Towards the automation of the Local Analytic Sector subtraction

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Automation of subtraction at NLO based on ✓

- Frixione-Kunst-Signer (FKS) subtraction — Frixione, Kunszt, Signer
- Catani-Seymour (CS) Dipole subtraction — Catani, Seymour, Dittmaier, et al.
- Nagy-Soper subtraction — Bevilacqua, Czakon, Kubocz, Worek

The long way to automation of subtraction at NNLO …

- Antenna subtraction — Gehrmann De Ridder, Gehrmann, Glover, Heinrich, et al.
- CoLoRFul subtraction — Del Duca, Duhr, Kardos, Somogyi, Troscanyi, et al.
- Sector-improved residue subtraction — Czakon et al.
- Nested soft-collinear subtraction — Melnikov et al.
- Local analytic sector subtraction — Magnea, Maina, Torrielli, U. et al.
- qT-slicing — Catani, Grazzini, et al.
- N-jettiness slicing — Boughezal, Petriello, et al.
- Projection to Born — Cacciari, Salam, Zanderighi, et al.
- Sector decomposition — Anastasiou, Binoth, et al.
- ε-prescription — Frixione, Grazzini
- Unsubtraction — Rodrigo et al.
- Geometric — Herzog

Structure of subtraction at NLO

$$\frac{d\sigma_{\rm NLO} - d\sigma_{\rm LO}}{dX} = \int d\Phi_n V\,\delta_{X_n} + \int d\Phi_{n+1} R\,\delta_{X_{n+1}} = \text{finite}.$$

X = IRC safe observable $\qquad \delta_{X_m} = \delta(X - X_m)$

X_m = observable computed with m-body kinematics

V has explicit poles in ϵ, R diverges in phase space integration

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V has explicit poles in ϵ, R diverges in phase space integration

- Introduce counterterm K and its integral I

$$\int d\Phi_{n+1}\, K\,\delta_{X_n} = \int d\Phi_n\, I\,\delta_{X_n}$$

$$\frac{d\sigma_{\rm NLO} - d\sigma_{\rm LO}}{dX} = \int d\Phi_n (V + I)\,\delta_{X_n} + \int d\Phi_{n+1}\Big(R\,\delta_{X_{n+1}} - K\,\delta_{X_n}\Big)$$

V+I is finite in ϵ, R-K converges in phase space integration

Local Analytic Sector subtraction at NLO

- Separate the phase space with sector functions

FKS SECTOR FUNCTIONS

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sigma} \qquad \sigma_{ij} = \frac{1}{\mathcal{E}_i \, w_{ij}} = \frac{s_{qj}}{s_{ij}} \qquad \sigma = \sum_{i\in\mathcal{F}} \sum_{j\neq i} \sigma_{ij}$$

$\mathcal{E}_i \to 0$ when particle i becomes soft

$w_{ij} \to 0$ when particle i and j become collinear

Sum rule: $$\sum_{i\in\mathcal{F}} \sum_{j\neq i} \mathcal{W}_{ij} = 1$$

allows to get rid of sector functions for analytical integration

Local Analytic Sector subtraction at NLO

- Separate the phase space with sector functions
- Identify counterterms through IRC limits

Sector $\mathcal{W}_{ij}$

"few" singular limits survive: $\mathbf{S}_i$ and $\mathbf{C}_{ij}$

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})R\,\mathcal{W}_{ij} \;=\; R\,\mathcal{W}_{ij} - K_{ij} \;=\; \text{finite}$$

Counterterm

$$K = \sum_{i,j\neq j}\Big[\bar{\mathbf{S}}_i + \bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_i)\Big]\,R\;\mathcal{W}_{ij}$$

Barred limits are defined such that satisfy

$$\mathbf{S}_i\bar{\mathbf{S}}_i R = \mathbf{S}_i R \qquad \mathbf{C}_{ij}\bar{\mathbf{C}}_{ij} R = \mathbf{C}_{ij} R$$

$$\mathbf{S}_i\bar{\mathbf{S}}_i\bar{\mathbf{C}}_{ij} R = \mathbf{S}_i\bar{\mathbf{C}}_{ij} R \qquad \mathbf{C}_{ij}\bar{\mathbf{S}}_i\bar{\mathbf{C}}_{ij} R = \mathbf{C}_{ij}\bar{\mathbf{S}}_i R$$

and allow a proper mapping of momenta from R to B phase space

Barred limits = Limits + Mapping

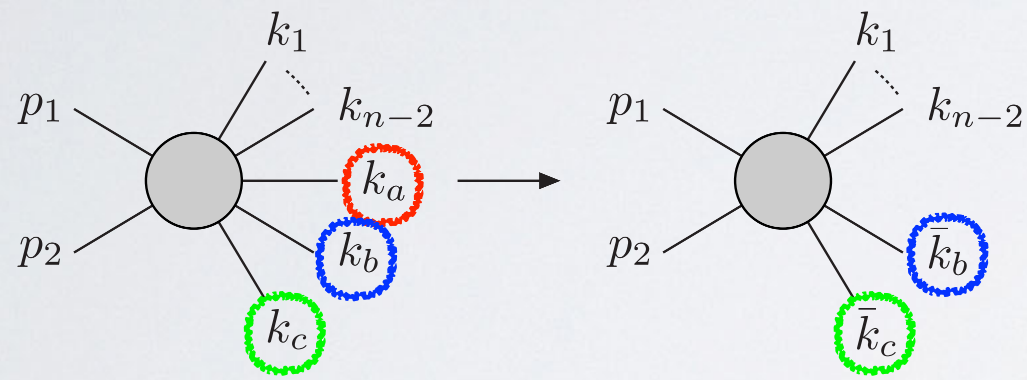
Local Analytic Sector subtraction at NLO

- Separate the phase space with sector functions
- Identify counterterms through IRC limits
- Mapping of outgoing momenta **{*k*}**

CATANI-SEYMOUR MAPPINGS

Based on 3 momenta

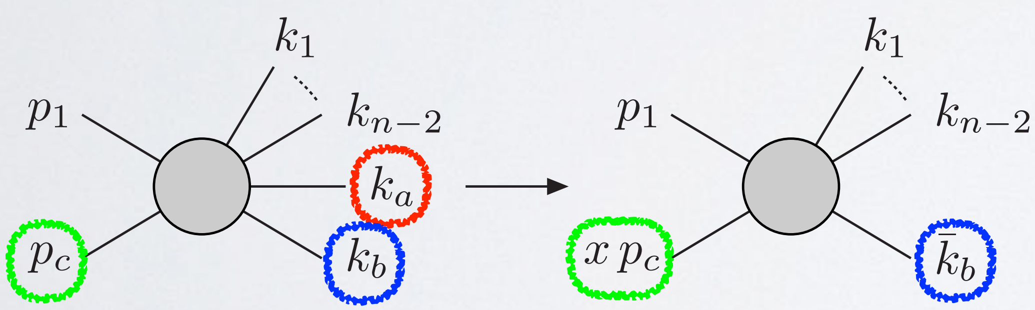
$$\{k\} \longrightarrow \{\bar{k}\}^{(abc)}, k_a$$



$$k_a = (y, z, \phi)$$

$$\bar{s}^{(abc)} = 2\,\bar{k}_b^{(abc)} \cdot \bar{k}_c^{(abc)}$$

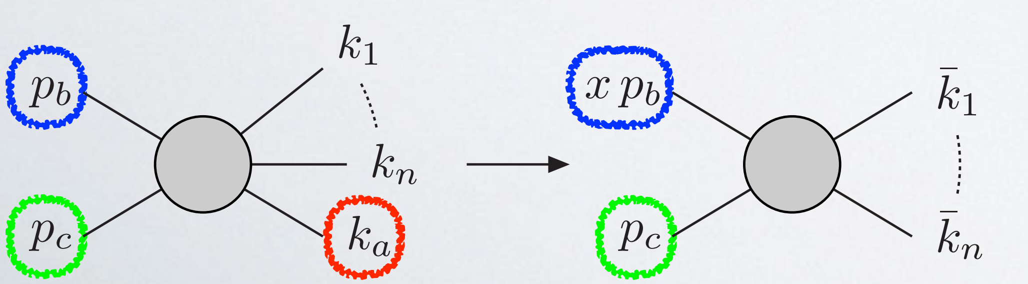
$$\begin{cases} s_{ab} = y\,\bar{s}^{(abc)} \\ s_{ac} = z(1-y)\,\bar{s}^{(abc)} \\ s_{bc} = (1-z)(1-y)\,\bar{s}^{(abc)} \end{cases}$$



$$k_a = (x, z, \phi)$$

$$\bar{s}^{(abc)} = 2\,\bar{k}_b^{(abc)} \cdot p_c$$

$$\begin{cases} s_{ab} = (1-x)\,\bar{s}^{(abc)} \\ s_{ac} = z\,\bar{s}^{(abc)} \\ s_{bc} = (1-z)\,\bar{s}^{(abc)} \end{cases}$$



$$k_a = (x, v, \phi)$$

$$\bar{s}^{(abc)} = 2\,p_b \cdot p_c$$

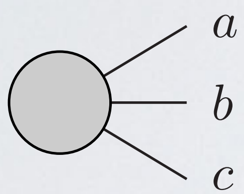
$$\begin{cases} s_{ab} = v(1-x)\,\bar{s}^{(abc)} \\ s_{ac} = (1-v)(1-x)\,\bar{s}^{(abc)} \\ s_{bc} = \bar{s}^{(abc)} \end{cases}$$

Local Analytic Sector subtraction at NLO

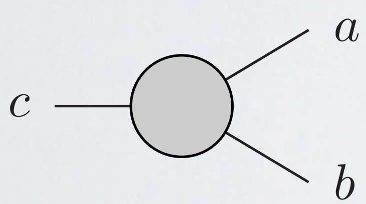
- Separate the phase space with sector functions
- Identify counterterms through IRC limits
- Mapping of outgoing momenta $\{k\}$

$$\{k\} \longrightarrow \{\bar{k}\}^{(abc)}, k_a$$

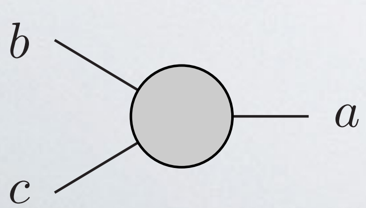
Based on 3 momenta



$$\int d\Phi_{n+1}(p_1, p_2; \{k\}) = \int d\Phi_n\left(p_1, p_2; \{\bar{k}\}^{(abc)}\right) \int d\Phi_1\left(\bar{s}^{(abc)}; y, z, \phi\right)$$



$$\int d\Phi_{n+1}(p_1, p_c; \{k\}) = \int \int d\Phi_n\left(p_1, x\, p_c; \{\bar{k}\}^{(abc)}\right) d\Phi_1\left(\bar{s}^{(abc)}; x, z, \phi\right)$$



$$\int d\Phi_{n+1}(p_b, p_c; \{k\}) = \int \int d\Phi_n\left(x\, p_b, p_c; \{\bar{k}\}^{(abc)}\right) d\Phi_1\left(\bar{s}^{(abc)}; x, v, \phi\right)$$

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Based on 3 momenta

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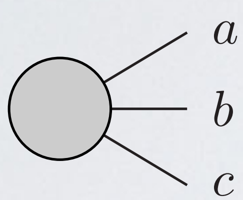
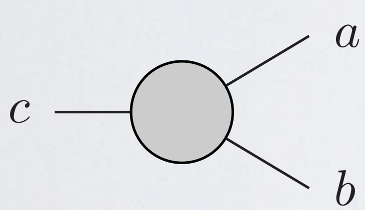


$$\int d\Phi_{n+1}(p_1, p_2; \{k\}) = \int d\Phi_n\Big(p_1, p_2; \{\bar{k}\}^{(abc)}\Big) \int d\Phi_1\Big(\bar{s}^{(abc)}; y, z, \phi\Big)$$

$$\int d\Phi_1(s; y, z, \phi) = N\, s^{1-\epsilon} \int_0^\pi d\phi \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz\, \left[y(1-y)^2\, z(1-z)\right]^{-\epsilon} (1-y)$$



$$\int d\Phi_{n+1}(p_1, p_c; \{k\}) = \int \int d\Phi_n\Big(p_1, x\, p_c; \{\bar{k}\}^{(abc)}\Big) d\Phi_1\Big(\bar{s}^{(abc)}; x, z, \phi\Big)$$

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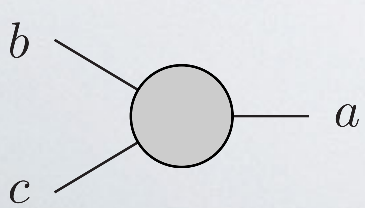

$$\int d\Phi_{n+1}(p_b, p_c; \{k\}) = \int \int d\Phi_n\Big(x\, p_b, p_c; \{\bar{k}\}^{(abc)}\Big) d\Phi_1\Big(\bar{s}^{(abc)}; x, v, \phi\Big)$$

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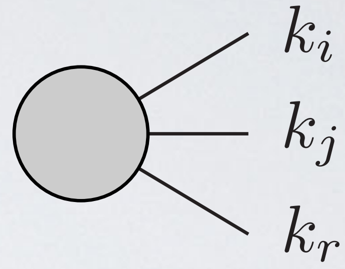
$$\overline{\mathbf{S}}_i\,R \;=\; -2\,\mathcal{N}_1\sum_{k\neq i}\sum_{\substack{l\neq i\\ l>k}}\mathcal{I}^{(i)}_{kl}\,\bar{B}^{(ikl)}_{kl}$$

$$\overline{\mathbf{C}}_{ij}\,R \;=\; \frac{\mathcal{N}_1}{s_{ij}}P^{\mu\nu}_{ij,F}(z)\,\bar{B}^{(ijr)}_{\mu\nu}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,R \;=\; \mathcal{N}_1\,2\,C_{f_j}\,\mathcal{I}^{(i)}_{jr}\,\bar{B}^{(ijr)}$$



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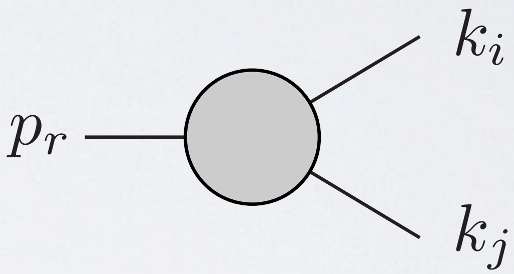

$$\overline{\mathbf{C}}_{ij}\,R \;=\; \frac{1}{x}\frac{\mathcal{N}_1}{s_{ij}}P^{\mu\nu}_{[ij]i,I}(x)\,\bar{B}^{(irj)}_{\mu\nu}$$

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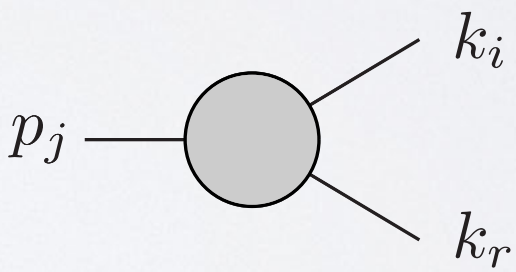

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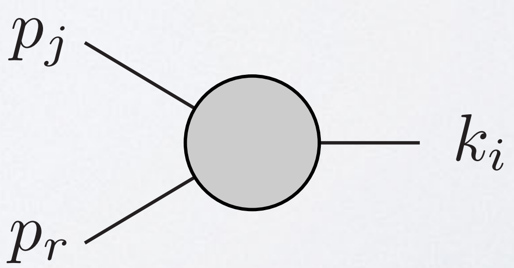

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$$\overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, R = \mathcal{N}_1\, 2\, C_{f_j}\, \mathcal{I}^{(i)}_{jr}\, (1-v)\, \bar{B}^{(ijr)}$$

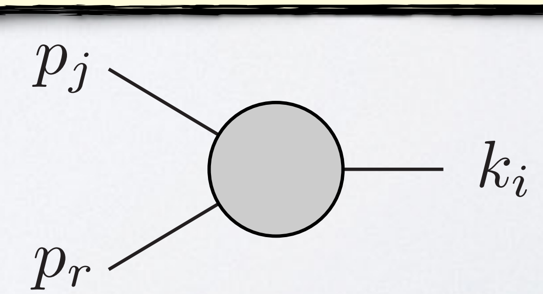
$$\mathcal{I}^{(i)}_{cd} = \delta_{f_i g}\, \frac{s_{cd}}{s_{ic}\, s_{id}}$$

$$P^{\mu\nu}_{ab,\star}(\xi) = -P_{ab}(\xi)\, g^{\mu\nu} + Q_{ab,\star}(\xi) \left[-g^{\mu\nu} + (d-2) \frac{\tilde{k}^{\mu}_{ij} \tilde{k}^{\nu}_{ij}}{\tilde{k}^2_{ij}} \right]$$

$$\begin{aligned} P_{ab}(\xi) &= \delta_{\{f_a f_b\}\{q\bar{q}\}}\, T_R \left[1 - \frac{2\xi(1-\xi)}{1-\epsilon} \right] \\ &+ \delta_{f_a \{q,\bar{q}\}} \delta_{f_b g}\, C_F \left[2\frac{\xi}{1-\xi} + (1-\epsilon)(1-\xi) \right] \\ &+ \delta_{f_a g} \delta_{f_b \{q,\bar{q}\}}\, C_F \left[2\frac{1-\xi}{\xi} + (1-\epsilon)\xi \right] \\ &+ \delta_{f_a g} \delta_{f_b g}\, 2 C_A \left[\frac{1-\xi}{\xi} + \frac{\xi}{1-\xi} + \xi(1-\xi) \right] \end{aligned}$$

$$Q_{ab,F}(\xi) = \delta_{\{f_a f_b\}\{q\bar{q}\}}\, T_R \frac{2\xi(1-\xi)}{1-\epsilon} - \delta_{f_a g} \delta_{f_b g}\, 2 C_A\, \xi(1-\xi)$$

$$Q_{ab,I}(\xi) = -\delta_{f_a g} \delta_{f_b \{q,\bar{q}\}}\, 2 C_F \frac{1-\xi}{\xi} - \delta_{f_a g} \delta_{f_b g}\, 2 C_A \frac{1-\xi}{\xi}$$



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$$P_{ab,\star}^{\mu\nu}(\xi) = -P_{ab}(\xi)\, g^{\mu\nu} + Q_{ab,\star}(\xi) \left[-g^{\mu\nu} + (d-2) \frac{\tilde{k}_{ij}^{\mu} \tilde{k}_{ij}^{\nu}}{\tilde{k}_{ij}^2} \right]$$

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Integration gives just Gamma functions !!

Damping factors

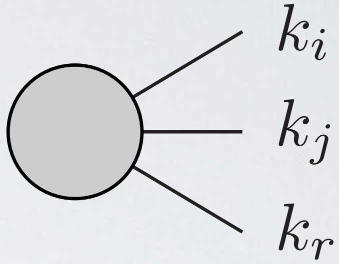
- Modulate non singular contribution of counterterms
- Analogous to α parameter in CS or δ and ξ in FKS
- Smooth modulation

Damping factors

$$\overline{\mathbf{S}}_i\, R = -2\,\mathcal{N}_1 \sum_{k\neq i} \sum_{\substack{l\neq i\\ l>k}} \left[(1-y)^\alpha (1-z)^\alpha\, \delta_{\substack{k\in\mathcal{F}\\ l\in\mathcal{F}}} + x^\alpha (1-z)^\alpha\, \delta_{\substack{k\in\mathcal{I}\\ l\in\mathcal{F}}} + x^\alpha\, \delta_{\substack{k\in\mathcal{I}\\ l\in\mathcal{I}}}\right] \mathcal{I}_{kl}^{(i)}\, \bar{B}_{kl}^{(ikl)}$$

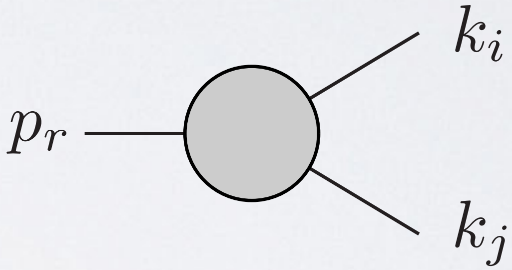
$$\overline{\mathbf{C}}_{ij}\, R = (1-y)^\beta\, \frac{\mathcal{N}_1}{s_{ij}} P_{ij,F}^{\mu\nu}(z)\, \bar{B}_{\mu\nu}^{(ijr)}$$

$$\overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, R = (1-y)^\beta\, (1-z)^\alpha\, \mathcal{N}_1\, 2\, C_{f_j}\, \mathcal{I}_{jr}^{(i)}\, \bar{B}^{(ijr)}$$



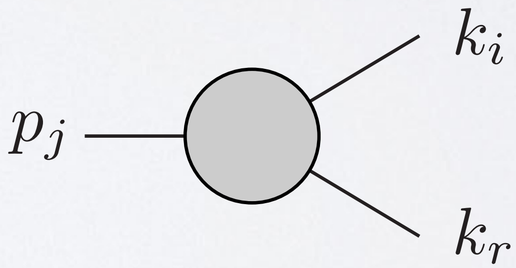
$$\overline{\mathbf{C}}_{ij}\, R = x^\beta\, \frac{\mathcal{N}_1}{s_{ij}} P_{ij,F}^{\mu\nu}(z)\, \bar{B}_{\mu\nu}^{(ijr)}$$

$$\overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, R = x^\beta\, (1-z)^\alpha\, \mathcal{N}_1\, 2\, C_{f_j}\, \mathcal{I}_{jr}^{(i)}\, \bar{B}^{(ijr)}$$



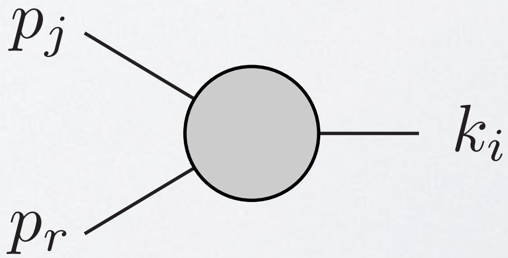
$$\overline{\mathbf{C}}_{ij}\, R = (1-z)^\gamma\, \frac{1}{x} \frac{\mathcal{N}_1}{s_{ij}} P_{[ij]i,I}^{\mu\nu}(x)\, \bar{B}_{\mu\nu}^{(irj)}$$

$$\overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, R = x^\alpha\, (1-z)^\gamma\, \mathcal{N}_1\, 2\, C_{f_j}\, \mathcal{I}_{jr}^{(i)}\, \frac{1}{1-z}\, \bar{B}^{(irj)}$$



$$\overline{\mathbf{C}}_{ij}\, R = (1-v)^\gamma\, \frac{1}{x} \frac{\mathcal{N}_1}{s_{ij}} P_{[ij]i,I}^{\mu\nu}(x)\, \bar{B}_{\mu\nu}^{(ijr)}$$

$$\overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, R = x^\alpha\, (1-v)^\gamma\, \mathcal{N}_1\, 2\, C_{f_j}\, \mathcal{I}_{jr}^{(i)}\, (1-v) \bar{B}^{(ijr)}$$

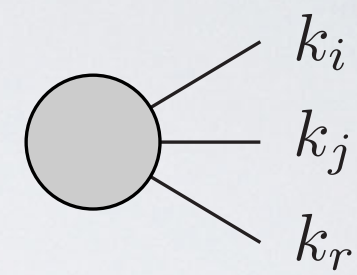


Damping factors

$$\overline{\mathbf{S}}_i\, R \;=\; -2\,\mathcal{N}_1 \sum_{k\neq i}\sum_{\substack{l\neq i\\ l>k}} \Big[(1-y)^{\alpha}(1-z)^{\alpha}\,\delta_{\substack{k\in\mathcal{F}\\ l\in\mathcal{F}}} + x^{\alpha}(1-z)^{\alpha}\,\delta_{\substack{k\in\mathcal{I}\\ l\in\mathcal{F}}} + x^{\alpha}\,\delta_{\substack{k\in\mathcal{I}\\ l\in\mathcal{I}}}\Big]\mathcal{I}_{kl}^{(i)}\,\bar{B}_{kl}^{(ikl)}$$

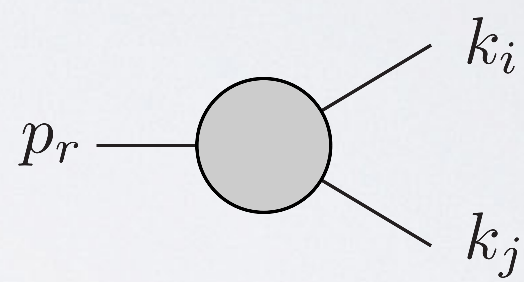
$$\overline{\mathbf{C}}_{ij}\, R \;=\; (1-y)^{\beta}\,\frac{\mathcal{N}_1}{s_{ij}} P_{ij,F}^{\mu\nu}(z)\,\bar{B}_{\mu\nu}^{(ijr)}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\, R \;=\; (1-y)^{\beta}\,(1-z)^{\alpha}\,\mathcal{N}_1\, 2\, C_{f_j}\,\mathcal{I}_{jr}^{(i)}\,\bar{B}^{(ijr)}$$



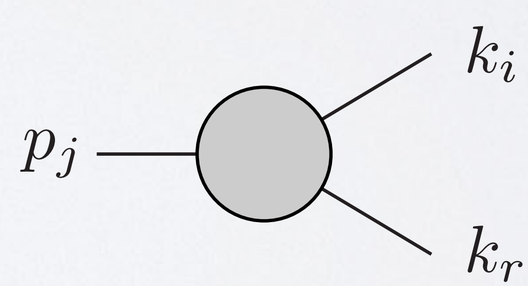
$$\overline{\mathbf{C}}_{ij}\, R \;=\; x^{\beta}\,\frac{\mathcal{N}_1}{s_{ij}} P_{ij,F}^{\mu\nu}(z)\,\bar{B}_{\mu\nu}^{(ijr)}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\, R \;=\; x^{\beta}\,(1-z)^{\alpha}\,\mathcal{N}_1\, 2\, C_{f_j}\,\mathcal{I}_{jr}^{(i)}\,\bar{B}^{(ijr)}$$



$$\overline{\mathbf{C}}_{ij}\, R \;=\; (1-z)^{\gamma}\,\frac{1}{x}\frac{\mathcal{N}_1}{s_{ij}} P_{[ij]i,I}^{\mu\nu}(x)\,\bar{B}_{\mu\nu}^{(irj)}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\, R \;=\; x^{\alpha}\,(1-z)^{\gamma}\,\mathcal{N}_1\, 2\, C_{f_j}\,\mathcal{I}_{jr}^{(i)}\,\frac{1}{1-z}\,\bar{B}^{(irj)}$$



$$\overline{\mathbf{C}}_{ij}\, R \;=\; (1-v)^{\gamma}\,\frac{1}{x}\frac{\mathcal{N}_1}{s_{ij}} P_{[ij]i,I}^{\mu\nu}(x)\,\bar{B}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\, R \;=\; x^{\alpha}\,(1-v)^{\gamma}\,\mathcal{N}_1\, 2\, C_{f_j}\,\mathcal{I}_{jr}^{(i)}\,($$

Integration still gives just Gamma functions !!

No initial state partons

$$I^F = \boxed{I_{\rm poles}} + I^F_{\rm fin}$$

$$I_{\rm poles} = \frac{\alpha_s}{2\pi}\left\{\frac{1}{\epsilon^2}\Sigma_C\,\mathbf{B} + \frac{1}{\epsilon}\left[\Sigma_\gamma\,\mathbf{B} + \sum_{c,d\neq c}\mathbf{L}_{cd}\,\mathbf{B}_{cd}\right]\right\}$$

$$\gamma_a = \frac{3}{2}C_F\,\delta_{f_a\{q,\bar{q}\}} + \frac{1}{2}\beta_0\,\delta_{f_a g} \qquad \Sigma_\gamma = \sum_a \gamma_a \qquad \Sigma_C = \sum_a C_{f_a}$$

$$L_{ab} = \ln\frac{s_{ab}}{\mu^2}$$

No initial state partons

$$I^F = I_{\rm poles} + I^F_{\rm fin}$$

$$I_{\rm poles} = \frac{\alpha_s}{2\pi}\left\{\frac{1}{\epsilon^2}\Sigma_C\,\mathbf{B} + \frac{1}{\epsilon}\left[\Sigma_\gamma\,\mathbf{B} + \sum_{c,d\neq c}\mathbf{L}_{cd}\,\mathbf{B}_{cd}\right]\right\} = -\mathbf{V}_{\rm poles}$$

$$\gamma_a = \frac{3}{2}C_F\,\delta_{f_a\{q,\bar{q}\}} + \frac{1}{2}\beta_0\,\delta_{f_a g} \qquad \Sigma_\gamma = \sum_a \gamma_a \qquad \Sigma_C = \sum_a C_{f_a}$$

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No initial state partons

$$I^F = I_{\rm poles} + \boxed{I^F_{\rm fin}}$$

$$I_{\rm poles} = \frac{\alpha_s}{2\pi}\left\{\frac{1}{\epsilon^2}\Sigma_C\,\mathbf{B} + \frac{1}{\epsilon}\left[\Sigma_\gamma\,\mathbf{B} + \sum_{c,d\neq c}\mathbf{L}_{cd}\,\mathbf{B}_{cd}\right]\right\} = -\mathbf{V}_{\rm poles}$$

$$I^F_{\rm fin} = \frac{\alpha_s}{2\pi}\left\{\left[\Sigma_\phi - \sum_{j\in\mathcal{F}}\gamma_j^{\rm hc}\,\mathbf{L}_{jr}\right]\mathbf{B} + \sum_{c,d\neq c}\mathbf{L}_{cd}\left(2-\frac{1}{2}\mathbf{L}_{cd}\right)\mathbf{B}_{cd}\right.$$
$$\left. + \left[\Sigma_C\,\mathcal{A}_2^{(0)}(\alpha)\left(2\,\mathcal{A}_2^{(0)}(\beta) + \mathcal{A}_2^{(0)}(\alpha) + 4\right) - \Sigma_C\,\mathcal{A}_2^{(1)}(\alpha) + \Sigma_\gamma^{\rm hc}\,\mathcal{A}_2^{(0)}(\beta)\right]\mathbf{B}\right\}$$

$$\gamma_a^{\rm hc} = \gamma_a - 2C_{f_a}$$

$$\phi_a = \frac{13}{3}C_F\,\delta_{f_a\{q,\bar q\}} + \frac{4}{3}\beta_0\,\delta_{f_a g} + \left(\frac{2}{3}-\frac{7}{2}\zeta_2\right)C_{f_a}$$

$$\Sigma_\phi = \sum_a \phi_a$$

$$\mathcal{A}_1^{(0)}(\xi) = \gamma_E + \Psi^{(0)}(\xi+1)$$

$$\mathcal{A}_2^{(0)}(\xi) = \gamma_E - 1 + \Psi^{(0)}(\xi+2)$$

$$\mathcal{A}_2^{(1)}(\xi) = \Psi^{(1)}(\xi) + 1 - \zeta_2$$

One initial state parton

$$\int d\Phi_{n+1} K^{IF} = \int d\Phi_n(p_a)\Big(I^F + \boxed{I^I_{\rm fin}}\Big) + \int_0^1 \frac{dx}{x}\int d\Phi_n(xp_a)\; J^I(x)$$

$$I^I_{\rm fin} = \frac{\alpha_{\rm s}}{2\pi}\, 2\, C_{f_a}\left[-2 + 2\zeta_2 - \mathcal{A}_2^{(0)}(\alpha)\Big(\mathcal{A}_1^{(0)}(\gamma) - \mathcal{A}_2^{(0)}(\beta) - 1\Big) + \mathcal{A}_2^{(1)}(\alpha)\right] B$$

One initial state parton

$$\int d\Phi_{n+1} K^{IF} = \int d\Phi_n(p_a)\Big(I^F + I^I_{\rm fin}\Big) + \int_0^1 \frac{dx}{x}\int d\Phi_n(xp_a)\, J^I(x)$$

$$I^I_{\rm fin} = \frac{\alpha_s}{2\pi}\, 2\, C_{f_a}\left[-2 + 2\zeta_2 - \mathcal{A}_2^{(0)}(\alpha)\Big(\mathcal{A}_1^{(0)}(\gamma) - \mathcal{A}_2^{(0)}(\beta) - 1\Big) + \mathcal{A}_2^{(1)}(\alpha)\right] B$$

$$\begin{aligned} J^I(x) = \frac{\alpha_s}{2\pi}\Bigg\{ &-\frac{1}{\epsilon} P_a(x) + P_a^{(1)}(x) + \left(\frac{x^{1+\beta}}{1-x}\right)_+ \left[\Sigma_\gamma - \gamma_a + 2(\Sigma_C - C_{f_a})\mathcal{A}_2^{(0)}(\alpha)\right] \\ &+ 2\, C_{f_a}\left[\left(\frac{x\ln(1-x)}{1-x}\right)_+ - \left(\frac{x}{1-x}\right)_+ \mathcal{A}_1^{(0)}(\gamma) + \left(\frac{x^{1+\alpha}}{1-x}\right)_+ \Big(\mathcal{A}_1^{(0)}(\gamma) - \mathcal{A}_2^{(0)}(\alpha) - 1\Big)\right]\Bigg\} B \\ &- \frac{\alpha_s}{2\pi}\sum_{j\in\mathcal{F}} 2\, L_{aj}\, B_{aj} \end{aligned}$$

One initial state parton

$$\int d\Phi_{n+1} K^{IF} = \int d\Phi_n(p_a)\Big(I^F + I^I_{\rm fin}\Big) + \int_0^1 \frac{dx}{x}\int d\Phi_n(xp_a)\, J^I(x)$$

$$I^I_{\rm fin} = \frac{\alpha_s}{2\pi}\, 2\, C_{f_a}\left[-2 + 2\zeta_2 - \mathcal{A}_2^{(0)}(\alpha)\Big(\mathcal{A}_1^{(0)}(\gamma) - \mathcal{A}_2^{(0)}(\beta) - 1\Big) + \mathcal{A}_2^{(1)}(\alpha)\right] B$$

$$J^I(x) = \frac{\alpha_s}{2\pi}\left\{-\frac{1}{\epsilon} P_a(x) + P_a^{(1)}(x) + \left(\frac{x^{1+\beta}}{1-x}\right)_+ \Big[\Sigma_\gamma - \gamma_a + 2(\Sigma_C - C_{f_a})\mathcal{A}_2^{(0)}(\alpha)\Big]\right.$$
$$\left. + 2\, C_{f_a}\left[\left(\frac{x\ln(1-x)}{1-x}\right)_+ - \left(\frac{x}{1-x}\right)_+ \mathcal{A}_1^{(0)}(\gamma) + \left(\frac{x^{1+\alpha}}{1-x}\right)_+ \Big(\mathcal{A}_1^{(0)}(\gamma) - \mathcal{A}_2^{(0)}(\alpha) - 1\Big)\right]\right\} B$$
$$- \frac{\alpha_s}{2\pi}\sum_{j\in\mathcal{F}} 2\, L_{aj}\, B_{aj}$$

$$P_a(x) = \delta_{f_a g}\left\{T_R\Big[x^2 + (1-x)^2\Big] + 2C_A\left[\left(\frac{1}{1-x}\right)_+ + \frac{1-x}{x} - 1 + x(1-x)\right] + \frac{\beta_0}{2}\,\delta(1-x)\right\}$$
$$+ \delta_{f_a\{q\bar q\}}\, C_F\left[\left(2\,\frac{1-x}{x} + x\right)_+ + \left(2\,\frac{x}{1-x} + 1 - x\right)_+\right]$$

$$P_a^{(m)}(x) = \delta_{f_a g}\left\{\left[T_R\Big(x^2 + (1-x)^2\Big) + 2C_A\left(\frac{1-x}{x} + x(1-x)\right)\right]\Big(m\ln(1-x) + L_{ar} - \mathcal{A}_1^{(0)}(\gamma)\Big) + T_R\, 2x(1-x)\right\}$$
$$+ \delta_{f_a\{q\bar q\}}\, C_F\left\{\left[\frac{1 + (1-x)^2}{x} + 1 - x\right]\Big(m\,\ln(1-x) + L_{ar} - \mathcal{A}_1^{(0)}(\gamma)\Big) + x\right\}$$

Two initial state partons

$$\int d\Phi_{n+1} K^{IIF} = \int d\Phi_n(p_a, p_b)\Big(I^F + \boxed{I^{II}_{\text{fin}}}\Big) + \left[\int_0^1 \frac{dx}{x}\int d\Phi_n(xp_a, p_b)\; J^{II}(x) \;\; + \Big(a \leftrightarrow b\Big)\right]$$

$$I^{II}_{\text{fin}} = \frac{\alpha_s}{2\pi}\Bigg\{ (C_{f_a}+C_{f_b})\left[-4+4\zeta_2 - \mathcal{A}^{(0)}_2(\alpha)\Big(2\mathcal{A}^{(0)}_1(\gamma) - 2\mathcal{A}^{(0)}_2(\beta) + \mathcal{A}^{(0)}_2(\alpha)\Big) + 3\mathcal{A}^{(1)}_2(\alpha)\right] B$$
$$+ \left[-4+4\zeta_2+4\mathcal{A}^{(1)}_2(\alpha)\right] B_{ab}\Bigg\}$$

Two initial state partons

$$\int d\Phi_{n+1} K^{IIF} = \int d\Phi_n(p_a,p_b)\Big(I^F + I^{II}_{\rm fin}\Big) + \Big[\int_0^1 \frac{dx}{x}\int d\Phi_n(xp_a,p_b)\, J^{II}(x) + \big(a \leftrightarrow b\big)\Big]$$

$$I^{II}_{\rm fin} = \frac{\alpha_s}{2\pi}\Big\{ (C_{f_a}+C_{f_b})\Big[-4+4\zeta_2 - \mathcal{A}_2^{(0)}(\alpha)\Big(2\mathcal{A}_1^{(0)}(\gamma) - 2\mathcal{A}_2^{(0)}(\beta) + \mathcal{A}_2^{(0)}(\alpha)\Big) + 3\mathcal{A}_2^{(1)}(\alpha)\Big] B$$
$$+ \Big[-4+4\zeta_2+4\mathcal{A}_2^{(1)}(\alpha)\Big] B_{ab}\Big\}$$

$$J^{II}(x) = \frac{\alpha_s}{2\pi}\Big\{-\frac{1}{\epsilon}P_a(x) + P_a^{(2)}(x) + \left(\frac{x^{1+\beta}}{1-x}\right)_+ \Big[\Sigma_\gamma - \gamma_a + 2(\Sigma_C - C_{f_a})\mathcal{A}_2^{(0)}(\alpha)\Big]$$
$$+2C_{f_a}\Big[2\left(\frac{x\ln(1-x)}{1-x}\right)_+ - \left(\frac{x^{1+\alpha}\ln(1-x)}{1-x}\right)_+$$
$$-\left(\frac{x}{1-x}\right)_+\mathcal{A}_1^{(0)}(\gamma) + \left(\frac{x^{1+\alpha}}{1-x}\right)_+\Big(\mathcal{A}_1^{(0)}(\gamma) - \mathcal{A}_2^{(0)}(\alpha) - 1\Big)\Big]\Big\} B$$
$$-\frac{\alpha_s}{2\pi}\Big\{2\Big[\left(\frac{x^{1+\alpha}\ln(1-x)}{1-x}\right)_+ + \left(\frac{x^{1+\alpha}}{1-x}\right)_+\Big(\mathcal{A}_2^{(0)}(\alpha)+1\Big)\Big] B_{ab} + \sum_{j\in\mathcal{F}} 2\, L_{aj}\, B_{aj}\Big\}$$

Implementation in MADNkLO:

Hirschi, Deutschmann, Lionetti, et al.

Python framework to automatically generate all ingredients for NLO and NNLO

- Cancellation of phase-space singularities checked up to pp → 3j
- Validation with MG5 on physical cross-sections

	No damping	$\alpha = \beta = \gamma = 1$	$\alpha = \beta = \gamma = 2$
$e^+e^- \to \mathrm{j\,j}$ [fb] $\sqrt{s} = 1$ TeV	$V + I = 26.647 \pm 9 \cdot 10^{-5}$ $R - K = -6.665 \pm 0.006$ $\sigma_{\mathrm{NLO-LO}} = 19.982 \pm 0.006$	$V + I = 19.985 \pm 7 \cdot 10^{-5}$ $R - K = -0.005 \pm 0.006$ $\sigma_{\mathrm{NLO-LO}} = 19.980 \pm 0.006$	$V + I = 6.662 \pm 2 \cdot 10^{-5}$ $R - K = 13.317 \pm 0.006$ $\sigma_{\mathrm{NLO-LO}} = 19.979 \pm 0.006$
$pp \to Z$ [pb] $\sqrt{s} = 13$ TeV	$V + I = 3981.2 \pm 0.6$ $R - K = 2829.2 \pm 0.1$ $\sigma_{\mathrm{NLO-LO}} = 6810.4 \pm 0.6$	$V + I = -3472.9 \pm 1.4$ $R - K = 10284.3 \pm 0.3$ $\sigma_{\mathrm{NLO-LO}} = 6811.4 \pm 1.4$	$V + I = -9163.2 \pm 1.2$ $R - K = 15974.5 \pm 0.5$ $\sigma_{\mathrm{NLO-LO}} = 6811.2 \pm 1.3$
$pp \to Z\,\mathrm{j}$ [pb] $\sqrt{s} = 13$ TeV	$V + I = 7172 \pm 2$ $R - K = -3396 \pm 15$ $\sigma_{\mathrm{NLO-LO}} = 3776 \pm 15$	$V + I = 5243 \pm 2$ $R - K = -1420 \pm 15$ $\sigma_{\mathrm{NLO-LO}} = 3823 \pm 15$	$V + I = 3623 \pm 2$ $R - K = 152 \pm 13$ $\sigma_{\mathrm{NLO-LO}} = 3775 \pm 14$

PDF set: PDF4LHC15

$\mu_R = \mu_F = M_Z$

Structure of subtraction at NNLO

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n VV\, \delta_{X_n} + \int d\Phi_{n+1} RV\, \delta_{X_{n+1}} + \int d\Phi_{n+2}\, RR\, \delta_{X_{n+2}}$$

VV and *RV* have poles in ϵ, *RV* and *RR* diverge in phase space

Structure of subtraction at NNLO

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n VV\, \delta_{X_n} + \int d\Phi_{n+1} RV\, \delta_{X_{n+1}} + \int d\Phi_{n+2}\, RR\, \delta_{X_{n+2}}$$

VV and RV have poles in ϵ, RV and RR diverge in phase space

Let's insert counterterms !

$$\begin{aligned}\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = & \int d\Phi_n \Big(VV \qquad\qquad\qquad \Big)\, \delta_{X_n} \\ & + \int d\Phi_{n+1} \Big[\Big(RV \qquad \Big)\, \delta_{X_{n+1}} \qquad\qquad\qquad \Big] \\ & + \int d\Phi_{n+2} \Big[RR\, \delta_{X_{n+2}} \qquad\qquad\qquad\qquad \Big]\end{aligned}$$

Structure of subtraction at NNLO

$$\frac{d\sigma_{\rm NNLO}-d\sigma_{\rm NLO}}{dX} = \int d\Phi_n VV\,\delta_{X_n} + \int d\Phi_{n+1} RV\,\delta_{X_{n+1}} + \int d\Phi_{n+2}\,RR\,\delta_{X_{n+2}}$$

VV and RV have poles in ϵ, RV and RR diverge in phase space

Let's insert counterterms !

$$\frac{d\sigma_{\rm NNLO}-d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \Big(VV \qquad\qquad\qquad\qquad \Big)\,\delta_{X_n}$$
$$+\int d\Phi_{n+1}\Big[\Big(RV \qquad\qquad \Big)\,\delta_{X_{n+1}} \qquad\qquad\qquad\qquad \Big]$$
$$+\int d\Phi_{n+2}\Big[RR\,\delta_{X_{n+2}} \qquad \text{Counterterms for } RR \qquad \Big]$$

Counterterms for *RR*

Counterterms for *RR*

- Partition of phase space through sector functions

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sigma}$$

$$\sigma_{ijkl} = \frac{1}{(\mathcal{E}_i\, w_{ij})^\alpha}\,\frac{1}{(\mathcal{E}_k + \delta_{kj}\,\mathcal{E}_i)\, w_{kl}} \qquad \alpha > 1$$

$$\sigma = \sum_{\substack{i,\,j\neq i\\ k\neq i,\,l\neq i,k}} \sigma_{ijkl} \qquad \sum_{i,j\neq i}\;\sum_{\substack{k\neq i\\ l\neq i,k}} \mathcal{W}_{ijkl} = 1 \qquad RR = \sum_{i,j\neq i}\;\sum_{\substack{k\neq i\\ l\neq i,k}} RR\,\mathcal{W}_{ijkl}$$

In each sector **"few"** singular limits survive

$$\mathcal{W}_{ijjk} \;:\; \mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ij} \quad \mathbf{C}_{ijk} \quad \mathbf{SC}_{ijk}$$

$$\mathcal{W}_{ijkj} \;:\; \mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ik} \quad \mathbf{C}_{ijk} \quad \mathbf{SC}_{ijk} \quad \mathbf{SC}_{kij}$$

$$\mathcal{W}_{ijkl} \;:\; \mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ik} \quad \mathbf{C}_{ijkl} \quad \mathbf{SC}_{ikl} \quad \mathbf{SC}_{kij}$$

Counterterms for *RR*

- Partition of phase space through sector functions

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sigma}$$

$$\sigma_{ijkl} = \frac{1}{(\mathcal{E}_i\, w_{ij})^{\alpha}} \frac{1}{(\mathcal{E}_k + \delta_{kj}\, \mathcal{E}_i)\, w_{kl}} \qquad \alpha > 1$$

$$\sigma = \sum_{\substack{i,\, j \neq i \\ k \neq i,\, l \neq i,k}} \sigma_{ijkl}$$

$\mathbf{S}_i : \quad \mathcal{E}_i \to 0$

$\mathbf{C}_{ij} : \quad w_{ij} \to 0$

$\mathbf{S}_{ij} : \quad \mathcal{E}_i, \mathcal{E}_j \to 0 \;$ uniformly

$\mathbf{C}_{ijk} : \quad w_{ij}, w_{jk}, w_{ij} \to 0 \;$ uniformly

$\mathbf{C}_{ijkl} : \quad w_{ij}, w_{kl} \to 0 \;$ uniformly

$\mathbf{SC}_{ijk} : \quad \mathcal{E}_i, w_{jk} \to 0 \;$ uniformly

In each sector ”

$\mathcal{W}_{ijjk}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ij}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	
$\mathcal{W}_{ijkj}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	$\mathbf{SC}_{kij}$
$\mathcal{W}_{ijkl}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijkl}$	$\mathbf{SC}_{ikl}$	$\mathbf{SC}_{kij}$

1-unresolved limits

2-unresolved limits

Counterterms for RR

- Partition of phase space through sector functions

$$RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\, \mathcal{W}_{ijkl}$$

$$\begin{array}{lcllllll}
\mathcal{W}_{ijjk} & : & \mathbf{S}_i & \mathbf{C}_{ij} & \mathbf{S}_{ij} & \mathbf{C}_{ijk} & \mathbf{SC}_{ijk} & \\
\mathcal{W}_{ijkj} & : & \mathbf{S}_i & \mathbf{C}_{ij} & \mathbf{S}_{ik} & \mathbf{C}_{ijk} & \mathbf{SC}_{ijk} & \mathbf{SC}_{kij} \\
\mathcal{W}_{ijkl} & : & \mathbf{S}_i & \mathbf{C}_{ij} & \mathbf{S}_{ik} & \mathbf{C}_{ijkl} & \mathbf{SC}_{ikl} & \mathbf{SC}_{kij}
\end{array}$$

All limits commute

Counterterms for *RR*

- Partition of phase space through sector functions

$$RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\, \mathcal{W}_{ijkl}$$

$$\begin{array}{lcllll} \mathcal{W}_{ijjk} & : & \mathbf{S}_i & \mathbf{C}_{ij} & \mathbf{S}_{ij} & \mathbf{C}_{ijk} & \mathbf{SC}_{ijk} & \\ \mathcal{W}_{ijkj} & : & \mathbf{S}_i & \mathbf{C}_{ij} & \mathbf{S}_{ik} & \mathbf{C}_{ijk} & \mathbf{SC}_{ijk} & \mathbf{SC}_{kij} \\ \mathcal{W}_{ijkl} & : & \mathbf{S}_i & \mathbf{C}_{ij} & \mathbf{S}_{ik} & \mathbf{C}_{ijkl} & \mathbf{SC}_{ikl} & \mathbf{SC}_{kij} \end{array}$$

$$\mathbf{L}^{(1)}_{ij} = \mathbf{S}_i + \mathbf{C}_{ij}(1 - \mathbf{S}_i)$$

$$\begin{aligned} \mathbf{L}^{(2)}_{ijjk} &= \mathbf{S}_{ij} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ij}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk}) \\ \mathbf{L}^{(2)}_{ijkj} &= \mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ijk} + \mathbf{SC}_{kij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk}) \\ \mathbf{L}^{(2)}_{ijkl} &= \mathbf{S}_{ik} + \mathbf{C}_{ijkl}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ikl} + \mathbf{SC}_{kij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl}) \end{aligned}$$

Counterterms for *RR*

- Partition of phase space through sector functions

$$RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\, \mathcal{W}_{ijkl}$$

$$\mathcal{W}_{ijjk} \; : \; \mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ij} \quad \mathbf{C}_{ijk} \quad \mathbf{SC}_{ijk}$$

$$\mathcal{W}_{ijkj} \; : \; \mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ik} \quad \mathbf{C}_{ijk} \quad \mathbf{SC}_{ijk} \quad \mathbf{SC}_{kij}$$

$$\mathcal{W}_{ijkl} \; : \; \mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ik} \quad \mathbf{C}_{ijkl} \quad \mathbf{SC}_{ikl} \quad \mathbf{SC}_{kij}$$

$$\mathbf{L}^{(1)}_{ij} = \mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)$$

$$\mathbf{L}^{(2)}_{ijjk} = \mathbf{S}_{ij} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ij}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})$$

$$\mathbf{L}^{(2)}_{ijkj} = \mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ijk}+\mathbf{SC}_{kij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})$$

$$\mathbf{L}^{(2)}_{ijkl} = \mathbf{S}_{ik} + \mathbf{C}_{ijkl}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ikl}+\mathbf{SC}_{kij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl})$$

- Identify counterterms through IR limits

$$\left(1-\mathbf{L}^{(1)}_{ij}\right)\left(1-\mathbf{L}^{(2)}_{ijkl}\right) RR\, \mathcal{W}_{ijkl} = \left[RR - \mathbf{L}^{(1)}_{ij} RR - \mathbf{L}^{(2)}_{ijkl} RR + \mathbf{L}^{(1)}_{ij}\mathbf{L}^{(2)}_{ijkl}\, RR\right]\mathcal{W}_{ijkl} = \text{finite}$$

Counterterms for *RR*

- Partition of phase space through sector functions $$RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\, \mathcal{W}_{ijkl}$$
- Identify counterterms through IR limits

$$\left(1-\mathbf{L}_{ij}^{(\mathbf{1})}\right)\left(1-\mathbf{L}_{ijkl}^{(\mathbf{2})}\right) RR\, \mathcal{W}_{ijkl} \;=\; \left[RR - \mathbf{L}_{ij}^{(\mathbf{1})} RR - \mathbf{L}_{ijkl}^{(\mathbf{2})} RR + \mathbf{L}_{ij}^{(\mathbf{1})} \mathbf{L}_{ijkl}^{(\mathbf{2})}\, RR\right] \mathcal{W}_{ijkl} \;=\; \text{finite}$$

Counterterms for *RR*

- Partition of phase space through sector functions $$RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\, \mathcal{W}_{ijkl}$$
- Identify counterterms through IR limits

$$\left(1-\mathbf{L}^{(\mathbf{1})}_{ij}\right)\left(1-\mathbf{L}^{(\mathbf{2})}_{ijkl}\right) RR\,\mathcal{W}_{ijkl} \;=\; \left[RR-\mathbf{L}^{(\mathbf{1})}_{ij}RR-\mathbf{L}^{(\mathbf{2})}_{ijkl}RR+\mathbf{L}^{(\mathbf{1})}_{ij}\mathbf{L}^{(\mathbf{2})}_{ijkl}\,RR\right]\mathcal{W}_{ijkl} \;=\; \text{finite}$$

- Mapping of momenta $\mathbf{L}^{(\mathbf{1})}_{ij}\,,\ \mathbf{L}^{(\mathbf{2})}_{ijkl}\,,\ \mathbf{L}^{(\mathbf{1})}_{ij}\mathbf{L}^{(\mathbf{2})}_{ijkl}$ → $\overline{\mathbf{L}}^{(\mathbf{1})}_{ij}\,,\ \overline{\mathbf{L}}^{(\mathbf{2})}_{ijkl}\,,\ \overline{\mathbf{L}}^{(\mathbf{1})}_{ij}\overline{\mathbf{L}}^{(\mathbf{2})}_{ijkl}$

Barred limits are defined such that

* satisfy $\left[RR-\overline{\mathbf{L}}^{(\mathbf{1})}_{ij}RR-\overline{\mathbf{L}}^{(\mathbf{2})}_{ijkl}RR+\overline{\mathbf{L}}^{(\mathbf{1})}_{ij}\overline{\mathbf{L}}^{(\mathbf{2})}_{ijkl}\,RR\right]\mathcal{W}_{ijkl} \;=\; \text{finite}$
* carry the **same symmetries** as unbarred limits

→ to use sum rules of sector functions

Counterterms for *RR*

- Partition of phase space through sector functions $$RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\, \mathcal{W}_{ijkl}$$
- Identify counterterms through IR limits

$$\left(1-\mathbf{L}^{(\mathbf{1})}_{ij}\right)\left(1-\mathbf{L}^{(\mathbf{2})}_{ijkl}\right)RR\,\mathcal{W}_{ijkl} \;=\; \left[RR-\mathbf{L}^{(\mathbf{1})}_{ij}RR-\mathbf{L}^{(\mathbf{2})}_{ijkl}RR+\mathbf{L}^{(\mathbf{1})}_{ij}\mathbf{L}^{(\mathbf{2})}_{ijkl}\,RR\right]\mathcal{W}_{ijkl} \;=\; \text{finite}$$

- Mapping of momenta

CRUCIAL FOR INTEGRAION !!!

NESTED CATANI-SEYMOUR MAPPINGS

$$\{k\} \to \{\bar{k}\}^{(abc)} \to \{\bar{k}\}^{(abc,def)}$$

Counterterms for *RR*

- Partition of phase space through sector functions $RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\, \mathcal{W}_{ijkl}$
- Identify counterterms through IR limits

$$\left(1 - \mathbf{L}^{(\mathbf{1})}_{ij}\right)\left(1 - \mathbf{L}^{(\mathbf{2})}_{ijkl}\right) RR\, \mathcal{W}_{ijkl} \;=\; \left[RR - \mathbf{L}^{(\mathbf{1})}_{ij} RR - \mathbf{L}^{(\mathbf{2})}_{ijkl} RR + \mathbf{L}^{(\mathbf{1})}_{ij}\mathbf{L}^{(\mathbf{2})}_{ijkl}\, RR\right]\mathcal{W}_{ijkl} \;=\; \text{finite}$$

- Mapping of momenta

CRUCIAL FOR INTEGRAION !!!

NESTED CATANI-SEYMOUR MAPPINGS

$$\{k\} \rightarrow \{\bar{k}\}^{(abc)} \rightarrow \{\bar{k}\}^{(abc,def)}$$

Properties:

* simple phase-space factorisation

* simple expressions for invariants

* flexibility → freedom in choosing (abc,def)

Counterterms for *RR*

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \Big(VV \Big) \delta_{X_n} + \int d\Phi_{n+1} \Big[\Big(RV \Big) \delta_{X_{n+1}} \Big] + \int d\Phi_{n+2} \Big[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \Big(K^{(\mathbf{2})} + K^{(\mathbf{12})} \Big) \delta_{X_n} \Big]$$

$$K^{(\mathbf{1})} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(\mathbf{1})}_{ij}\, RR\, \mathcal{W}_{ijkl}$$

$K^{(1)}$ → 3 barred limits

$$K^{(\mathbf{2})} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(\mathbf{2})}_{ijkl}\, RR\, \mathcal{W}_{ijkl}$$

$K^{(2)}$ → 11 barred limits

$$K^{(\mathbf{12})} = - \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(\mathbf{1})}_{ij}\, \overline{\mathbf{L}}^{(\mathbf{2})}_{ijkl}\, RR\, \mathcal{W}_{ijkl}$$

$K^{(12)}$ → 21 barred limits

We have verified analytically that

* Each barred limit carries the right symmetries ✓
* $RR + K^{(1)} + K^{(2)} + K^{(12)}$ → no phase-space singularities ✓

verified sector by sector

Integrated counterterms for *RR*

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \left(VV + I^{(\mathbf{2})} \right) \delta_{X_n}$$
$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} \right) \delta_{X_n} \right]$$
$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})} \right) \delta_{X_n} \right]$$

$$I^{(\mathbf{1})} = \int d\Phi_{\rm rad}\, K^{(\mathbf{1})}$$

- Integration of $I^{(1)}$ trivial

$$I^{(\mathbf{12})} = \int d\Phi_{\rm rad}\, K^{(\mathbf{12})}$$

- Integration of $I^{(12)}$ easy

$$I^{(\mathbf{2})} = \int d\Phi_{\rm rad,2}\, K^{(\mathbf{2})}$$

- Integration of $I^{(2)}$ feasible via hypergeometric functions

We have verified analytically that

- $RV + I^{(1)}$ → no ε poles ✓
- $I^{(1)} + I^{(12)}$ → no phase-space singularities ✓ verified sector by sector

Counterterm for *RV*

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \left(VV + I^{(\mathbf{2})}\right) \delta_{X_n}$$
$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})}\right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})}\right) \delta_{X_n}\right]$$
$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})}\right) \delta_{X_n}\right]$$

$$K^{(\mathbf{RV})} = \sum_{i,j\neq i} \left[\overline{\mathbf{L}}_{ij}^{(\mathbf{1})}\, RV + \Delta_{ij}\right] \mathcal{W}_{ij}$$

$$\overline{\mathbf{L}}_{ij}^{(\mathbf{1})} = \overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij}(1 - \overline{\mathbf{S}}_i)$$

It is defined such that

- RV - $K^{(RV)}$ → no IR singularities

$$\mathbf{S}_i\, \overline{\mathbf{L}}_{ij}^{(\mathbf{1})}\, RV = \mathbf{S}_i\, RV \qquad \mathbf{S}_i\, \Delta_{ij} = 0$$
$$\mathbf{C}_{ij}\, \overline{\mathbf{L}}_{ij}^{(\mathbf{1})}\, RV = \mathbf{C}_{ij}\, RV \qquad \mathbf{C}_{ij}\, \Delta_{ij} = 0$$

- $I^{(12)}$ - $K^{(RV)}$ → no ε poles

Integrated counterterm for *RV*

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \left(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})}\right) \delta_{X_n}$$
$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})}\right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})}\right) \delta_{X_n}\right]$$
$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})}\right) \delta_{X_n}\right]$$

$$I^{(\mathbf{RV})} = \int d\Phi_{\rm rad'}\, K^{(\mathbf{RV})}$$

We have verified analytically that

$VV + I^{(2)} + I^{(RV)}$ → no ε poles ✓

The subtraction formula at NNLO

massless final state radiation

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \left(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})} \right) \delta_{X_n}$$
$$+ \int d\Phi_{n+1} \Big[\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})} \right) \delta_{X_n} \Big]$$
$$+ \int d\Phi_{n+2} \Big[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})} \right) \delta_{X_n} \Big]$$

Sector function are summed up for sake of compactness

The subtraction formula at NNLO

massless final state radiation

$$
\begin{aligned}
\frac{d\sigma_{\mathrm{NNLO}} - d\sigma_{\mathrm{NLO}}}{dX} = & \int d\Phi_n \left(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})} \right) \delta_{X_n} \\
& + \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})} \right) \delta_{X_n} \right] \\
& + \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})} \right) \delta_{X_n} \right]
\end{aligned}
$$

Sector function are summed up for sake of compactness

$$RR\,\delta_{X_{n+2}} - K^{(\mathbf{1})}\,\delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})} \right) \delta_{X_n}$$

$$RR\,\delta_{X_{n+2}} - \boxed{K^{(\mathbf{1})}}\,\delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})} \right) \delta_{X_n}$$

$$K^{(\mathbf{1})} = K^{(\mathbf{1})}_{\mathrm{s}} + K^{(\mathbf{1})}_{\mathrm{hc}}$$

$$K^{(\mathbf{1})}_{\mathrm{s}} = -\mathcal{N}_1 \sum_{i} \sum_{\substack{c \neq i \\ d \neq i,c}} \mathcal{I}^{(\mathbf{i})}_{\mathbf{cd}}\, \bar{\mathbf{R}}^{(icd)}_{cd}$$

$$K^{(\mathbf{1})}_{\mathrm{hc}} = \mathcal{N}_1 \sum_{i,\, j>i} \left[\sum_{k \neq i,j} \overline{\mathcal{W}}^{(ijr)}_{jk} \frac{\mathbf{P}^{\mathrm{hc},\mu\nu}_{\mathbf{ij(r)}}}{\mathbf{s_{ij}}} \bar{\mathbf{R}}^{(ijr)}_{\mu\nu} + \sum_{\substack{k \neq i,j \\ l \neq i,j,k}} \overline{\mathcal{W}}^{(ijr')}_{kl} \frac{\mathbf{P}^{\mathrm{hc},\mu\nu}_{\mathbf{ij(r')}}}{\mathbf{s_{ij}}} \bar{\mathbf{R}}^{(ijr')}_{\mu\nu} \right]$$

$$RR\,\delta_{X_{n+2}} - \boxed{K^{(\mathbf{1})}}\,\delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})} \right) \delta_{X_n}$$

$$K^{(\mathbf{1})} = K_{\mathrm{s}}^{(\mathbf{1})} + K_{\mathrm{hc}}^{(\mathbf{1})}$$

$$K_{\mathrm{s}}^{(\mathbf{1})} = -\mathcal{N}_1 \sum_i \sum_{\substack{c \neq i \\ d \neq i,c}} \mathcal{I}_{\mathbf{cd}}^{(\mathbf{i})} \bar{\mathbf{R}}_{cd}^{(icd)}$$

Mapped momenta in R phase space

$$K_{\mathrm{hc}}^{(\mathbf{1})} = \mathcal{N}_1 \sum_{i,\, j>i} \left[\sum_{k \neq i,j} \overline{\mathcal{W}}_{jk}^{(ijr)} \frac{\mathbf{P}_{\mathbf{ij(r)}}^{\mathrm{hc},\mu\nu}}{\mathbf{s_{ij}}} \bar{\mathbf{R}}_{\mu\nu}^{(ijr)} + \sum_{\substack{k \neq i,j \\ l \neq i,j,k}} \overline{\mathcal{W}}_{kl}^{(ijr')} \frac{\mathbf{P}_{\mathbf{ij(r')}}^{\mathrm{hc},\mu\nu}}{\mathbf{s_{ij}}} \bar{\mathbf{R}}_{\mu\nu}^{(ijr')} \right]$$

Momenta in RR phase space

$$RR\,\delta_{X_{n+2}} - K^{(\mathbf{1})}\,\delta_{X_{n+1}} - \left(\boxed{K^{(\mathbf{2})}} + K^{(\mathbf{12})} \right) \delta_{X_n}$$

$$K^{(\mathbf{2})} = K^{(\mathbf{2})}_{\rm ss} + K^{(\mathbf{2})}_{\rm hcc} + K^{(\mathbf{2})}_{\rm hchc} + K^{(\mathbf{2})}_{\rm shc}$$

$$K^{(\mathbf{2})}_{\rm ss} = \sum_{i,j>i} \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c\neq i,j\\ d\neq i,j,c}} \Bigg[\sum_{\substack{e\neq i,j,c,d\\ f\neq i,j,c,d}} \mathcal{I}^{(\mathbf{i})}_{\mathbf{cd}} \mathcal{I}^{(\mathbf{j})}_{\mathbf{ef}}\, \bar{\mathbf{B}}^{(icd,jef)}_{cdef} + 4 \sum_{e\neq i,j,c,d} \mathcal{I}^{(\mathbf{i})}_{\mathbf{cd}} \mathcal{I}^{(\mathbf{j})}_{\mathbf{ed}}\, \bar{\mathbf{B}}^{(icd,jed)}_{cded} + 2\, \mathcal{I}^{(\mathbf{i})}_{\mathbf{cd}} \mathcal{I}^{(\mathbf{j})}_{\mathbf{cd}}\, \bar{\mathbf{B}}^{(icd,jcd)}_{cdcd} + \frac{1}{2} \Big(2\, \mathcal{I}^{(\mathbf{ij})}_{\mathbf{cd}} - \mathcal{I}^{(\mathbf{ij})}_{\mathbf{cc}} - \mathcal{I}^{(\mathbf{ij})}_{\mathbf{dd}} \Big) \bar{\mathbf{B}}^{(ijcd)}_{cd} \Bigg]$$

$$K^{(\mathbf{2})}_{\rm hcc} = \sum_{i,\,j>i} \sum_{k>j} \mathcal{N}_1^2\, \frac{\mathbf{P}^{\rm hc,\mu\nu}_{\mathbf{ijk(r)}}}{\mathbf{s}^{\mathbf{2}}_{\mathbf{ijk}}}\, \bar{\mathbf{B}}^{(ijkr)}_{\mu\nu}$$

$$K^{(\mathbf{2})}_{\rm hchc} = \sum_{i,\,j>i} \sum_{\substack{k\neq j\\ k>i}} \sum_{\substack{l\neq j\\ l>k}} \mathcal{N}_1^2\, \frac{\mathbf{P}^{\rm hc,\mu\nu}_{\mathbf{ij(r')}}}{\mathbf{s_{ij}}}\, \frac{\mathbf{P}^{\rm hc,\rho\sigma}_{\mathbf{kl(r')}}}{\mathbf{s_{kl}}}\, \bar{\mathbf{B}}^{(ijr',klr')}_{\mu\nu\rho\sigma}$$

$$K^{(\mathbf{2})}_{\rm shc} = -\sum_{i,j\neq i} \sum_{\substack{k\neq i\\ k>j}} \mathcal{N}_1^2\, \frac{\mathbf{P}^{\rm hc,\mu\nu}_{\mathbf{jk(r)}}}{\mathbf{s_{jk}}} \Bigg\{ \sum_{c,d\neq i,j,k,r} \mathcal{I}^{(\mathbf{i})}_{\mathbf{cd}}\, \bar{\mathbf{B}}^{(jkr,icd)}_{\mu\nu,cd} + 2 \sum_{c\neq i,j,k,r} \mathcal{I}^{(\mathbf{i})}_{\mathbf{cr}}\, \bar{\mathbf{B}}^{(jkr,icr)}_{\mu\nu,cr}$$
$$+ \sum_{c\neq i,j,k} \mathcal{I}^{(\mathbf{i})}_{\mathbf{jc}} \left[\frac{C_{f_{[jk]}} + C_{f_j} - C_{f_k}}{C_{f_{[jk]}}}\, \bar{\mathbf{B}}^{(krj,icj)}_{\mu\nu,[jk]c} + \bar{\tilde{\mathbf{B}}}^{(krj,icj)}_{\mu\nu,[jk]c} \left(\delta_{f_j q}\delta_{f_k \bar{q}} - \delta_{f_j \bar{q}}\delta_{f_k q} \right) \right]$$
$$+ \sum_{c\neq i,j,k} \mathcal{I}^{(\mathbf{i})}_{\mathbf{kc}} \left[\frac{C_{f_{[jk]}} + C_{f_k} - C_{f_j}}{C_{f_{[jk]}}}\, \bar{\mathbf{B}}^{(jrk,ick)}_{\mu\nu,[jk]c} + \bar{\tilde{\mathbf{B}}}^{(jrk,ick)}_{\mu\nu,[jk]c} \left(\delta_{f_k q}\delta_{f_j \bar{q}} - \delta_{f_k \bar{q}}\delta_{f_j q} \right) \right]$$
$$+ \left[\left(C_{f_{[jk]}} + C_{f_j} - C_{f_k} \right) \mathcal{I}^{(\mathbf{i})}_{\mathbf{jr}}\, \bar{\mathbf{B}}^{(krj,irj)}_{\mu\nu} + \left(C_{f_{[jk]}} + C_{f_k} - C_{f_j} \right) \mathcal{I}^{(\mathbf{i})}_{\mathbf{kr}}\, \bar{\mathbf{B}}^{(jrk,irk)}_{\mu\nu} \right] \Bigg\}$$

$$RR\,\delta_{X_{n+2}} - K^{(\mathbf{1})}\,\delta_{X_{n+1}} - \left(\boxed{K^{(\mathbf{2})}} + K^{(\mathbf{12})} \right)\delta_{X_n}$$

$$K^{(\mathbf{2})} = K^{(\mathbf{2})}_{\rm ss} + K^{(\mathbf{2})}_{\rm hcc} + K^{(\mathbf{2})}_{\rm hchc} + K^{(\mathbf{2})}_{\rm shc}$$

$$K^{(\mathbf{2})}_{\rm ss} = \sum_{i,j>i} \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c\neq i,j\\ d\neq i,j,c}} \Bigg[\sum_{\substack{e\neq i,j,c,d\\ f\neq i,j,c,d}} \mathcal{I}^{(\mathbf{i})}_{\mathbf{cd}} \mathcal{I}^{(\mathbf{j})}_{\mathbf{ef}}\, \bar{\mathbf{B}}^{(icd,jef)}_{cdef} + 4 \sum_{e\neq i,j,c,d} \mathcal{I}^{(\mathbf{i})}_{\mathbf{cd}} \mathcal{I}^{(\mathbf{j})}_{\mathbf{ed}}\, \bar{\mathbf{B}}^{(icd,jed)}_{cded} + 2\, \mathcal{I}^{(\mathbf{i})}_{\mathbf{cd}} \mathcal{I}^{(\mathbf{j})}_{\mathbf{cd}}\, \bar{\mathbf{B}}^{(icd,jcd)}_{cdcd} + \frac{1}{2} \left(2\, \mathcal{I}^{(\mathbf{ij})}_{\mathbf{cd}} - \mathcal{I}^{(\mathbf{ij})}_{\mathbf{cc}} - \mathcal{I}^{(\mathbf{ij})}_{\mathbf{dd}} \right) \bar{\mathbf{B}}^{(ijcd)}_{cd} \Bigg]$$

$$K^{(\mathbf{2})}_{\rm hcc} = \sum_{i,\,j>i} \sum_{k>j} \mathcal{N}_1^2 \frac{\mathbf{P}^{\rm hc,\mu\nu}_{\mathbf{ijk(r)}}}{\mathbf{s}^{\mathbf{2}}_{\mathbf{ijk}}} \bar{\mathbf{B}}^{(ijkr)}_{\mu\nu}$$

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$$K^{(\mathbf{2})}_{\rm hchc} = \sum_{i,\,j>i} \sum_{\substack{k\neq j\\ k>i}} \sum_{\substack{l\neq j\\ l>k}} \mathcal{N}_1^2 \frac{\mathbf{P}^{\rm hc,\mu\nu}_{\mathbf{ij(r')}}}{\mathbf{s_{ij}}} \frac{\mathbf{P}^{\rm hc,\rho\sigma}_{\mathbf{kl(r')}}}{\mathbf{s_{kl}}} \bar{\mathbf{B}}^{(ijr',klr')}_{\mu\nu\rho\sigma}$$

$$K^{(\mathbf{2})}_{\rm shc} = -\sum_{i,j\neq i} \sum_{\substack{k\neq i\\ k>j}} \mathcal{N}_1^2 \frac{\mathbf{P}^{\rm hc,\mu\nu}_{\mathbf{jk(r)}}}{\mathbf{s_{jk}}} \Bigg\{ \sum_{c,d\neq i,j,k,r} \mathcal{I}^{(\mathbf{i})}_{\mathbf{cd}}\, \bar{\mathbf{B}}^{(jkr,icd)}_{\mu\nu,cd} + 2 \sum_{c\neq i,j,k,r} \mathcal{I}^{(\mathbf{i})}_{\mathbf{cr}}\, \bar{\mathbf{B}}^{(jkr,icr)}_{\mu\nu,cr}$$
$$+ \sum_{c\neq i,j,k} \mathcal{I}^{(\mathbf{i})}_{\mathbf{jc}} \left[\frac{C_{f_{[jk]}} + C_{f_j} - C_{f_k}}{C_{f_{[jk]}}} \bar{\mathbf{B}}^{(krj,icj)}_{\mu\nu,[jk]c} + \bar{\tilde{\mathbf{B}}}^{(krj,icj)}_{\mu\nu,[jk]c} \left(\delta_{f_j q}\delta_{f_k \bar{q}} - \delta_{f_j \bar{q}}\delta_{f_k q} \right) \right]$$
$$+ \sum_{c\neq i,j,k} \mathcal{I}^{(\mathbf{i})}_{\mathbf{kc}} \left[\frac{C_{f_{[jk]}} + C_{f_k} - C_{f_j}}{C_{f_{[jk]}}} \bar{\mathbf{B}}^{(jrk,ick)}_{\mu\nu,[jk]c} + \bar{\tilde{\mathbf{B}}}^{(jrk,ick)}_{\mu\nu,[jk]c} \left(\delta_{f_k q}\delta_{f_j \bar{q}} - \delta_{f_k \bar{q}}\delta_{f_j q} \right) \right]$$
$$+ \left[\left(C_{f_{[jk]}} + C_{f_j} - C_{f_k} \right) \mathcal{I}^{(\mathbf{i})}_{\mathbf{jr}}\, \bar{\mathbf{B}}^{(krj,irj)}_{\mu\nu} + \left(C_{f_{[jk]}} + C_{f_k} - C_{f_j} \right) \mathcal{I}^{(\mathbf{i})}_{\mathbf{kr}}\, \bar{\mathbf{B}}^{(jrk,irk)}_{\mu\nu} \right] \Bigg\}$$

$$RR\,\delta_{X_{n+2}} - K^{(1)}\,\delta_{X_{n+1}} - \left(\boxed{K^{(2)}} + K^{(12)} \right)\delta_{X_n}$$

$$K^{(2)} = K^{(2)}_{\rm ss} + K^{(2)}_{\rm hcc} + K^{(2)}_{\rm hchc} + K^{(2)}_{\rm shc}$$

$$K^{(2)}_{\rm ss} = \sum_{i,j>i} \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c\neq i,j\\ d\neq i,j,c}} \Bigg[\sum_{\substack{e\neq i,j,c,d\\ f\neq i,j,c,d}} \mathcal{I}^{(\mathbf{i})}_{\mathbf{cd}} \mathcal{I}^{(\mathbf{j})}_{\mathbf{ef}}\, \bar{\mathbf{B}}^{(icd,jef)}_{cdef} + 4 \sum_{e\neq i,j,c,d} \mathcal{I}^{(\mathbf{i})}_{\mathbf{cd}} \mathcal{I}^{(\mathbf{j})}_{\mathbf{ed}}\, \bar{\mathbf{B}}^{(icd,jed)}_{cded}$$

$$+ 2\, \mathcal{I}^{(\mathbf{i})}_{\mathbf{cd}} \mathcal{I}^{(\mathbf{j})}_{\mathbf{cd}}\, \bar{\mathbf{B}}^{(icd,jcd)}_{cdcd} + \frac{1}{2} \boxed{\left(2\, \mathcal{I}^{(\mathbf{ij})}_{\mathbf{cd}} - \mathcal{I}^{(\mathbf{ij})}_{\mathbf{cc}} - \mathcal{I}^{(\mathbf{ij})}_{\mathbf{dd}} \right)} \bar{\mathbf{B}}^{(ijcd)}_{cd} \Bigg]$$

$$K^{(2)}_{\rm hcc} = \sum_{i,\,j>i} \sum_{k>j} \mathcal{N}_1^2 \boxed{\frac{\mathbf{P}^{{\rm hc},\mu\nu}_{\mathbf{ijk(r)}}}{\mathbf{s}^2_{\mathbf{ijk}}}} \bar{\mathbf{B}}^{(ijkr)}_{\mu\nu}$$

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$$K^{(2)}_{\rm hchc} = \sum_{i,\,j>i} \sum_{\substack{k\neq j\\ k>i}} \sum_{\substack{l\neq j\\ l>k}} \mathcal{N}_1^2 \frac{\mathbf{P}^{{\rm hc},\mu\nu}_{\mathbf{ij(r')}}}{\mathbf{s_{ij}}} \frac{\mathbf{P}^{{\rm hc},\rho\sigma}_{\mathbf{kl(r')}}}{\mathbf{s_{kl}}} \bar{\mathbf{B}}^{(ijr',klr')}_{\mu\nu\rho\sigma}$$

$$K^{(2)}_{\rm shc} = -\sum_{i,j\neq i} \sum_{\substack{k\neq i\\ k>j}} \mathcal{N}_1^2 \frac{\mathbf{P}^{{\rm hc},\mu\nu}_{\mathbf{jk(r)}}}{\mathbf{s_{jk}}} \Bigg\{ \sum_{c,d\neq i,j,k,r} \mathcal{I}^{(\mathbf{i})}_{\mathbf{cd}}\, \bar{\mathbf{B}}^{(jkr,icd)}_{\mu\nu,cd} + 2 \sum_{c\neq i,j,k,r} \mathcal{I}^{(\mathbf{i})}_{\mathbf{cr}}\, \bar{\mathbf{B}}^{(jkr,icr)}_{\mu\nu,cr}$$

$$+ \sum_{c\neq i,j,k} \mathcal{I}^{(\mathbf{i})}_{\mathbf{jc}} \left[\frac{C_{f_{[jk]}} + C_{f_j} - C_{f_k}}{C_{f_{[jk]}}} \bar{\mathbf{B}}^{(krj,icj)}_{\mu\nu,[jk]c} + \boxed{\bar{\tilde{\mathbf{B}}}^{(krj,icj)}_{\mu\nu,[jk]c}} \left(\delta_{f_j q}\delta_{f_k \bar{q}} - \delta_{f_j \bar{q}}\delta_{f_k q} \right) \right]$$

$$+ \sum_{c\neq i,j,k} \mathcal{I}^{(\mathbf{i})}_{\mathbf{kc}} \left[\frac{C_{f_{[jk]}} + C_{f_k} - C_{f_j}}{C_{f_{[jk]}}} \bar{\mathbf{B}}^{(jrk,ick)}_{\mu\nu,[jk]c} + \boxed{\bar{\tilde{\mathbf{B}}}^{(jrk,ick)}_{\mu\nu,[jk]c}} \left(\delta_{f_k q}\delta_{f_j \bar{q}} - \delta_{f_k \bar{q}}\delta_{f_j q} \right) \right]$$

$$+ \left[\left(C_{f_{[jk]}} + C_{f_j} - C_{f_k} \right) \mathcal{I}^{(\mathbf{i})}_{\mathbf{jr}}\, \bar{\mathbf{B}}^{(krj,irj)}_{\mu\nu} + \left(C_{f_{[jk]}} + C_{f_k} - C_{f_j} \right) \mathcal{I}^{(\mathbf{i})}_{\mathbf{kr}}\, \bar{\mathbf{B}}^{(jrk,irk)}_{\mu\nu} \right] \Bigg\}$$

$$\bar{\tilde{B}} = \langle \mathcal{M}_{\bar{B}} | T^A_c \tilde{T}^A_{[jk]} | \mathcal{M}_{\bar{B}} \rangle$$

$$(\tilde{T}^A_{[jk]})_{BC} = d_{ABC}$$

$$RR\,\delta_{X_{n+2}} - K^{(\mathbf{1})}\,\delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + \boxed{K^{(\mathbf{12})}} \right) \delta_{X_n}$$

$$K^{(\mathbf{12})} = K_{\mathrm{s}}^{(\mathbf{12})} + K_{\mathrm{hc}}^{(\mathbf{12})}$$

$$RR\,\delta_{X_{n+2}} - K^{(\mathbf{1})}\,\delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + \boxed{K^{(\mathbf{12})}} \right) \delta_{X_n}$$

$$K^{(\mathbf{12})} = \boxed{K_{\rm s}^{(\mathbf{12})}} + K_{\rm hc}^{(\mathbf{12})}$$

$$\begin{aligned}
K_{\rm s}^{(\mathbf{12})} = -\sum_{i,j\neq i} \mathcal{N}_1^{\,2} \Bigg\{ \sum_{\substack{c\neq i,j \\ d\neq i,j,c}} \Bigg[& \frac{1}{2} \sum_{\substack{e\neq i,j,c,d \\ f\neq i,j,c,d,e}} \mathcal{I}_{\mathbf{cd}}^{(\mathbf{i})}\, \bar{\mathcal{I}}_{\mathbf{ef}}^{(\mathbf{j})}\, \bar{\mathbf{B}}_{cdef}^{(icd,jef)} \\
& + \sum_{e\neq i,j,c,d} \mathcal{I}_{\mathbf{cd}}^{(\mathbf{i})} \left(\bar{\mathcal{I}}_{\mathbf{ed}}^{(\mathbf{j})}\, \bar{\mathbf{B}}_{cded}^{(icd,jed)} + \bar{\mathcal{I}}_{\mathbf{ed}}^{(\mathbf{j})}\, \bar{\mathbf{B}}_{cded}^{(idc,jed)} \right) + \mathcal{I}_{\mathbf{cd}}^{(\mathbf{i})}\, \bar{\mathcal{I}}_{\mathbf{cd}}^{(\mathbf{j})}\, \bar{\mathbf{B}}_{cdcd}^{(icd,jcd)} \\
& - C_A \left(\mathcal{I}_{\mathbf{jc}}^{(\mathbf{i})}\, \bar{\mathcal{I}}_{\mathbf{cd}}^{(\mathbf{j})}\, \bar{\mathbf{B}}_{cd}^{(icj,cjd)} + \mathcal{I}_{\mathbf{jd}}^{(\mathbf{i})}\, \bar{\mathcal{I}}_{\mathbf{cd}}^{(\mathbf{j})}\, \bar{\mathbf{B}}_{cd}^{(ijd,cjd)} - \mathcal{I}_{\mathbf{cd}}^{(\mathbf{i})}\, \bar{\mathcal{I}}_{\mathbf{cd}}^{(\mathbf{j})}\, \bar{\mathbf{B}}_{cd}^{(icd,jcd)} \right) \Bigg] \\
- \sum_{\substack{k\neq i \\ k>j}} \Bigg[& \sum_{\substack{c\neq i,j,k \\ d\neq i,j,k,c}} \mathcal{I}_{\mathbf{cd}}^{(\mathbf{i})} \frac{\bar{\mathbf{P}}_{\mathbf{jk(r)}}^{\rm hc,\mu\nu}}{\bar{\mathbf{s}}_{\mathbf{jk}}^{(\mathbf{icd})}} \bar{\mathbf{B}}_{\mu\nu,cd}^{(icd,jkr)} \\
& + \sum_{c\neq i,j,k} \mathcal{I}_{\mathbf{jc}}^{(\mathbf{i})} \frac{C_{f_{[jk]}} + C_{f_j} - C_{f_k}}{2\,C_{f_{[jk]}}} \left(\frac{\bar{\mathbf{P}}_{\mathbf{jk(r)}}^{\rm hc,\mu\nu}}{\bar{\mathbf{s}}_{\mathbf{jk}}^{(\mathbf{ijc})}} \bar{\mathbf{B}}_{\mu\nu,[jk]c}^{(ijc,krj)} + \frac{\bar{\mathbf{P}}_{\mathbf{jk(r)}}^{\rm hc,\mu\nu}}{\bar{\mathbf{s}}_{\mathbf{jk}}^{(\mathbf{icj})}} \bar{\mathbf{B}}_{\mu\nu,[jk]c}^{(icj,krj)} \right) + (j\leftrightarrow k) \\
& + \sum_{c\neq i,j,k} \frac{\mathcal{I}_{\mathbf{jc}}^{(\mathbf{i})}}{2} \left(\frac{\bar{\mathbf{P}}_{\mathbf{jk(r)}}^{\rm hc,\mu\nu}}{\bar{\mathbf{s}}_{\mathbf{jk}}^{(\mathbf{ijc})}} \bar{\tilde{B}}_{\mu\nu,[jk]c}^{(ijc,krj)} + \frac{\bar{\mathbf{P}}_{\mathbf{jk(r)}}^{\rm hc,\mu\nu}}{\bar{\mathbf{s}}_{\mathbf{jk}}^{(\mathbf{icj})}} \bar{\tilde{B}}_{\mu\nu,[jk]c}^{(icj,krj)} \right) \left(\delta_{f_j q}\delta_{f_k \bar{q}} - \delta_{f_j \bar{q}}\delta_{f_k q} \right) + (j\leftrightarrow k) \Bigg] \\
& + \frac{C_{f_{[jk]}} - C_{f_j} - C_{f_k}}{2} \mathcal{I}_{\mathbf{jk}}^{(\mathbf{i})} \left(\frac{\bar{\mathbf{P}}_{\mathbf{jk(r)}}^{\rm hc,\mu\nu}}{\bar{\mathbf{s}}_{\mathbf{jk}}^{(\mathbf{ijk})}} \bar{\mathbf{B}}_{\mu\nu}^{(ijk,jkr)} + \frac{\bar{\mathbf{P}}_{\mathbf{jk(r)}}^{\rm hc,\mu\nu}}{\bar{\mathbf{s}}_{\mathbf{jk}}^{(\mathbf{ikj})}} \bar{\mathbf{B}}_{\mu\nu}^{(ikj,jkr)} \right) \Bigg\}
\end{aligned}$$

$$RR\,\delta_{X_{n+2}} - K^{(\mathbf{1})}\,\delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + \boxed{K^{(\mathbf{12})}} \right) \delta_{X_n}$$

$$K^{(\mathbf{12})} = \boxed{K_{\mathrm{s}}^{(\mathbf{12})}} + K_{\mathrm{hc}}^{(\mathbf{12})}$$

$$
\begin{aligned}
K_{\mathrm{s}}^{(\mathbf{12})} = -\sum_{i,j\neq i} \mathcal{N}_1^2 \Bigg\{ & \sum_{\substack{c\neq i,j \\ d\neq i,j,c}} \Bigg[\frac{1}{2} \sum_{\substack{e\neq i,j,c,d \\ f\neq i,j,c,d,e}} \mathcal{I}_{\mathbf{cd}}^{(\mathbf{i})}\, \bar{\mathcal{I}}_{\mathbf{ef}}^{(\mathbf{j})}\, \bar{\mathbf{B}}_{cdef}^{(icd,jef)} \\
& + \sum_{e\neq i,j,c,d} \mathcal{I}_{\mathbf{cd}}^{(\mathbf{i})} \left(\bar{\mathcal{I}}_{\mathbf{ed}}^{(\mathbf{j})}\, \bar{\mathbf{B}}_{cded}^{(icd,jed)} + \bar{\mathcal{I}}_{\mathbf{ed}}^{(\mathbf{j})}\, \bar{\mathbf{B}}_{cded}^{(idc,jed)} \right) + \mathcal{I}_{\mathbf{cd}}^{(\mathbf{i})}\, \bar{\mathcal{I}}_{\mathbf{cd}}^{(\mathbf{j})}\, \bar{\mathbf{B}}_{cdcd}^{(icd,jcd)} \\
& - C_A \left(\mathcal{I}_{\mathbf{jc}}^{(\mathbf{i})}\, \bar{\mathcal{I}}_{\mathbf{cd}}^{(\mathbf{j})}\, \bar{\mathbf{B}}_{cd}^{(icj,cjd)} + \mathcal{I}_{\mathbf{jd}}^{(\mathbf{i})}\, \bar{\mathcal{I}}_{\mathbf{cd}}^{(\mathbf{j})}\, \bar{\mathbf{B}}_{cd}^{(ijd,cjd)} - \mathcal{I}_{\mathbf{cd}}^{(\mathbf{i})}\, \bar{\mathcal{I}}_{\mathbf{cd}}^{(\mathbf{j})}\, \bar{\mathbf{B}}_{cd}^{(icd,jcd)} \right) \Bigg] \\
& - \sum_{\substack{k\neq i \\ k>j}} \Bigg[\sum_{\substack{c\neq i,j,k \\ d\neq i,j,k,c}} \mathcal{I}_{\mathbf{cd}}^{(\mathbf{i})} \frac{\bar{\mathbf{P}}_{\mathbf{jk(r)}}^{\mathrm{hc},\mu\nu}}{\bar{\mathbf{s}}_{\mathbf{jk}}^{(\mathbf{icd})}} \bar{\mathbf{B}}_{\mu\nu,cd}^{(icd,jkr)} \\
& + \sum_{c\neq i,j,k} \mathcal{I}_{\mathbf{jc}}^{(\mathbf{i})} \frac{C_{f_{[jk]}} + C_{f_j} - C_{f_k}}{2\, C_{f_{[jk]}}} \left(\frac{\bar{\mathbf{P}}_{\mathbf{jk(r)}}^{\mathrm{hc},\mu\nu}}{\bar{\mathbf{s}}_{\mathbf{jk}}^{(\mathbf{ijc})}} \bar{\mathbf{B}}_{\mu\nu,[jk]c}^{(ijc,krj)} + \frac{\bar{\mathbf{P}}_{\mathbf{jk(r)}}^{\mathrm{hc},\mu\nu}}{\bar{\mathbf{s}}_{\mathbf{jk}}^{(\mathbf{icj})}} \bar{\mathbf{B}}_{\mu\nu,[jk]c}^{(icj,krj)} \right) + (j \leftrightarrow k) \\
& + \sum_{c\neq i,j,k} \frac{\mathcal{I}_{\mathbf{jc}}^{(\mathbf{i})}}{2} \left(\frac{\bar{\mathbf{P}}_{\mathbf{jk(r)}}^{\mathrm{hc},\mu\nu}}{\bar{\mathbf{s}}_{\mathbf{jk}}^{(\mathbf{ijc})}} \bar{\tilde{B}}_{\mu\nu,[jk]c}^{(ijc,krj)} + \frac{\bar{\mathbf{P}}_{\mathbf{jk(r)}}^{\mathrm{hc},\mu\nu}}{\bar{\mathbf{s}}_{\mathbf{jk}}^{(\mathbf{icj})}} \bar{\tilde{B}}_{\mu\nu,[jk]c}^{(icj,krj)} \right) \left(\delta_{f_j q}\delta_{f_k \bar{q}} - \delta_{f_j \bar{q}}\delta_{f_k q} \right) + (j \leftrightarrow k) \Bigg] \\
& + \frac{C_{f_{[jk]}} - C_{f_j} - C_{f_k}}{2} \mathcal{I}_{\mathbf{jk}}^{(\mathbf{i})} \left(\frac{\bar{\mathbf{P}}_{\mathbf{jk(r)}}^{\mathrm{hc},\mu\nu}}{\bar{\mathbf{s}}_{\mathbf{jk}}^{(\mathbf{ijk})}} \bar{\mathbf{B}}_{\mu\nu}^{(ijk,jkr)} + \frac{\bar{\mathbf{P}}_{\mathbf{jk(r)}}^{\mathrm{hc},\mu\nu}}{\bar{\mathbf{s}}_{\mathbf{jk}}^{(\mathbf{ikj})}} \bar{\mathbf{B}}_{\mu\nu}^{(ikj,jkr)} \right) \Bigg\}
\end{aligned}
$$

Momenta in RR phase space

Mapped momenta in B phase space

Mapped momenta in R phase space

$$RR\,\delta_{X_{n+2}} - K^{(\mathbf{1})}\,\delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})} \right) \delta_{X_n}$$

$$K^{(\mathbf{12})} = K_{\rm s}^{(\mathbf{12})} + K_{\rm hc}^{(\mathbf{12})}$$

$$K_{\rm hc}^{(\mathbf{12})} = \frac{\mathcal{N}_1^2}{\mathbf{s_{ij}}} \sum_{i,\,j>i} \Bigg\{ \mathbf{P}^{{\rm hc},\mu\nu}_{\mathbf{ij(r)}} \sum_{k\neq i,j} \Bigg[\sum_{c,d\neq i,j,k,r} \bar{\mathcal{I}}^{(\mathbf{k})}_{\mathbf{cd}}\, \bar{\mathbf{B}}^{(ijr,kcd)}_{\mu\nu,cd} + 2 \sum_{c\neq i,j,k,r} \bar{\mathcal{I}}^{(\mathbf{k})}_{\mathbf{cr}}\, \bar{\mathbf{B}}^{(ijr,kcr)}_{\mu\nu,cr} + 2 \sum_{c\neq i,j,k} \bar{\mathcal{I}}^{(\mathbf{k})}_{\mathbf{jc}}\, \bar{\mathbf{B}}^{(ijr,kcj)}_{\mu\nu,jc} \Bigg]$$

$$+ \sum_{\substack{c\neq i,j \\ d\neq i,j,c}} \Bigg\{ \delta_{\{f_i f_j\}\{q\bar{q}\}}\, T_R\, \bar{\mathcal{I}}^{(\mathbf{j})}_{\mathbf{cd}} + (d-2)\, \frac{\mathbf{Q_{ij(r)}}}{\tilde{\mathbf{k}}^2} \left[\frac{\tilde{\mathbf{k}}\cdot\bar{\mathbf{k}}_{\mathbf{c}}^{(\mathbf{ijr})}}{\bar{\mathbf{s}}_{\mathbf{jc}}^{(\mathbf{ijr})}} - \frac{\tilde{\mathbf{k}}\cdot\bar{\mathbf{k}}_{\mathbf{d}}^{(\mathbf{ijr})}}{\bar{\mathbf{s}}_{\mathbf{jd}}^{(\mathbf{ijr})}} \right]^{\mathbf{2}} \Bigg\} \bar{\mathbf{B}}^{(ijr,jcd)}_{cd}$$

$$- \sum_{k\neq i,j} \frac{\mathbf{1}}{\bar{\mathbf{s}}_{\mathbf{jk}}^{(\mathbf{ijr})}} \Bigg[\left(\mathbf{P}^{\rm hc}_{\mathbf{ij(r)}} + \mathbf{Q_{ij(r)}} \right) \bar{\mathbf{P}}^{{\rm hc},\mu\nu}_{\mathbf{jk(r)}}\, \bar{\mathbf{B}}^{(ijr,jkr)}_{\mu\nu}$$

$$+ \delta_{f_k g}\, 2\, C_A\, (d-2)\, \mathbf{Q_{ij(r)}}\, \frac{\bar{\mathbf{s}}_{\mathbf{jr}}^{(\mathbf{ijr})}\bar{\mathbf{s}}_{\mathbf{kr}}^{(\mathbf{ijr})}}{(\bar{\mathbf{s}}_{\mathbf{jr}}^{(\mathbf{ijr})} + \bar{\mathbf{s}}_{\mathbf{kr}}^{(\mathbf{ijr})})^{\mathbf{2}}}\, \frac{\bar{\tilde{\mathbf{k}}}^\mu \bar{\tilde{\mathbf{k}}}^\nu}{\bar{\tilde{\mathbf{k}}}^{\mathbf{2}}}\, \bar{\mathbf{B}}^{(ijr,jkr)}_{\mu\nu}$$

$$- \delta_{f_k\{q,\bar{q}\}}\, \frac{C_F}{2}\, (d-2)\, \mathbf{Q_{ij(r)}}\, \frac{\bar{\mathbf{s}}_{\mathbf{jr}}^{(\mathbf{ijr})}}{\bar{\mathbf{s}}_{\mathbf{jr}}^{(\mathbf{ijr})} + \bar{\mathbf{s}}_{\mathbf{kr}}^{(\mathbf{ijr})}}\, \bar{\mathbf{B}}^{(ijr,jkr)} \Bigg]$$

$$- \sum_{k\neq i,j} \sum_{\substack{l\neq i,j,k \\ l>k}} \mathbf{P}^{{\rm hc},\mu\nu}_{\mathbf{ij(r')}}\, \frac{\bar{\mathbf{P}}^{{\rm hc},\rho\sigma}_{\mathbf{kl(r')}}}{\bar{\mathbf{s}}_{\mathbf{kl}}^{(\mathbf{ijr'})}}\, \bar{\mathbf{B}}^{(ijr',klr')}_{\mu\nu\rho\sigma} \Bigg\}$$

The subtraction formula at NNLO

massless final state radiation

$$
\begin{aligned}
\frac{d\sigma_{\mathrm{NNLO}} - d\sigma_{\mathrm{NLO}}}{dX} &= \int d\Phi_n \left(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})} \right) \delta_{X_n} \\
&+ \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})} \right) \delta_{X_n} \right] \\
&+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})} \right) \delta_{X_n} \right]
\end{aligned}
$$

Sector function are summed up for sake of compactness

$$\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})} \right) \delta_{X_n} \;\; =$$

$$\sum_{i,j \neq i} \mathcal{W}_{ij} \Big(RV + \;\; I^{(\mathbf{1})} \;\; \Big) \delta_{X_{n+1}} + \left(\sum_{i,j \neq i} \mathcal{W}^{\mathrm{s}}_{ij} \;\; IK^{(\mathbf{12-RV})}_{S_i} + \sum_{i,j>i} \;\; IK^{(\mathbf{12-RV})}_{HC_{ij}} \;\; \right) \delta_{X_n}$$

$$\left(RV + I^{(\mathbf{1})}\right)\delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})}\right)\delta_{X_n} =$$

$$\sum_{i,j\neq i} \mathcal{W}_{ij}\left(RV + \boxed{I^{(\mathbf{1})}}\right)\delta_{X_{n+1}} + \left(\sum_{i,j\neq i} \mathcal{W}^{\mathrm{s}}_{ij}\; IK^{(\mathbf{12-RV})}_{S_i} + \sum_{i,j>i} IK^{(\mathbf{12-RV})}_{HC_{ij}}\right)\delta_{X_n}$$

$$I^{(\mathbf{1})} = I^{(\mathbf{1})}_{\text{poles}} + I^{(\mathbf{1})}_{\text{fin}}$$

$$\left(RV + I^{(\mathbf{1})}\right)\delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})}\right)\delta_{X_n} =$$

$$\sum_{i,j\neq i}\mathcal{W}_{ij}\left(RV + \boxed{I^{(\mathbf{1})}}\right)\delta_{X_{n+1}} + \left(\sum_{i,j\neq i}\mathcal{W}^{\mathrm{s}}_{ij}\; IK^{(\mathbf{12-RV})}_{S_i} + \sum_{i,j>i} IK^{(\mathbf{12-RV})}_{HC_{ij}}\right)\delta_{X_n}$$

$$I^{(\mathbf{1})} = \boxed{I^{(\mathbf{1})}_{\mathrm{poles}}} + I^{(\mathbf{1})}_{\mathrm{fin}}$$

$$I^{(\mathbf{1})}_{\mathrm{poles}} = \frac{\alpha_s}{2\pi}\left\{\frac{1}{\epsilon^2}\Sigma_C\,\mathbf{R} + \frac{1}{\epsilon}\left[\Sigma_\gamma\,\mathbf{R} + \sum_{c,d\neq c}\mathbf{L}_{cd}\,\mathbf{R}_{cd}\right]\right\}$$

$$\left(RV + I^{(\mathbf{1})}\right)\delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})}\right)\delta_{X_n} \ =$$

$$\sum_{i,j\neq i} \mathcal{W}_{ij}\left(RV + \boxed{I^{(\mathbf{1})}}\right)\delta_{X_{n+1}} + \left(\sum_{i,j\neq i} \mathcal{W}^{\mathrm{s}}_{ij} \ IK^{(\mathbf{12}-\mathbf{RV})}_{S_i} + \sum_{i,j>i} \ IK^{(\mathbf{12}-\mathbf{RV})}_{HC_{ij}}\right)\delta_{X_n}$$

$$I^{(\mathbf{1})} \quad = \boxed{I^{(\mathbf{1})}_{\text{poles}}} + I^{(\mathbf{1})}_{\text{fin}}$$

$$I^{(\mathbf{1})}_{\text{poles}} \quad = \quad \frac{\alpha_s}{2\pi}\left\{\frac{1}{\epsilon^2}\Sigma_C\,\mathbf{R} + \frac{1}{\epsilon}\left[\Sigma_\gamma\,\mathbf{R} + \sum_{c,d\neq c}\mathbf{L}_{cd}\,\mathbf{R}_{cd}\right]\right\} \ = \ -\,\mathbf{RV}_{\mathbf{poles}}$$

$$\left(RV + I^{(\mathbf{1})}\right)\delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})}\right)\delta_{X_n} =$$

$$\sum_{i,j\neq i}\mathcal{W}_{ij}\left(RV + \boxed{I^{(1)}}\right)\delta_{X_{n+1}} + \left(\sum_{i,j\neq i}\mathcal{W}^{\mathrm{s}}_{ij}\; IK^{(\mathbf{12-RV})}_{S_i} + \sum_{i,j>i} IK^{(\mathbf{12-RV})}_{HC_{ij}}\right)\delta_{X_n}$$

$$I^{(\mathbf{1})} = I^{(\mathbf{1})}_{\text{poles}} + \boxed{I^{(\mathbf{1})}_{\text{fin}}}$$

$$I^{(\mathbf{1})}_{\text{poles}} = \frac{\alpha_s}{2\pi}\left\{\frac{1}{\epsilon^2}\Sigma_C\,\mathbf{R} + \frac{1}{\epsilon}\left[\Sigma_\gamma\,\mathbf{R} + \sum_{c,d\neq c}\mathbf{L}_{cd}\,\mathbf{R}_{cd}\right]\right\} = -\,\mathbf{RV}_{\mathbf{poles}}$$

$$I^{(\mathbf{1})}_{\text{fin}} = \frac{\alpha_s}{2\pi}\left\{\left[\Sigma_\phi - \sum_j \gamma^{\text{hc}}_j\,\mathbf{L}_{jr}\right]\mathbf{R} + \sum_{c,d\neq c}\mathbf{L}_{cd}\left(2 - \frac{1}{2}\mathbf{L}_{cd}\right)\mathbf{R}_{cd}\right\}$$

$$\left(RV + I^{(\mathbf{1})}\right)\delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})}\right)\delta_{X_n} \;=$$

$$\sum_{i,j\neq i}\mathcal{W}_{ij}\left(RV + \; I^{(\mathbf{1})}\;\right)\delta_{X_{n+1}} + \left(\sum_{i,j\neq i}\mathcal{W}^{\mathrm{s}}_{ij}\boxed{IK^{(\mathbf{12-RV})}_{S_i}} + \sum_{i,j>i} IK^{(\mathbf{12-RV})}_{HC_{ij}}\right)\delta_{X_n}$$

$$\begin{aligned}
IK^{(\mathbf{12-RV})}_{S_i} = \; & 4\,\alpha_s^2 \sum_{\substack{c\neq i\\ d\neq i,c}} \mathcal{I}^{(\mathbf{i})}_{\mathbf{cd}} \Bigg\{ \sum_{\substack{e\neq i\\ f\neq i,e}} \left(L_{ef} - \frac{1}{4}L^2_{ef}\right)\bar{\mathbf{B}}^{(icd)}_{cdef} + 2\sum_{e\neq i,d}\left(L_{ed} - \frac{1}{4}L^2_{ed}\right)\left(\bar{\mathbf{B}}^{(icd)}_{cded} - \bar{\mathbf{B}}^{(idc)}_{cded}\right) \\
& + \sum_{e\neq i,d} \ln^2\frac{\bar{s}^{(icd)}_{de}}{s_{de}}\bar{\mathbf{B}}^{(icd)}_{cded} - \frac{1}{2}\ln^2\frac{\bar{s}^{(icd)}_{cd}}{s_{cd}}\bar{\mathbf{B}}^{(icd)}_{cdcd} - 2\pi\sum_{e\neq i,c,d}\ln\frac{s_{id}s_{ie}}{\mu^2\, s_{de}}\bar{\mathbf{B}}^{(icd)}_{cde} \\
& + \Bigg[\left(6 - \frac{7}{2}\zeta_2\right)\left(\Sigma_C + 2C_{f_d} - 2C_{f_c}\right) + \sum_p \left(\phi^{\mathrm{hc}}_p - \gamma^{\mathrm{hc}}_p\, L_{pr}\right) \\
& + C_A\left(6 - \zeta_2 - \ln\frac{s_{ic}}{s_{cd}}\ln\frac{s_{id}}{s_{cd}} - 2\ln\frac{s_{ic}s_{id}}{\mu^2 s_{cd}}\right)\Bigg]\bar{\mathbf{B}}^{(icd)}_{cd}\Bigg\} \\
& + 4\alpha_s^2\sum_{p\neq i}\left(\phi^{\mathrm{hc}}_p - \gamma^{\mathrm{hc}}_p L_{pr}\right)\Bigg[\sum_{c\neq i,p}\mathcal{I}^{(\mathbf{i})}_{\mathbf{cp}}\left(\bar{\mathbf{B}}^{(icp)}_{cp} - \bar{\mathbf{B}}^{(ipc)}_{cp}\right) + \sum_{c\neq i,r}\mathcal{I}^{(\mathbf{i})}_{\mathbf{cr}}\left(\bar{\mathbf{B}}^{(icr)}_{cr} - \bar{\mathbf{B}}^{(irc)}_{cr}\right) \\
& - \mathcal{I}^{(\mathbf{i})}_{\mathbf{pr}}\left(\bar{\mathbf{B}}^{(ipr)}_{pr} - \bar{\mathbf{B}}^{(irp)}_{pr}\right)\Bigg] \\
& + 8\pi\alpha_s\sum_{\substack{c\neq i\\ d\neq i,c}}\mathcal{I}^{(\mathbf{i})}_{\mathbf{cd}}\,\bar{\mathbf{V}}^{(icd)}_{\mathrm{fin},cd}
\end{aligned}$$

$$\left(RV + I^{(\mathbf{1})}\right)\delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})}\right)\delta_{X_n} \; = $$

$$\sum_{i,j\neq i} \mathcal{W}_{ij}\left(RV + \; I^{(\mathbf{1})}\right)\delta_{X_{n+1}} + \left(\sum_{i,j\neq i} \mathcal{W}^{\rm s}_{ij} \; IK^{(\mathbf{12-RV})}_{S_i} + \sum_{i,j>i} \boxed{IK^{(\mathbf{12-RV})}_{HC_{ij}}}\right)\delta_{X_n}$$

$$\begin{aligned}
IK^{(\mathbf{12-RV})}_{HC_{ij}} \; = \; & 4\alpha_{\rm s}^2 \, \frac{\tilde{\mathbf{P}}^{{\rm hc},\mu\nu}_{{\rm fin},\mathbf{ij(r)}}}{\mathbf{s_{ij}}} \, \bar{B}^{(ijr)}_{\mu\nu} \\
+ \; & 4\alpha_{\rm s}^2 \, \frac{\mathbf{P}^{{\rm hc},\mu\nu}_{\mathbf{ij(r)}}}{\mathbf{s_{ij}}} \Bigg\{ \sum_{c\neq i,j} \Bigg[\ln^2 \frac{\bar{s}^{(ijr)}_{jc}}{s_{[ij]r}} \, \bar{B}^{(ijr)}_{\mu\nu,[ij]c} - \frac{1}{2} \sum_{d\neq i,j,c} \left(4L_{cd} - L^2_{cd}\right) \bar{B}^{(ijr)}_{\mu\nu,cd} \Bigg] - \sum_{c\neq i,j,r} \ln^2 \frac{\bar{s}^{(ijr)}_{cr}}{s_{cr}} \, \bar{B}^{(ijr)}_{\mu\nu,cr} \\
& - \sum_{c\neq i,j,r} \Bigg[\frac{C_{f_{[ij]}} + C_{f_i} - C_{f_j}}{2C_{f_{[ij]}}} \, \mathcal{L}_{ijcr} \, \bar{B}^{(jri)}_{\mu\nu,[ij]c} + \frac{C_{f_{[ij]}} + C_{f_j} - C_{f_i}}{2C_{f_{[ij]}}} \, \mathcal{L}_{jicr} \, \bar{B}^{(irj)}_{\mu\nu,[ij]c} \Bigg] \\
& - \frac{1}{2} \sum_{c\neq i,j} \left(\delta_{f_i q}\delta_{f_j \bar{q}} - \delta_{f_i \bar{q}}\delta_{f_j q}\right) \left(\tilde{\mathcal{L}}_{ijcr} \tilde{\bar{B}}^{(jri)}_{\mu\nu,[ij]c} - \tilde{\mathcal{L}}_{jicr} \tilde{\bar{B}}^{(irj)}_{\mu\nu,[ij]c} \right) \\
& - \Bigg[\left(6 - \frac{7}{2}\zeta_2\right) \left(\Sigma_C - C_{f_{[ij]}} + C_{f_i} + C_{f_j}\right) + \frac{C_{f_{[ij]}} - C_{f_i} - C_{f_j}}{2} \left(4L_{ij} - L^2_{ij}\right) \\
& \quad - \frac{C_{f_{[ij]}} + C_{f_i} - C_{f_j}}{2} \left(4L_{ir} - L^2_{ir}\right) - \frac{C_{f_{[ij]}} + C_{f_j} - C_{f_i}}{2} \left(4L_{jr} - L^2_{jr}\right) + \Sigma^{\rm hc}_{\phi} \Bigg] \bar{B}^{(ijr)}_{\mu\nu} \Bigg\} \\
+ \; & 4\alpha_{\rm s}^2 \left\{ 2\, C_{f_j} \, \mathcal{I}^{(\mathbf{i})}_{\mathbf{jr}} \left[\left(\phi^{\rm hc}_j - \gamma^{\rm hc}_j L_{jr}\right) \left(\bar{B}^{(irj)} - \bar{B}^{(ijr)}\right) - C_{f_{[ij]}} \ln^2 \frac{s_{jr}}{s_{[ij]r}} \, \bar{B}^{(ijr)} \right] + (i \leftrightarrow j) \right\} \\
+ \; & 4\alpha_{\rm s}^2 \left[\frac{\mathbf{P}^{{\rm hc},\mu\nu}_{\mathbf{ij(r)}}}{\mathbf{s_{ij}}} \left(\gamma^{\rm hc}_i L_{ir} + \gamma^{\rm hc}_j L_{jr}\right) \bar{B}^{(ijr)}_{\mu\nu} + \sum_{p\neq i,j} \frac{\mathbf{P}^{{\rm hc},\mu\nu}_{\mathbf{ij(r')}}}{\mathbf{s_{ij}}} \, \gamma^{\rm hc}_p \, L_{pr'} \, \bar{B}^{(ijr')}_{\mu\nu} \right] - 8\pi\alpha_{\rm s} \, \frac{\mathbf{P}^{{\rm hc},\mu\nu}_{\mathbf{ij}}}{\mathbf{s_{ij}}} \, \bar{V}^{(ijr)}_{{\rm fin},\mu\nu}
\end{aligned}$$

$$\left(RV + I^{(\mathbf{1})}\right)\delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})}\right)\delta_{X_n} \;=$$

$$\sum_{i,j\neq i} \mathcal{W}_{ij}\left(RV + \; I^{(\mathbf{1})}\right)\delta_{X_{n+1}} + \left(\sum_{i,j\neq i}\mathcal{W}^{\mathrm{s}}_{ij}\; IK^{(\mathbf{12-RV})}_{S_i} + \sum_{i,j>i} \boxed{IK^{(\mathbf{12-RV})}_{HC_{ij}}}\right)\delta_{X_n}$$

$$IK^{(\mathbf{12-RV})}_{HC_{ij}} = 4\alpha_s^2 \frac{\boxed{\tilde{\mathbf{P}}^{\mathrm{hc},\mu\nu}_{\mathrm{fin},\mathbf{ij(r)}}}}{\mathbf{s_{ij}}}\bar{B}^{(ijr)}_{\mu\nu}$$

$$+\, 4\alpha_s^2 \frac{\mathbf{P}^{\mathrm{hc},\mu\nu}_{\mathbf{ij(r)}}}{\mathbf{s_{ij}}}\Bigg\{\sum_{c\neq i,j}\left[\ln^2\frac{\bar{s}^{(ijr)}_{jc}}{s_{[ij]r}}\,\bar{B}^{(ijr)}_{\mu\nu,[ij]c} - \frac{1}{2}\sum_{d\neq i,j,c}\left(4L_{cd}-L^2_{cd}\right)\bar{B}^{(ijr)}_{\mu\nu,cd}\right] - \sum_{c\neq i,j,r}\ln^2\frac{\bar{s}^{(ijr)}_{cr}}{s_{cr}}\,\bar{B}^{(ijr)}_{\mu\nu,cr}$$

$$-\sum_{c\neq i,j,r}\left[\frac{C_{f_{[ij]}}+C_{f_i}-C_{f_j}}{2C_{f_{[ij]}}}\,\mathcal{L}_{ijcr}\,\bar{B}^{(jri)}_{\mu\nu,[ij]c} + \frac{C_{f_{[ij]}}+C_{f_j}-C_{f_i}}{2C_{f_{[ij]}}}\,\mathcal{L}_{jicr}\,\bar{B}^{(irj)}_{\mu\nu,[ij]c}\right]$$

$$-\frac{1}{2}\sum_{c\neq i,j}\left(\delta_{f_i q}\delta_{f_j \bar{q}} - \delta_{f_i \bar{q}}\delta_{f_j q}\right)\left(\tilde{\mathcal{L}}_{ijcr}\tilde{\bar{B}}^{(jri)}_{\mu\nu,[ij]c} - \tilde{\mathcal{L}}_{jicr}\tilde{\bar{B}}^{(irj)}_{\mu\nu,[ij]c}\right)$$

$$+\, 4\alpha_s^2 \Big\{\, 2 \;\ldots$$

$$+\, 4\alpha_s^2 \Big[\, \frac{\mathbf{P}}{\;\ldots}$$

$$\tilde{\mathbf{P}}^{\mathrm{hc},\mu\nu}_{\mathrm{fin},\mathbf{ij(r)}} = \mathbf{P}^{\mu\nu}_{\mathbf{ij(r)}}\left[\frac{C_{f_{[ij]}}-C_{f_i}-C_{f_j}}{2}\left(7\zeta_2 - L^2_{ij}\right) + \left(C_{f_{[ij]}}+C_{f_i}-C_{f_j}\right)\mathcal{L}^{ij}_2 + \left(C_{f_{[ij]}}-C_{f_i}+C_{f_j}\right)\mathcal{L}^{ji}_2\right]$$

$$-\, \mathbf{g}^{\mu\nu}\frac{C_A}{2}\left[2C_{f_j}\mathcal{I}^{(\mathbf{i})}_{\mathbf{jr}}\left(5\zeta_2 - \ln^2\frac{s_{ij}s_{ir}}{\mu^2 s_{jr}}\right) + 2C_{f_i}\mathcal{I}^{(\mathbf{j})}_{\mathbf{ir}}\left(5\zeta_2 - \ln^2\frac{s_{ij}s_{jr}}{\mu^2 s_{ir}}\right)\right]$$

$$-\, \mathbf{P}^{\mu\nu}_{\mathbf{ij(r)}}\,\delta_{\{f_i f_j\}\{q\bar{q}\}}\left[\frac{7}{3}C_A - 8C_F + \frac{5}{3}\beta_0 + (3C_F - \beta_0)L_{ij}\right]$$

$$+\, \mathbf{g}^{\mu\nu}\left(\delta_{f_i\{q,\bar{q}\}}\delta_{f_j g} + \delta_{f_i g}\delta_{f_j\{q,\bar{q}\}}\right)C_F(C_A - C_F) - (d-2)\frac{\tilde{\mathbf{k}}^{\mu}_{\mathbf{ij}}\tilde{\mathbf{k}}^{\nu}_{\mathbf{ij}}}{\tilde{\mathbf{k}}^{\mathbf{2}}_{\mathbf{ij}}}\,\delta_{f_i g}\delta_{f_j g}\,\frac{C_A}{2}(3C_A - \beta_0)$$

$$\left(RV + I^{(\mathbf{1})}\right)\delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})}\right)\delta_{X_n} \;=$$

$$\sum_{i,j\neq i} \mathcal{W}_{ij}\left(RV + \; I^{(\mathbf{1})}\;\right)\delta_{X_{n+1}} + \left(\sum_{i,j\neq i} \mathcal{W}_{ij}^{\rm s}\; IK_{S_i}^{(\mathbf{12-RV})} + \sum_{i,j>i} \boxed{IK_{HC_{ij}}^{(\mathbf{12-RV})}}\right)\delta_{X_n}$$

$$IK_{HC_{ij}}^{(\mathbf{12-RV})} = 4\alpha_s^2 \frac{\boxed{\tilde{\mathbf{P}}^{\rm hc,\mu\nu}_{\rm fin,\mathbf{ij(r)}}}}{\mathbf{s_{ij}}} \bar{B}^{(ijr)}_{\mu\nu}$$

$$+\; 4\alpha_s^2 \frac{\mathbf{P}^{\rm hc,\mu\nu}_{\mathbf{ij(r)}}}{\mathbf{s_{ij}}} \Bigg\{ \sum_{c\neq i,j} \left[\ln^2 \frac{\bar{s}^{(ijr)}_{jc}}{s_{[ij]r}} \bar{B}^{(ijr)}_{\mu\nu,[ij]c} - \frac{1}{2} \sum_{d\neq i,j,c} \left(4L_{cd} - L_{cd}^2\right) \bar{B}^{(ijr)}_{\mu\nu,cd}\right] - \sum_{c\neq i,j,r} \ln^2 \frac{\bar{s}^{(ijr)}_{cr}}{s_{cr}} \bar{B}^{(ijr)}_{\mu\nu,cr}$$

$$- \sum_{c\neq i,j,r} \left[\frac{C_{f_{[ij]}}+C_{f_i}-C_{f_j}}{2C_{f_{[ij]}}} \boxed{\mathcal{L}_{ijcr}} \bar{B}^{(jri)}_{\mu\nu,[ij]c} + \frac{C_{f_{[ij]}}+C_{f_j}-C_{f_i}}{2C_{f_{[ij]}}} \boxed{\mathcal{L}_{jicr}} \bar{B}^{(irj)}_{\mu\nu,[ij]c} \right]$$

$$- \frac{1}{2} \sum_{c\neq i,j} \left(\delta_{f_i q}\delta_{f_j \bar{q}} - \delta_{f_i \bar{q}}\delta_{f_j q}\right) \left(\boxed{\tilde{\mathcal{L}}_{ijcr}} \bar{\tilde{B}}^{(jri)}_{\mu\nu,[ij]c} - \boxed{\tilde{\mathcal{L}}_{jicr}} \bar{\tilde{B}}^{(irj)}_{\mu\nu,[ij]c} \right)$$

$$\tilde{\mathbf{P}}^{\rm hc,\mu\nu}_{\rm fin,\mathbf{ij(r)}} = \mathbf{P}^{\mu\nu}_{\mathbf{ij(r)}} \left[\frac{C_{f_{[ij]}}-C_{f_i}-C_{f_j}}{2}\left(7\zeta_2 - L_{ij}^2\right) + \left(C_{f_{[ij]}}+C_{f_i}-C_{f_j}\right)\boxed{\mathcal{L}_2^{ij}} + \left(C_{f_{[ij]}}-C_{f_i}+C_{f_j}\right)\boxed{\mathcal{L}_2^{ji}}\right.$$

$$\ldots 5\zeta_2 - \ln^2 \frac{s_{ij}s_{ir}}{\mu^2 s_{jr}}\Bigg) + 2C_{f_i}\mathcal{I}^{(\mathbf{j})}_{\mathbf{ir}} \left(5\zeta_2 - \ln^2 \frac{s_{ij}s_{jr}}{\mu^2 s_{ir}}\right)\Bigg]$$

$$\ldots C_A - 8C_F + \frac{5}{3}\beta_0 + (3C_F - \beta_0)L_{ij}\Bigg]$$

$$\ldots \delta_{f_j\{q,\bar{q}\}}\Big) C_F (C_A - C_F) - (d-2)\frac{\tilde{\mathbf{k}}^\mu_{\mathbf{ij}}\tilde{\mathbf{k}}^\nu_{\mathbf{ij}}}{\tilde{\mathbf{k}}^{\mathbf{2}}_{\mathbf{ij}}} \delta_{f_i g}\delta_{f_j g} \frac{C_A}{2}(3C_A - \beta_0)$$

$$\mathcal{L}_2^{ij} = L_{ij} \ln \frac{s_{ir}}{s_{[ij]r}} + \frac{1}{2}\ln^2 \frac{s_{ir}}{s_{[ij]r}} + {\rm Li}_2\left(\frac{s_{jr}}{s_{[ij]r}}\right)$$

$$\mathcal{L}_{ijcr} = 4\,L_{ic} - 4\,L_{ir} - L_{ic}^2 + L_{ir}^2 + \ln^2 \frac{\bar{s}^{(jri)}_{ic}}{\bar{s}^{(jri)}_{ir}} - \ln^2 \frac{s_{ir}\bar{s}^{(jri)}_{ic}}{\bar{s}^{(jri)}_{ir}\, s_{ic}}$$

$$\tilde{\mathcal{L}}_{ijcr} = 4\,L_{ic} - L_{ic}^2 + \ln^2 \frac{\bar{s}^{(jri)}_{ic}}{\mu^2} - \ln^2 \frac{\bar{s}^{(jri)}_{ic}}{s_{ic}}$$

The subtraction formula at NNLO

massless final state radiation

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \left(\boxed{VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})}} \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})} \right) \delta_{X_n} \right]$$

$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})} \right) \delta_{X_n} \right]$$

Sector function are summed up for sake of compactness

$$VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})} \quad = \quad VV + \; I^{(\mathbf{2+RV})}_{\text{poles}} \; + \; I^{(\mathbf{2+RV})}_{\text{fin}}$$

$$VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})} \quad = \quad VV + \boxed{I^{(\mathbf{2+RV})}_{\text{poles}}} + \quad I^{(\mathbf{2+RV})}_{\text{fin}}$$

$$
\begin{aligned}
I^{(\mathbf{2+RV})}_{\text{poles}} = & \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\{ \frac{1}{\epsilon^4}\frac{\Sigma_C^2}{2}\,\mathbf{B} + \frac{1}{\epsilon^3}\,\Sigma_C \Bigg[\left(\Sigma_\gamma - \frac{3}{8}\beta_0\right)\mathbf{B} + \sum_{c,d\neq c} \mathbf{L}_{cd}\,\mathbf{B}_{cd}\Bigg] \\
& + \frac{1}{\epsilon^2}\Bigg[\left(\frac{\widehat{\gamma}_K^{(2)}}{16}\Sigma_C + (2\Sigma_\gamma - \beta_0)\frac{\Sigma_\gamma}{4}\right)\mathbf{B} + \left(\Sigma_\gamma - \frac{\beta_0}{4}\right)\sum_{c,d\neq c}\mathbf{L}_{cd}\,\mathbf{B}_{cd} + \frac{1}{4}\sum_{\substack{c,d\neq c\\ e,f\neq e}}\mathbf{L}_{ef}\,\mathbf{B}_{cdef}\Bigg)\Bigg] \\
& + \frac{1}{\epsilon}\Bigg[\frac{\Sigma_\gamma^{(2)}}{2}\,\mathbf{B} + \frac{\widehat{\gamma}_K^{(2)}}{8}\sum_{c,d\neq c}\mathbf{L}_{cd}\,\mathbf{B}_{cd}\Bigg]\Bigg\} \\
& + \left(\frac{\alpha_s}{2\pi}\right)\Bigg\{\frac{1}{\epsilon^2}\Sigma_C\,\mathbf{V} + \frac{1}{\epsilon}\Bigg[\Sigma_\gamma\,\mathbf{V} + \sum_{c,d\neq c}\mathbf{L}_{cd}\,\mathbf{V}_{cd}\Bigg]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\gamma_a^{(2)} = & \ \delta_{f_a\{q,\bar q\}}\,C_F\left[\left(\frac{3}{8} - 3\zeta_2 + 6\zeta_3\right)C_F + \left(\frac{41}{36} - \frac{13}{2}\zeta_3\right)C_A + \left(\frac{65}{72} + \frac{3}{4}\zeta_2\right)\beta_0\right] \\
& + \delta_{f_a g}\left\{C_A\left[-\frac{11}{4}C_F + \left(-\frac{1}{9} - \frac{1}{2}\zeta_3\right)C_A\right] + \beta_0\left[\frac{3}{4}C_F + \left(\frac{16}{9} - \frac{1}{4}\zeta_2\right)C_A\right]\right\}
\end{aligned}
$$

$$\Sigma_\gamma^{(2)} = \sum_a \gamma_a^{(2)} \qquad\qquad \widehat{\gamma}_K^{(2)} = \left(\frac{67}{18} - \zeta_2\right)C_A - \frac{10}{9}N_f T_R$$

$$VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})} = VV + I^{(\mathbf{2+RV})}_{\text{poles}} + I^{(\mathbf{2+RV})}_{\text{fin}}$$

$$I^{(\mathbf{2+RV})}_{\text{poles}} = \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\{ \frac{1}{\epsilon^4}\frac{\Sigma_C^{\,2}}{2}\,\mathbf{B} + \frac{1}{\epsilon^3}\,\Sigma_C \Bigg[\left(\Sigma_\gamma - \frac{3}{8}\beta_0\right)\mathbf{B} + \sum_{c,d\neq c}\mathbf{L}_{cd}\,\mathbf{B}_{cd}\Bigg]$$

$$+ \frac{1}{\epsilon^2}\Bigg[\left(\frac{\widehat{\gamma}_K^{(2)}}{16}\Sigma_C + (2\Sigma_\gamma - \beta_0)\frac{\Sigma_\gamma}{4}\right)\mathbf{B} + \left(\Sigma_\gamma - \frac{\beta_0}{4}\right)\sum_{c,d\neq c}\mathbf{L}_{cd}\,\mathbf{B}_{cd} + \frac{1}{4}\sum_{\substack{c,d\neq c\\ e,f\neq e}}\mathbf{L}_{ef}\,\mathbf{B}_{cdef}\Bigg)\Bigg]$$

$$+ \frac{1}{\epsilon}\Bigg[\frac{\Sigma_\gamma^{(2)}}{2}\,\mathbf{B} + \frac{\widehat{\gamma}_K^{(2)}}{8}\sum_{c,d\neq c}\mathbf{L}_{cd}\,\mathbf{B}_{cd}\Bigg]\Bigg\}$$

$$+ \left(\frac{\alpha_s}{2\pi}\right)\Bigg\{\frac{1}{\epsilon^2}\,\Sigma_C\,\mathbf{V} + \frac{1}{\epsilon}\Bigg[\Sigma_\gamma\,\mathbf{V} + \sum_{c,d\neq c}\mathbf{L}_{cd}\,\mathbf{V}_{cd}\Bigg]\Bigg\}$$

$$= -\mathbf{VV}_{\mathbf{poles}}$$

$$\gamma_a^{(2)} = \delta_{f_a\{q,\bar{q}\}}\,C_F\left[\left(\frac{3}{8} - 3\zeta_2 + 6\zeta_3\right)C_F + \left(\frac{41}{36} - \frac{13}{2}\zeta_3\right)C_A + \left(\frac{65}{72} + \frac{3}{4}\zeta_2\right)\beta_0\right]$$

$$+ \delta_{f_a g}\left\{C_A\left[-\frac{11}{4}C_F + \left(-\frac{1}{9} - \frac{1}{2}\zeta_3\right)C_A\right] + \beta_0\left[\frac{3}{4}C_F + \left(\frac{16}{9} - \frac{1}{4}\zeta_2\right)C_A\right]\right\}$$

$$\Sigma_\gamma^{(2)} = \sum_a \gamma_a^{(2)} \qquad \widehat{\gamma}_K^{(2)} = \left(\frac{67}{18} - \zeta_2\right)C_A - \frac{10}{9}N_f T_R$$

$$VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})} \quad = \quad VV + \quad I^{(\mathbf{2+RV})}_{\mathrm{poles}} \quad + \quad \boxed{I^{(\mathbf{2+RV})}_{\mathrm{fin}}}$$

$$\begin{aligned}
I^{(\mathbf{2+RV})}_{\mathrm{fin}} = \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\{ & \Bigg[I^{(0)} + \sum_j I^{(1)}_j \mathbf{L}_{jr} + \sum_j I^{(2)}_j \mathbf{L}^2_{jr} + \frac{1}{2} \sum_{j,l\neq j} \gamma_j^{\mathrm{hc}} \gamma_l^{\mathrm{hc}} \mathbf{L}_{jr'} \mathbf{L}_{lr'} \Bigg] \mathbf{B} \\
& + \sum_j \Big[I^{(0)}_{jr} + I^{(1)}_{jr} \mathbf{L}_{jr} \Big] \mathbf{B}_{jr} - 2\,(1-\zeta_2) \sum_{j,c\neq j,r} \gamma_j^{\mathrm{hc}} \left(2 - \mathbf{L}_{cr}\right) \mathbf{B}_{cr} \\
& + \sum_{c,d\neq c} \mathbf{L}_{cd} \Bigg[I^{(0)}_{cd} + I^{(1)}_{cd} \mathbf{L}_{cd} + \frac{\beta_0}{12} \mathbf{L}^2_{cd} + (4 - \mathbf{L}_{cd}) \sum_j \gamma_j^{\mathrm{hc}} \mathbf{L}_{jr} \Bigg] \mathbf{B}_{cd} \\
& + \sum_{c,d\neq c} \Bigg[-2 + \zeta_2 + 2\,\zeta_3 - \frac{5}{4}\zeta_4 + 2(1-\zeta_3)\,\mathbf{L}_{cd} \Bigg] \mathbf{B}_{cdcd} \\
& + (1-\zeta_2) \sum_{\substack{c,d\neq c \\ e\neq d}} \mathbf{L}_{cd} \mathbf{L}_{ed} \mathbf{B}_{cded} + \sum_{\substack{c,d\neq c \\ e,f\neq e}} \mathbf{L}_{cd} \mathbf{L}_{ef} \left[1 - \frac{1}{2} \mathbf{L}_{cd} \left(1 - \frac{1}{8} \mathbf{L}_{ef} \right) \right] \mathbf{B}_{cdef} \\
& + \pi \sum_{\substack{c,d\neq c \\ e\neq c,d}} \left[\ln\frac{s_{ce}}{s_{de}} \mathbf{L}^2_{cd} + \frac{1}{3} \ln^3 \frac{s_{ce}}{s_{de}} + 2\,\mathrm{Li}_3\left(-\frac{s_{ce}}{s_{de}} \right) \right] \mathbf{B}_{cde} \Bigg\} \\
+ \left(\frac{\alpha_s}{2\pi}\right) \Bigg\{ & \Bigg[\Sigma_\phi - \sum_j \gamma_j^{\mathrm{hc}} \mathbf{L}_{jr} \Bigg] \mathbf{V}^{\mathrm{fin}} + \sum_{c,d\neq c} \mathbf{L}_{cd} \left(2 - \frac{1}{2} \mathbf{L}_{cd} \right) \mathbf{V}^{\mathrm{fin}}_{cd} \Bigg\}
\end{aligned}$$

$$VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})} \quad = \quad VV + \quad I^{(\mathbf{2+RV})}_{\text{poles}} \quad + \quad I^{(\mathbf{2+RV})}_{\text{fin}}$$

$$
\begin{aligned}
I^{(\mathbf{2+RV})}_{\text{fin}} = \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\{ & \Bigg[I^{(0)} + \sum_j I^{(1)}_j \mathbf{L}_{jr} + \sum_j I^{(2)}_j \mathbf{L}^2_{jr} + \frac{1}{2} \sum_{j,l\neq j} \gamma^{\text{hc}}_j \gamma^{\text{hc}}_l \mathbf{L}_{jr'} \mathbf{L}_{lr'} \Bigg] \mathbf{B} \\
& + \sum_j \left[I^{(0)}_{jr} + I^{(1)}_{jr} \mathbf{L}_{jr} \right] \mathbf{B}_{jr} - 2\,(1-\zeta_2) \sum_{j,c\neq j,r} \gamma^{\text{hc}}_j \,(2 - \mathbf{L}_{cr})\, \mathbf{B}_{cr} \\
& + \sum_{c,d\neq c} \mathbf{L}_{cd} \left[I^{(0)}_{cd} + I^{(1)}_{cd} \mathbf{L}_{cd} + \frac{\beta_0}{12} \mathbf{L}^2_{cd} + (4 - \mathbf{L}_{cd}) \sum_j \gamma^{\text{hc}}_j \mathbf{L}_{jr} \right] \mathbf{B}_{cd} \\
& + \sum_{c,d\neq c} \left[-2 + \zeta_2 + 2\,\zeta_3 - \frac{5}{4}\zeta_4 + 2(1-\zeta_3)\, \mathbf{L}_{cd} \right] \mathbf{B}_{cdcd} \\
& + (1-\zeta_2) \sum_{\substack{c,d\neq c \\ e\neq d}} \mathbf{L}_{cd} \mathbf{L}_{ed} \mathbf{B}_{cded} + \sum_{\substack{c,d\neq c \\ e,f\neq e}} \mathbf{L}_{cd} \mathbf{L}_{ef} \left[1 - \frac{1}{2} \mathbf{L}_{cd} \left(1 - \frac{1}{8} \mathbf{L}_{ef} \right) \right] \mathbf{B}_{cdef} \\
& + \pi \sum_{\substack{c,d\neq c \\ e\neq c,d}} \left[\ln \frac{s_{ce}}{s_{de}} \mathbf{L}^2_{cd} + \frac{1}{3} \ln^3 \frac{s_{ce}}{s_{de}} + 2\,\mathrm{Li}_3\left(- \frac{s_{ce}}{s_{de}} \right) \right] \mathbf{B}_{cde} \Bigg\} \\
+ \left(\frac{\alpha_s}{2\pi}\right) \Bigg\{ & \left[\Sigma_\phi - \sum_j \gamma^{\text{hc}}_j \mathbf{L}_{jr} \right] \mathbf{V}^{\text{fin}} + \sum_{c,d\neq c} \mathbf{L}_{cd} \left(2 - \frac{1}{2} \mathbf{L}_{cd} \right) \mathbf{V}^{\text{fin}}_{cd} \Bigg\}
\end{aligned}
$$

$$VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})} = VV + I^{(\mathbf{2+RV})}_{\text{poles}} + I^{(\mathbf{2+RV})}_{\text{fin}}$$

$$I^{(\mathbf{2+RV})}_{\text{fin}} = \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\{ \Bigg[I^{(0)} + \sum_j I^{(1)}_j \mathbf{L}_{jr} + \sum_j I^{(2)}_j \mathbf{L}^2_{jr} + \frac{1}{2} \sum_{j,l\neq j} \gamma^{\text{hc}}_j \gamma^{\text{hc}}_l \mathbf{L}_{jr'} \mathbf{L}_{lr'} \Bigg] \mathbf{B}$$

$$+ \sum_j \left[I^{(0)}_{jr} + I^{(1)}_{jr} \mathbf{L}_{jr} \right. \ldots$$

$$+ \sum_{c,d\neq c} \mathbf{L}_{cd} \left[I^{(0)}_{cd} + I^{(1)}_{cd} \ldots \right.$$

$$+ \sum_{c,d\neq c} \Big[-2 + \zeta_2 + 2\,\zeta \ldots$$

$$+ (1-\zeta_2) \sum_{\substack{c,d\neq c\\ e\neq d}} \mathbf{L}_{cd} \mathbf{L}_{ed} \ldots$$

$$+ \pi \sum_{\substack{c,d\neq c\\ e\neq c,d}} \Big[\ln\frac{s_{ce}}{s_{de}} \mathbf{L}^2_{cd} + \ldots$$

$$+ \left(\frac{\alpha_s}{2\pi}\right) \Bigg\{ \Bigg[\Sigma_\phi - \sum_j \gamma^{\text{hc}}_j \mathbf{L}_{jr} \Bigg] \mathbf{V} \ldots$$

$$I^{(0)} = N_q^2 C_F^2 \left[\frac{101}{8} - \frac{141}{8}\zeta_2 + \frac{245}{16}\zeta_4\right] + N_g N_q C_F \left[C_A \left(\frac{13}{3} - \frac{125}{6}\zeta_2 + \frac{245}{8}\zeta_4\right) + \beta_0 \left(\frac{77}{12} - \frac{53}{12}\zeta_2\right)\right]$$
$$+ N_g^2 \left[C_A^2 \left(\frac{20}{9} - \frac{13}{3}\zeta_2 + \frac{245}{16}\zeta_4\right) + \beta_0^2 \left(\frac{73}{72} - \frac{1}{8}\zeta_2\right) + C_A \beta_0 \left(-\frac{1}{9} - \frac{11}{3}\zeta_2\right)\right]$$
$$+ N_q C_F \left[C_F \left(\frac{53}{32} - \frac{57}{8}\zeta_2 + \frac{1}{2}\zeta_3 + \frac{21}{4}\zeta_4\right) + C_A \left(\frac{677}{432} + \frac{5}{3}\zeta_2 - \frac{25}{2}\zeta_3 + \frac{47}{8}\zeta_4\right) + \beta_0 \left(\frac{5669}{864} - \frac{85}{24}\zeta_2 - \frac{11}{12}\zeta_3\right)\right]$$
$$+ N_g \left[C_F C_A \left(-\frac{737}{48} + 11\zeta_3\right) + C_F \beta_0 \left(\frac{67}{16} - 3\zeta_3\right) + \beta_0^2 \left(\frac{73}{72} - \frac{3}{8}\zeta_2\right) + C_A^2 \left(-\frac{4289}{216} + \frac{15}{2}\zeta_2 - 14\zeta_3 + \frac{89}{8}\zeta_4\right) + C_A \beta_0 \left(\frac{647}{54} - \frac{53}{8}\zeta_2 - \frac{11}{12}\zeta_3\right)\right]$$

$$I^{(1)}_j = \delta_{f_a\{q,\bar q\}} C_F \left[N_q C_F \left(\frac{5}{2} - \frac{7}{4}\zeta_2\right) + N_g C_A \left(\frac{1}{3} - \frac{7}{4}\zeta_2\right) + \frac{2}{3} N_g \beta_0 + C_F \left(-\frac{3}{8} - 4\zeta_2 + 2\zeta_3\right) + C_A \left(\frac{25}{12} - 3\zeta_2 + 3\zeta_3\right) + \beta_0 \left(\frac{1}{24} + \zeta_2\right)\right]$$
$$+ \delta_{f_a g} \left[N_q C_F C_A (10 - 7\zeta_2) - N_q C_F \beta_0 \left(\frac{5}{2} - \frac{7}{4}\zeta_2\right) + N_g C_A^2 \left(\frac{4}{3} - 7\zeta_2\right) + N_g C_A \beta_0 \left(\frac{7}{3} + \frac{7}{4}\zeta_2\right) - \frac{2}{3}(N_g + 1)\beta_0^2 + \frac{11}{4} C_F C_A - \frac{3}{4} C_F \beta_0 + C_A^2 \left(\frac{28}{3} - \frac{23}{2}\zeta_2 + 5\zeta_3\right) - C_A \beta_0 \left(\frac{2}{3} - \frac{5}{2}\zeta_2\right)\right]$$

$$I^{(2)}_j = \frac{1}{8}(15\,C_A - 7\,\beta_0 - 15) C_{f_j} - \frac{1}{4}(5\,C_A - 2\,\beta_0)\gamma_j + 2\,\zeta_2\, C^2_{f_j}$$

$$I^{(0)}_{jr} = (-1 + 3\zeta_2 - 2\zeta_3) C_A - \frac{1}{2}(13 + 10\zeta_2 + 2\zeta_3) C_{f_j} + (5 + 2\zeta_3)\gamma_j$$

$$I^{(1)}_{jr} = (1 - \zeta_2) C_A + \frac{1}{2}(4 + 7\zeta_2) C_{f_j} - (2 + \zeta_2)\gamma_j$$

$$I^{(0)}_{cd} = \left(\frac{20}{9} - 2\zeta_2 - \frac{7}{2}\zeta_3\right) C_A + \frac{31}{9}\beta_0 + 2\,\Sigma_\phi + 8\,(1 - \zeta_2)\, C_{f_d}$$

$$I^{(1)}_{cd} = -\left(\frac{1}{3} - \frac{1}{2}\zeta_2\right) C_A - \frac{11}{12}\beta_0 - \frac{1}{2}\Sigma_\phi$$

State-of-the-art at NLO

- Initial and final state radiation in the massless case
- Damping factors to improve convergence
- Final state radiation with massive particles
- Numerical implementation in MADNkLO

Outlook

- Complete treatment of the massive cases

State-of-the-art at NNLO

massless final state radiation

- Verified cancellation of phase space singularities for *RR* and *RV*
- Verified cancellation of virtual singularities for *RV* and *VV*

Outlook

- Numerical implementation in MADNkLO
- Initial state radiation at NNLO
- The massive case at NNLO

Thanks for your attention

Back-up slides

Counterterms for *RR*

- Partition of phase space through sector functions

$$RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\, \mathcal{W}_{ijkl}$$

- Identify counterterms through IR limits

$$\left(1-\mathbf{L}^{(1)}_{ij}\right)\left(1-\mathbf{L}^{(2)}_{ijkl}\right) RR\, \mathcal{W}_{ijkl} \;=\; \left[RR - \mathbf{L}^{(1)}_{ij} RR - \mathbf{L}^{(2)}_{ijkl} RR + \mathbf{L}^{(1)}_{ij}\mathbf{L}^{(2)}_{ijkl}\, RR\right]\mathcal{W}_{ijkl} \;=\; \text{finite}$$

- Mapping of momenta

CRUCIAL FOR INTEGRAION !!!

NESTED CATANI-SEYMOUR MAPPINGS

$$\{k\} \to \{\bar{k}\}^{(abc)} \to \{\bar{k}\}^{(abc,def)}$$

$$\bar{k}^{(abc)}_b = k_a + k_b - \frac{s_{ab}}{s_{ac}+s_{bc}}\, k_c \qquad \bar{k}^{(abc)}_c = \frac{s_{abc}}{s_{ac}+s_{bc}}\, k_c \qquad k_a \to w, y, z$$

$$d\Phi_{n+1}\left(\{k\}\right) = d\Phi_n\left(\{\bar{k}\}^{(abc)}\right) d\Phi_{\text{rad}}\left(\bar{s}^{(abc)}_{bc}; w, y, z\right)$$

$$\int d\Phi_{\text{rad}}(s; w, y, z) = N_\epsilon\, s^{1-\epsilon} \int_0^1 dw \int_0^1 dy \int_0^1 dz\, \left[w(1-w)\right]^{-\epsilon-\frac{1}{2}} \left[y(1-y)^2\, z(1-z)\right]^{-\epsilon} (1-y)$$

$$s_{ab} = y\, \bar{s}^{(abc)}_{bc} \qquad s_{ac} = z(1-y)\bar{s}^{(abc)}_{bc} \qquad s_{bc} = (1-z)(1-y)\bar{s}^{(abc)}_{bc} \qquad s_{cd} = (1-y)s^{(abc)}_{cd}$$

$$s_{ad} = y(1-z)s^{(abc)}_{cd} + z s^{(abc)}_{bd} - 2\,(1-2w)\left[yz(1-z)s^{(abc)}_{bd}\, s^{(abc)}_{cd}\right]^{1/2}$$

$$s_{bd} = yz s^{(abc)}_{cd} + (1-z)s^{(abc)}_{bd} + 2\,(1-2w)\left[yz(1-z)s^{(abc)}_{bd}\, s^{(abc)}_{cd}\right]^{1/2}$$