbol{r} \mp i\sqrt{m^2 + \boldsymbol{p}^2}\, t \right)$$

Resulting spinors u,v form a complete basis in which any wave function can be expanded

$$\tilde{\Phi}(\vec{k}) = \sum_r \left[A_r(\vec{k}) \frac{u_r(\vec{k})}{\sqrt{2k^0}} + B_r^*(-\vec{k}) \frac{v_r(-\vec{k})}{\sqrt{2k^0}} \right]$$

negative energy states

Example where E<0 important: bound muon decay

PHYSICAL REVIEW D **102,** 073001 (2020)

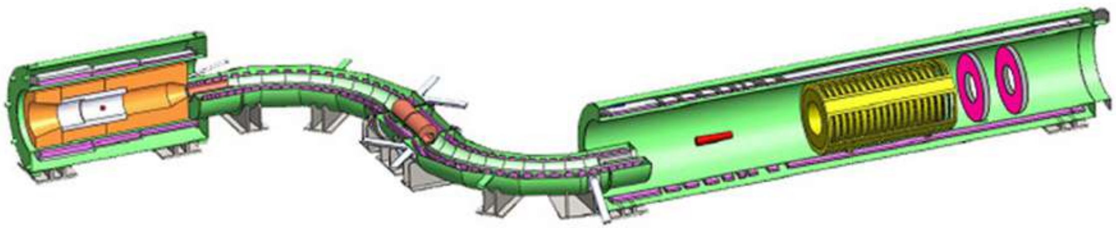
Decay of a bound muon into a bound electron

M. Jamil Aslam,[1,2] Andrzej Czarnecki,[1] Guangpeng Zhang,[1] and Anna Morozova[1]

Bound muon decay: why do we care?

Muon-electron conversion searches

Mu2e
Fermilab



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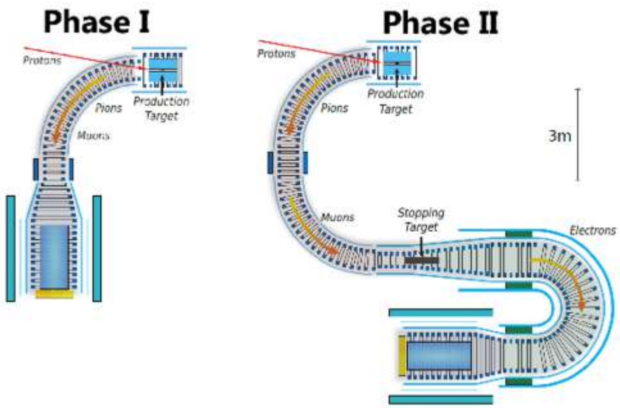

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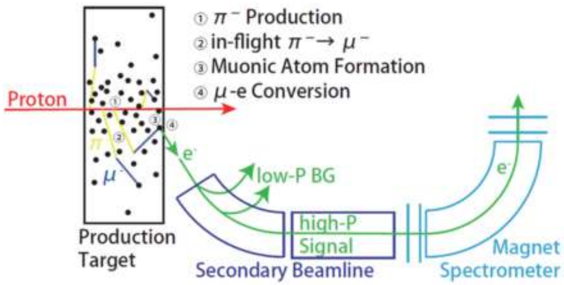
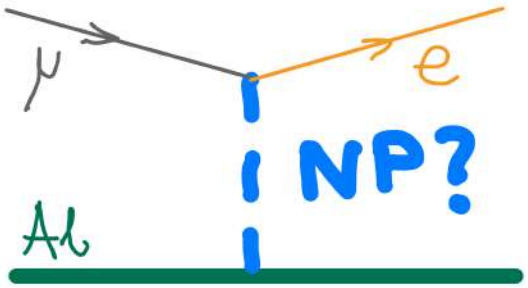
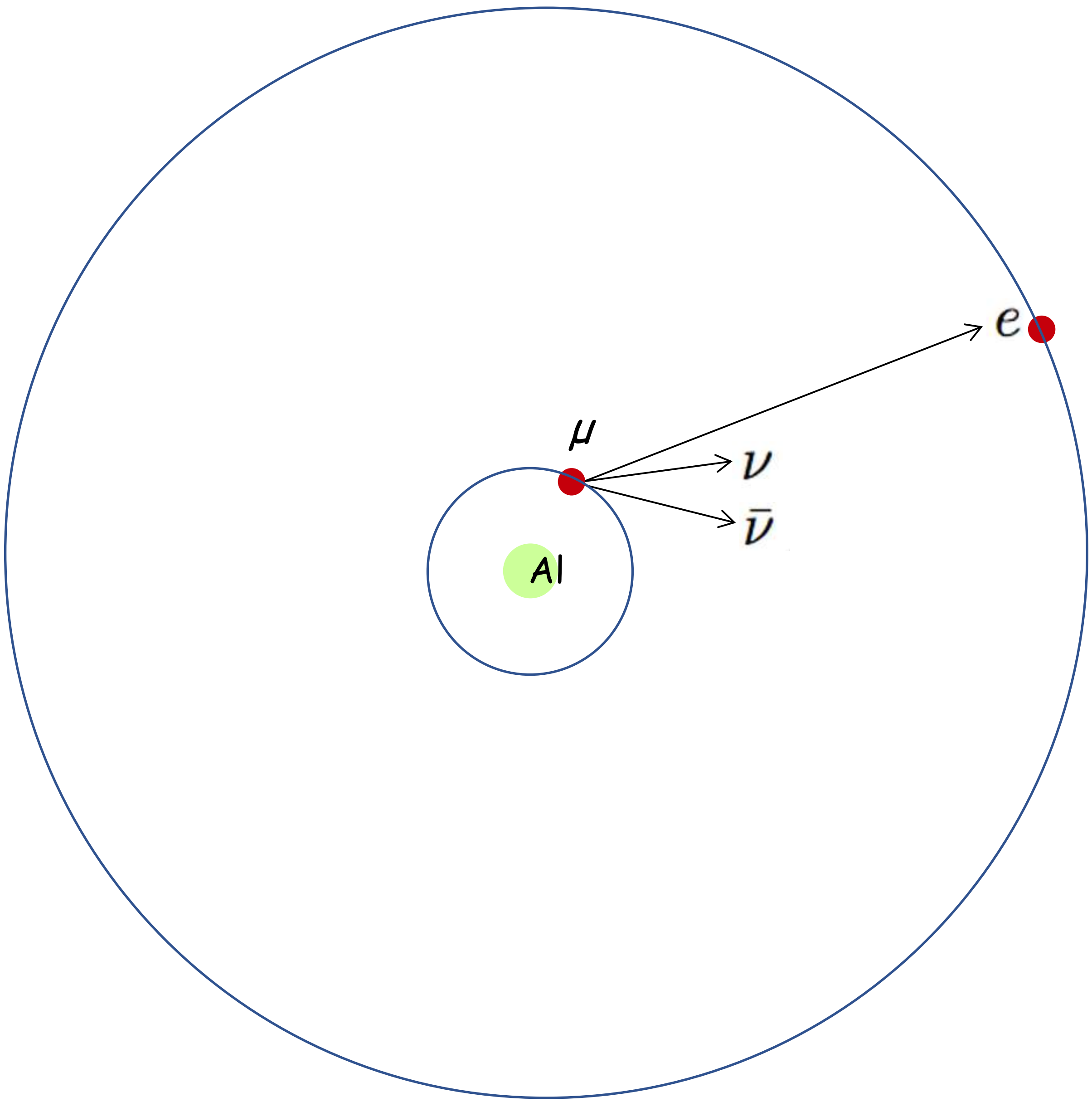




Bound muon decay into a bound electron



What is the probability of binding of the daughter electron?

Bound muon decay into a bound electron

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Atomic alchemy: Weak decays of muonic and pionic atoms into other atoms

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Bound particle wave function can be expanded in plane waves

$$\tilde{\Phi}(\vec{k}) = \sum_r \left[A_r(\vec{k}) \frac{u_r(\vec{k})}{\sqrt{2k^0}} + B_r^*(-\vec{k}) \frac{v_r(-\vec{k})}{\sqrt{2k^0}} \right]$$

negative energy states:
neglected in the "alchemy" paper:

The integral $\int [d^3k/(2\pi)^3] \sum_r |B_r(\vec{k})|^2$ gives the probability to find a three particle Fock state ($e^+e^-e^-$) in the atom. Even for $Z = 80$ this fraction is tiny ($\approx 0.2\%$), so we only consider the one-Fock contribution characterized by $A_r(\vec{k})$.

Bound muon decay: our study

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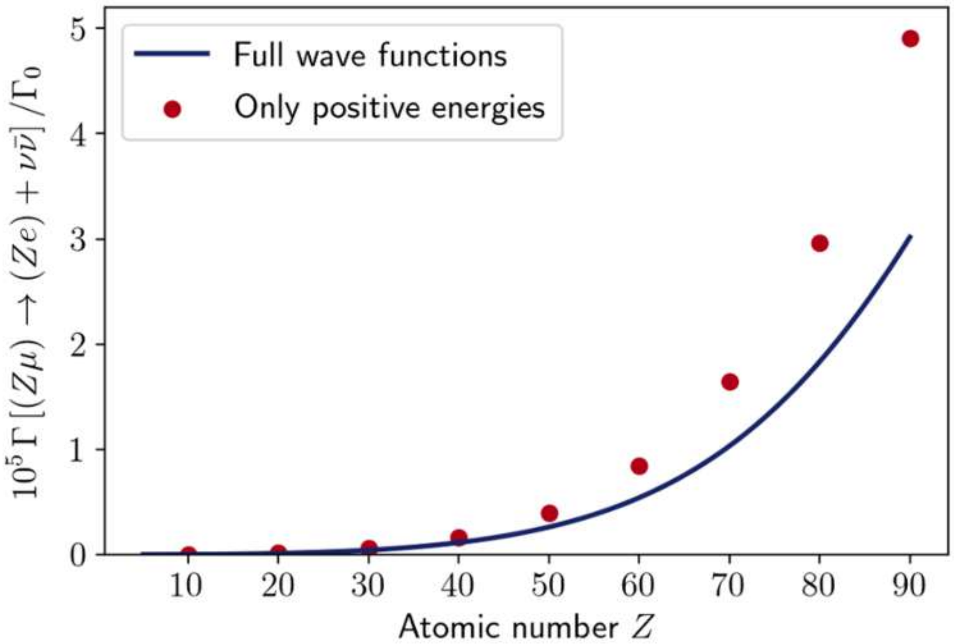


Why such large effect of negative energies?

Tentative explanation:

The decay happens where the muon and the electron wave functions overlap.
This is a tiny fraction of the electron's range, close to the nucleus.
In that region, E < 0 is relatively likely.

Check: position space calculation

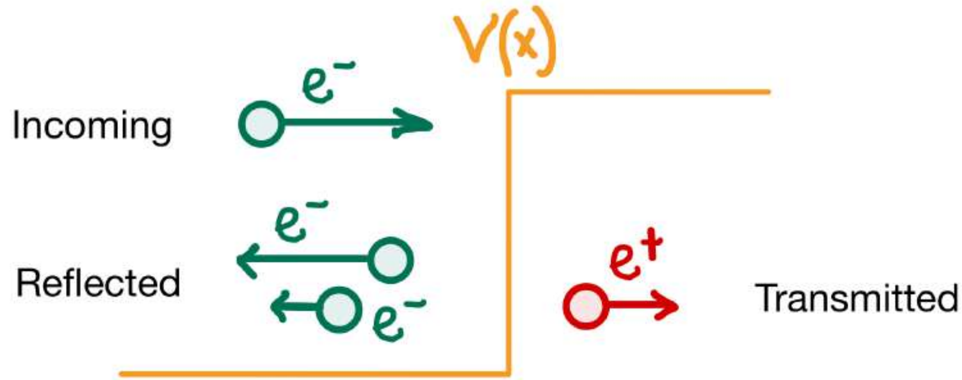
$$\mathcal{M} = \frac{g}{\sqrt{2}} \int \mathrm{d}^3\boldsymbol{r}\, \exp\left(i\boldsymbol{q}\cdot\boldsymbol{r}\right)\bar{\Phi}_\mu(\boldsymbol{r})\not{\epsilon}^{\lambda_A\star} L \Phi_e(\boldsymbol{r})$$

$$\frac{1}{\Gamma_0}\Gamma[(Z\mu^-) \to (Ze^-) + \nu\bar{\nu}] = 128 \int_0^{z_{\max}} (N_a^2 + N_b^2 + F_a^2 + F_b^2) k_A z^3 \mathrm{d}z.$$

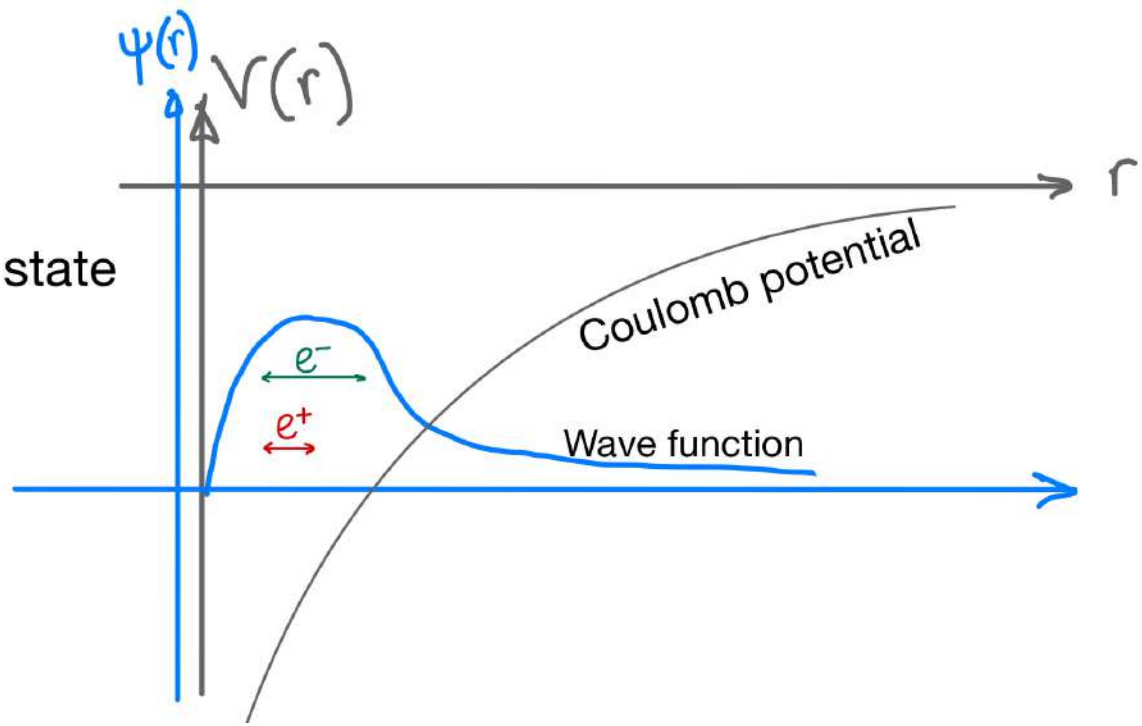
Our position space and momentum space evaluations agree (if both A and B included).

Origin of negative energy states in the bound-particle wave function

Remember Klein paradox?



Bound state



Bethe's work on the Dirac case

First step towards understanding the origin of E < 0 states: their **probability**

Effective coupling constant if the nucleus has Z protons: $\alpha_Z = Z\alpha$

Decompose H-atom ground state wave function in plane waves,

$$\psi\left(\boldsymbol{r}\right) = \sum_{\tau} \int \frac{\mathrm{d}^3 k}{\left(2\pi\right)^3} \phi\left(\boldsymbol{k}, \tau\right) u_\tau\left(\boldsymbol{k}\right) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

1,2,3,4

$$E_{1,2} = E, \quad E_{3,4} = -E \qquad E = \sqrt{m^2 + k^2}$$

We want to set up an integral equation for the Fourier coefficients.

$$\phi\left(\boldsymbol{k}, \sigma\right) = \int \mathrm{d}^3 r e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \left[u_\sigma^\dagger \psi\left(\boldsymbol{r}\right)\right]$$

Integral equation for Fourier coefficients

Start with the usual Dirac equation with V(r): Coulomb potential

$$[V(r) + \boldsymbol{\alpha} \cdot \boldsymbol{k} + \beta m]\,\psi = W\psi \qquad W \simeq m - \frac{\alpha_Z^2 m}{2}$$

$$(W - E_\sigma)\phi(\boldsymbol{k},\sigma) = -4\pi\alpha_Z \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{(k-q)^2} \sum_\tau \phi(\boldsymbol{q},\tau)\left[u_\sigma^\dagger(\boldsymbol{k}) u_\tau(\boldsymbol{q})\right]$$

Fourier coefficient of the negative energy part can be found approximately for small Z,

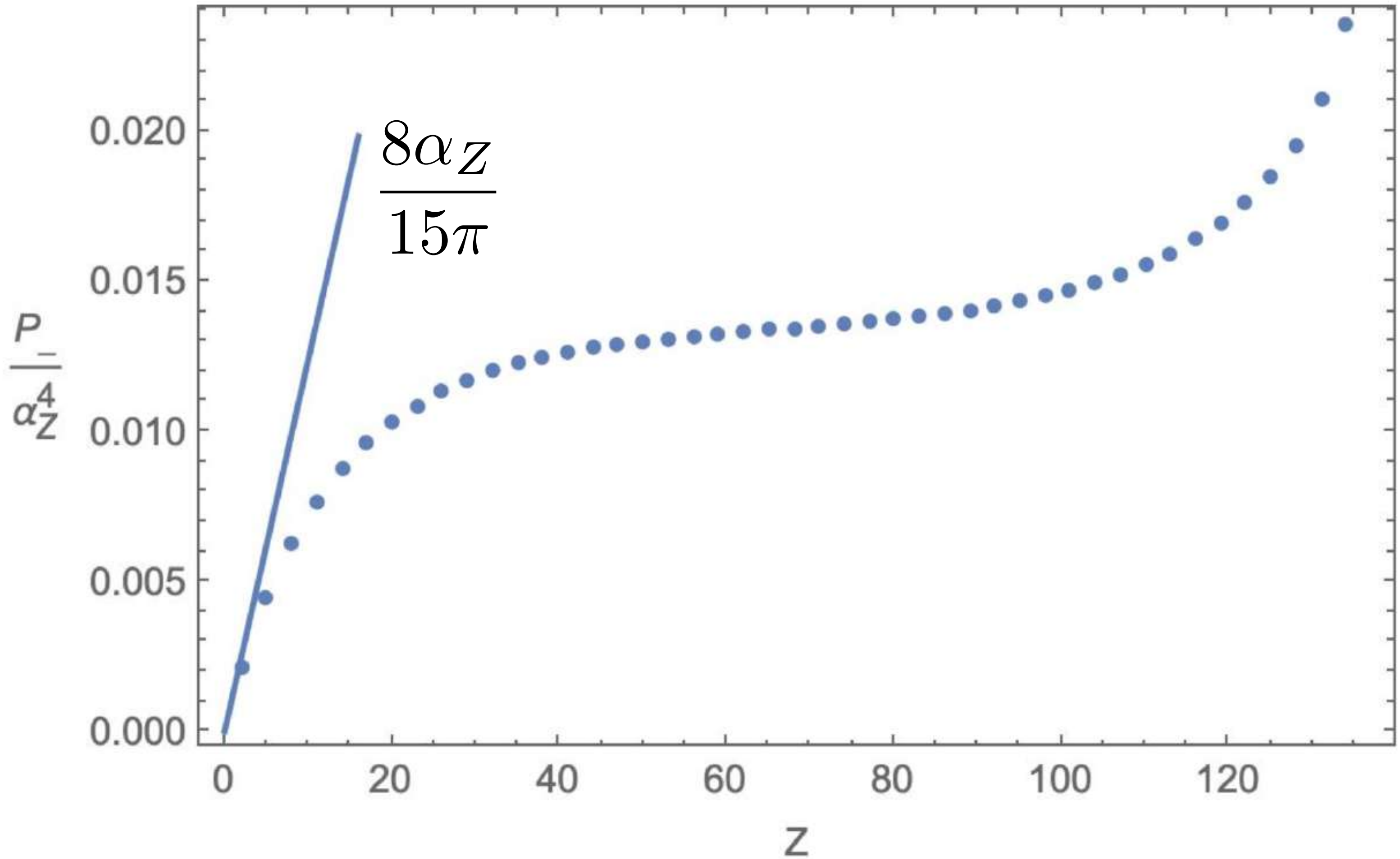
$$\phi_-(\boldsymbol{k}) = -\frac{4\pi\alpha_Z}{(m+E)\,k^2}\sqrt{\frac{E-m}{2E}}\,\psi(0)$$

The only sensitivity to the position wave function: at the origin, $\psi(0)$

Pairs created only near the origin, where the potential is strong.
We use the result to compute the probability of E < 0: $P_-(Z) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left|\phi_-(\boldsymbol{k})\right|^2$

Probability of negative energy states

$$P_-(Z) = \int \frac{d^3k}{(2\pi)^3} \left|\phi_-(\boldsymbol{k})\right|^2 = \frac{8\alpha_Z^5}{15\pi}.$$



Two-component form of Klein-Gordon equation

Feshbach & Villars

How can we separate the E < 0 part of the KG wave function?
(It has only one component, not four.)

$$\left[(i\partial_t - V)^2 + \nabla^2 - m^2\right] \psi(\boldsymbol{r}, t) = 0$$

Convert this 2nd order equation into a system of two 1st order equations,

$$\Psi(\boldsymbol{r}, t) = \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \frac{1}{\sqrt{2m}} \begin{pmatrix} m + i\partial_t - V \\ m - i\partial_t + V \end{pmatrix} \psi(\boldsymbol{r}, t)$$

$$\Psi(\boldsymbol{r}, t) = \Psi(\boldsymbol{r}) e^{-iWt} \qquad W = m - \frac{m\alpha_Z^2}{2} + \mathcal{O}\left(\alpha_Z^4\right)$$

Fourier transform and decompose into positive/negative energy parts,

$$\Psi(\boldsymbol{p}) = u(p)\, \Psi_0^{(+)}(\boldsymbol{p}) + v(p)\, \Psi_0^{(-)}(\boldsymbol{p})$$

Two-component form of the KG equation

Feshbach & Villars

$$\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

$$i\partial_t \Psi = m \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi - \frac{\nabla^2}{2m} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \Psi \equiv H_{\mathrm{f}} \Psi$$

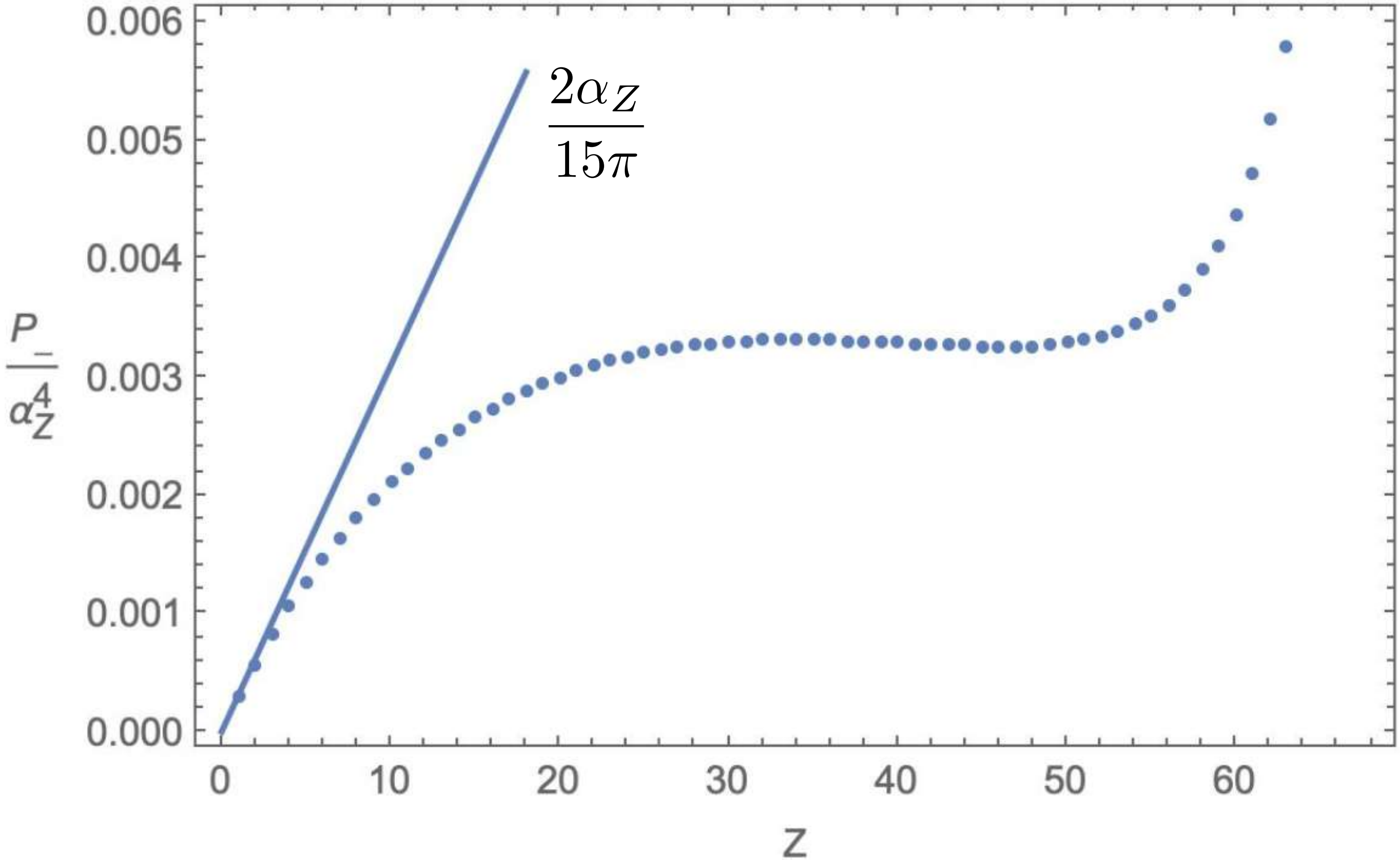
Integral equation form:

$$(W + E_p)\, v = -4\pi\alpha_Z \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{(E_p - E_q)\, u}{2\sqrt{E_p E_q}\, (\boldsymbol{p} - \boldsymbol{q})^2}$$

Probability of E < 0 for spin 0

$$P_-(Z) = \int \frac{d^3p}{(2\pi)^3} |v(p)|^2$$

$$P_-(Z \to 0) = \frac{2\alpha_Z^5}{15\pi}$$



Summary

We now know the probability of finding a $E < 0$ state in the H-atom, for spins 0 and ½

Remaining qualitative questions

- what Feynman diagrams leads to large corrections at moderate Z?
- can one explain the ratio of 4 between Dirac and Klein-Gordon cases?
- why is the critical charge twice smaller in the Klein-Gordon case?