The Quark and Gluon Form Factor to Three Loops in Massless QCD

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PHYSICS AT THE TERA SCALE
Helmholtz Alliance

In collaboration with
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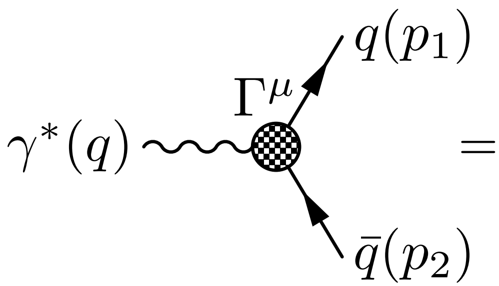
Outline

- Definition of the quark and gluon form factor in massless QCD
- (Brief) history and status of the form factors
- Computational techniques and results
- Applications
- Conclusion

Quark Form Factor

- Quark form factor $\mathcal{F}^q$: $\gamma^* \to q\bar{q}\,,$ massless, on-shell quarks



$$= -i\, e\, \bar{u}(p_1)\, \Gamma^{\mu}_{q\bar{q}}\; u(p_2)\,, \qquad \Gamma^{\mu}_{q\bar{q}} = \gamma^{\mu}\, \mathcal{F}^q$$

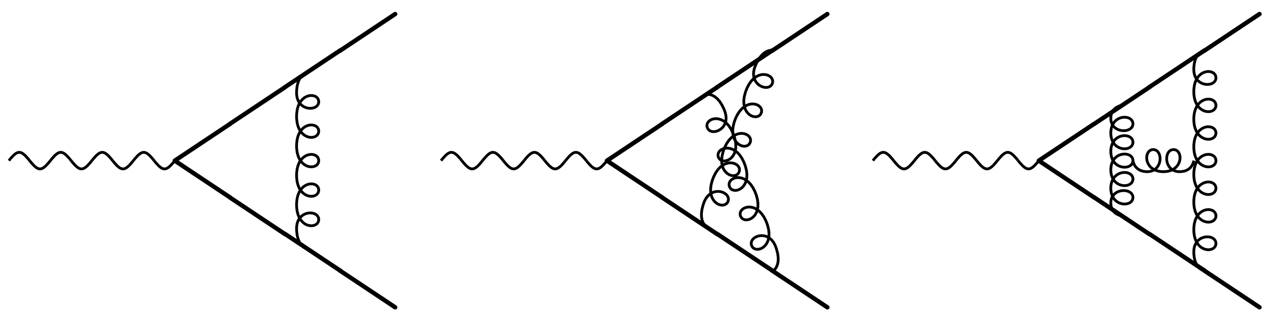
- Can project on $\mathcal{F}^q$ via

$$\mathcal{F}^q = -\frac{1}{4(1-\epsilon)q^2}\,\mathrm{Tr}\left(\not{p}_2\, \Gamma^{\mu}_{q\bar{q}}\; \not{p}_1\, \gamma_{\mu}\right)$$

- Perturbative expansion ($s_{12} \equiv q^2$)

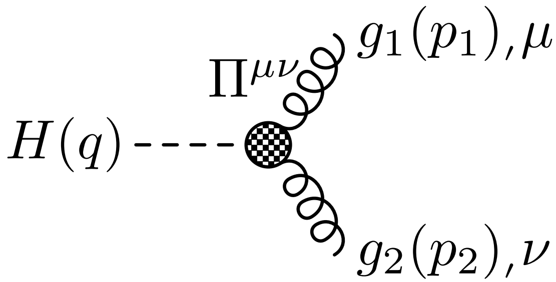
$$\mathcal{F}^q(\alpha_s^b, s_{12}) = 1 + \sum_{n=1}^{\infty} \left(\frac{\alpha_s^b}{4\pi}\right)^n \left(\frac{-s_{12}}{\mu_0^2}\right)^{-n\epsilon} S_\epsilon^n\, \mathcal{F}_n^q$$

- Sample diagrams



Gluon Form Factor

- Gluon form factor $\mathcal{F}^g$: $H \to gg$, from effective vertex $\mathscr{L}_{eff} = -\frac{\lambda}{4} H F_a^{\mu\nu} F^a_{\mu\nu}$



$$= i\,\lambda\,\Pi^{\mu\nu}_{gg} = i\,\lambda\,\mathcal{F}^g \; (g^{\mu\nu}\, p_1 \cdot p_2 - p_1^\nu\, p_2^\mu)$$

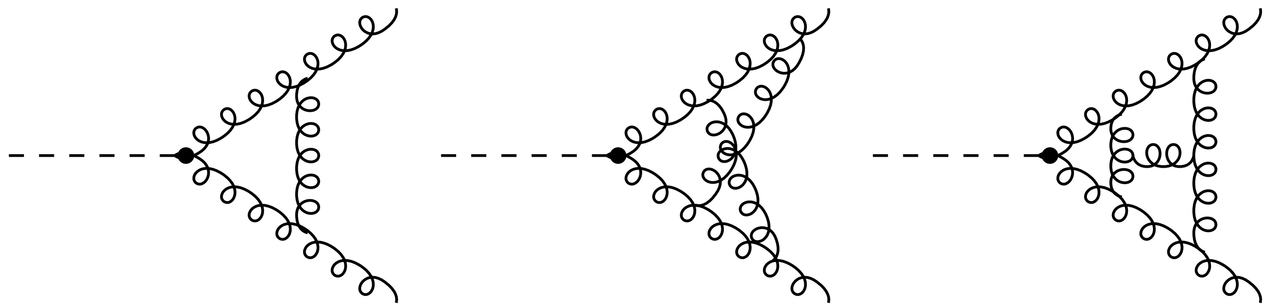
- Can project on $\mathcal{F}^g$ via

$$\mathcal{F}^g = \frac{p_1 \cdot p_2\; g_{\mu\nu} - p_{1,\mu}\, p_{2,\nu} - p_{1,\nu}\, p_{2,\mu}}{2(1-\epsilon)}\, \Pi^{\mu\nu}_{gg}$$

- Perturbative expansion ($s_{12} \equiv q^2$)

$$\mathcal{F}^g(\alpha_s^b, s_{12}) = \lambda^b \left[1 + \sum_{n=1}^{\infty} \left(\frac{\alpha_s^b}{4\pi}\right)^n \left(\frac{-s_{12}}{\mu_0^2}\right)^{-n\epsilon} S_\epsilon^n\, \mathcal{F}_n^g\right]$$

- Sample diagrams



History and status of the form factors I

- General multi-loop strategies
  - Regulate UV and IR divergences of amplitude dimensionally, $D = 4 - 2\epsilon$
  - Apply algebraic reduction methods, reduction is exact in D dimensions
  - Obtain amplitude as a linear combination of a small set of master integrals
  - At L loops, get poles up to 1/ϵ^{2L}
  - Computation of finite contribution at L loops
    requires $(L - m)$-loop result to $\mathcal{O}(\epsilon^{2m})$

- Two-loop form factors through $\mathcal{O}(\epsilon^0)$ known since long
  - $\mathcal{F}_2^q$ *[(Gonsalves'83); Kramer,Lampe'87; Matsuura,van Neerven'88; Matsuura,van der Maarck,van Neerven'89]*
  - $\mathcal{F}_2^g$ *[Harlander'00; Ravindran,Smith,van Neerven'04]*

- Also extension of $\mathcal{F}_2^q$ and $\mathcal{F}_2^g$ to all orders in ϵ *[Gehrmann,Maitre,TH'05]*
  - $\mathcal{F}_2^q$ and $\mathcal{F}_2^g$ through order $\mathcal{O}(\epsilon^2)$: First step towards three-loop accuracy

History and status of the form factors II

- Three-loop form factors $\mathcal{F}_3^q$ and $\mathcal{F}_3^g$: Pole terms known through $\mathcal{O}(\epsilon^{-1})$, and also the finite pieces of the fermionic corrections to $\mathcal{F}_3^q$ *[Moch,Vermaseren,Vogt'05]*
- Identification of masters for three-loop form factors *[Gehrmann,Heinrich,Studerus,TH'06]*
- Computation of three-loop master integrals *[Gehrmann,Heinrich,Studerus,TH'06; Heinrich,Maitre,TH'07]* *[Heinrich,Kosower,Smirnov,TH'09; Lee,Smirnov,Smirnov'10]*
- Recently the full $\mathcal{F}_3^q$ and $\mathcal{F}_3^g$ have become available independently *[Baikov,Chetyrkin,Smirnov,Smirnov,Steinhauser'09]* *[Gehrmann,Glover,Ikizlerli,Studerus,TH'10]*
- Extension of masters to two more orders in ϵ *[Lee,Smirnov,Smirnov'10; Lee,Smirnov'10]*
- Allows to obtain $\mathcal{F}_3^q$ and $\mathcal{F}_3^g$ through $\mathcal{O}(\epsilon^2)$ *[Gehrmann,Glover,Ikizlerli,Studerus,TH'10]*

The stage is set for the four-loop calculation

Computation of the three-loop form factors

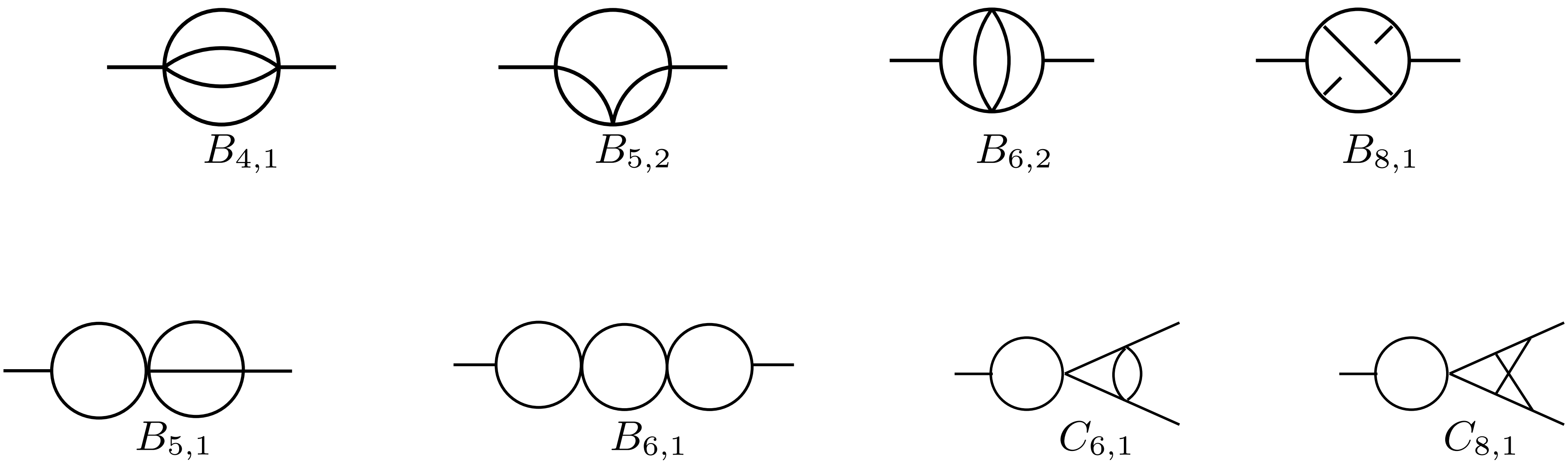
- Generate Feynman diagrams using QGRAPH. [Nogueira'93]
  244 diagrams contribute to $\mathcal{F}_3^q$, 1586 to $\mathcal{F}_3^g$.
- After projection on $\mathcal{F}_3^q$ and $\mathcal{F}_3^g$, obtain hundreds of scalar integrals for each diagram
- Up to 9 different propagators in each integral
- Up to $s = 4$ (quark FF) or $s = 5$ (gluon FF) powers of irreducible scalar products $l_i \cdot l_j$ or $l_i \cdot p_k$ in numerator
- Use integration-by-parts (IBP) and Lorentz-invariance (LI) identities to relate different integrals [Chetyrkin,Tkachov'81; Gehrmann,Remiddi'00]
- Yields huge system of linear equations, have > 900 000 equations already for $s \leq 4$
- Perform Laporta reduction with `AIR` (Maple), `FIRE` (Mathematica), and `Reduze` (C++) [Laporta'01; Anastasiou,Lazopoulos'04; Smirnov'08; Studerus'09]
  - Pure computing time is from few weeks to two months
- Express each integral as a linear combination of 22 master integrals

Master integrals I

- 8 of the 22 masters are two-point functions or factorizable vertex diagrams (all known)

[Tkachov'81; Chetyrkin,Tkachov'81; Gorishnii,Larin,Surguladze,Tkachov'89]

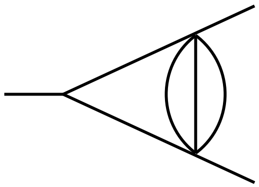
[Larin,Tkachov,Vermaseren'91; Bekavac'05; Lee,Smirnov,Smirnov'10]



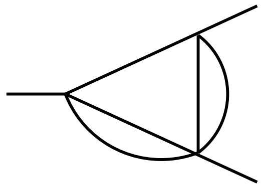
- In addition: 14 genuine three-loop vertex integrals

Master integrals II

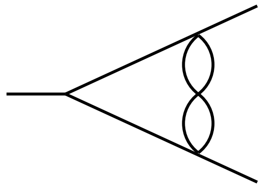
- 14 genuine three-loop vertex integrals



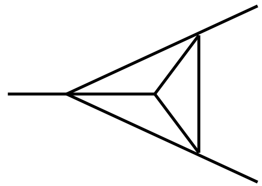
$A_{5,1}$



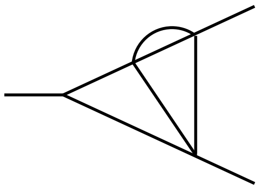
$A_{5,2}$



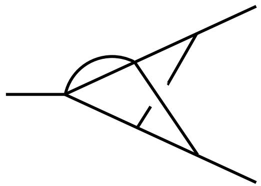
$A_{6,1}$



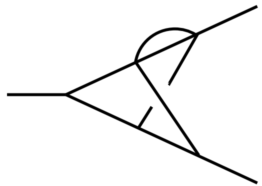
$A_{6,2}$



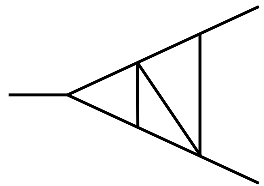
$A_{6,3}$



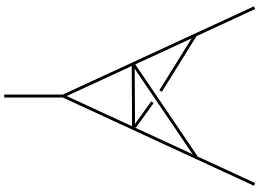
$A_{7,1}$



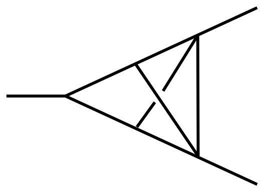
$A_{7,2}$



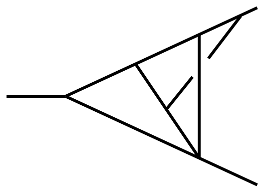
$A_{7,3}$



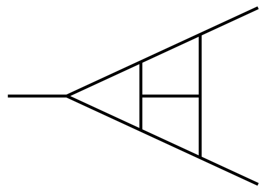
$A_{7,4}$



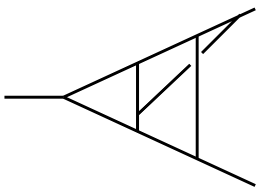
$A_{7,5}$



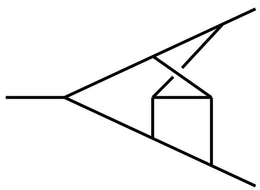
$A_{8,1}$



$A_{9,1}$



$A_{9,2}$



$A_{9,4}$

Criteria:

Number of propagators,

bubble insertions

planar vs. crossed topologies,

Number of lines at outermost vertices.

Computational techniques for masters I

- Kinematics: $p_1^2 = p_2^2 = 0,$ all propagators massless.
  Only one scale: $q^2 = (p_1 + p_2)^2$, which has to factor out
  - General form of the result: $A = i^3 S_\Gamma^3 \left[- q^2 - i\,\eta\right]^{L\cdot D/2 - n_p} \cdot f(\epsilon)$

- Required: Expansion of $f(\epsilon)$ about $\epsilon = 0$
  - Coefficients have increasing transcendentality T of Riemann ζ-function
  - Need all coefficients with $T \leq 6$, (i.e. π^6 and ζ_3^2) for finite piece at three loops
  - Need all coefficients with $T \leq 8$, (i.e. π^8, $\pi^2\zeta_3^2$, $\zeta_3\,\zeta_5$, $\zeta_{5,3}$) for $\mathcal{O}(\epsilon^2)$ at three loops

- Gamma functions $A_{6,1} = i^3\, S_\Gamma^3\, \left[-q^2 - i\,\eta\right]^{-3\epsilon}\, \frac{\Gamma^7(1-\epsilon)\,\Gamma^2(\epsilon)\,\Gamma(3\,\epsilon)\,\Gamma^2(1-3\,\epsilon)}{\Gamma^2(2-2\,\epsilon)\,\Gamma(2-4\,\epsilon)}$

- Hypergeometric functions, use HypExp or XSummer *[Maître,TH'05; Moch,Uwer'05]*

$$A_{6,3} = i^3\, S_\Gamma^3\, \left[-q^2 - i\,\eta\right]^{-3\,\epsilon}\, \frac{2\,\Gamma^6(1-\epsilon)}{(1-3\,\epsilon)\,\Gamma(3-4\,\epsilon)} \times \left[\frac{\Gamma(1-3\epsilon)\,\Gamma(3\epsilon)\,\Gamma(2\epsilon)\,\Gamma(\epsilon)\,\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\right.$$

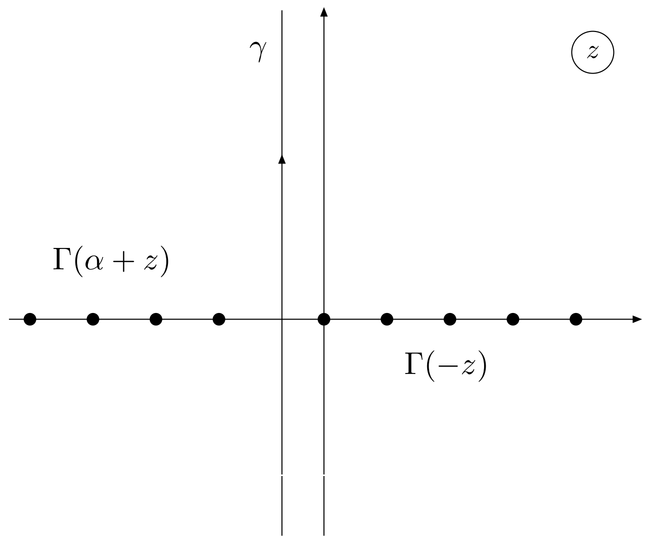
$$\left. + \frac{\Gamma(3\epsilon-1)\,\Gamma(1-\epsilon)}{(1-2\,\epsilon)}\; {}_3F_2(1, 1-\epsilon, 1-2\,\epsilon\,;\, 2-2\epsilon, 2-3\epsilon\,;\, 1)\right]$$

Computational techniques for masters II

- Multiple Mellin-Barnes representations

[Smirnov'99; Tausk'99]

$$\frac{1}{(A_1 + A_2)^\alpha} = \int \frac{dz}{2\pi i} \, A_1^z \, A_2^{-\alpha - z} \, \frac{\Gamma(-z)\,\Gamma(\alpha + z)}{\Gamma(\alpha)}$$



- Analytic continuation to $\epsilon = 0$,
  efficient extraction of poles [Czakon'05; Smirnov,Smirnov'09]

$$\begin{aligned} A_{6,2} &= -i\, S_\Gamma^3 \left[- q^2 - i\,\eta \right]^{-3\,\epsilon} \frac{\Gamma^3(1-\epsilon)\,\Gamma(3\,\epsilon)\,\Gamma^2(1-3\,\epsilon)}{\Gamma(1-2\,\epsilon)\,\Gamma(2-4\,\epsilon)} \int\limits_{c_1 - i\,\infty}^{c_1 + i\,\infty} \frac{dw_1}{2\pi i} \int\limits_{c_2 - i\,\infty}^{c_2 + i\,\infty} \frac{dw_2}{2\pi i} \\ &\times \frac{\Gamma(-1+3\,\epsilon-w_1)\,\Gamma(-1+2\,\epsilon-w_1)\,\Gamma(2-4\,\epsilon+w_1)\,\Gamma(-w_2)\,\Gamma(w_2-w_1)}{\Gamma(3\,\epsilon-w_1)\,\Gamma(2-4\,\epsilon+w_2)\,\Gamma(2-4\,\epsilon+w_1-w_2)} \\ &\times \Gamma(1-\epsilon+w_2)\,\Gamma(1-\epsilon+w_1-w_2)\,\Gamma(1-2\,\epsilon+w_2)\,\Gamma(1-2\,\epsilon+w_1-w_2) \end{aligned}$$

- Dimensional recurrence relations [Tarasov'98; Lee'09]

$$I^{(D+2)} = g_0(D)\; I^{(D)} + g_1(D)$$

  - Obtain coefficients of Laurent series with high numerical precision.
    Fit rational constants with PSLQ [Ferguson,Bailey,Arno'99]

Results I

- Structure of the one-loop form factors. $D = 4 - 2\epsilon$ and $S_R = e^{\epsilon\gamma_E}/\Gamma(1-\epsilon)$

$$\mathcal{F}_1^q/S_R = \quad C_F\, B_{2,1}\left[\frac{4}{(D-4)} + D - 3\right]$$

$$\mathcal{F}_1^g/S_R = \quad C_A\, B_{2,1}\left[\frac{4}{(D-4)} - \frac{4}{(D-2)} + 10 - D\right]$$

- Structure of the two-loop form factors

$$\mathcal{F}_2^q/S_R^2 = C_F^2\, X^q_{C_F^2} + C_F C_A\, X^q_{C_F C_A} + C_F N_F\, X^q_{C_F N_F}$$

$$\mathcal{F}_2^g/S_R^2 = C_A^2\, X^g_{C_A^2} + C_A N_F\, X^g_{C_A N_F} + C_F N_F\, X^g_{C_F N_F}$$

- Structure of the three-loop form factors

$$\begin{aligned}\mathcal{F}_3^q/S_R^3 = \; & C_F^3\, X^q_{C_F^3} + C_F^2 C_A\, X^q_{C_F^2 C_A} + C_F C_A^2\, X^q_{C_F C_A^2} + C_F^2 N_F\, X^q_{C_F^2 N_F} \\ & + C_F C_A N_F\, X^q_{C_F C_A N_F} + C_F N_F^2\, X^q_{C_F N_F^2} + C_F N_{F,V}\left(\frac{N_c^2 - 4}{N_c}\right) X^q_{C_F N_{F,V}}\end{aligned}$$

$$\begin{aligned}\mathcal{F}_3^g/S_R^3 = \; & C_A^3\, X^g_{C_A^3} + C_A^2 N_F\, X^g_{C_A^2 N_F} + C_A C_F N_F\, X^g_{C_A C_F N_F} + C_F^2 N_F\, X^g_{C_F^2 N_F} \\ & + C_A N_F^2\, X^g_{C_A N_F^2} + C_F N_F^2\, X^g_{C_F N_F^2}\end{aligned}$$

Results II

- Structure of the term $X^q_{C_F^3}$, unexpanded (22 terms, 3 pages), $D = 4 - 2\epsilon$

$$X^q_{C_F^3} = -B_{4,1}\Big(+\frac{489406D^3}{625} - \frac{43304589D^2}{3125} + \frac{615952127D}{7500} + \frac{34015}{4(2D-7)} - \frac{109222498}{75(2D-9)} + \frac{50720}{9(3D-10)} + \frac{6816654}{11(3D-14)}$$
$$+\frac{89728}{25(D-2)} - \frac{12581}{12(D-3)} + \frac{6489724}{15(D-4)} + \frac{19326056092}{7734375(5D-16)} - \frac{7186019918}{78125(5D-18)} + \frac{643118017984}{703125(5D-22)}$$
$$-\frac{1024}{3(D-2)^2} - \frac{779}{12(D-3)^2} + \frac{884312}{5(D-4)^2} + \frac{1187096}{15(D-4)^3} + \frac{745376}{15(D-4)^4} + \frac{91648}{5(D-4)^5} - \frac{53258146831}{562500}\Big)$$
$$+\dots$$
$$-s_{12}^3 A_{9,4}\Big(+\frac{567D^3}{80000} - \frac{125091D^2}{800000} + \frac{1808937D}{1600000} + \frac{4067}{2304(2D-7)} + \frac{232399}{998400(2D-9)}$$
$$-\frac{16}{75(D-2)} - \frac{225}{448(D-3)} + \frac{8388688}{3046875(5D-16)} - \frac{574016}{234375(5D-18)} - \frac{7557808}{4921875(5D-22)} - \frac{38866491}{16000000}\Big)$$

- Structure of the term $X^q_{C_F^3}$, expanded in ϵ

$$S_R^3 X^q_{C_F^3} = -\frac{4}{3\epsilon^6} - \frac{6}{\epsilon^5} + \frac{1}{\epsilon^4}(2\zeta_2 - 25) + \frac{1}{\epsilon^3}\left(-3\zeta_2 + \frac{100\zeta_3}{3} - 83\right) + \frac{1}{\epsilon^2}\left(\frac{213\zeta_2^2}{10} - \frac{77\zeta_2}{2} + 138\zeta_3 - \frac{515}{2}\right)$$
$$+\frac{1}{\epsilon}\left(\frac{1461\zeta_2^2}{20} - \frac{214\zeta_2\zeta_3}{3} - \frac{467\zeta_2}{2} + \frac{2119\zeta_3}{3} + \frac{644\zeta_5}{5} - \frac{9073}{12}\right)$$
$$+\left(-\frac{53675}{24} - \frac{13001\zeta_2}{12} + \frac{12743\zeta_2^2}{40} - \frac{9095\zeta_2^3}{252} + 2669\zeta_3 + 61\zeta_3\zeta_2 - \frac{1826\zeta_3^2}{3} + \frac{4238\zeta_5}{5}\right)$$
$$+\epsilon\left(-\frac{343393}{48} - \frac{11896\zeta_7}{7} + \frac{22349\zeta_5}{3} + \frac{40835\zeta_3}{6} - 1203\zeta_3^2 - \frac{105553\zeta_2}{24} - \frac{7858\zeta_2\zeta_5}{15} + \frac{6083\zeta_2\zeta_3}{6} + \frac{36693\zeta_2^2}{40} - \frac{3931\zeta_2^2\zeta_3}{6} + \frac{321227\zeta_2^3}{840}\right)$$
$$+\epsilon^2\Big(-\frac{2512115}{96} + \frac{4160\zeta_{5,3}}{3} + \frac{45168\zeta_7}{7} + \frac{716537\zeta_5}{15} - \frac{137417\zeta_3}{12} - \frac{33148\zeta_3\zeta_5}{3} + \frac{12749\zeta_3^2}{6} - \frac{797995\zeta_2}{48}$$
$$-\frac{12361\zeta_2\zeta_5}{5} + \frac{18469\zeta_2\zeta_3}{2} + 1985\zeta_2\zeta_3^2 + \frac{7653\zeta_2^2}{80} - \frac{15491\zeta_2^2\zeta_3}{20} + \frac{1147979\zeta_2^3}{240} - \frac{742087277\zeta_2^4}{50400}\Big) + \mathcal{O}(\epsilon^3)$$

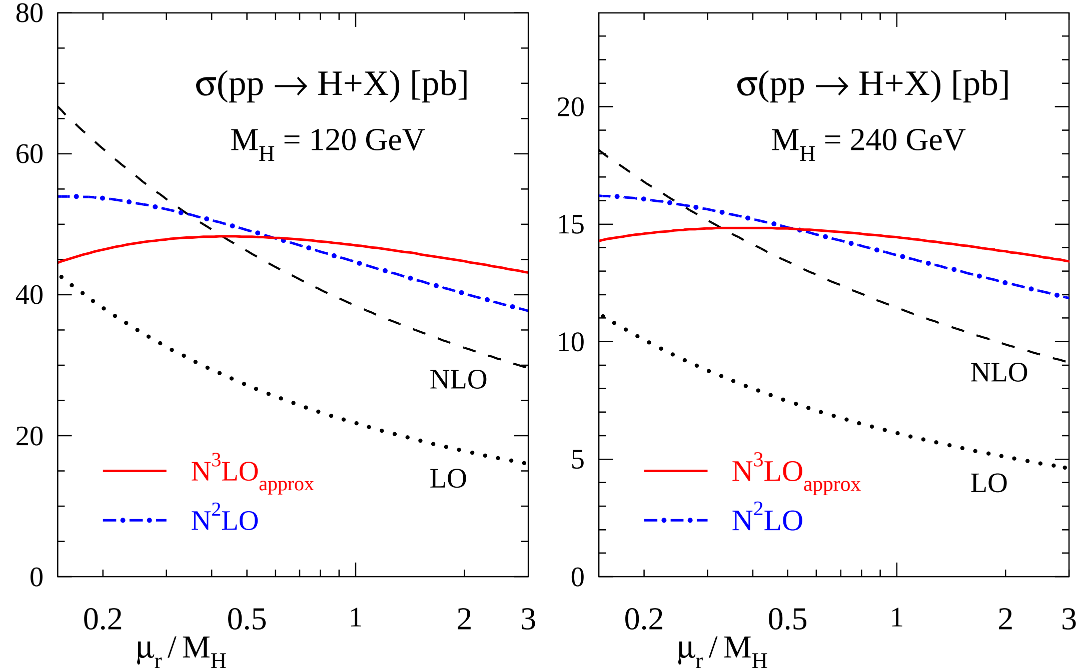
Applications of the form factors

- Both form factors have applications in many collider processes
- Quark form factor
  - Deep-inelastic scattering *[Moch,Vermaseren,Vogt'04-'05]*
  - Drell-Yan process $q\bar{q} \to W^{\pm}\,, Z^0\,, \gamma^*$ *[Hamberg,Matsuura,van Neerven'91]*
  - Two-parton contribution to $e^+e^- \to$ jets
- Gluon form factor
  - Higgs-production: $gg \to H$

[Dawson'91; Djouadi,Graudenz,Spira,Zerwas'91-'93]
[Harlander,Kilgore'01-'02; Catani,de Florian,Grazzini'01]
[Anastasiou,Melnikov'02; Ravindran,Smith,van Neerven'03]
[Anastasiou,Melnikov,Petriello'05; Moch,Vogt'05]

  - N^3LO without finite term
  - $\sigma_{\mathrm{tot.}}$ approximated to $\mathcal{O}(1\%)$ by $\sigma_{\mathrm{tot.}}^{m_t\to\infty}$ up to $M_H \approx 2m_t$

[Krämer,Laenen,Spira '96, see also e.g. Harlander,Ozeren'09]
[Pak,Rogal,Steinhauser'09; Anastasiou,Bucherer, Kunszt'09]



Applications of the form factors

- The quark and gluon form factor are the simplest quantities with IR divergences at higher orders in massless QFT $\Rightarrow$ Analytic result most desirable.
  - Prediction of the IR pole structure of QCD amplitudes

    [Magnea,Sterman'90; Catani'98; Sterman,Tejeda-Yeomans'02; Gehrmann,Gehrmann-de Ridder,Glover'04-'05]

    [Becher,Neubert'09; Gardi,Magnea'09; Dixon'09; Dixon,Gardi,Magnea'09]
  - Relation between form factors, cusp (soft) ADM and quark / gluon collinear ADM ($i = q, g$ and $C_q = C_F$, $C_g = C_A$ for the cusp ADM)

$$
\begin{aligned}
Poles(F_1^i) &= -\frac{C_i \gamma_0^{\text{cusp}}}{2\epsilon^2} + \frac{\gamma_0^i}{\epsilon} \\
Poles(F_2^i) &= \frac{3 C_i \gamma_0^{\text{cusp}} \beta_0}{8\epsilon^3} + \frac{1}{\epsilon^2}\left(-\frac{\beta_0 \gamma_0^i}{2} - \frac{C_i \gamma_1^{\text{cusp}}}{8}\right) + \frac{\gamma_1^i}{2\epsilon} + \frac{\left(F_1^i\right)^2}{2} \\
Poles(F_3^i) &= -\frac{11 \beta_0^2 C_i \gamma_0^{\text{cusp}}}{36\epsilon^4} + \frac{1}{\epsilon^3}\left(\frac{5\beta_0 C_i \gamma_1^{\text{cusp}}}{36} + \frac{\beta_0^2 \gamma_0^i}{3} + \frac{2 C_i \gamma_0^{\text{cusp}} \beta_1}{9}\right) + \frac{1}{\epsilon^2}\left(-\frac{\beta_0 \gamma_1^i}{3} - \frac{C_i \gamma_2^{\text{cusp}}}{18} - \frac{\beta_1 \gamma_0^i}{3}\right) + \frac{\gamma_2^i}{3\epsilon} - \frac{\left(F_1^i\right)^3}{3} + F_2^q F_1^q \\
Poles(F_4^i) &= \frac{25 \beta_0^3 C_i \gamma_0^{\text{cusp}}}{96\epsilon^5} - \frac{\beta_0\left(24 \beta_0^2 \gamma_0^i + 13 \beta_0 C_i \gamma_1^{\text{cusp}} + 40 C_i \gamma_0^{\text{cusp}} \beta_1\right)}{96\epsilon^4} + \frac{1}{\epsilon^3}\left(\frac{7\beta_0 C_i \gamma_2^{\text{cusp}}}{96} + \frac{3\beta_1 C_i \gamma_1^{\text{cusp}}}{32} + \frac{\beta_0^2 \gamma_1^i}{4} + \frac{\beta_1 \beta_0 \gamma_0^i}{2}\right. \\
&\quad \left. + \frac{5 C_i \gamma_0^{\text{cusp}} \beta_2}{32}\right) + \frac{1}{\epsilon^2}\left(-\frac{\beta_1 \gamma_1^i}{4} - \frac{C_i \gamma_3^{\text{cusp}}}{32} - \frac{\beta_0 \gamma_2^i}{4} - \frac{\beta_2 \gamma_0^i}{4}\right) + \frac{\gamma_3^i}{4\epsilon} + \frac{\left(F_1^i\right)^4}{4} + \left(F_1^i\right)^2 F_2^i - \frac{\left(F_2^i\right)^2}{2} - F_1^i F_3^i
\end{aligned}
$$

Assume Casimir scaling (universal cusp ADM). Need $\mathcal{O}(\epsilon)$ parts of 3-loop FFs for $\gamma_3^{q,g}$

Applications of the form factors

- Large Sudakov Logs can be resummed using the framework of SCET
- Matching coefficients for Drell-Yan and Higgs production can be obtained from quark and gluon form factor via on-shell matching of QCD onto SCET

$$C^{(q,\,g)}(\alpha_s(\mu^2), s_{12}, \mu^2) = \lim_{\epsilon \to 0} Z^{-1}{}_{(q,\,g)}(\alpha_s(\mu^2), \epsilon, s_{12}, \mu)\; F^{(q,\,g)}(\alpha_s(\mu^2), \epsilon, s_{12}, \mu^2)$$

- The matching coefficients have the perturbative expansion

$$C^{(q,\,g)}(\alpha_s(\mu^2), s_{12}, \mu^2) = 1 + \sum_{n=1}^{\infty} \left(\frac{\alpha_s(\mu^2)}{4\pi} \right)^n C_n^{(q,\,g)}(s_{12}, \mu^2)$$

- Numerically, for $s_{12} = -\mu^2$, $N_F = 5$ and with $\widetilde{\alpha}_s = \alpha_s(\mu^2)/(4\pi)$

$$\begin{aligned} C^{(q)} &= 1 - 8.473\,\widetilde{\alpha}_s - 48.61\,\widetilde{\alpha}_s^2 - 1390\,\widetilde{\alpha}_s^3 \stackrel{\mu=M_Z}{=} 1 - 0.080 - 0.004 - 0.001 \\ C^{(g)} &= 1 + 4.935\,\widetilde{\alpha}_s - 24.04\,\widetilde{\alpha}_s^2 - 4066\,\widetilde{\alpha}_s^3 \stackrel{\mu=M_Z}{=} 1 + 0.047 - 0.002 - 0.003 \end{aligned}$$

Conclusion

- We computed the quark and gluon form factors to three loops in massless QCD
- Calculation requires dedicated computer algebra tools for generation, reduction, and computation of master integrals
- Result is given as linear combination of 22 master integrals
- The three-loop result is also available through to $\mathcal{O}(\epsilon^2)$
- Together with $\mathcal{O}(\epsilon^6)$ of one- and $\mathcal{O}(\epsilon^4)$ of two-loop form factors, the stage is set for the four-loop calculation
- Many applications, of which we discussed
  - infrared pole structure of QCD amplitudes
  - matching from QCD onto SCET

Backup slides

More applications of the form factors

- Determination of resummation coefficients

  [Collins,Soper,Sterman'84-'85; Magnea'00; Moch,Vermaseren,Vogt'05]

- Check of exponential ansatz for planar n-point MHV amplitudes in $N = 4$ Super-Yang-Mills *[Anastastiou,Bern,Dixon,Kosower'03; Bern,Dixon,Smirnov'05]*

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon)\, M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right]$$

  - $M_n^{(1)}(\epsilon)$: one-loop amplitude, exact in ϵ. $a = \frac{N_c\, \alpha_s}{2\pi} (4\pi e^{-\gamma_E})^\epsilon$
  - $f^{(l)}(\epsilon) = f_0^{(l)} + f_1^{(l)}\, \epsilon + f_2^{(l)}\, \epsilon^2$
  - $C^{(l)}$ independent of n,   and $E_n^{(l)}(\epsilon = 0) = 0\,.$