DESY Summer School 2021

Lasers and Optics:

Ultrafast Optical Imaging of Ultrasmall

Lecture 1

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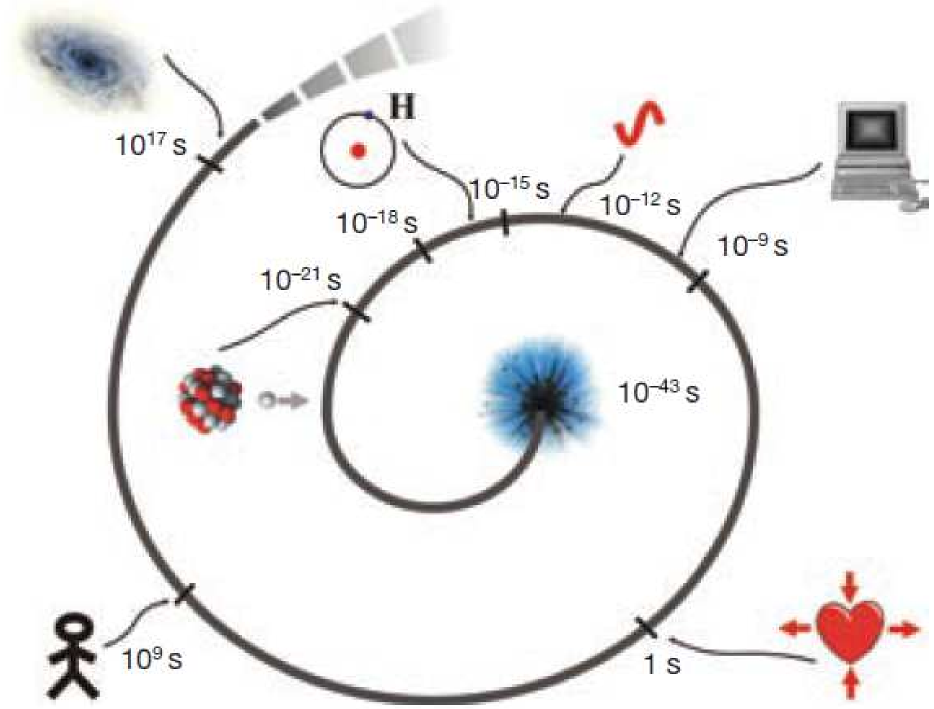
Slides: in part from Franz X. Kärtner & Noah Chang

Outline

- Lecture 1
  - Why ultrafast optics?
  - General techniques in solving optics problems?
    - Impulse response approach
    - Eigenfunction approach
    - Laplace/Fourier transfrom approach
  - Diffraction and dispersion theory
  - Solution of flattop-beam propagation with 3 approaches
- Lecture 2
  - Brief history of lasers
  - How lasers work?
  - How do we generate pulses with lasers?
    - ns-to-ps pulse generation with Q-switching
    - ps-to-fs pulse generation with mode-locking

The long and short of time



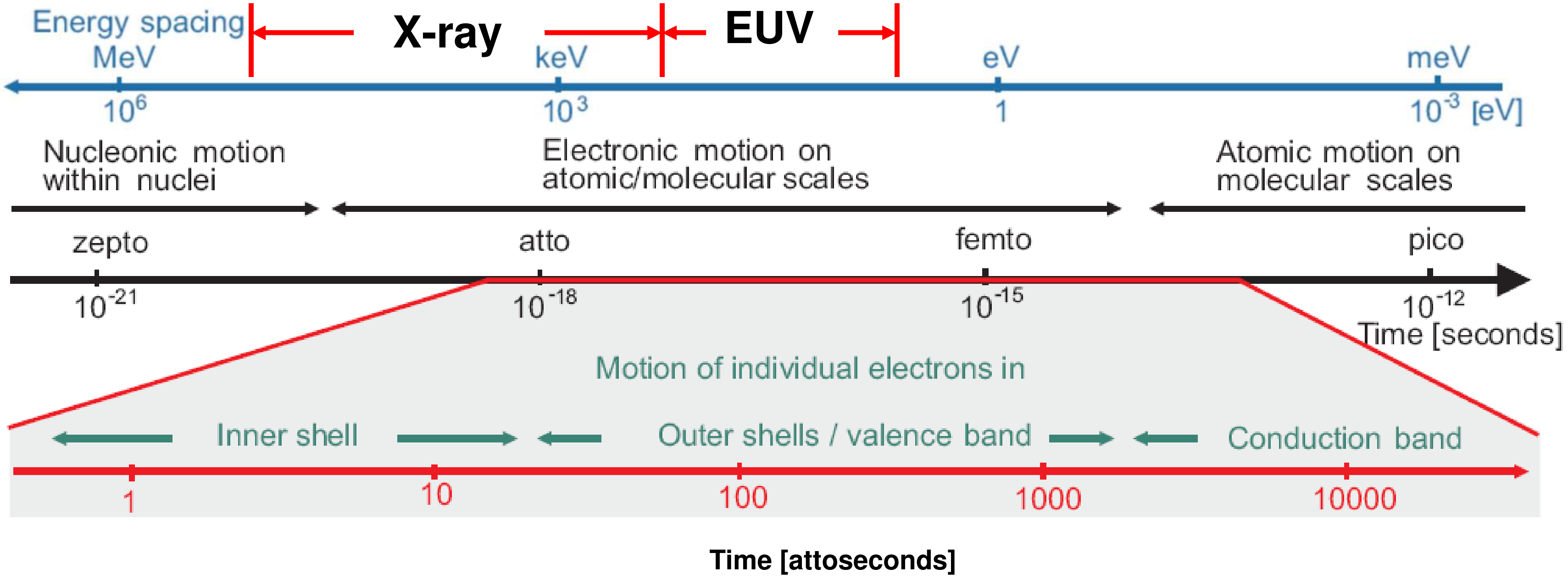
Notable time scales, in seconds		Length by other measures
5×10^{17} s	Estimated age of universe	14 billion years
2×10^{9} s	Average human lifetime	70 years
1 s	Length of a heartbeat	1 second
0.3×10^{-9} s	Current computer clock frequency	0.3 nanosecond
10^{-12} s	Length of a typical THz pulse	1 picosecond
3×10^{-15} s	Cycle length of laser	3 femtoseconds
1.5×10^{-16} s	Electron circles proton in Hydrogen atom	0.15 femtosecond
10^{-18} s	Next horizon for controllable laser pulses?	1 attosecond
10^{-21} s	Strong nuclear reactions	1 zeptosecond
10^{-43} s	Birth flash of the Big Bang	Planck time

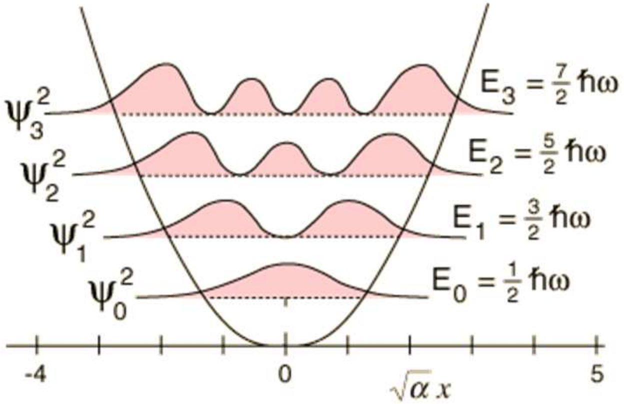
A. E. Kaplan, "No-when: the long and short of time," Optics and Photonics News (2006, February)

Time scale of physical events

- Mean life of W and Z bozons: 0.3x10^{-24} s (yoctosecond)
- Half-life of helium-9's outer neutron in the second nuclear halo: 7x10^{-21} s (Zeptosecond)
- Shortest event ever created: 53 attosecond (10^{-18}s) x-ray pulse (2017)
- Smallest time uncertainty established: 850 zeptoseconds
- Bohr orbit period in hydrogen atom: 150 attoseconds
- Single oscillation of 600 nm light: 2 fs (10^{-15}s)
- Vibrational modes of a molecule: ps (10^{-12}s) timescale
- Electron transfer in photosynthesis: ps timescale
- Period of phonon vibrations in a solid: ps timescale
- Mean time between atomic collisions in ambient air: 0.1 ns (10^{-9}s)
- Period of mid-range sound vibrations: ms

Physics on femto- to attosecond time scales?





The larger the energy separation between the two eigenstates, the faster is the particle's motion in the superposition state.

$|\Psi\rangle = \frac{1}{\sqrt{2}}\,|\Psi_0\rangle + \frac{1}{\sqrt{2}}\,|\Psi_1\rangle$ → Wave packet: Change in position of particles center of mass

$\Delta E = E_1 - E_0$ → Energy spacing

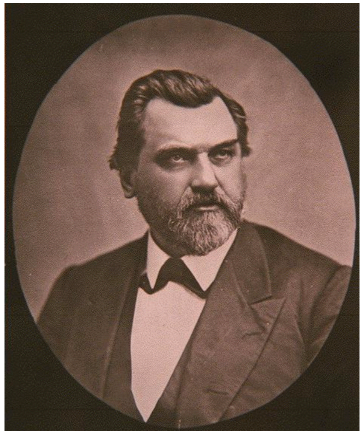
$\langle x\rangle = \langle\phi_0|x|\phi_1\rangle Cos\left[\frac{\Delta E}{\hbar}t\right]$ → Expectation value of position

$T_{osc} = 2\pi\frac{\hbar}{\Delta E}$ → Oscillation period

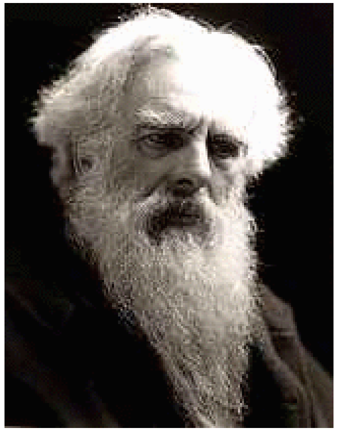
"Attosecond physics" F. Krausz and M. Ivanov, Rev. Mod. Phys. 81, 163 (2009)

Birth of ultrafast technology

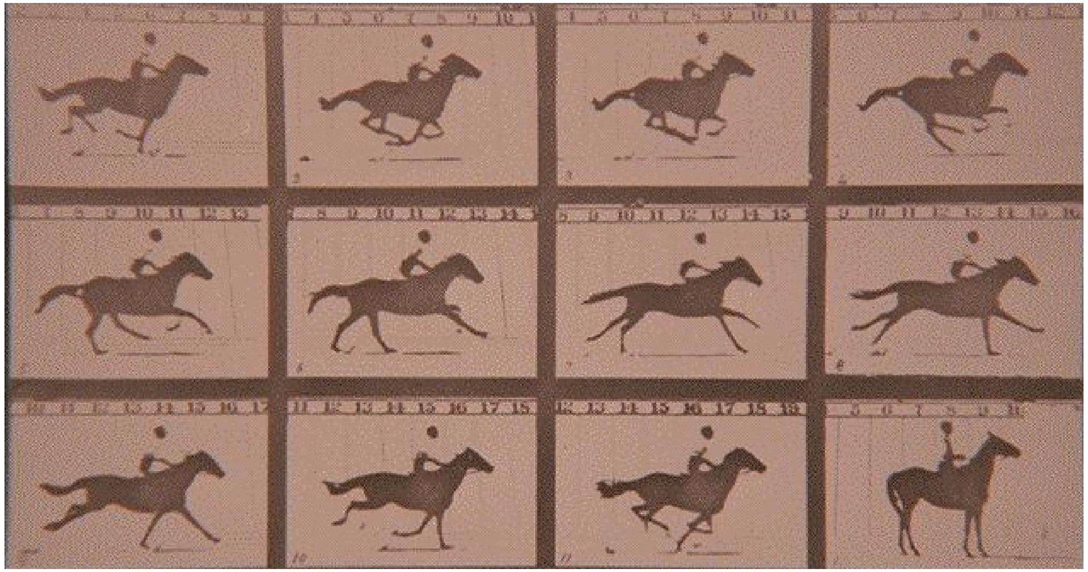
$25,000 bet: Do all four hooves of a running horse ever simultaneously leave the ground? (1872)



Leland Stanford



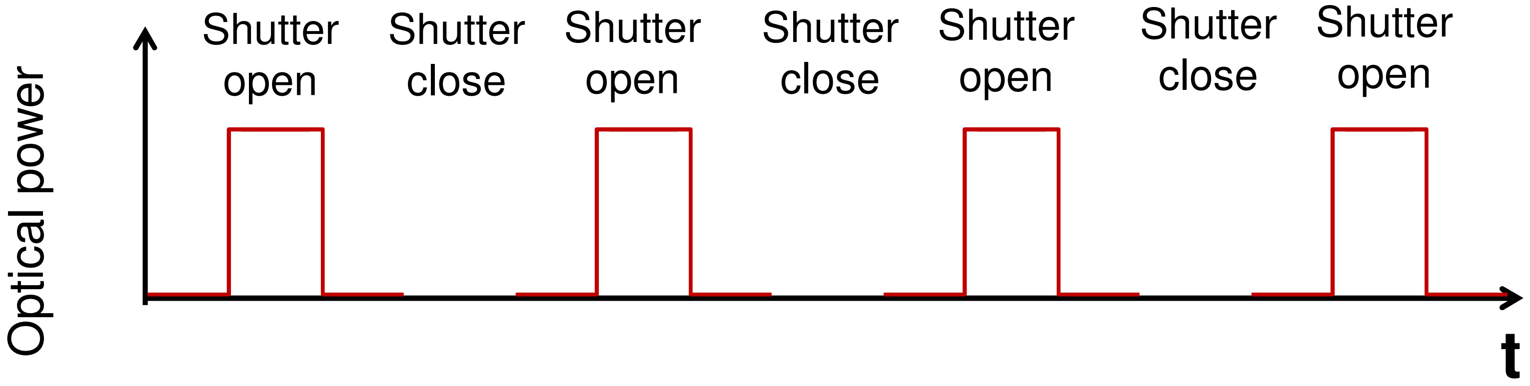
Eadweard Muybridge



Adapted from Rick Trebino's course slides

What do we need to probe a fast event?

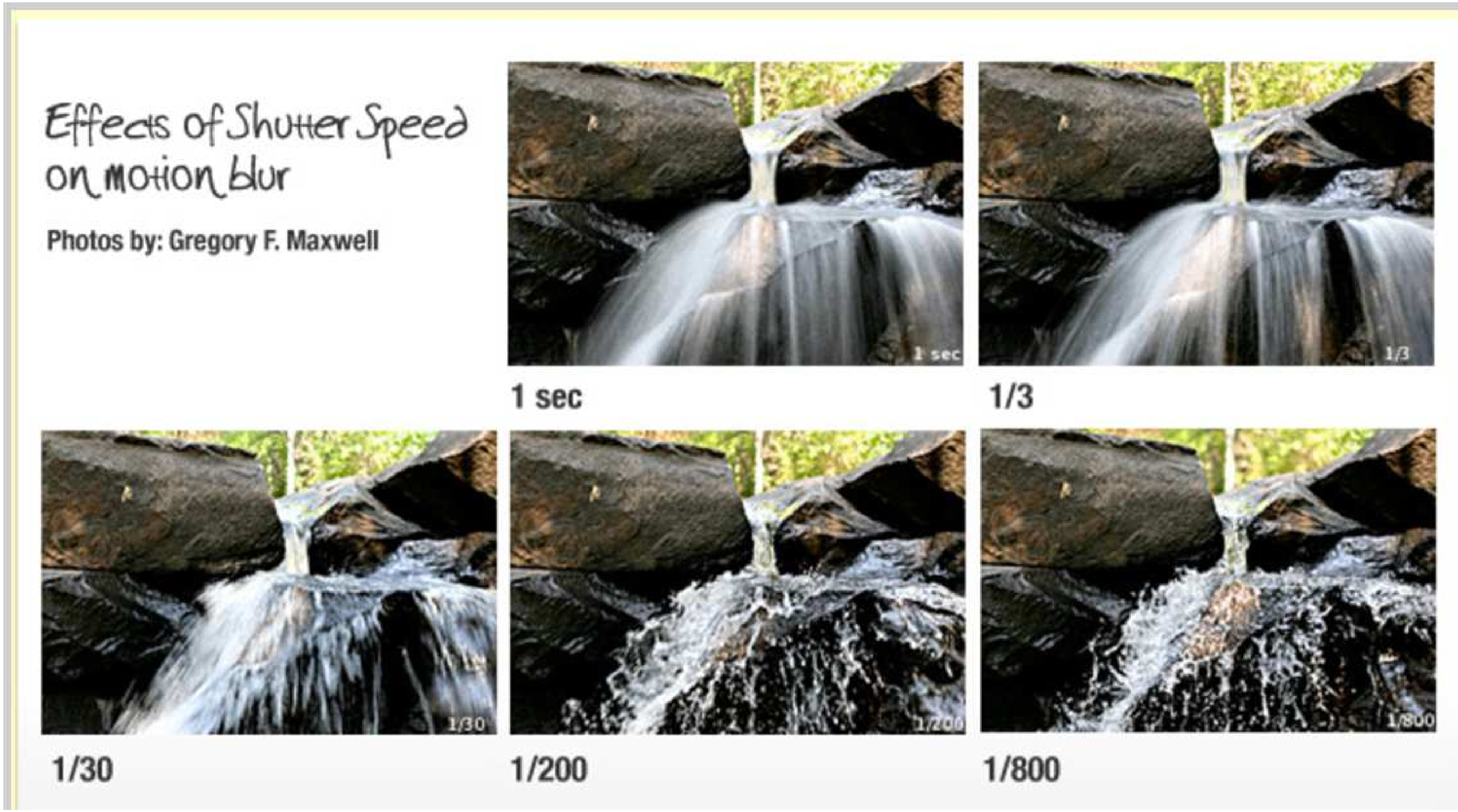
- The light signal received by the camera film is a train of optical pulse.



- We need a FASTER event to freeze the motion. Here the FASTER event is shutter opening and closing.

- If we have an optical pulse source, we can record images of a running horse in a dark room.

Effect of Shutter speed on photography



Effects of Shutter Speed on motion blur

Photos by: Gregory F. Maxwell

Understanding Ultrafast Processes

Is there a time during galloping, when all feet are off the ground? (1872) Leland Stanford

Eadweard Muybridge, * 9. April 1830 in Kingston upon Thames; † 8. Mai 1904, Britisch pioneer of photography

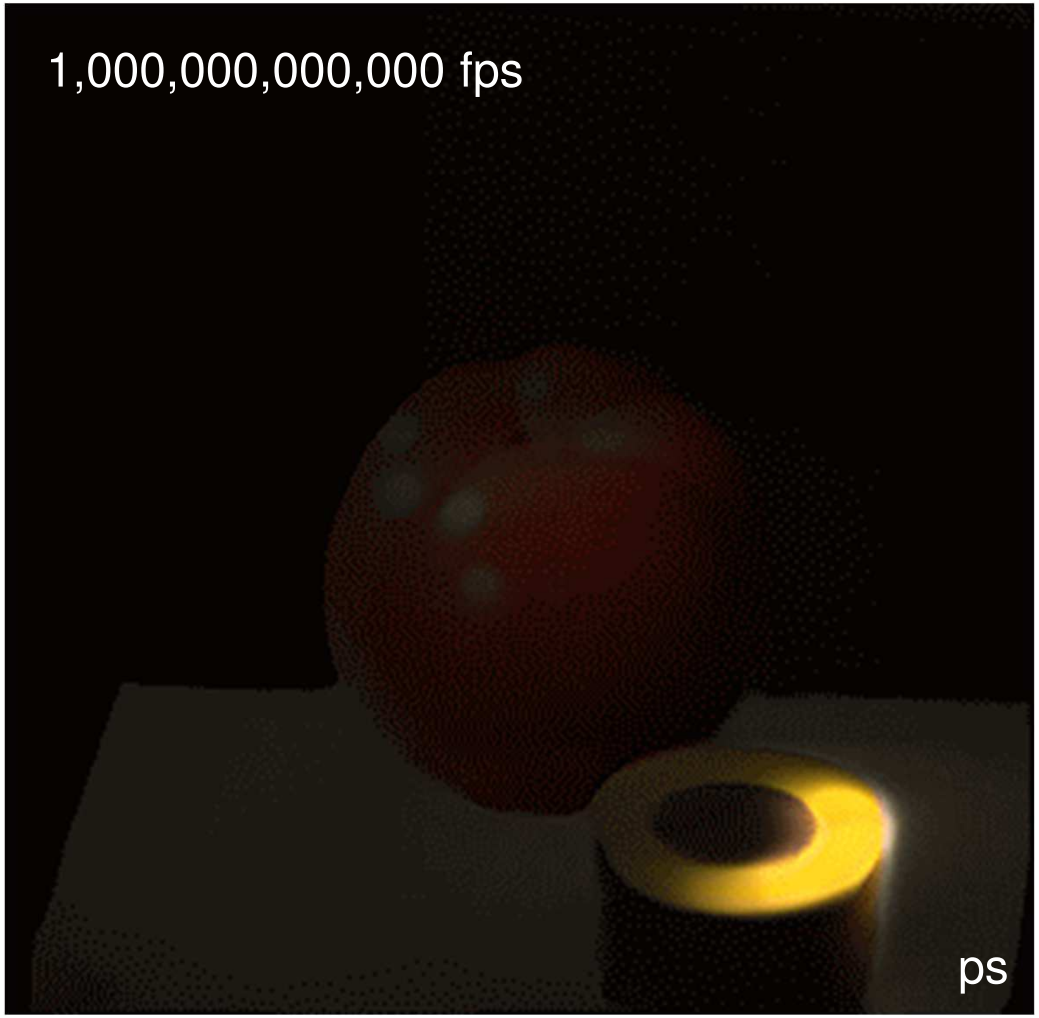
What happens when a bullet rips through an apple?

Harold Edgerton, * 6. April 1903 in Fremont, Nebraska, USA; † 4. Januar 1990 in Cambridge, MA, american electrical engineer, inventor strobe photography.

http://www.eadweardmuybridge.co.uk/

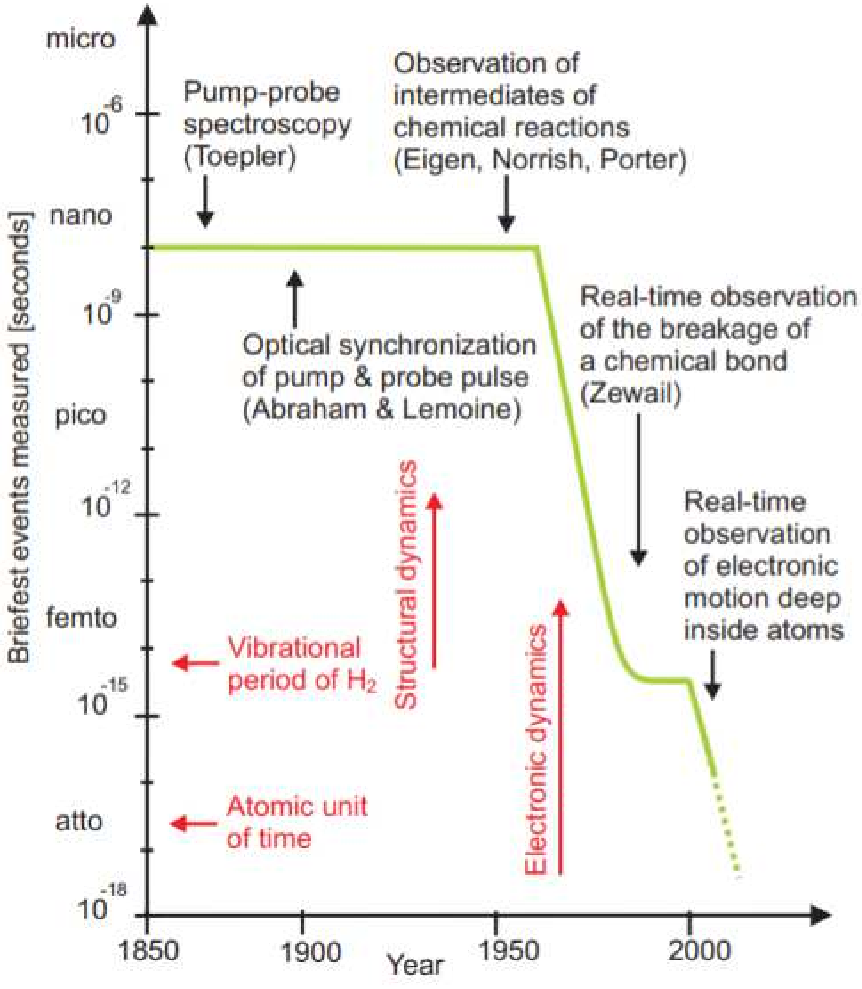
http://web.mit.edu/edgerton/

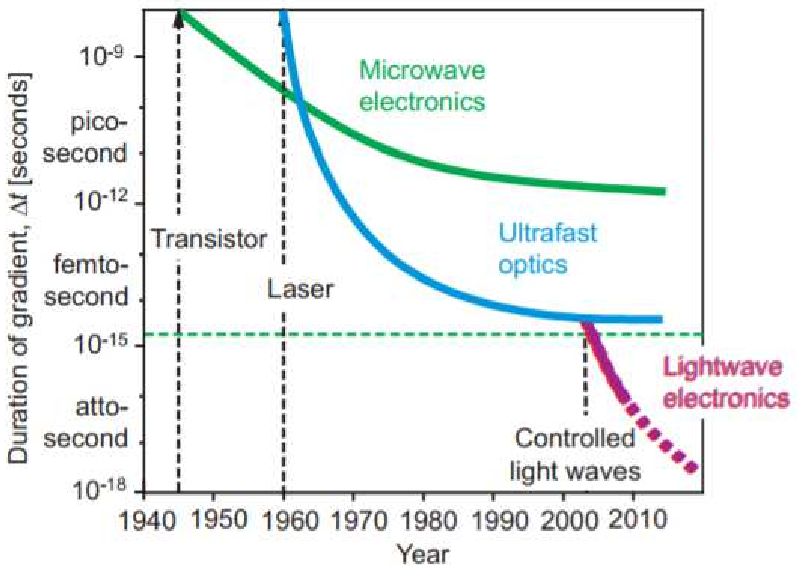
A trillion frames per second movie

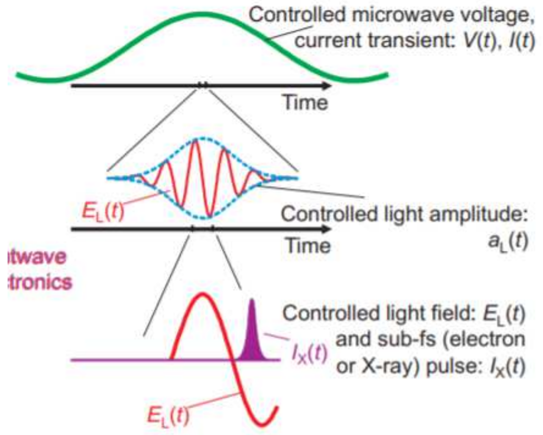


Ref: A. Velten, R. Raskar, and M. Bawendi, "Picosecond Camera for Time-of-Flight Imaging," in *Imaging Systems Applications*, OSA Technical Digest (CD) (Optical Society of America, 2011)

Evolution of ultrafast science







“Attosecond physics“ F. Krausz and M. Ivanov, Rev. Mod. Phys. 81, 163 (2009)

The size of things

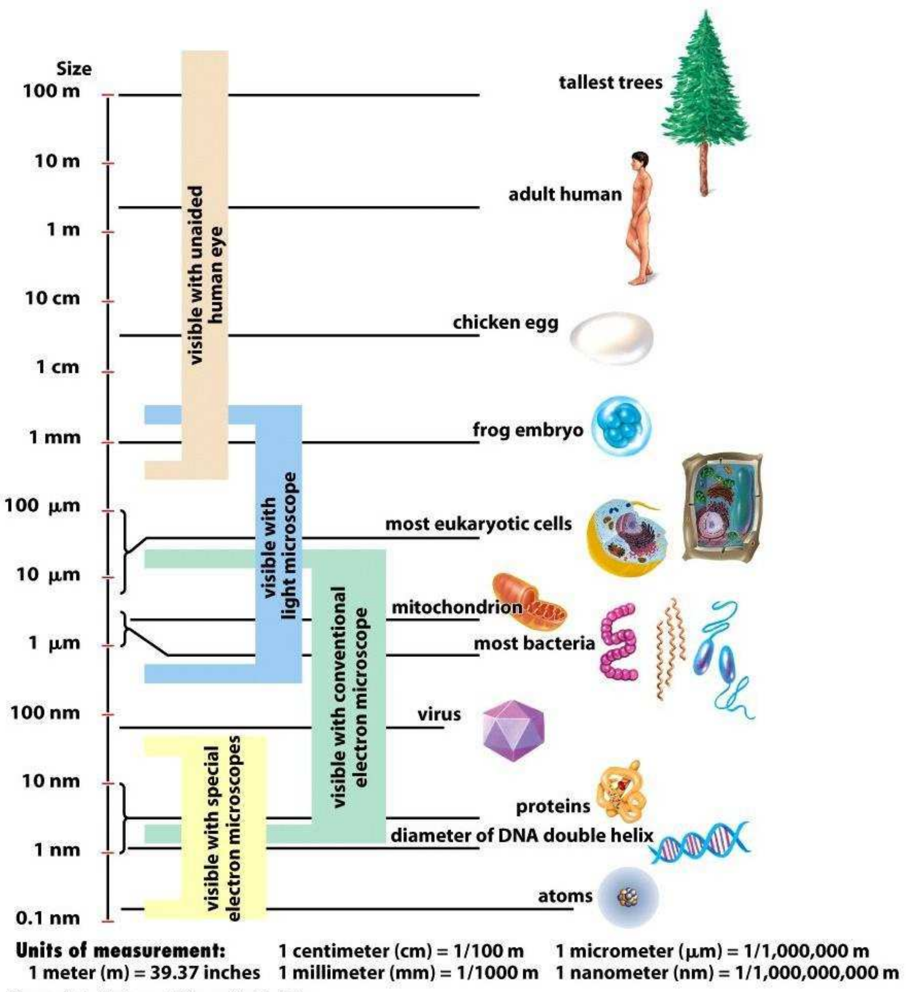
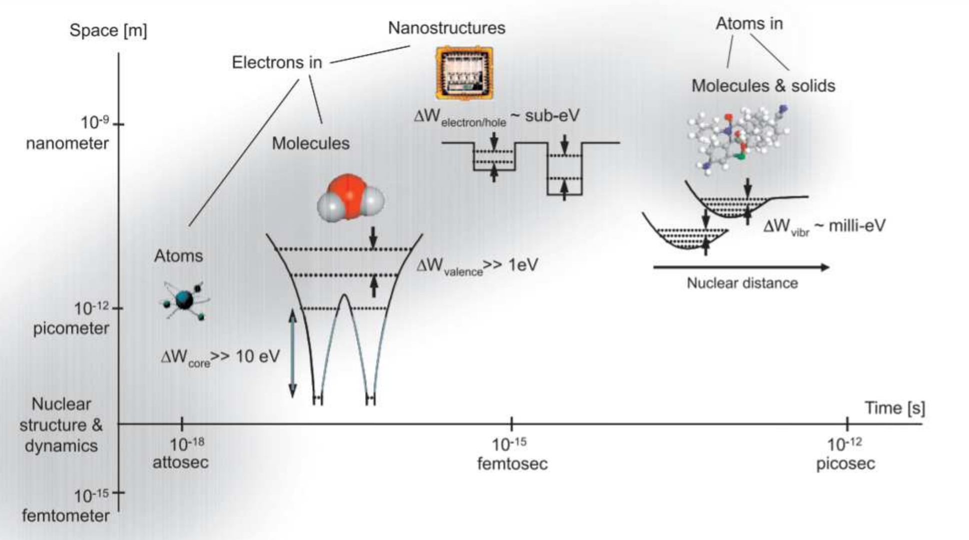


Figure 4-1 Biology: Life on Earth, 8/e
© 2008 Pearson Prentice Hall, Inc.



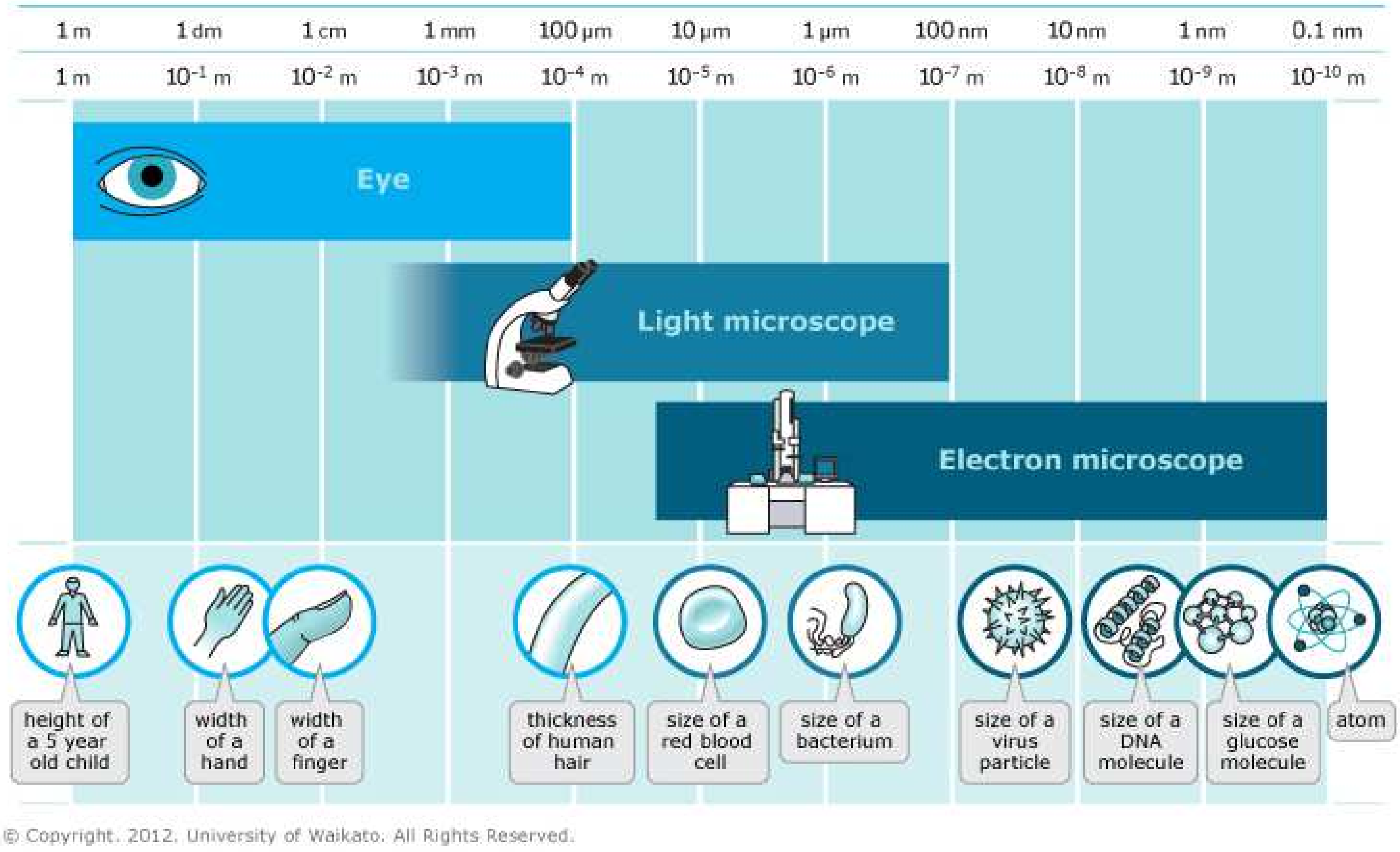
In general as the process gets faster:

a) the interaction energy increases

b) the system become smaller

“Attosecond physics“ F. Krausz and M. Ivanov, Rev. Mod. Phys. 81, 163 (2009)

Resolving power of microscopes

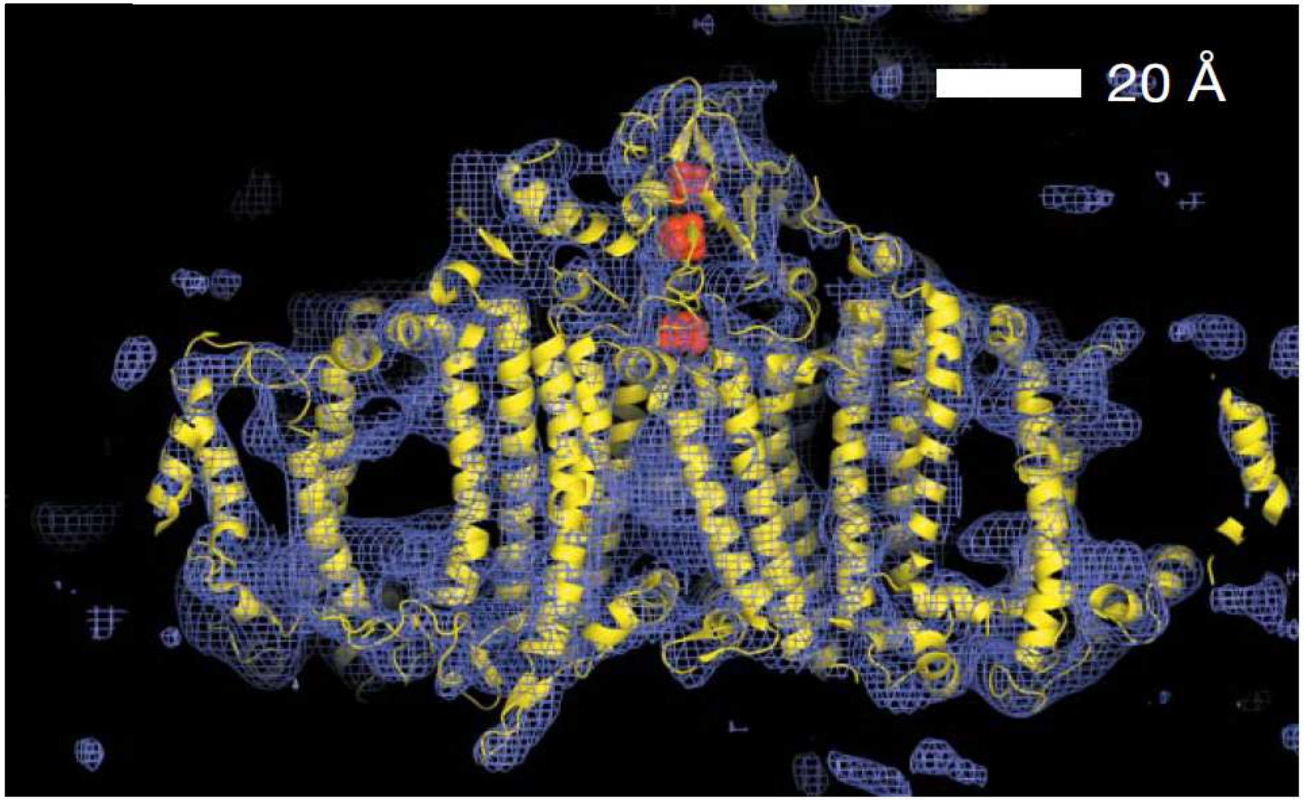


Ref: https://www.sciencelearn.org.nz/resources/495-magnification-and-resolution

Todays Frontiers in Space and Time

Structure, Dynamics and Function of Atoms and Molecules

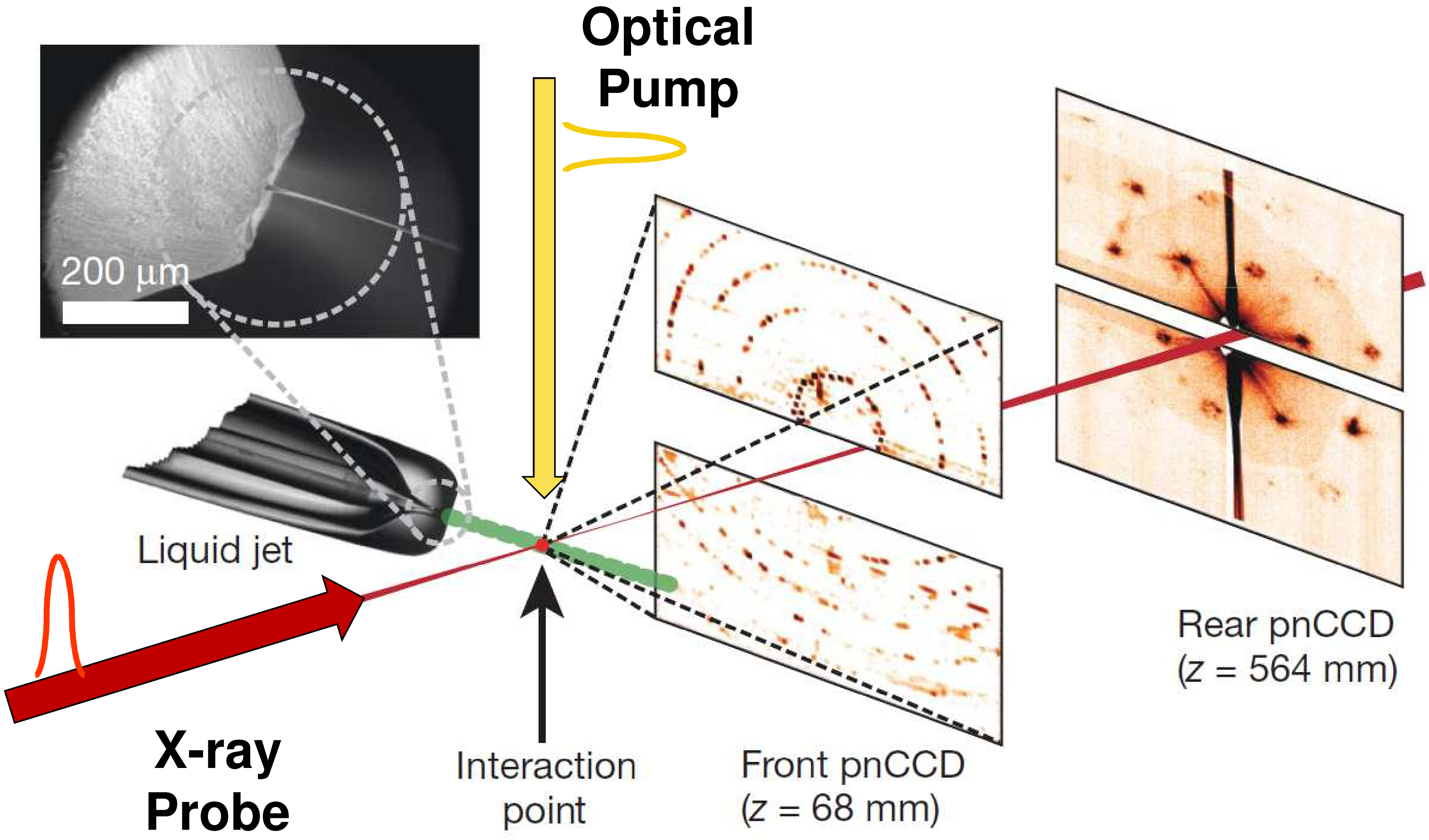
Struture of Photosystem I



Aim: Exploring matter at the sub-nm & sub-femtosecond space and time scales

Chapman, et al. Nature 470, 73, 2011

X-ray Imaging (Time Resolved)

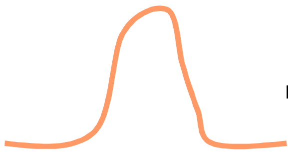


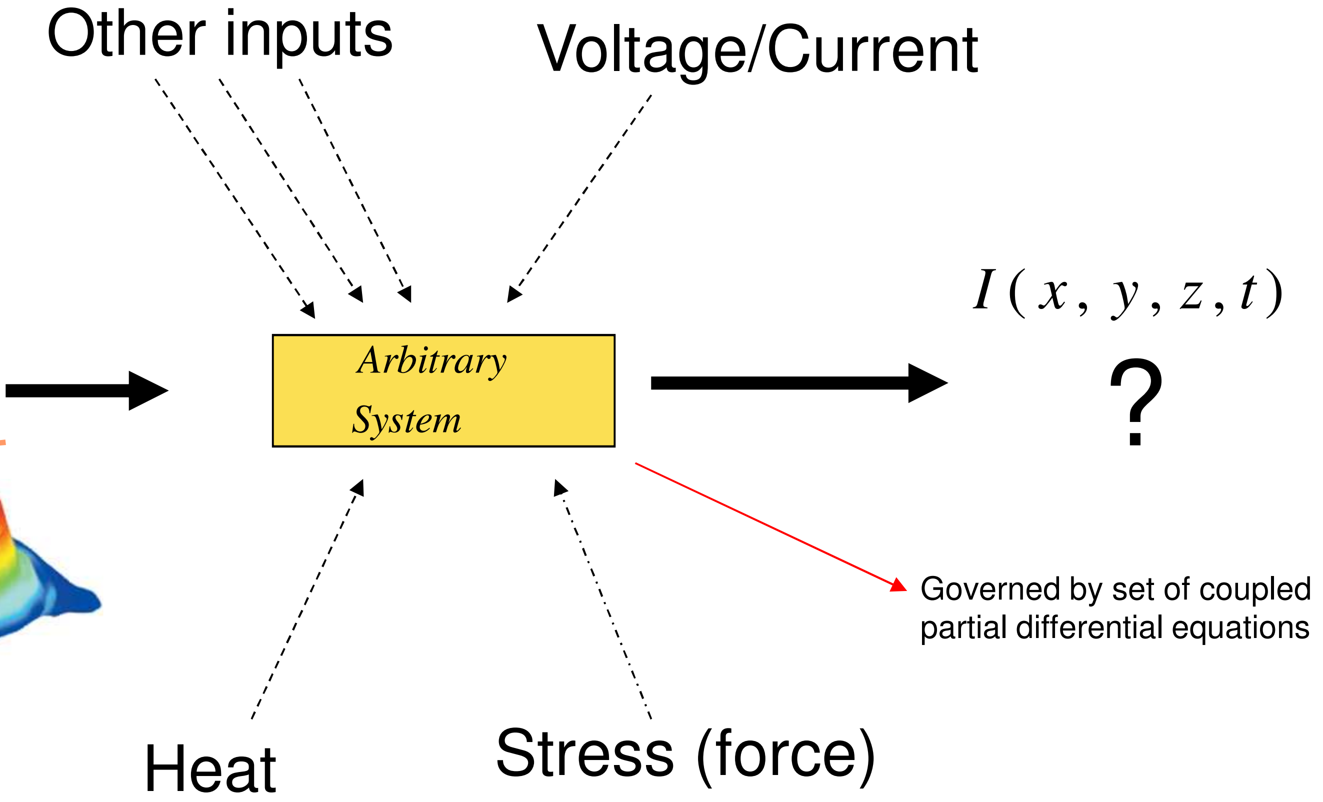
Imaging before destruction: Femtosecond Serial X-ray crystalography

Chapman, et al. Nature 470, 73, 2011

One of the Main Questions in Optics

$I_0(x_0, y_0, z_0, t_0)$





Other inputs

Voltage/Current

Arbitrary System

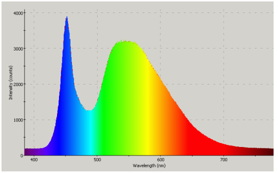
$I(x, y, z, t)$

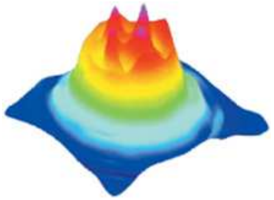
?

Governed by set of coupled partial differential equations

Heat

Stress (force)





The general problem can be very complex:

- <u>Linear optical effects</u>
- Thermo-optical effects
- Electro-optical effects
- Nonlinear optical effects
- Quantum effects

Simplify

Two simpler questions for today's lecture (still requires math):

- What is the effect on beam profile? (Diffraction theory)
- What is the effect on pulse width? (Dispersion theory)

Opto- and thermo-mechanical effects (Example case)

$$\rho C\frac{\partial T(x,y,z,t)}{\partial t}-K_x\frac{\partial^2 T(x,y,z)}{\partial x^2}-K_y\frac{\partial^2 T(x,y,z)}{\partial y^2}-K_z\frac{\partial^2 T(x,y,z)}{\partial z^2}=Q(x,y,z,t),$$

$$-K\hat{\mathbf{n}}\cdot\nabla T+h(T-T_0)|_{\text{boundary}}+\varepsilon\sigma(T^4-T_\infty^4)=0,$$

Temperature

$$\nabla^2\mathbf{U}+\frac{1}{1-2\nu}\nabla(\nabla\cdot\mathbf{U})=\frac{2(1+\nu)}{1-2\nu}\alpha\nabla T.$$

$$\epsilon_{ij}=\frac{1}{E}\left[(1+\nu)\sigma_{ij}-\nu\mathrm{Tr}\left[\overline{\sigma}\right]\delta_{ij}\right]+\alpha T\delta_{ij}$$

$$\sigma_{ij}=\frac{E}{1-\nu}\left[\epsilon_{ij}+\frac{\nu}{1-2\nu}\mathrm{Tr}\left[\overline{\epsilon}\right]\delta_{ij}-\frac{1+\nu}{1-2\nu}\alpha T\delta_{ij}\right].$$

Stress & strain

$$\mathrm{OPD}(x,y)=\int_0^l\frac{\partial n}{\partial T}T(x,y)dz+n_0\Delta u(x,y)+\sum_{i,j=1}^{3}\int_0^l\frac{\partial n}{\partial\varepsilon_{ij}}\varepsilon_{ij}(x,y)dz$$

Thermal lens

dispersion

loss

$$\boxed{\begin{aligned}T_R\frac{\partial A(T,t')}{\partial T} &= -lA(T,t')+j\sum_{n=2}^{\infty}D_n\left(j\frac{\partial}{\partial t'}\right)^n A(T,t')\\ &+ g(T)\left(1+\frac{1}{\Omega_g^2}\frac{\partial^2}{\partial t'^2}\right)A(T,t') \qquad (5.21)\\ &- q(T,t')A(T,t')+j\delta|A(T,t')|^2A(T,t').\end{aligned}}$$

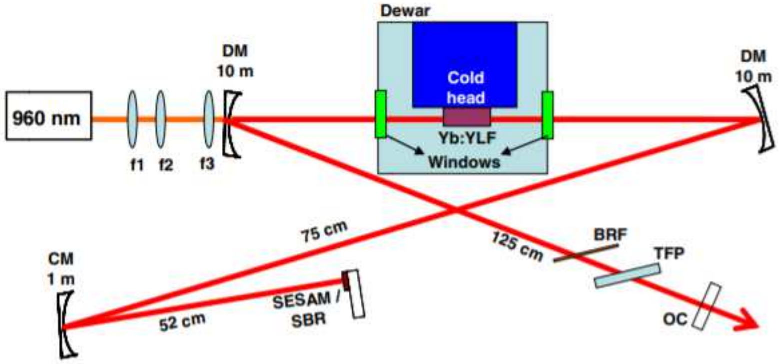
gain

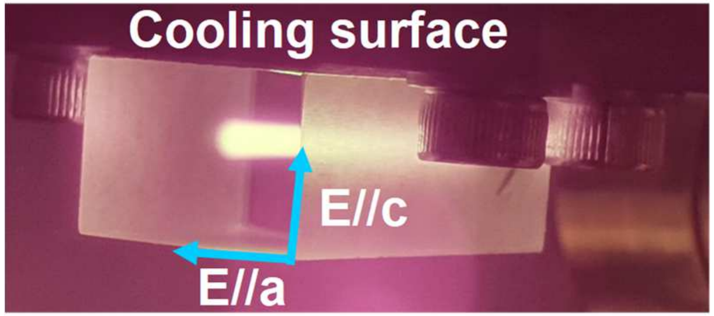
Mode-locking element

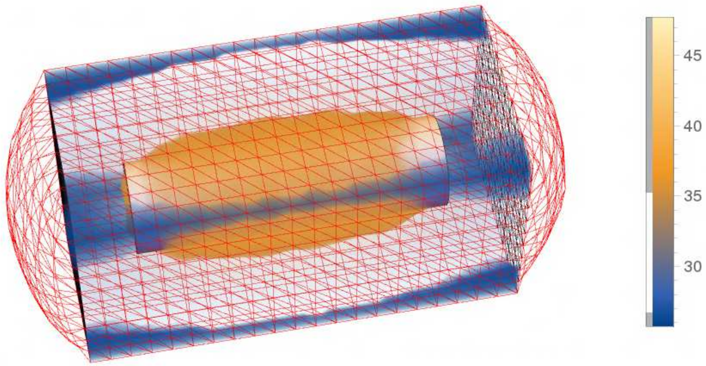
Gain dispersion

Self-phase modulation

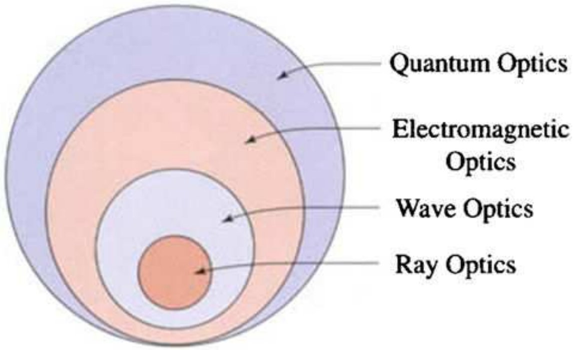
Laser dynamics

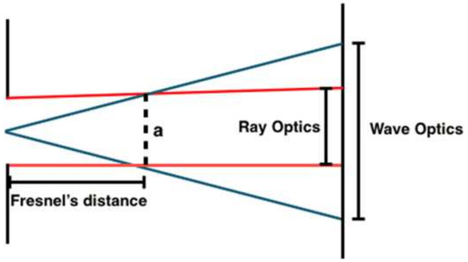


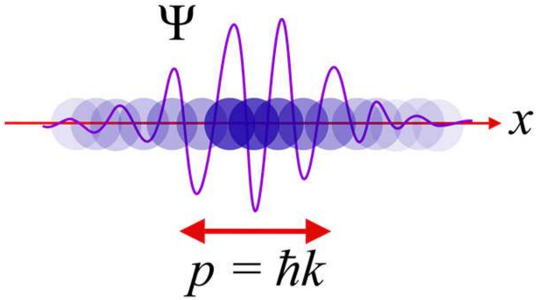




Overview of Optics & Photonics

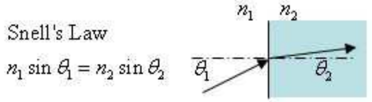


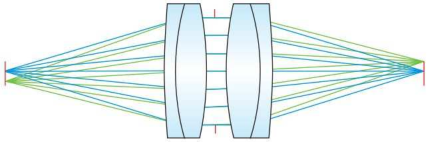




Ray Optics (Geometrical Optics)

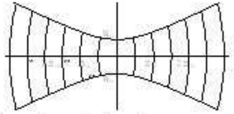
- Focus on **location** & **direction** of light rays
- Limit of Wave Optics where $\lambda \to 0$



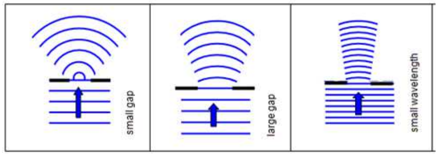


Wave Optics (Gaussian Beam)

- Scalar wave theory
  (Single scalar wavefunction describes light)

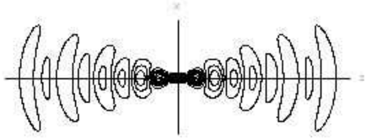


$$U(\mathbf{r}) = A\frac{W}{W(z)}\exp\left[-\frac{\rho^2}{W^2(z)}\right]\exp\left[-jkz - jk\frac{\rho^2}{2R(z)} + j\zeta(z)\right]$$

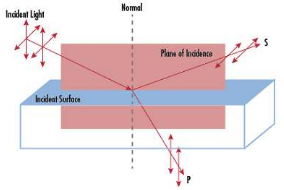


E&M Optics (Geometrical Optics)

- Two mutually coupled vector waves ($\mathbf{E}$ & $\mathbf{M}$)

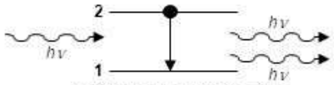


E-field of Gaussian Beam

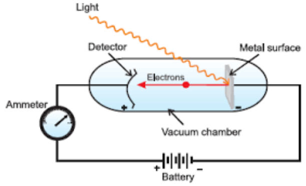


Quantum Optics (Photon Optics)

- Describes certain optical phenomena that are characteristically quantum mechanical

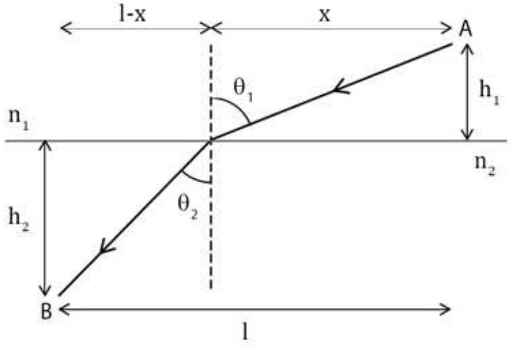


Stimulated Emission



Snell‘s Law

Derivation via Fermat‘s principle



The refractive index of a medium is given by,

$$n = \frac{c}{v}$$

where, *c* is the speed of light in air and *v* is the speed of light in a medium

Consider a light ray traveling from point A in a medium with refractive index n_1 to point B in a medium with refractive index n_2. The time (*t*) to travel between the two points is the distance in each medium divided by the speed of light in that medium.

$$t = \frac{\sqrt{x^2 + h_1^2}}{c/n_1} + \frac{\sqrt{(l-x)^2 + h_2^2}}{c/n_2}$$

To minimize the time, we set the derivative of the time with respect to *x* equal to zero. We also use the definition of sine as the opposite side over hypotenuse to relate the lengths to the angles of incidence and refraction.

$$\frac{dt}{dx} = \frac{n_1 x}{c\sqrt{x^2 + h_1^2}} + \frac{-n_2(l-x)}{c\sqrt{(l-x)^2 + h_2^2}} = 0$$

$$\Rightarrow \frac{n_1 x}{\sqrt{x^2 + h_1^2}} = \frac{n_2(l-x)}{\sqrt{(l-x)^2 + h_2^2}}$$

$$\Rightarrow n_1 \sin\theta_1 = n_2 \sin\theta_2$$

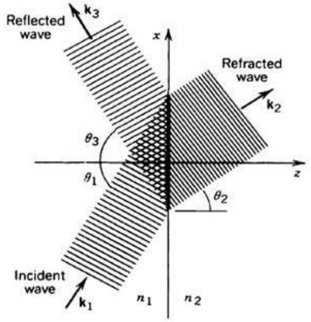
ScienceFacts

Derivation from Maxwell‘s Equations

- Boundary conditions of fields
  - ☐ The conditions have to be satisfied *everywhere* and *at all times* on the boundary

(i) $\varepsilon_1 E_{1\perp} = \varepsilon_2 E_{2\perp}$ (ii) $\vec{E}_{1\parallel} = \vec{E}_{2\parallel}$

(iii) $B_{1\perp} = B_{2\perp}$ (iv) $\frac{1}{\mu_1}\vec{B}_{1\parallel} = \frac{1}{\mu_2}\vec{B}_{2\parallel}$



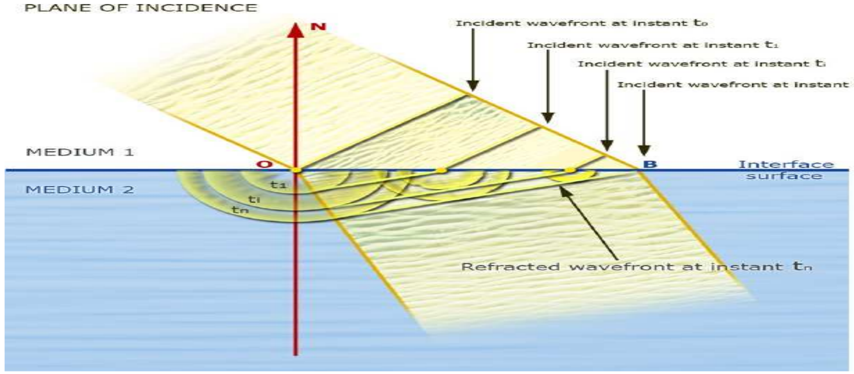
- Derivation of Snell’s law
  - ☐ Phase matching: continuity of E-field across the boundaries

$$E = A \cdot \exp(-i\omega t) \cdot \exp(ik_x x + ik_z z)$$

$$\left.\begin{array}{l} E_1 \cdot \exp(ik_{1,x} x) + E_3 \cdot \exp(ik_{3,x} x) = E_2 \cdot \exp(ik_{2,x} x) \\ |\bar{k}_1| = |\bar{k}_3| = \frac{n_1}{n_2} |\bar{k}_2| \qquad k_x = |\bar{k}| \cdot \sin\theta \end{array}\right\} \frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1} \quad \text{Snell’s law}$$

Derivation from energy and momentum conservation

Derivation from Huygens principle



Ray Propagation

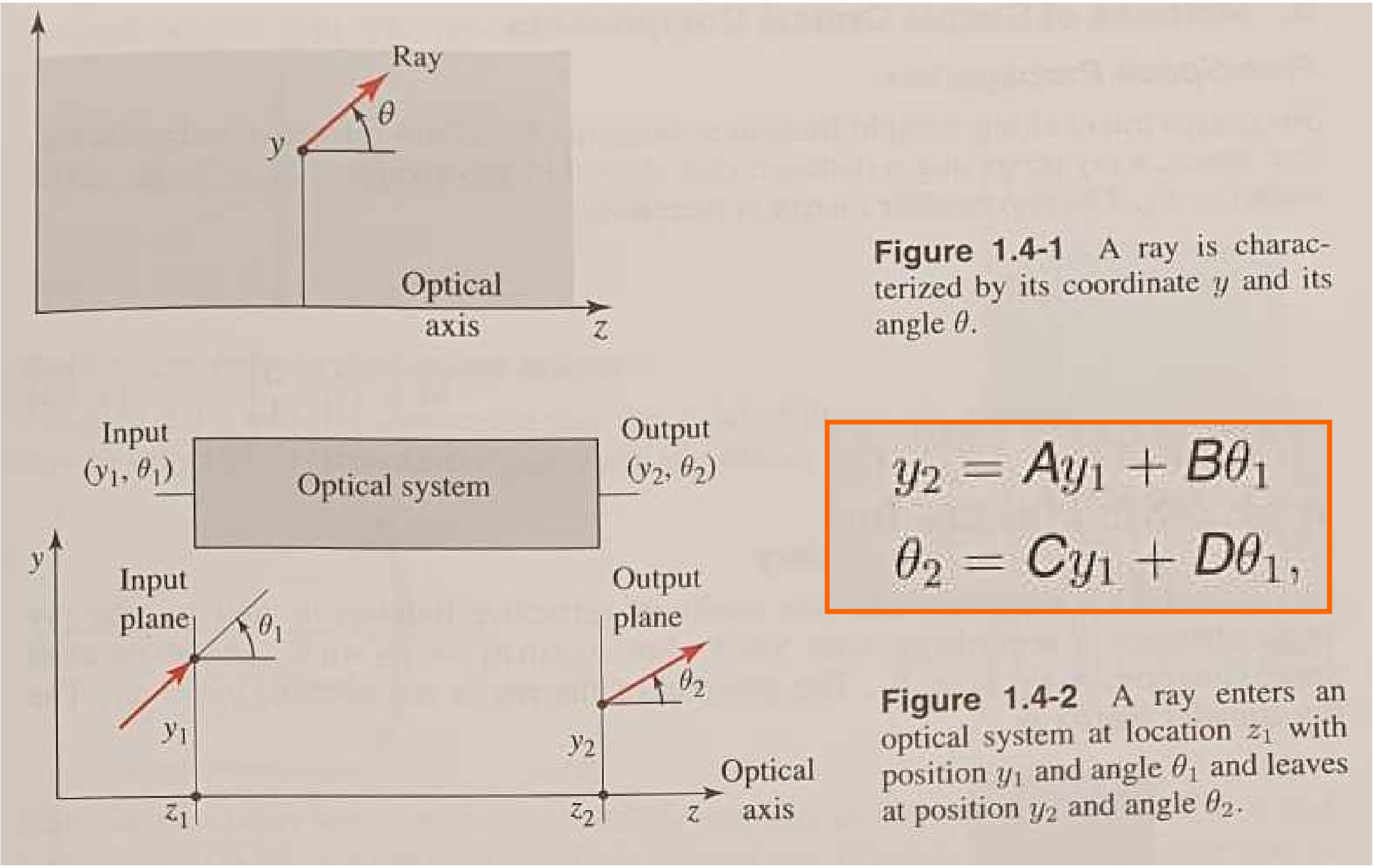
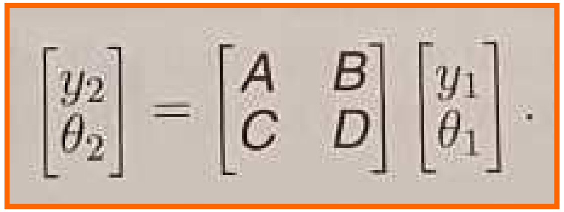


Figure 1.4-1 A ray is characterized by its coordinate y and its angle θ.

$$y_2 = Ay_1 + B\theta_1$$
$$\theta_2 = Cy_1 + D\theta_1,$$

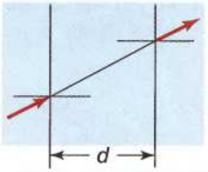
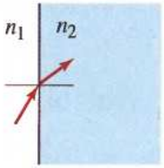
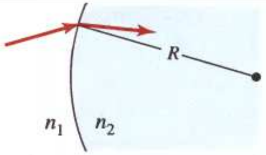
Figure 1.4-2 A ray enters an optical system at location z_1 with position y_1 and angle θ_1 and leaves at position y_2 and angle θ_2.

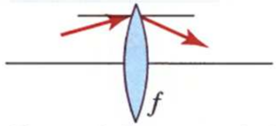
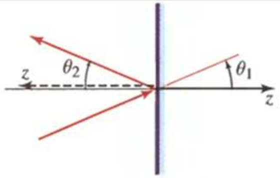
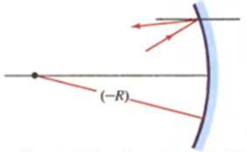


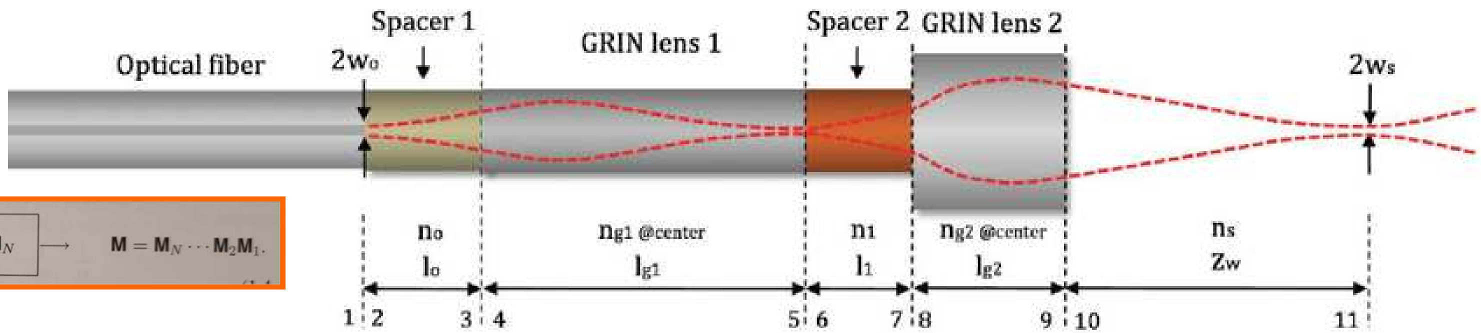
$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}.$$

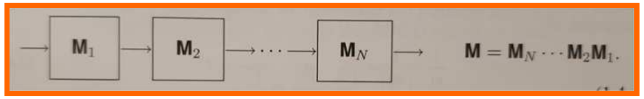
Saleh and Teich, Fundementals of Photonics.

Table of Ray Matrices

Component name	Diagram	Matrix
Free-Space Propagation	 	$\mathbf{M} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$
Refraction at a Planar Boundary	 	$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}$
Refraction at Spherical Boundary	 Convex: $R > 0$; concave: $R < 0$	$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ -\frac{(n_2 - n_1)}{n_2 R} & \frac{n_1}{n_2} \end{bmatrix}$

Component name	Diagram	Matrix
Transmission through thin lens	 Convex: $f > 0$; concave: $f < 0$	$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$
Reflection from planar mirror	 	$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection from spherical mirror	 Concave: $R < 0$; convex: $R > 0$	$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}$





$$\mathbf{M} = \mathbf{M}_N \cdots \mathbf{M}_2 \mathbf{M}_1.$$

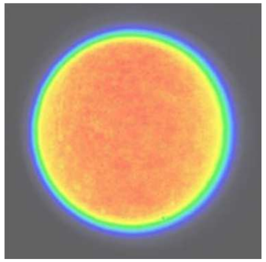
$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}.$$

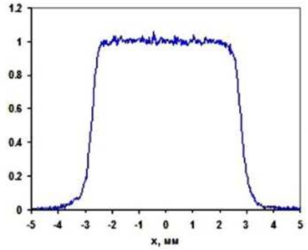
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & z_w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_{g2}}{n_s} \end{pmatrix} \begin{pmatrix} \cos(g_2 l_{g2}) & \frac{1}{g_2}\sin(g_2 l_{g2}) \\ -g_2 \sin(g l_{g2}) & \cos(g_2 l_{g2}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_{g2}} \end{pmatrix} \begin{pmatrix} 1 & l_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_{g1}}{n_1} \end{pmatrix} \begin{pmatrix} \cos(g_1 l_{g1}) & \frac{1}{g_1}\sin(g_1 l_{g1}) \\ -g_1 \sin(g_1 l_{g1}) & \cos(g_1 l_{g1}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_o}{n_{g1}} \end{pmatrix} \begin{pmatrix} 1 & l_o \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_f}{n_o} \end{pmatrix}$$

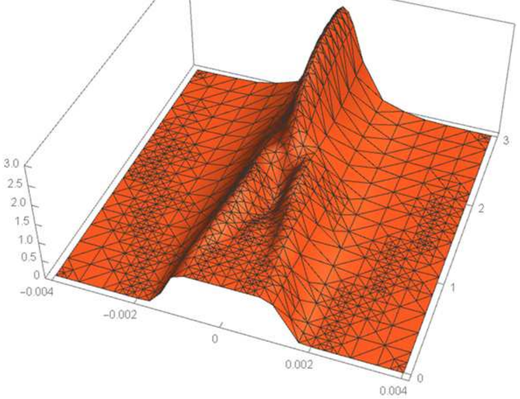
Saleh and Teich, Fundementals of Photonics.
Jung et al., Journal of Biomedical Optics 15(6):066027, (2010).

Outline of the following slides

$I_0(x_0, y_0, z_0)$ → Optical system → ? → $I(x, y, z)$

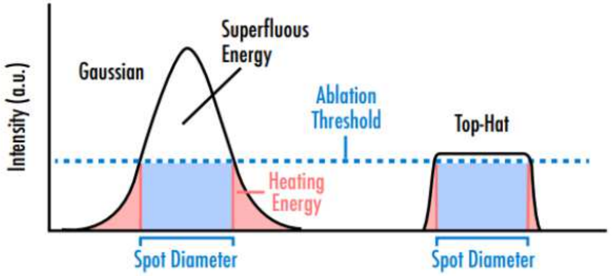






Review fundemental techniques for solving research and engineering problems in optics and photonics field:

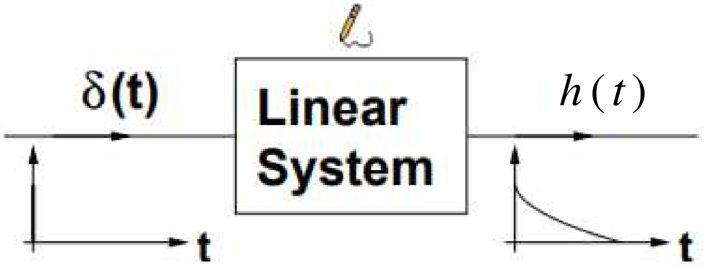
- Impulse response aproach
- Eigenfunction aproach
- Fourier/Laplace transfrom aproach
- …



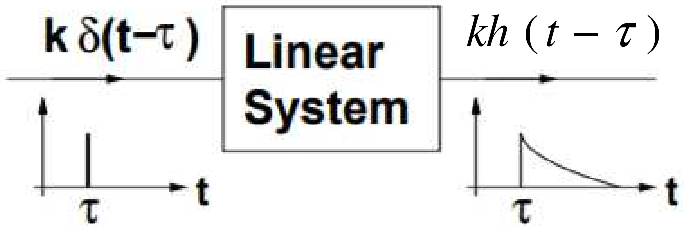
https://www.edmundoptics.eu/knowledge-center/application-notes/optics/why-use-a-flat-top-laser-beam/

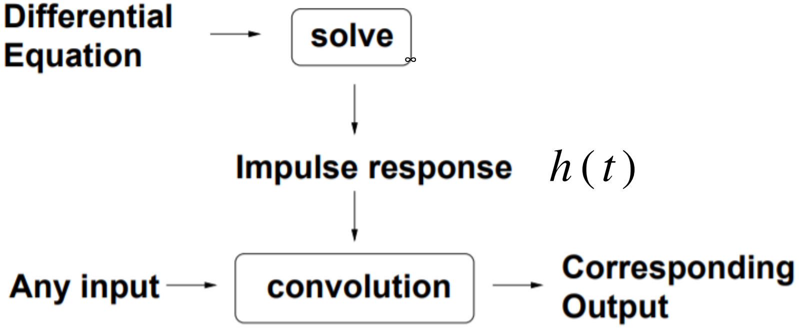
Review: Impulse Response Approach I

An impulse at time $t = 0$ produces the impulse response.



An impulse that has been scaled by k and delayed to time $t = \tau$ produces an impulse response scaled by k and starting at time τ.





Differential Equation → **solve**

↓

Impulse response $h(t)$

↓

Any input → **convolution** → **Corresponding Output**

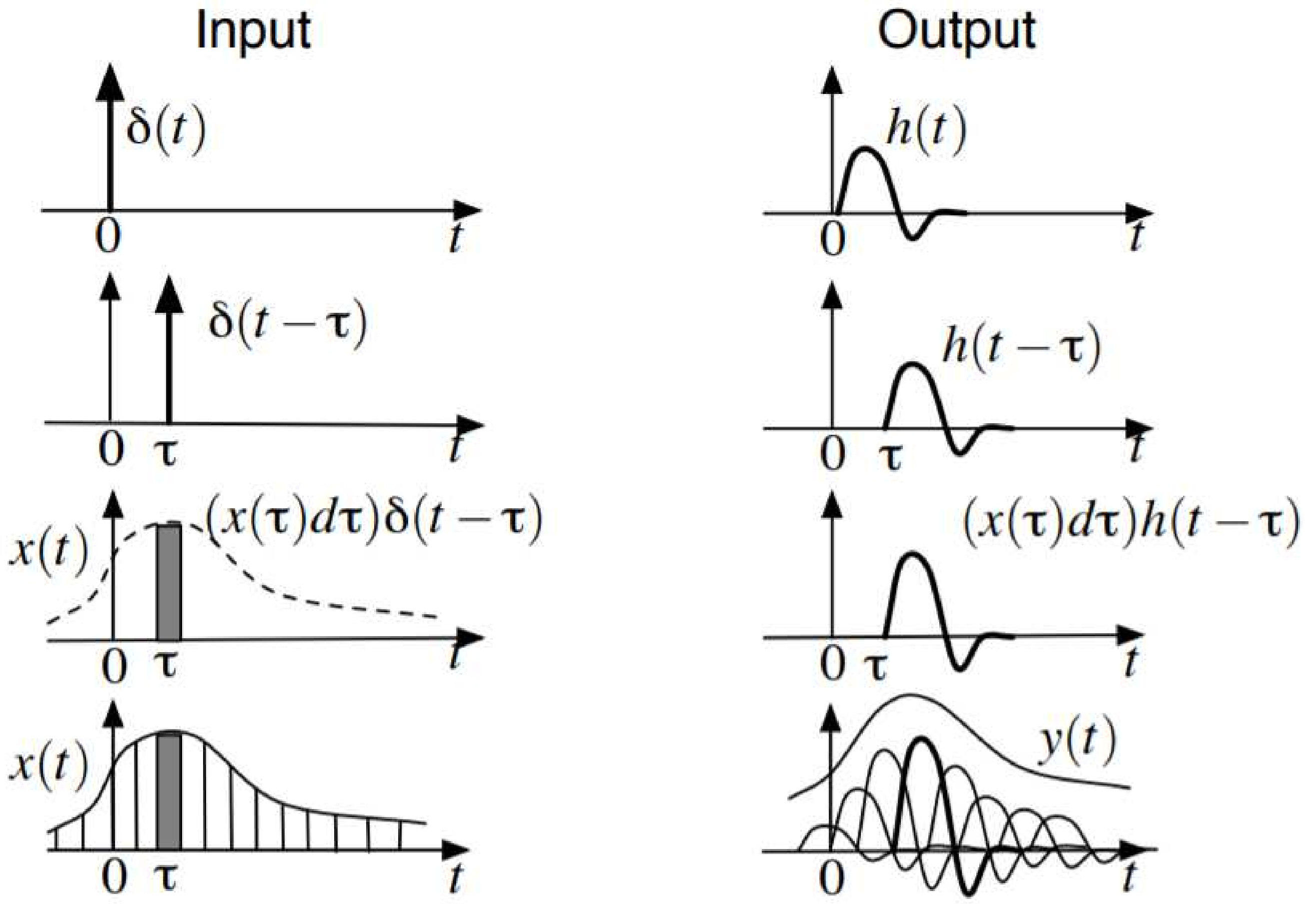
$x(t)$

$y(t)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

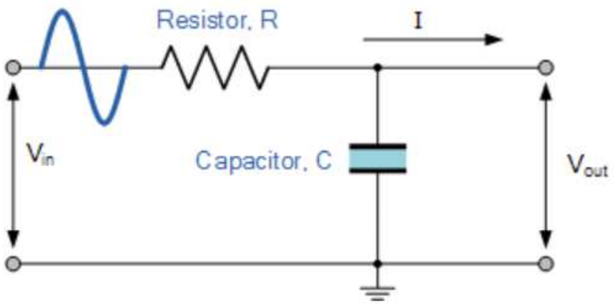
Ref: http://mi.eng.cam.ac.uk/~rwp/Maths/vid09/l2notes.pdf

Review: Impulse Response Approach II



$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \qquad y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Review: Impulse Response Approach III



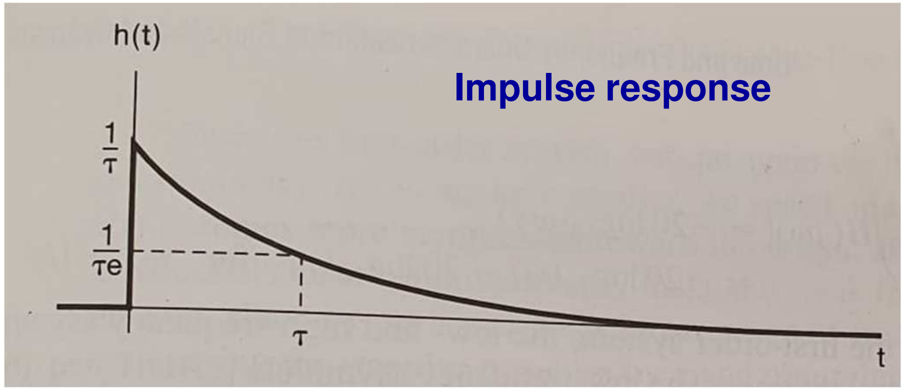
$$\tau \frac{dy(t)}{dt} + y(t) = x(t)$$

$$h(t) = \frac{1}{\tau} e^{-t/\tau}\, u(t)$$

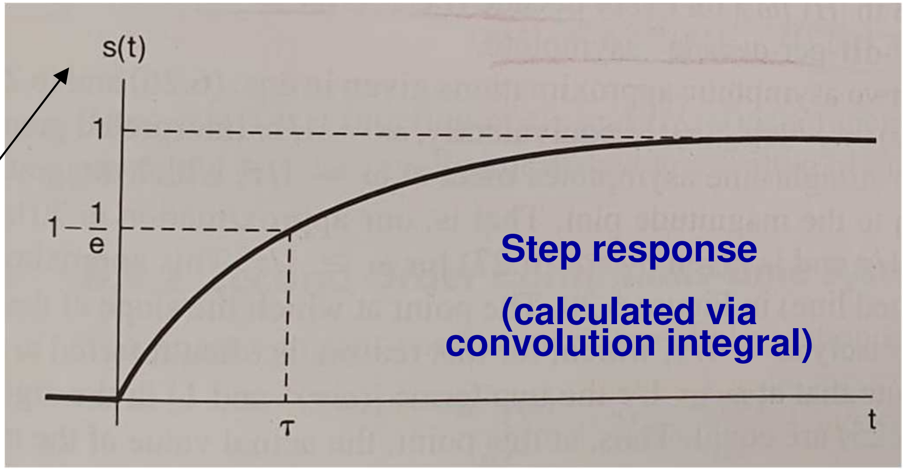
$$x(t) = u(t)$$

$$y(t) = u(t) * h(t)$$

$$y(t) = \left[1 - e^{-t/\tau}\right] u(t)$$



Impulse response



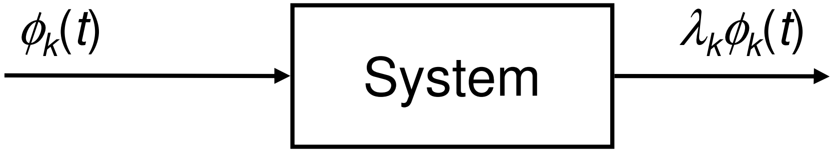
Step response

(calculated via convolution integral)

Ref: Oppenheim et al., Signals and Systems.

Review: Eigen Functions Approach I

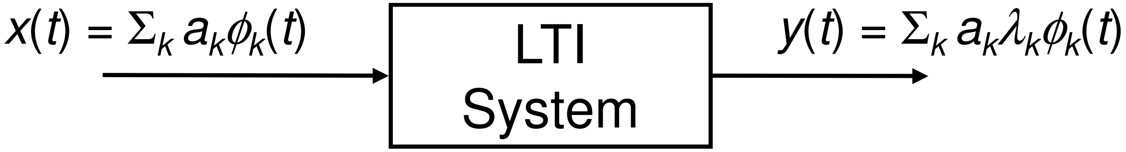
- Lets imagine what (basis) signals $\phi_k(t)$ have the property that:



$\phi_k(t)$ → System → $\lambda_k\phi_k(t)$

- i.e. the output signal is the same as the input signal, multiplied by the constant "gain" λ_k (which may be complex)

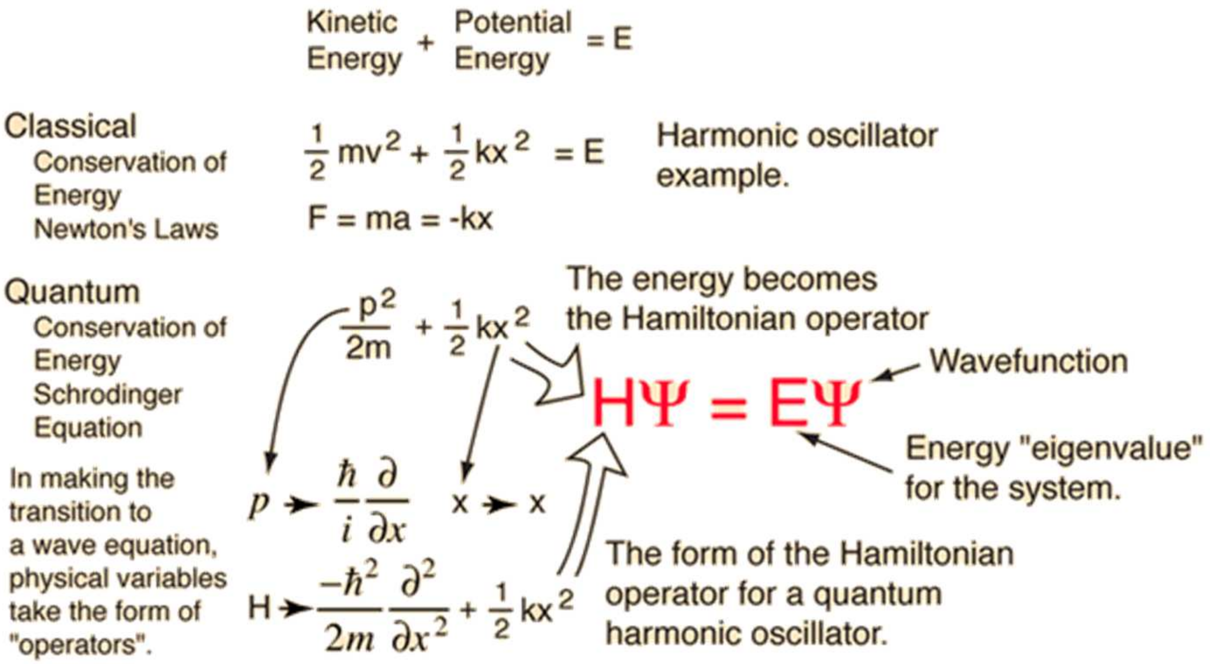
- For CT LTI systems, we also have that



$x(t) = \Sigma_k\, a_k\phi_k(t)$ → LTI System → $y(t) = \Sigma_k\, a_k\lambda_k\phi_k(t)$

- Therefore, to make use of this theory we need:
  1) **system identification** is determined by finding $\{\phi_k,\lambda_k\}$.
- 2) **response**, we also have to decompose $x(t)$ in terms of $\phi_k(t)$ by calculating the coefficients $\{a_k\}$.
- This is analogous to eigenvectors/eigenvalues matrix decomposition

Review: Eigen Functions Approach II



Kinetic Energy + Potential Energy = E

Classical
Conservation of Energy
Newton's Laws

$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E$

$F = ma = -kx$

Harmonic oscillator example.

Quantum
Conservation of Energy
Schrodinger Equation

$\frac{p^2}{2m} + \frac{1}{2}kx^2$

The energy becomes the Hamiltonian operator

$H\Psi = E\Psi$

Wavefunction

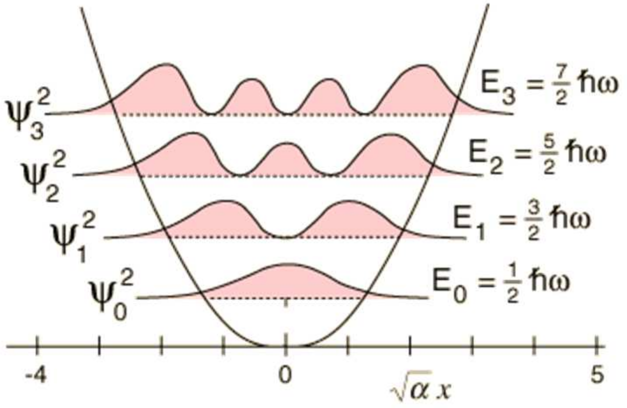
Energy "eigenvalue" for the system.

In making the transition to a wave equation, physical variables take the form of "operators".

$p \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial x}$ $\quad x \rightarrow x$

$H \rightarrow \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}kx^2$

The form of the Hamiltonian operator for a quantum harmonic oscillator.



$$\Psi_0 = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-y^2/2}$$

$$\Psi_1 = \left(\frac{\alpha}{\pi}\right)^{1/4} \sqrt{2}y\, e^{-y^2/2}$$

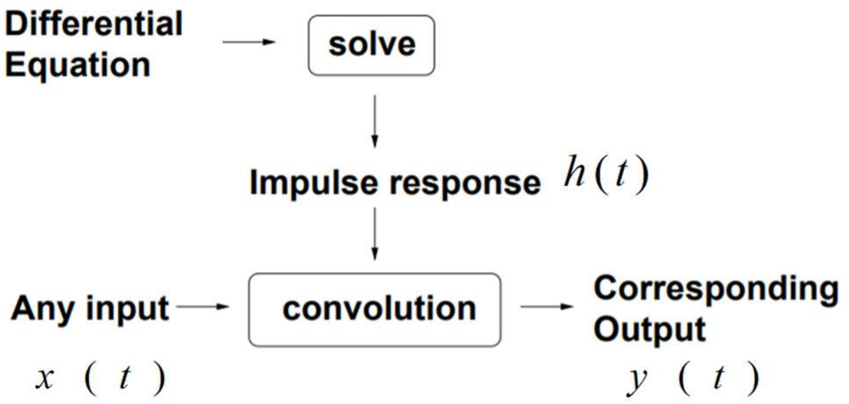
$$\Psi_2 = \left(\frac{\alpha}{\pi}\right)^{1/4} \frac{1}{\sqrt{2}}(2y^2 - 1)\, e^{-y^2/2}$$

$$\Psi_3 = \left(\frac{\alpha}{\pi}\right)^{1/4} \frac{1}{\sqrt{3}}(2y^3 - 3y) e^{-y^2/2}$$

$$\alpha = \frac{m\,\omega}{\hbar} \qquad y = \sqrt{\alpha}\,x$$

Ref: http://pchemandyou.blogspot.de/2008/04/particle-in-finite-box-and-harmonic.html

Review: Laplace/Fourier Transform Approach I



$$y(t) = x(t) * h(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

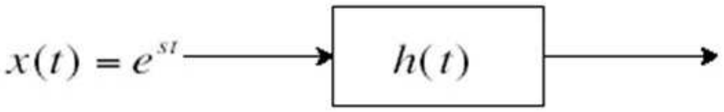
Laplace transform

$$\boxed{X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\, dt}$$

Inverse Laplace transform

$$\boxed{x(t) = \frac{1}{2\pi j}\int X(s)e^{st} ds}$$

Complex exponentials are the only eigenfunctions of any LTI systems

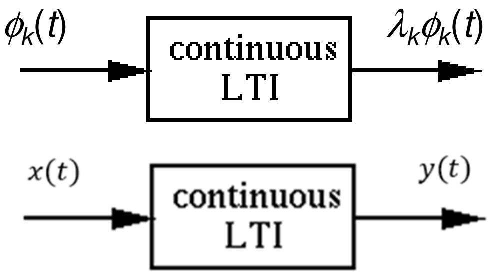


$$y(t) = \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau$$
$$= \left[\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau\right]e^{st}$$
$$= \underbrace{H(s)}_{\text{eigenvalue}}\underbrace{e^{st}}_{\text{eigenfunction}}$$

The eigenvalue of a complex exponentioal input (H(s)) is the Laplace transform of systems impulse response (h(t))

Review: Laplace/Fourier Transform Approach II

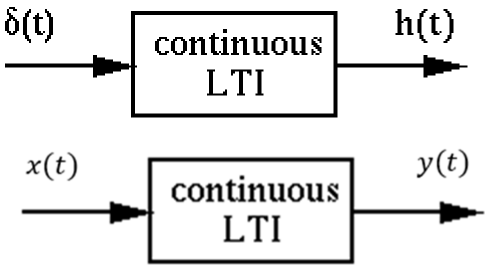
Eigen-function approach



$$x(t) = \Sigma_k\, a_k\phi_k(t)$$

$$y(t) = \Sigma_k\, a_k\lambda_k\phi_k(t)$$

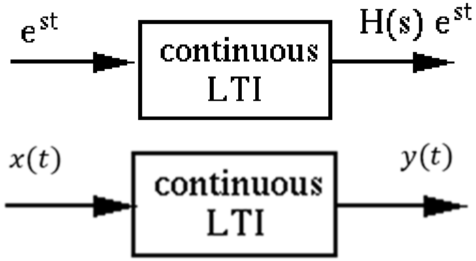
Impulse response approach



$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

$$y(t) = x(t)*h(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

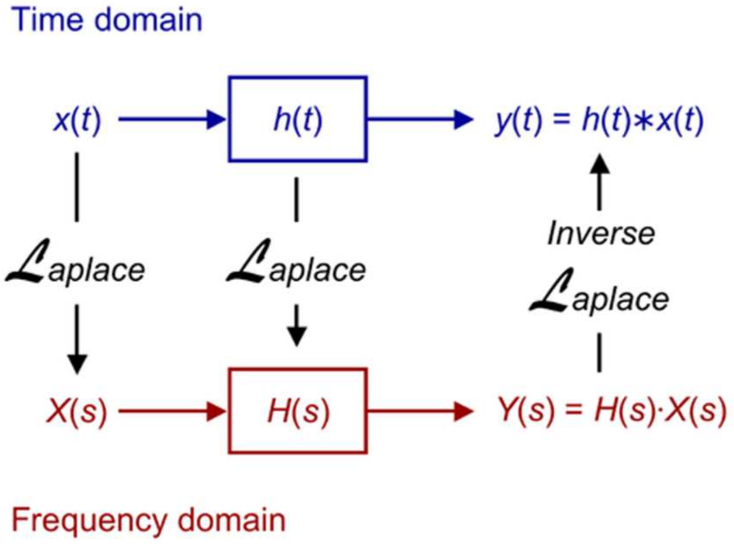
Complex exponential eigenfunction aproach



$$x(t) = \frac{1}{2\pi j}\int X(s)e^{st}ds$$

$$y(t) = \frac{1}{2\pi j}\int X(s)H(s)e^{st}ds = \frac{1}{2\pi j}\int Y(s)e^{st}ds$$

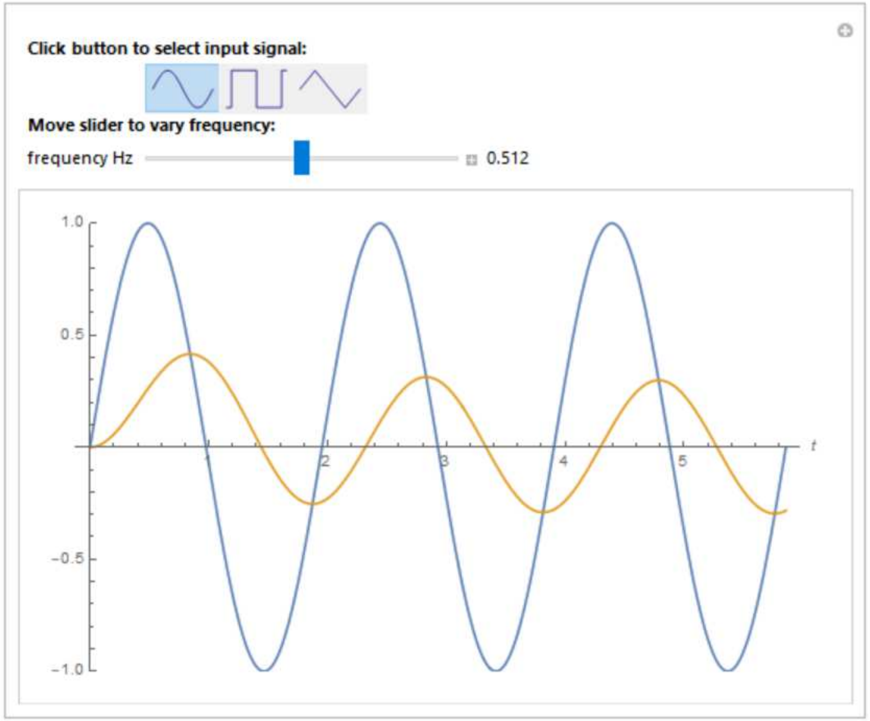
Laplace/Fourier transform apraoch



$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt$$

$$x(t) = \frac{1}{2\pi j}\int X(s)e^{st}ds$$

Review: Laplace/Fourier Transform Approach III



Click button to select input signal:

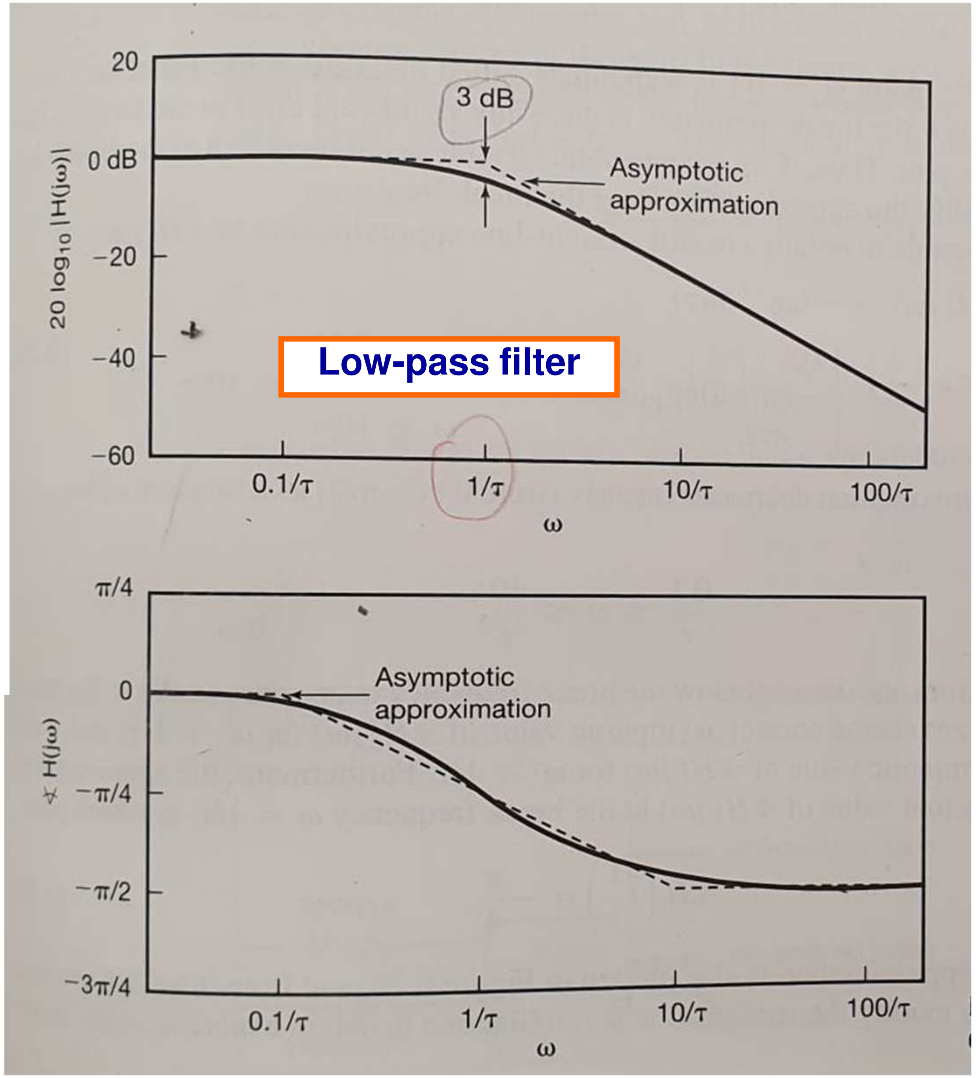
Move slider to vary frequency:

frequency Hz 0.512

$$H(s) = H(j\omega) = \frac{1}{j\omega\tau + 1}$$

$$20log_{10}|H(j\omega)| = -10log_{10}[(\omega\tau)^2 + 1]$$

Bode plot (frequency response)

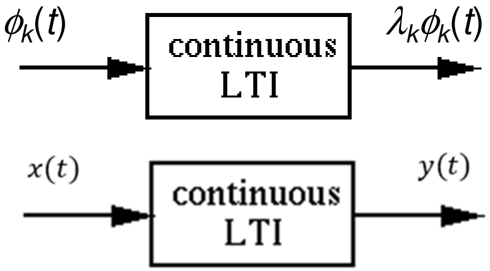


Low-pass filter

Ref: Oppenheim et al., Signals and Systems.

Review: Laplace/Fourier Transform Approach II

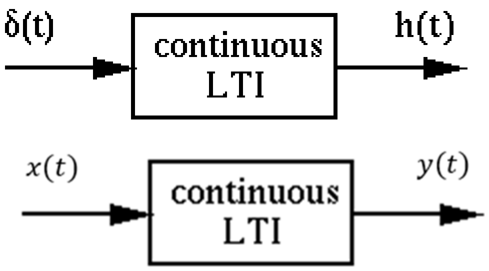
Eigen-function approach



$$x(t) = \Sigma_k\, a_k\phi_k(t)$$

$$y(t) = \Sigma_k\, a_k\lambda_k\phi_k(t)$$

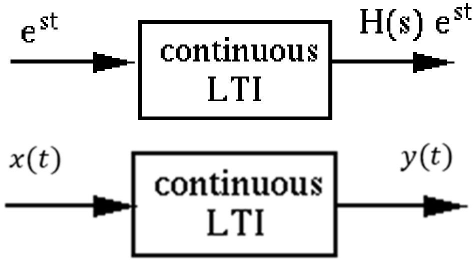
Impulse response approach



$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

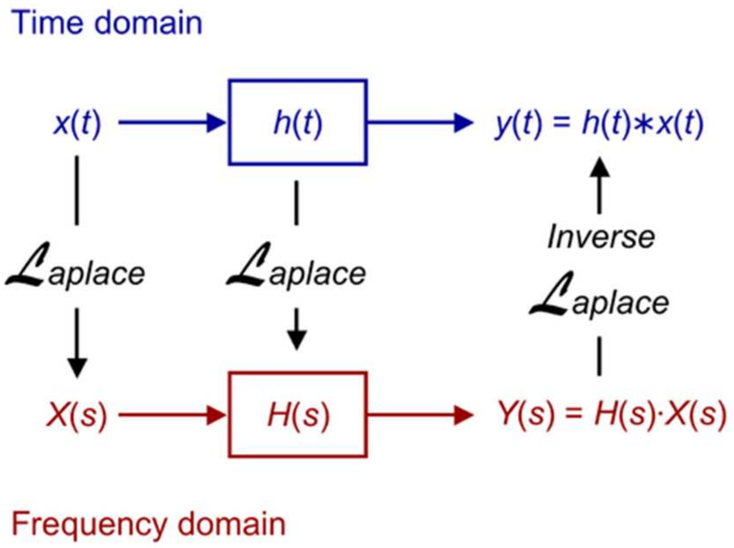
Complex exponential eigenfunction aproach



$$x(t) = \frac{1}{2\pi j}\int X(s)e^{st}ds$$

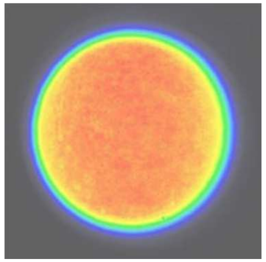
$$\boxed{y(t) = \frac{1}{2\pi j}\int X(s)H(s)e^{st}ds = \frac{1}{2\pi j}\int Y(s)e^{st}ds}$$

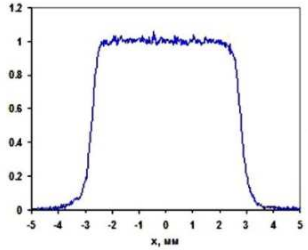
Laplace/Fourier transform apraoch

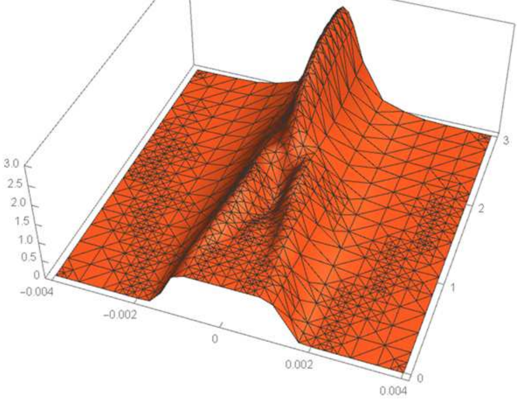


Outline of the following slides

$I_0(x_0, y_0, z_0)$ → Optical system → ? $I(x, y, z)$

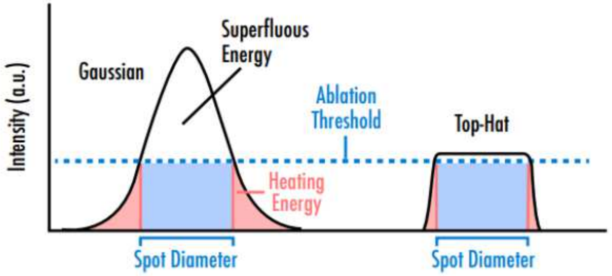






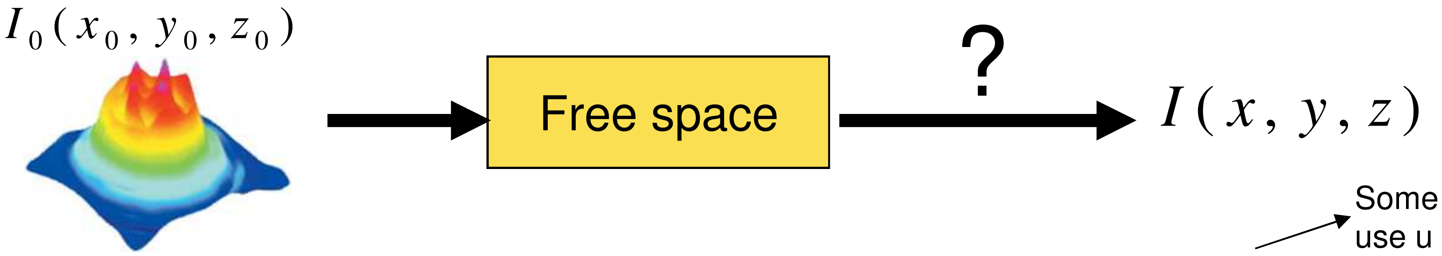
Review fundemental techniques for solving research and engineering problems in optics and photonics field:

- Impulse response aproach
- Eigenfunction aproach
- Fourier/Laplace transfrom aproach
- …



https://www.edmundoptics.eu/knowledge-center/application-notes/optics/why-use-a-flat-top-laser-beam/

Diffraction Theory: Impulse response approach



$$I_0(x_0,y_0,z_0) \Longrightarrow E(x_0,y_0,z_0) \Longrightarrow u_0(x_0,y_0,z_0) \qquad I = \frac{1}{2Z_F}|\underline{E}|^2 = \frac{1}{2}Z_F|\underline{H}|^2.$$

$$E = -jw(A + \frac{1}{w^2\mu_o\varepsilon_o}\nabla(\nabla.A)) \qquad A = \hat{n}\psi(x,y,z)e^{jwt} = \hat{n}u(x,y,z)e^{jwt}e^{-jkz}$$

Some math: use u (x,y,z) rather than I (x,y,z) to simplfy algebra….

1-2 hours of work starting from Maxwell's Equations….

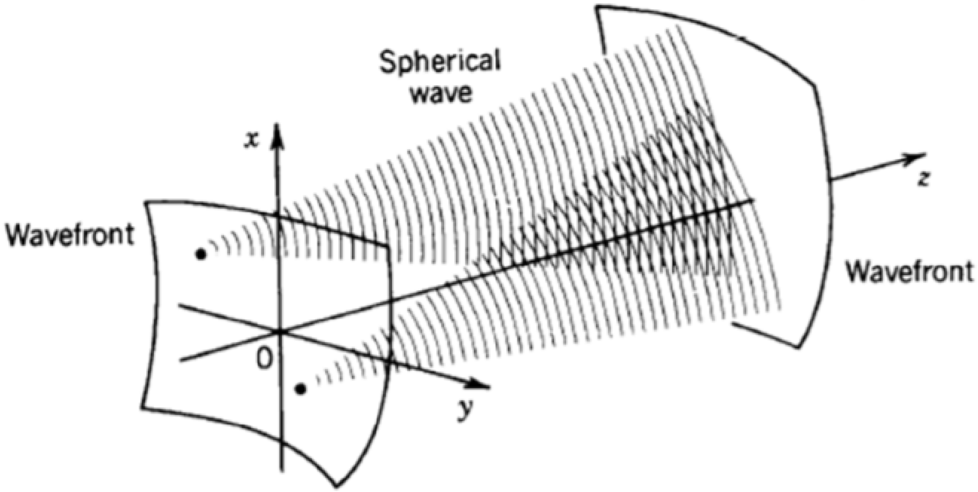
$$u_o(x_0,y_0,z_0) \longrightarrow \boxed{h(x,y,z)} \longrightarrow u_o(x_0,y_0,z_0) * h(x,y,z)$$

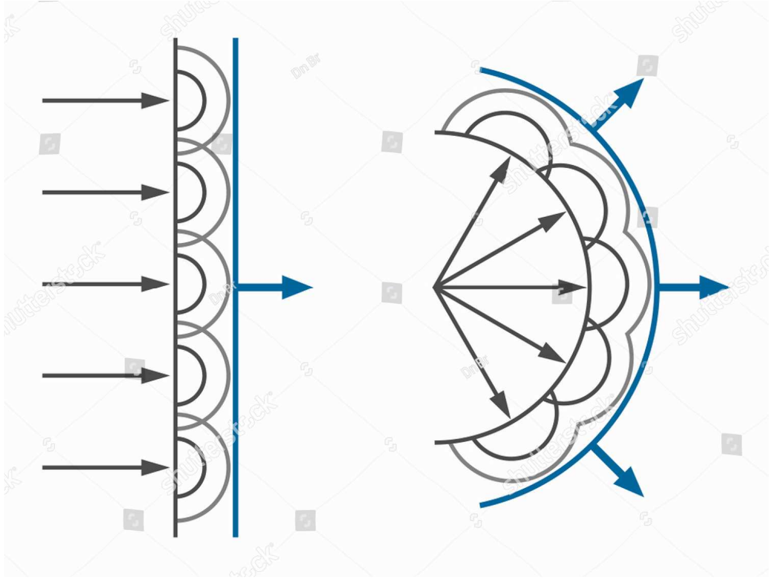
$$\boxed{h(x,y,z) = \frac{j}{\lambda z} Exp\ (-ik\frac{(x^2+y^2)}{2z})}$$

Impulse response of free space: EM disturbance that is due to a point source according to scalar diffraction theory

$$\boxed{u(x,y,z) = \iint u_o(x_0,y_0,z_0) \quad h(x-x_0, y-y_0, z)dx_0dy_0}$$

Impulse response approach of diffraction ≈ Huygens-Fresnel principle





The Huygens-Fresnel principle states that each point on a wavefront generates a spherical wave. The envelope of these secondary waves constitutes a new wavefront. Their superposition constitutes the wave in another plane.

The system's impulse-response function for propagation between the planes z = 0 and z = d is

$$h(x, y, z) \approx \frac{1}{r} Exp(-jkr) \qquad r = \sqrt{x^2 + y^2 + z^2}$$

Ref: Prof. Seok Ho Song, http://optics.hanyang.ac.kr/~shsong/P4-Fourier%20optics.pdf

Impulse response approach of diffraction (Fraunhoffer regime: far-field)

$$u_o(x_0, y_0, z_0) \longrightarrow \boxed{h(x, y, z)} \longrightarrow u_o(x_0, y_0, z_0) * h(x, y, z)$$

$$\boxed{h(x, y, z) = \frac{j}{\lambda z} Exp\left(-ik \frac{(x^2 + y^2)}{2z}\right)}$$

$$\boxed{u(x, y, z) = \iint u_o(x_0, y_0, z_0) \quad h(x - x_0, y - y_0, z)dx_0 dy_0}$$

When

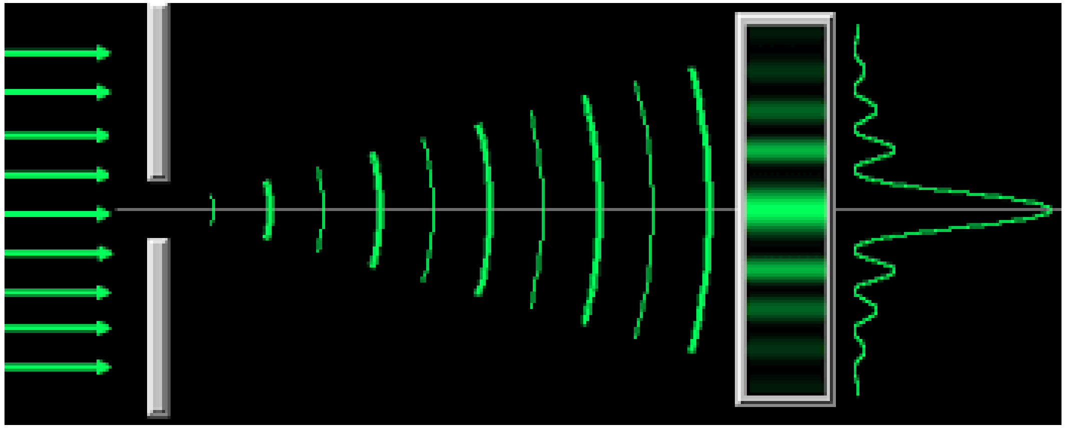
$$\frac{k(x_o{}^2 + y_o{}^2)}{(z - z_o)} \ll 1 \Rightarrow u(x,y,z) \approx \frac{j}{\lambda z} e^{-j\frac{k}{2z}(x^2+y^2)} \iint u_o(x_o, y_o) e^{j\frac{k}{z}(x_o x + y_o y)} dx_o dy_o$$

$$\boxed{u(x,y,z) \approx \frac{j}{\lambda z} e^{-j\frac{k}{2z}(x^2+y^2)} U_o\left(\frac{k_x}{z}, \frac{k_y}{z}\right)} \longrightarrow$$

Long propagation in free space takes spatial Fourier transfrom

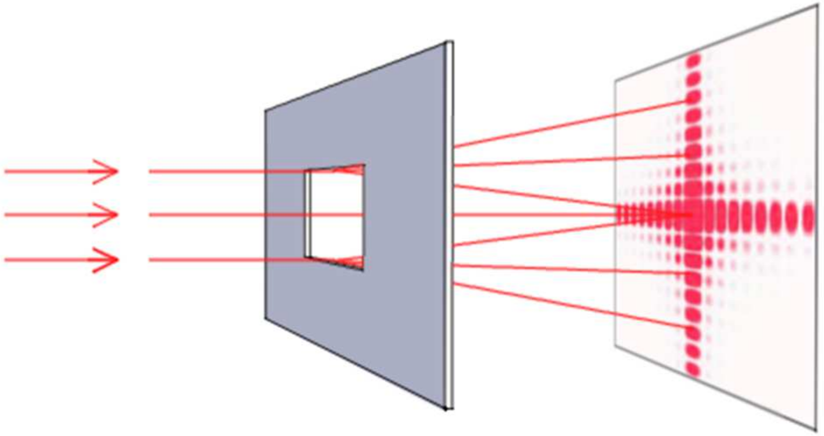
Impulse response approach of diffraction (Fraunhoffer regime: far-field)

$Rect(ax)$



$\sim \frac{1}{|a|} Sinc(\frac{k_x}{a})$

$Rect(ax)\,Rect(by)$

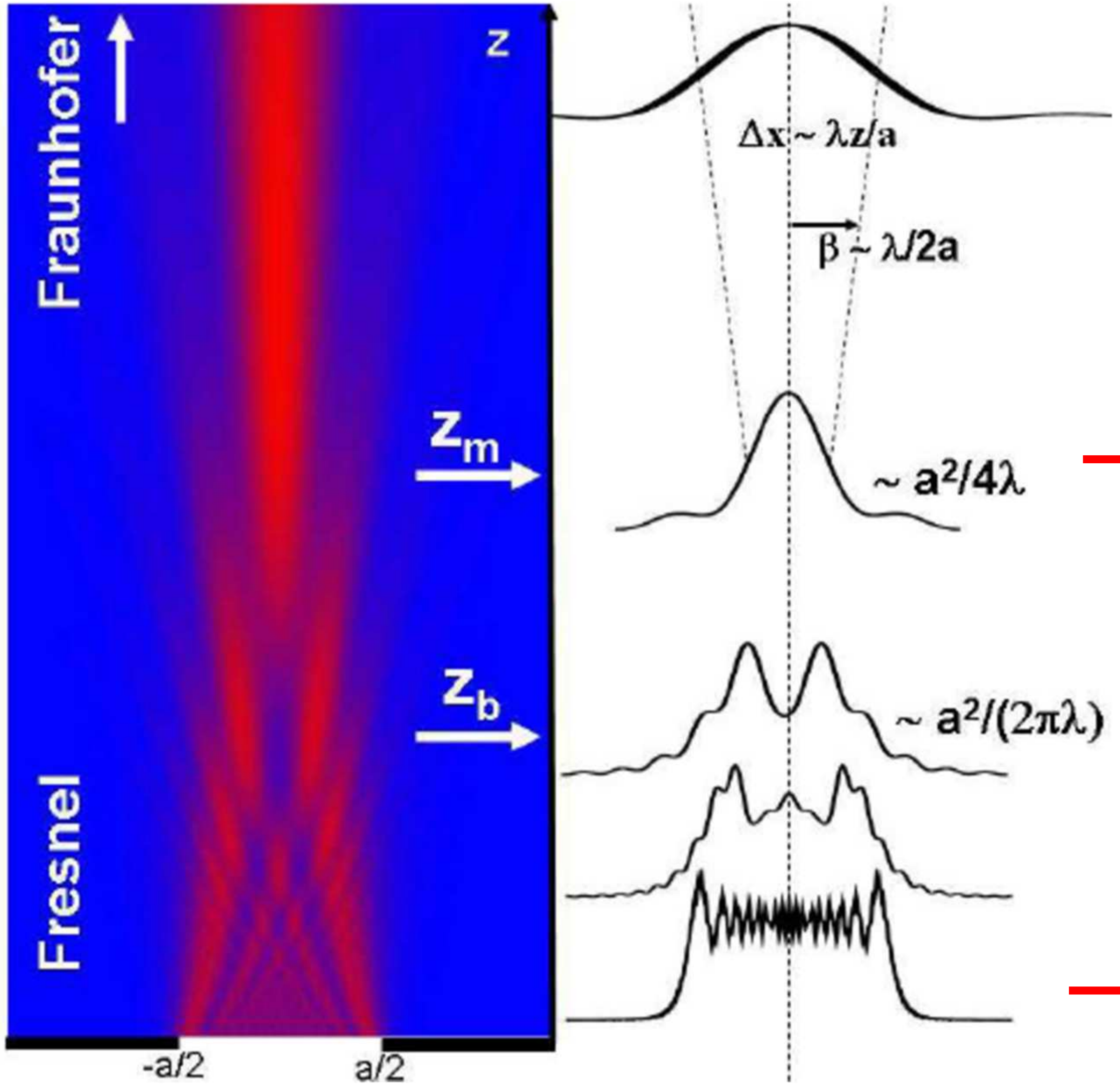


$\sim \frac{1}{|ab|} Sinc(\frac{k_x}{a}) Sinc(\frac{k_y}{b})$

$$u_o(x_0, y_0, z_0) \longrightarrow \boxed{\text{Free space}} \longrightarrow U_o(\frac{k_x}{z}, \frac{k_y}{z})$$

Spatial Fourier transfrom operation

Impulse response approach of diffraction: different regimes



→ Fourier transfrom

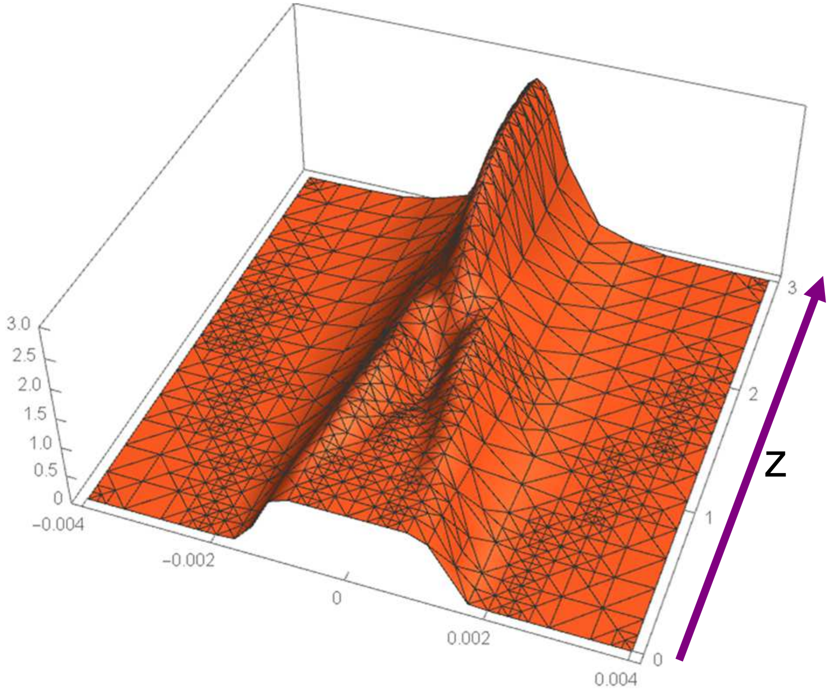
→ Minimum width

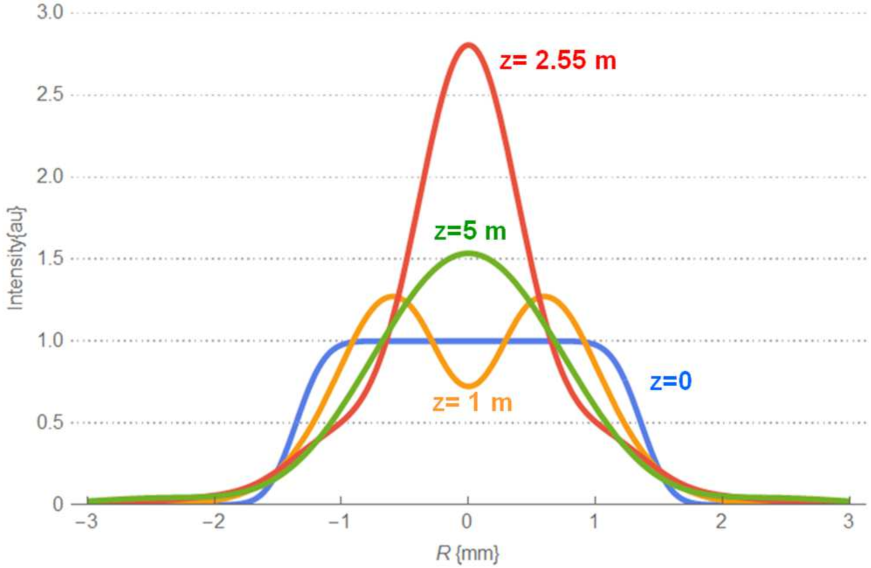
→ Dark spot

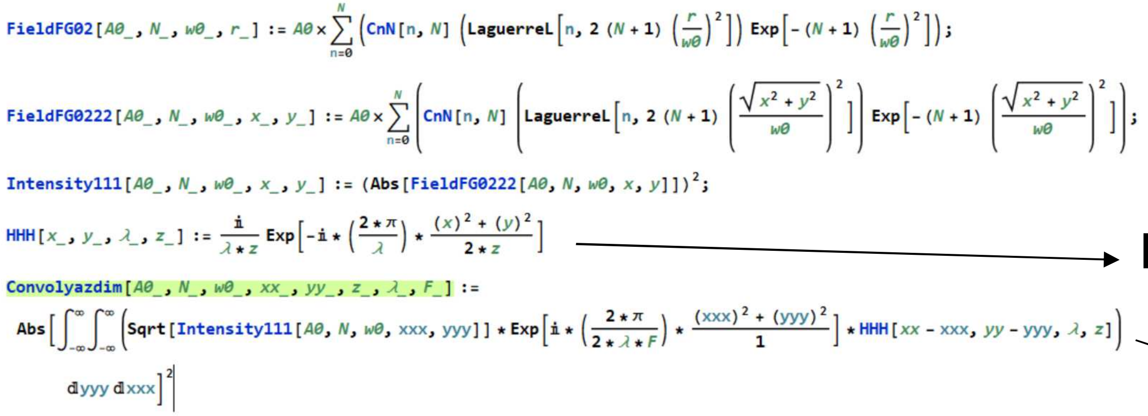
→ Shadow

Ref: „X-ray Coherent diffraction interpreted through the fractional Fourier transform“, The European Physical Journal B.

Flattop-Gaussian beam propagation: impulse response approach







```
FieldFG02[A0_, N_, w0_, r_] := A0 × Sum[(CnN[n, N] (LaguerreL[n, 2 (N + 1) (r/w0)^2]) Exp[- (N + 1) (r/w0)^2]), {n, 0, N}];

FieldFG0222[A0_, N_, w0_, x_, y_] := A0 × Sum[(CnN[n, N] (LaguerreL[n, 2 (N + 1) (Sqrt[x^2 + y^2]/w0)^2]) Exp[- (N + 1) (Sqrt[x^2 + y^2]/w0)^2]), {n, 0, N}];

Intensity111[A0_, N_, w0_, x_, y_] := (Abs[FieldFG0222[A0, N, w0, x, y]])^2;

HHH[x_, y_, λ_, z_] := ⅈ/(λ * z) Exp[-ⅈ * ((2 * π)/λ) * ((x)^2 + (y)^2)/(2 * z)]

Convolyazdim[A0_, N_, w0_, xx_, yy_, z_, λ_, F_] :=
 Abs[∫_{-∞}^{∞} ∫_{-∞}^{∞} (Sqrt[Intensity111[A0, N, w0, xxx, yyy]] * Exp[ⅈ * ((2 * π)/(2 * λ * F)) * ((xxx)^2 + (yyy)^2)/1] * HHH[xx - xxx, yy - yyy, λ, z])
   ⅆyyy ⅆxxx]^2
```

Impulse response appraoch works, but it is computationally expensive

Impulse response of free space

Convolution integral

Ref: „X-ray Coherent diffraction interpreted through the fractional Fourier transform“, The European Physical Journal B.

Diffraction Theory: Frequency response approach

Spatial domain

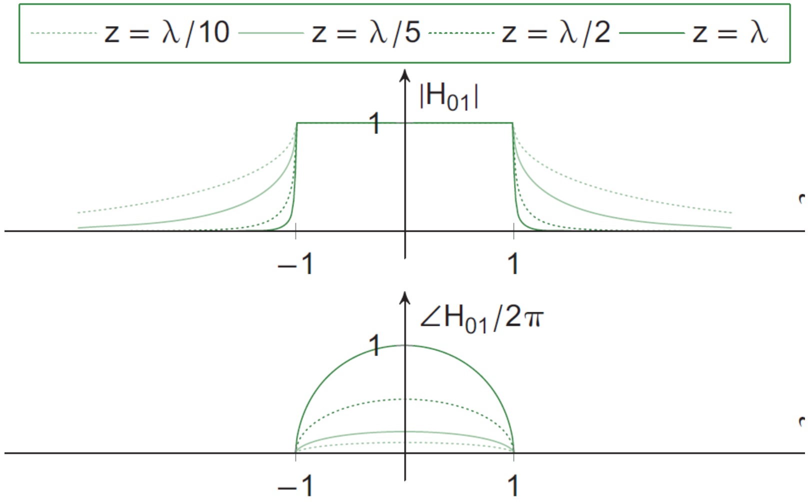
$$\begin{tikzcd} u_o(x_0, y_0, z_0) \arrow[r] \arrow[d, "\text{Fourier transform}"] & \boxed{h(x, y, z)} \arrow[r] \arrow[d, "\text{Fourier transform}"] & u_o(x_0, y_0, z_0) * h(x, y, z) \\ U(k_x, k_y, z_0) \arrow[r] & \boxed{H(k_x, k_y)} \arrow[r] & U(k_x, k_y, z_0) H(k_x, k_y) \arrow[u, "\text{Inverse Fourier transform}"'] \end{tikzcd}$$

Frequency domain

Frequency response of free space

$$\boxed{H(k_x, k_y, z) = \frac{j}{(2\pi)^2} Exp\left(-i\frac{z}{2k}(k_x{}^2 + k_y{}^2)\right)}$$

$$\boxed{U(k_x, k_y, z) = U(k_x, k_y, z_0) H(k_x, k_y)}$$



Ref: „Field Guide to Linear Systems in Optics“, SPIE.

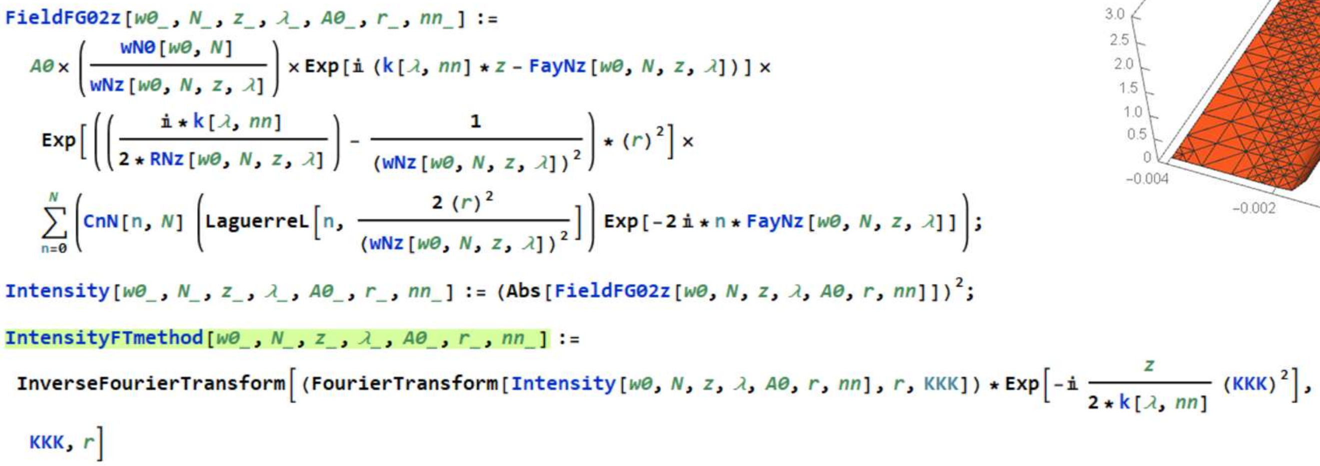
Flattop-Gaussian beam propagation: frequency response approach

Spatial domain

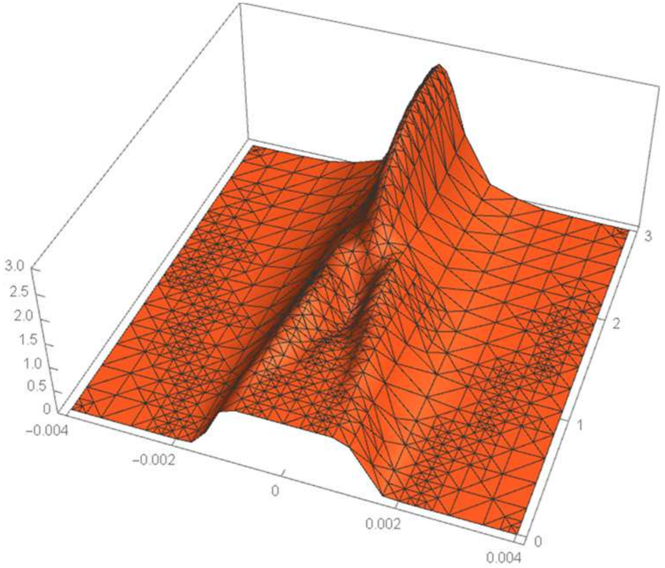
$$\begin{tikzcd}
u_o(x_0, y_0, z_0) \arrow[r] \arrow[d, "\text{Fourier transform}"] & \boxed{h(x, y, z)} \arrow[r] \arrow[d, "\text{Fourier transform}"] & u_o(x_0, y_0, z_0) * h(x, y, z) \\
U(k_x, k_y, z_0) \arrow[r] & \boxed{H(k_x, k_y)} \arrow[r] & U(k_x, k_y, z_0) H(k_x, k_y) \arrow[u, "\text{Inverse Fourier transform}"']
\end{tikzcd}$$

Frequency domain

Similar results with impulse response and frequency response approach



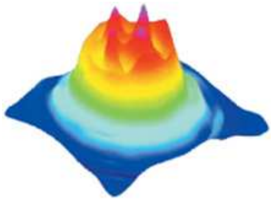
```
FieldFG02z[w0_, N_, z_, λ_, A0_, r_, nn_] :=
  A0 × (wN0[w0, N] / wNz[w0, N, z, λ]) × Exp[ⅈ (k[λ, nn] * z - FayNz[w0, N, z, λ])] ×
  Exp[((ⅈ * k[λ, nn] / (2 * RNz[w0, N, z, λ])) - 1 / (wNz[w0, N, z, λ])^2) * (r)^2] ×
  Sum[(CnN[n, N] (LaguerreL[n, 2 (r)^2 / (wNz[w0, N, z, λ])^2]) Exp[-2 ⅈ * n * FayNz[w0, N, z, λ]]), {n, 0, N}];

Intensity[w0_, N_, z_, λ_, A0_, r_, nn_] := (Abs[FieldFG02z[w0, N, z, λ, A0, r, nn]])^2;

IntensityFTmethod[w0_, N_, z_, λ_, A0_, r_, nn_] :=
  InverseFourierTransform[(FourierTransform[Intensity[w0, N, z, λ, A0, r, nn], r, KKK]) * Exp[-ⅈ z / (2 * k[λ, nn]) (KKK)^2],
  KKK, r]
```



→ Multiplication in frequency domain

Diffraction Theory: Eigenfunction approach

$I_0(x_0, y_0, z_0)$

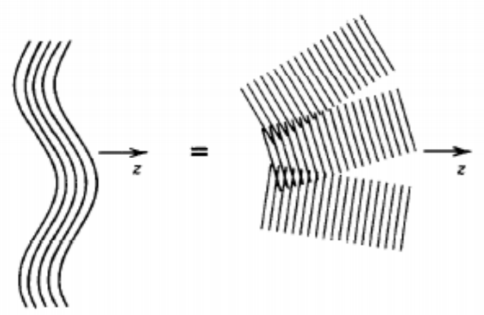


$$I_0(x_0, y_0, z_0) \longrightarrow \boxed{\text{Free space}} \xrightarrow{?} I(x, y, z)$$

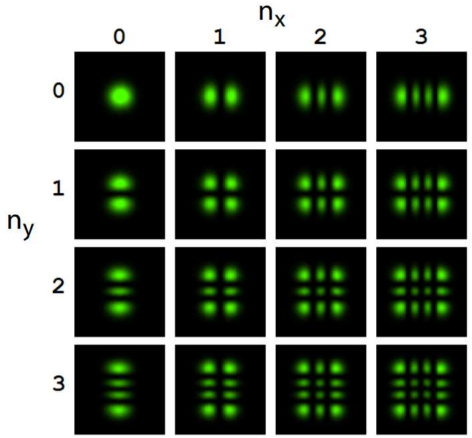
Eigenfunctions of free space

$$\phi_k(x,y) \longrightarrow \boxed{\text{Free space}} \longrightarrow \lambda_k\phi_k(x,y)$$

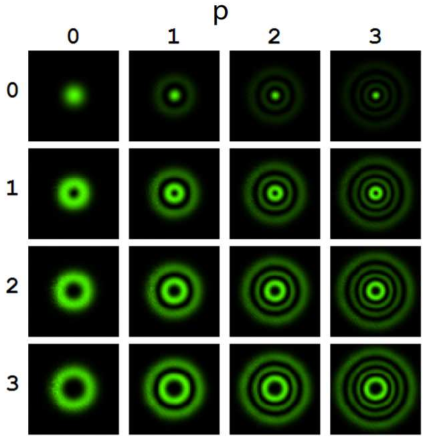
Plane waves



Hermite-Gaussian modes



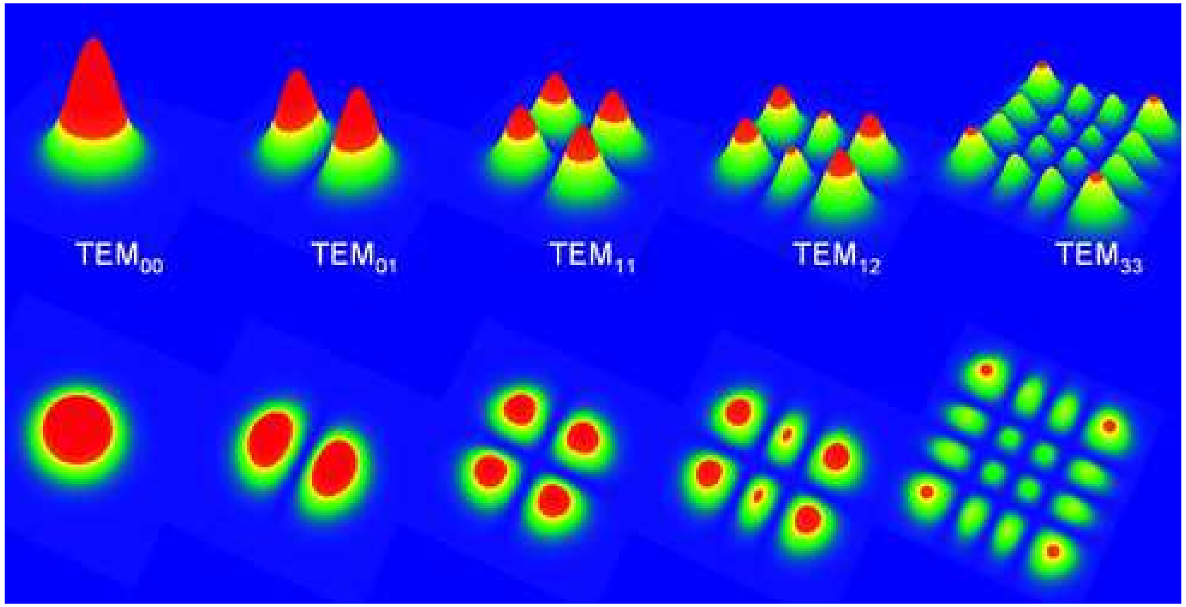
Laguerre-Gaussian modes



$$I(x,y,z_o) = \Sigma_k\, a_k\phi_k(x,y) \longrightarrow \boxed{\text{Free space}} \longrightarrow I(x,y,z) = \Sigma_k\, a_k\lambda_k\phi_k(x,y)$$

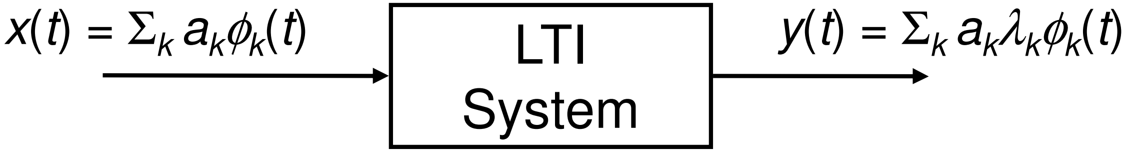
Ref: „Field Guide to Linear Systems in Optics“, SPIE.

Hermite-Gaussian Beams (Eigen-functions of free space)



$$\tilde{E}_0(r,z) \sim \frac{1}{w(z)} \exp\left[-\frac{r^2}{w^2(z)} - jk_0 \frac{r^2}{2R(z)} + j\zeta(z) \right]$$

Gaussian beams are also eigen-functions of general more complex paraxial wave systems

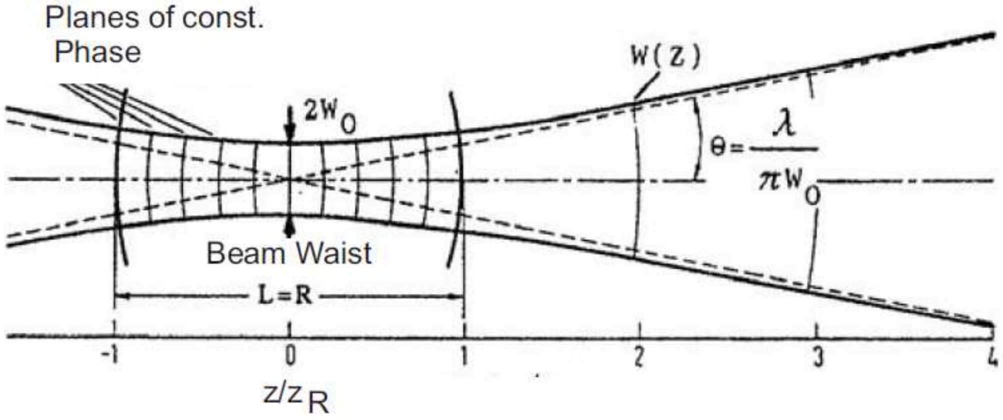


$x(t) = \Sigma_k a_k \phi_k(t)$

LTI System

$y(t) = \Sigma_k a_k \lambda_k \phi_k(t)$

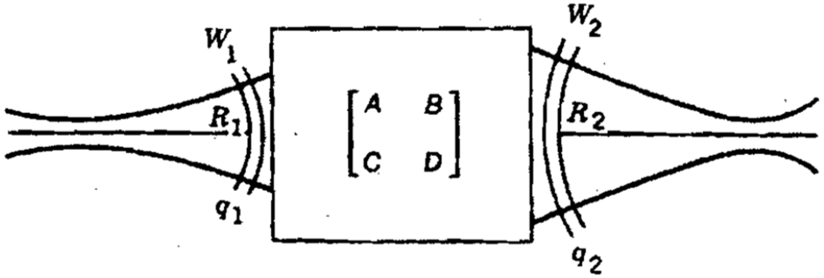
Hermite-Gaussian Beam Propagation



$$I(r,z) = \frac{2P}{\pi w^2(z)} \exp\left[-\frac{2r^2}{w^2(z)}\right].$$

$$w(z) = w_o\left[1+\left(\frac{\lambda z}{\pi w_o^2}\right)^2\right]^{1/2},$$

$$R(z) = z\left[1+\left(\frac{\pi w_o^2}{\lambda z}\right)^2\right].$$



Gaussian beam transformation by ABCD law

$$\frac{1}{q(z)} = \frac{1}{R(z)} - j\frac{\lambda}{\pi w^2(z)}.$$

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}$$

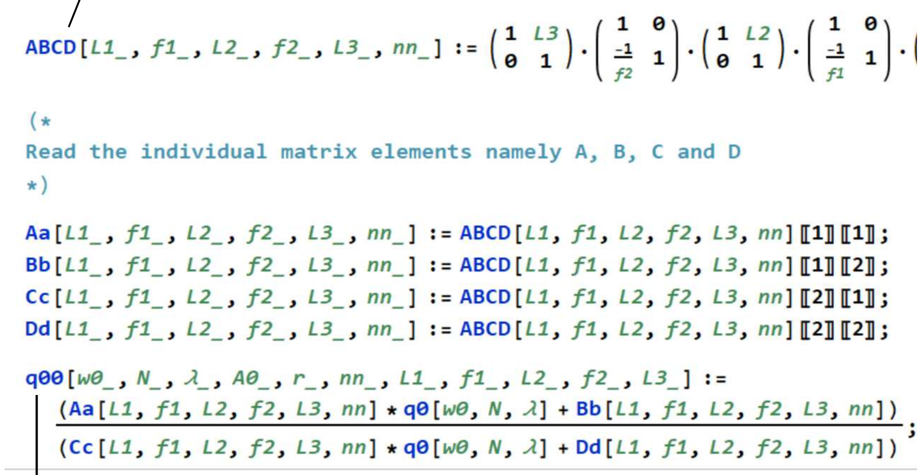
Flattop-Gaussian beam propagation: eigenfunction approach

Eigenfunctions of free space

$\phi_k(x,y)$ → Free space → $\lambda_k\phi_k(x,y)$

$I(x,y,z_o) = \Sigma_k\, a_k\phi_k(x,y)$ → Free space → $I(x,y,z) = \Sigma_k\, a_k\lambda_k\phi_k(x,y)$

ABCD matrix for the optical system

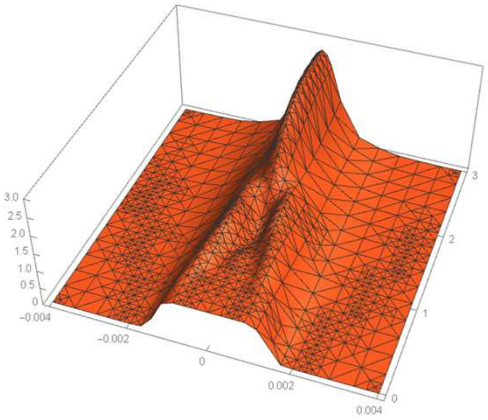


$$\text{ABCD}[L1\_,\, f1\_,\, L2\_,\, f2\_,\, L3\_,\, nn\_] := \begin{pmatrix}1 & L3\\ 0 & 1\end{pmatrix}.\begin{pmatrix}1 & 0\\ \frac{-1}{f2} & 1\end{pmatrix}.\begin{pmatrix}1 & L2\\ 0 & 1\end{pmatrix}.\begin{pmatrix}1 & 0\\ \frac{-1}{f1} & 1\end{pmatrix}.($$

```
(*
Read the individual matrix elements namely A, B, C and D
*)

Aa[L1_, f1_, L2_, f2_, L3_, nn_] := ABCD[L1, f1, L2, f2, L3, nn][[1]][[1]];
Bb[L1_, f1_, L2_, f2_, L3_, nn_] := ABCD[L1, f1, L2, f2, L3, nn][[1]][[2]];
Cc[L1_, f1_, L2_, f2_, L3_, nn_] := ABCD[L1, f1, L2, f2, L3, nn][[2]][[1]];
Dd[L1_, f1_, L2_, f2_, L3_, nn_] := ABCD[L1, f1, L2, f2, L3, nn][[2]][[2]];

q00[w0_, N_, λ_, A0_, r_, nn_, L1_, f1_, L2_, f2_, L3_] :=
  (Aa[L1, f1, L2, f2, L3, nn] * q0[w0, N, λ] + Bb[L1, f1, L2, f2, L3, nn])
  ─────────────────────────────────────────────────────────────────────────── ;
  (Cc[L1, f1, L2, f2, L3, nn] * q0[w0, N, λ] + Dd[L1, f1, L2, f2, L3, nn])
```

Q-parameter of the beam

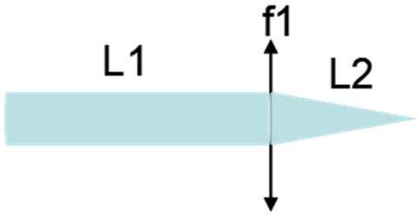
- **FGBs can be expressed as a finite sum of Laguerre-Gauss beams.**
- **Paraxial propagation of FBGs can be solved by using the propagation formulas of Laguerre-Gauss beams.**



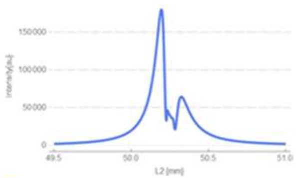
Eigenfunction aproach is the simplest option for this specific problem

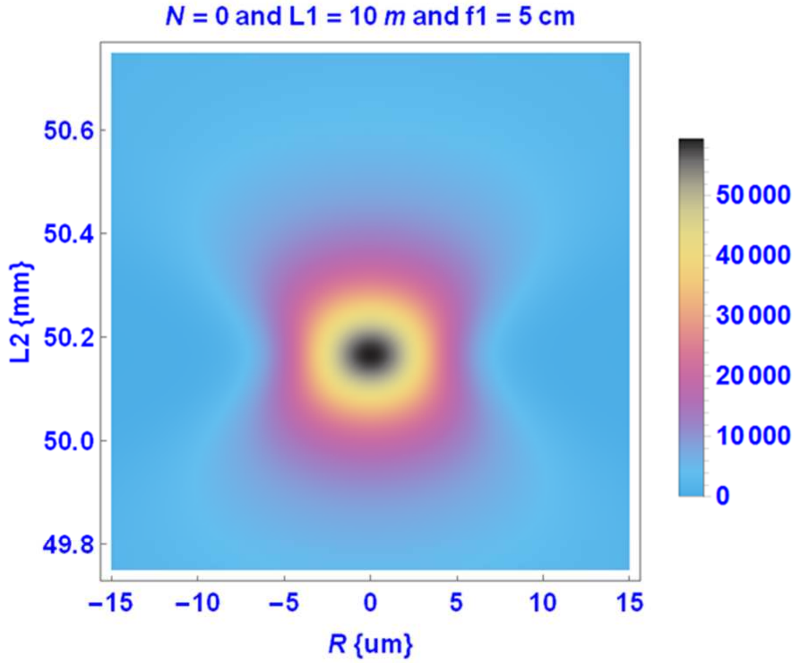
Ref: B. Lu et al., Journal of Modern Optics, (2001).
V. Bagini et al., JOSA-A, (1996).

Flattop beam propagation: solution with eigenfunction approach

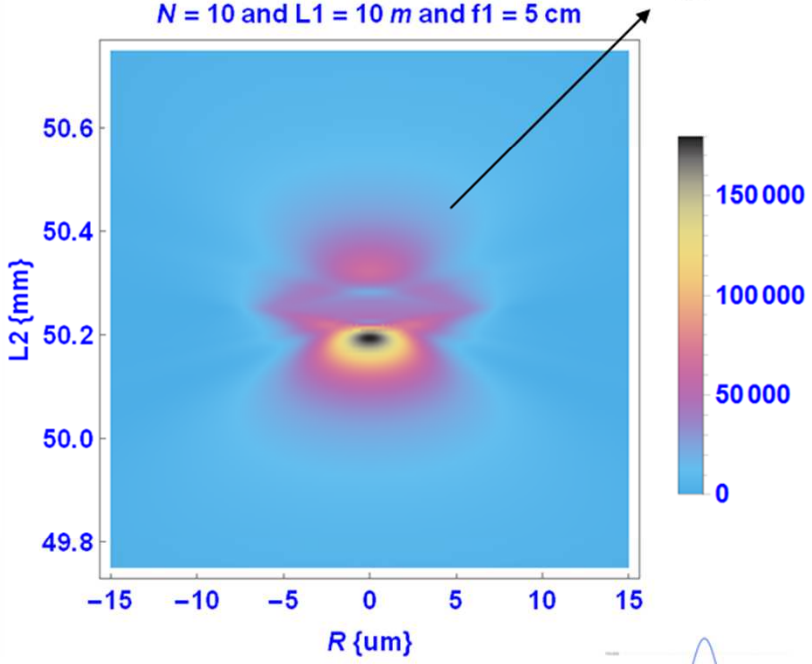


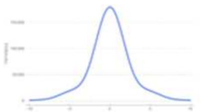
3 mm beam (after 100 mJ CTD), lambda= 1um
f1=5 cm, L2 (variable)
L1=10 m
N=0, N=10





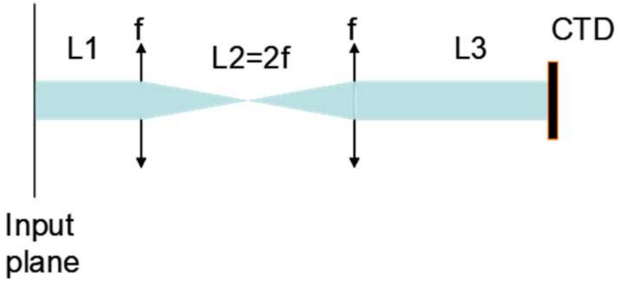
Focusing distance: 50.17 mm
Focused beam FWHM: 7.3 um
Depth of focus (FWHM): 0.25 mm
Relative peak intensity: 59,300

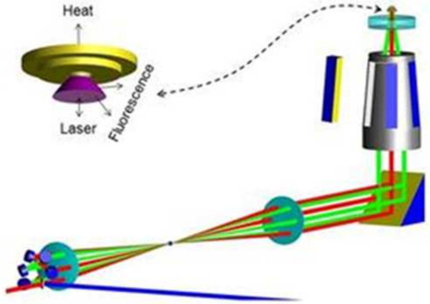




Focusing distance: 50.195 mm
Focused beam FWHM: 4.4 um
Depth of focus (FWHM): 0.9 mm
Relative peak intensity: 181,500

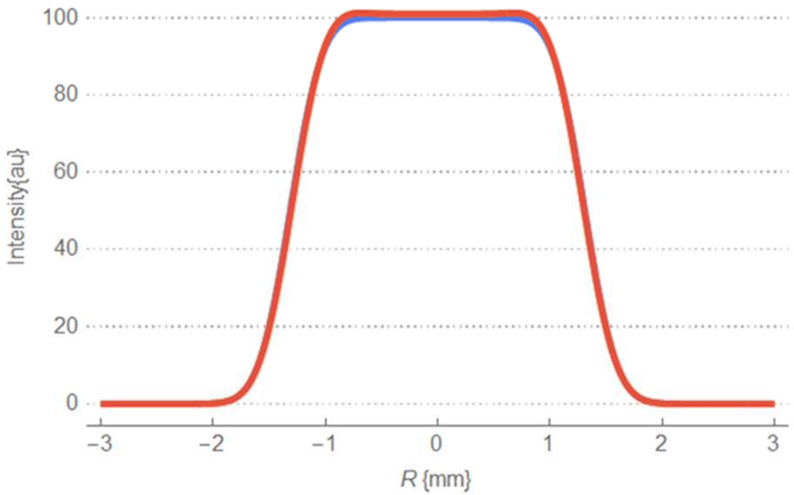
Flattop beam propagation: solution with eigenfunction approach

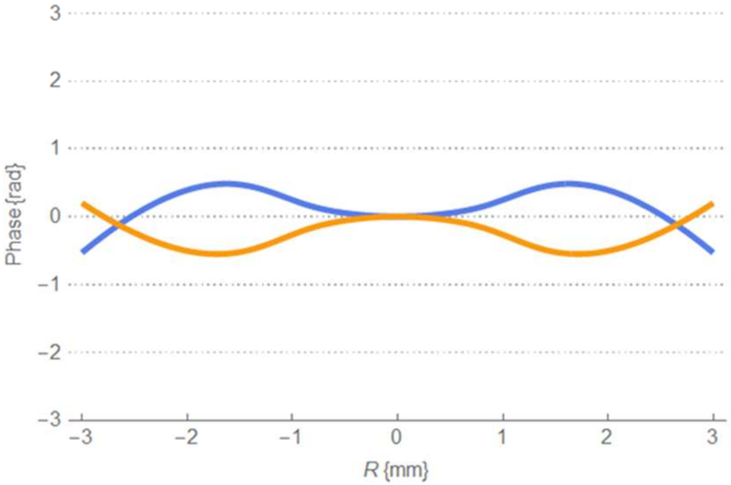




- **Perturbation 1: L1+L3 is not exactly 2f (60 cm), Take +- 1% of perturbation 1 (+- 6 mm error)**
- **Perturbation 2: A thermal lens of a focal length 5 m at the position of the CTD**
- **Solution: Put a negative lens after 3 passes to correct thermal lens**

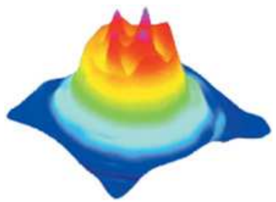
Intensity and Phase Profile With Correction





Diffraction Theory

$I_0(x_0, y_0, z_0)$



$I(x, y, z)$

Impulse response approach

$$h(x, y, z) = \frac{j}{\lambda z} Exp\left(-ik\frac{(x^2 + y^2)}{2z}\right)$$

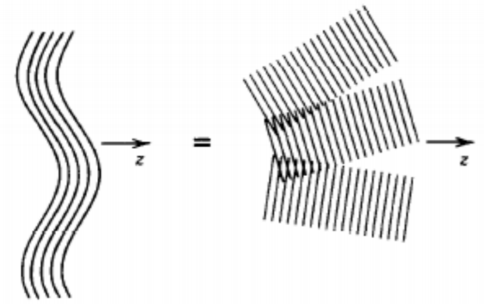
$$u(x, y, z) = u_o(x_0, y_0, z_0) * h(x, y, z)$$

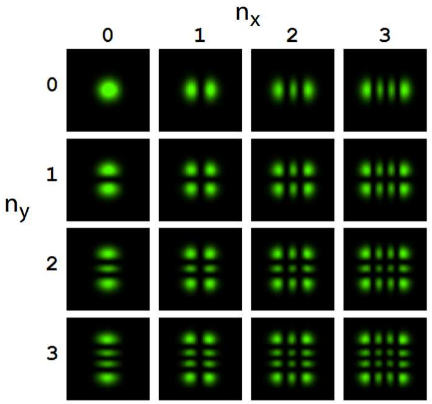
Frequency response approach

$$H(k_x, k_y, z) = \frac{j}{(2\pi)^2} Exp\left(-i\frac{z}{2k}(k_x^2 + k_y^2)\right)$$

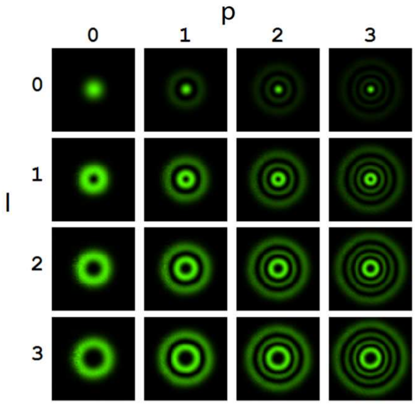
$$U(k_x, k_y, z) = U(k_x, k_y, z_0) H(k_x, k_y)$$

Eigen-function approach



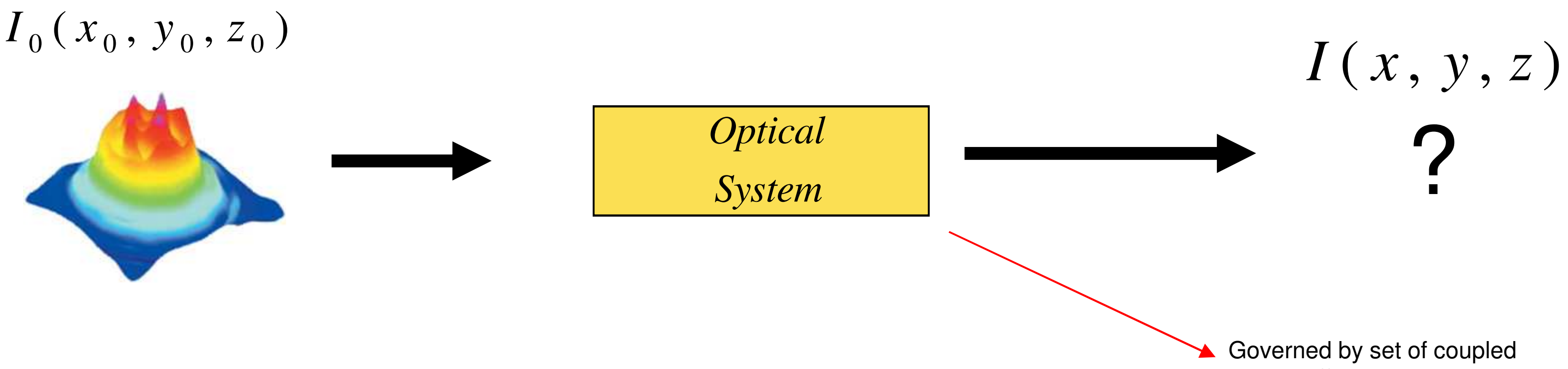


Hermite-Gaussian modes



Laguerre-Gaussian modes

Diffraction Theory



$I_0(x_0, y_0, z_0)$

Optical System

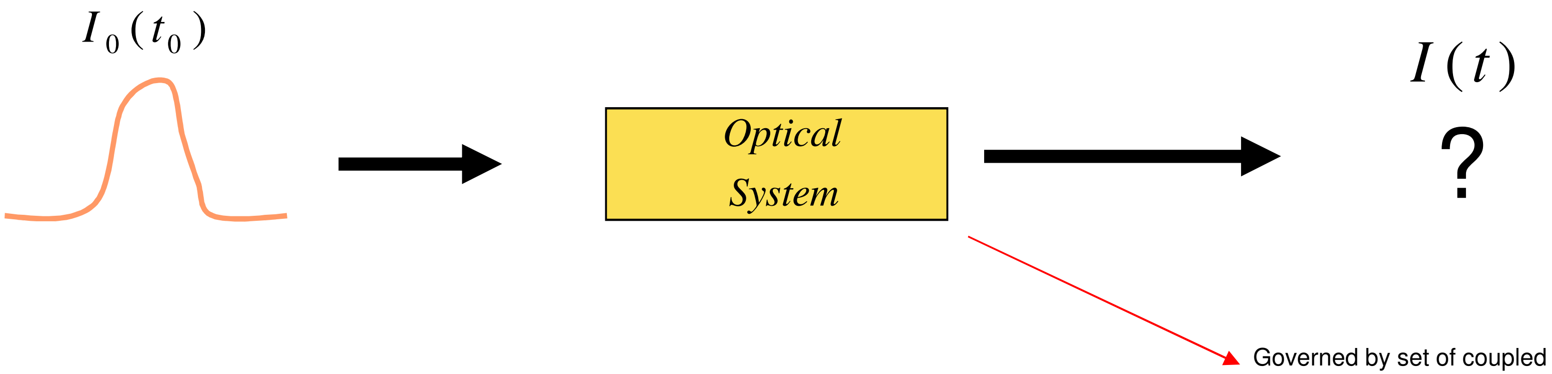
$I(x, y, z)$

?

Governed by set of coupled partial differential equations

If I know the beam profile at the input (beam shape), what is the beam profile at the output?

Dispersion Theory



$I_0(t_0)$

Optical System

$I(t)$

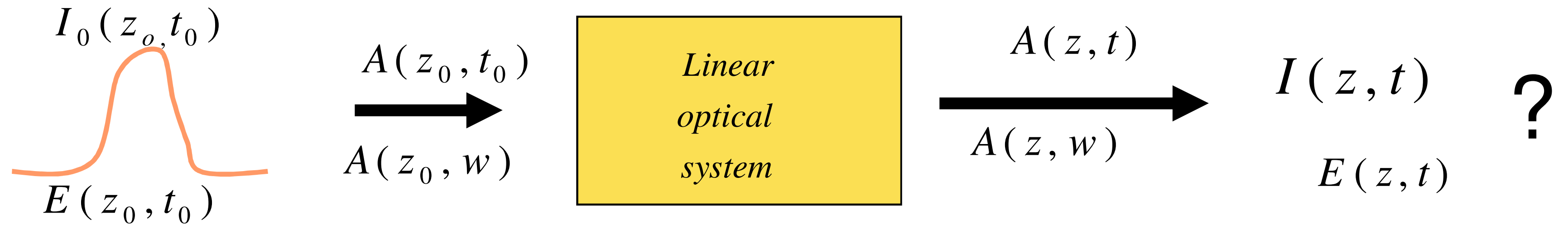
?

Governed by set of coupled partial differential equations

If I know the pulse shape at the input, what is the pulse shape at the output?

Dispersion Theory

$$I = \frac{1}{2Z_F}|\underline{E}|^2 = \frac{1}{2}Z_F|\underline{H}|^2.$$



$$I_0(z_0,t_0) \Rightarrow E(z_0,t_0) \Rightarrow A(z_0,t_0) \qquad E(t) = A(t)e^{jwt}e^{-jK|_{w=w_0}z}$$

$$K(w) = K\Big|_{w=w_0} + K'\Big|_{w=w_0} w + \frac{K''\Big|_{w=w_0}}{2}w^2 + \frac{K'''\Big|_{w=w_0}}{6}w^3 + h.o.t.$$

$$K(w) = k(w) + K\Big|_{w=w_0}$$

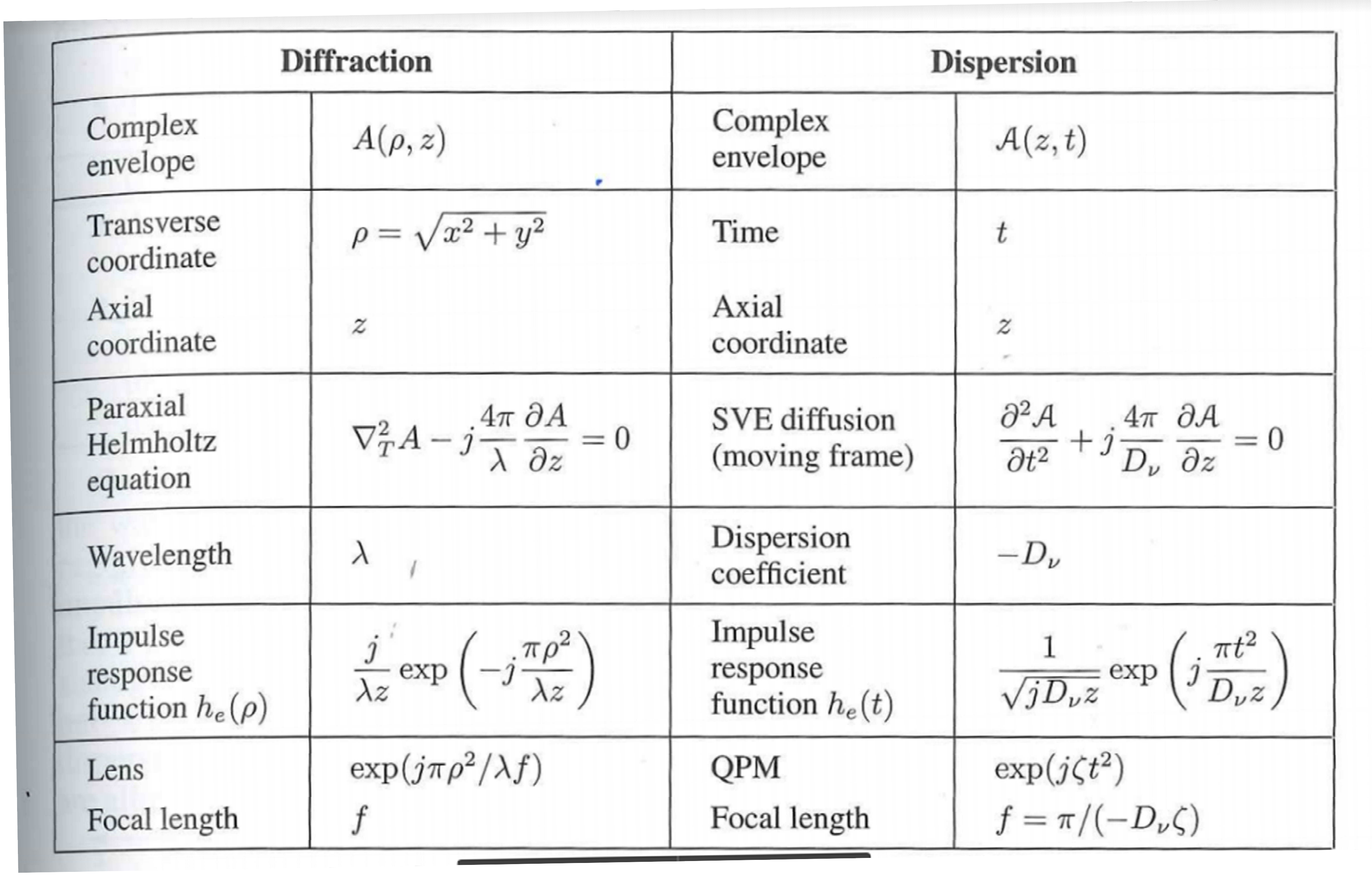
$$A(z,t) = A_o(z_0,t_0) * h(t) = \int A(z_0,t_0-\tau)h(\tau)d\tau$$

$$A(z,w) = A(z_0,w)H(w) = A(z_0,w)Exp[-jk(w)z]$$

$$h(t) = \frac{1}{2\pi}\int Exp[-jk(w)z]dw$$ **Impulse response**

$$H(w) = Exp[-jk(w)z]$$ **Frequency response**

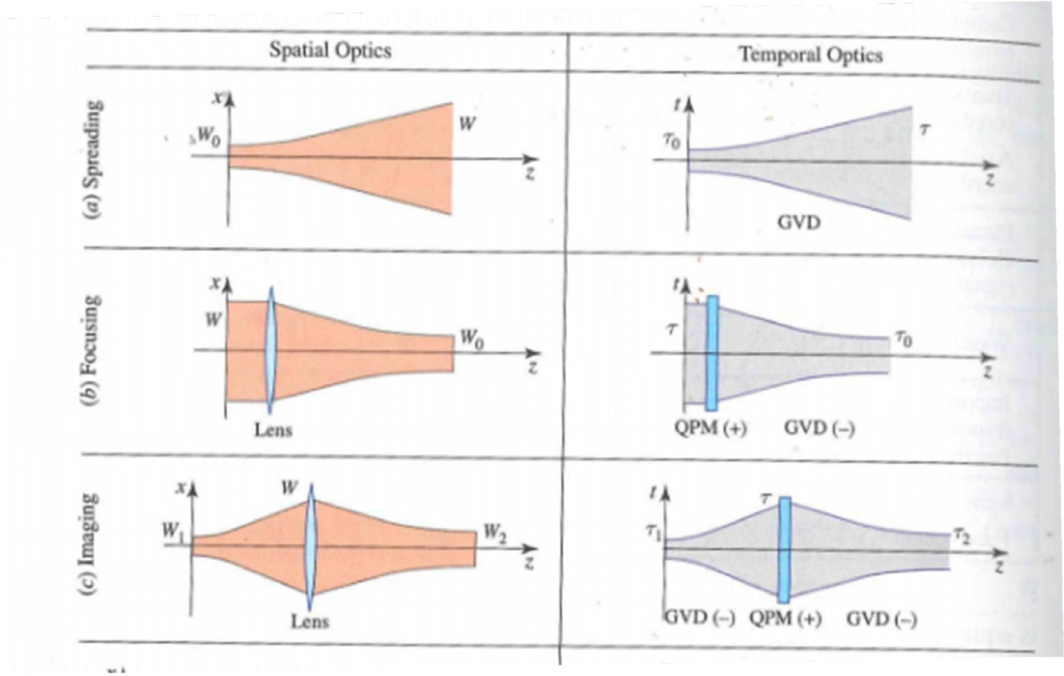
Diffraction and Dispersion Theory

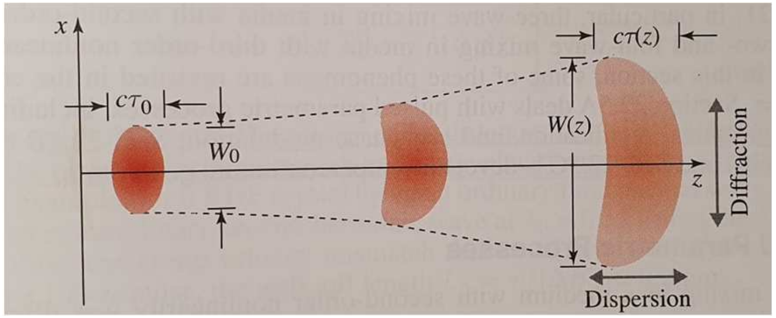


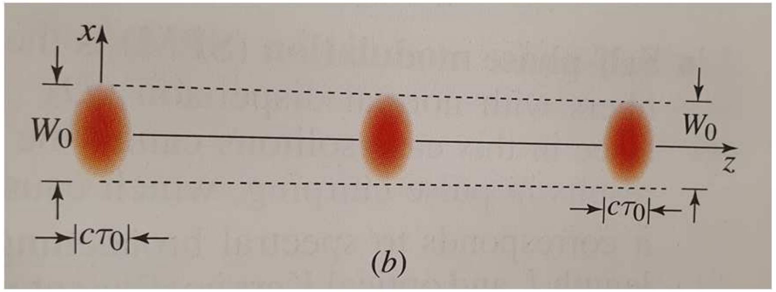
Diffraction		Dispersion	
Complex envelope	$A(\rho, z)$	Complex envelope	$\mathcal{A}(z, t)$
Transverse coordinate Axial coordinate	$\rho = \sqrt{x^2 + y^2}$ z	Time Axial coordinate	t z
Paraxial Helmholtz equation	$\nabla_T^2 A - j\frac{4\pi}{\lambda}\frac{\partial A}{\partial z} = 0$	SVE diffusion (moving frame)	$\frac{\partial^2 \mathcal{A}}{\partial t^2} + j\frac{4\pi}{D_\nu}\frac{\partial \mathcal{A}}{\partial z} = 0$
Wavelength	λ	Dispersion coefficient	$-D_\nu$
Impulse response function $h_e(\rho)$	$\frac{j}{\lambda z}\exp\left(-j\frac{\pi\rho^2}{\lambda z}\right)$	Impulse response function $h_e(t)$	$\frac{1}{\sqrt{jD_\nu z}}\exp\left(j\frac{\pi t^2}{D_\nu z}\right)$
Lens Focal length	$\exp(j\pi\rho^2/\lambda f)$ f	QPM Focal length	$\exp(j\zeta t^2)$ $f = \pi/(-D_\nu\zeta)$

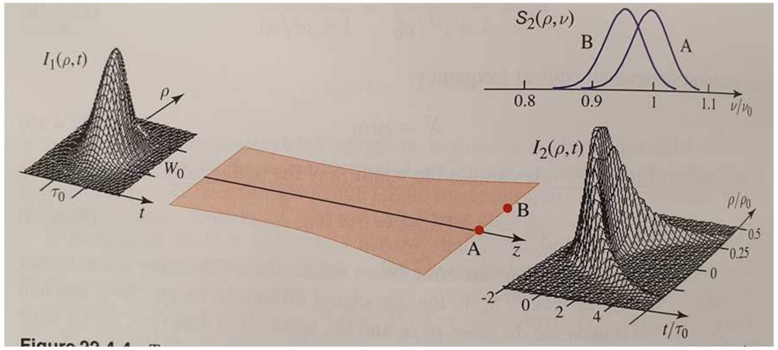
Ref: Saleh and Teich, „Fundamentals of Photonics“.

Analogy of Diffraction and Dispersion Theory









How do we generate, transport and make experiments with ultrafast optical pulses considering diffraction and dispersion effects (and others)?

Ref: Saleh and Teich, „Fundamentals of Photonics“.