Symmetry theories from string theory

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Based on

- 1908.08027 with B. Heidenreich and D. Regalado,
- 2005.12831 with F. Albertini, M. Del Zotto and S. Hosseini,
- 2112.02092 with F. Apruzzi, F. Bonetti, S. Hosseini and S. Schäfer-Nameki.

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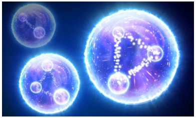

Simons Collaboration on Global Categorical Symmetries

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- We might not have a Lagrangian!

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Symmetries are *categorical*

The symmetries and anomalies of d-dimensional theories are encoded in a $(d+1)$-dimensional topological field theory.

In this talk I would like to:

- Motivate this answer.
- Identify these TFTs in some simple M-theory examples.

Discrete symmetries: an apology

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But there are virtues to discrete symmetries too: they can have much more interesting behaviour. For instance: [Córdova, Dumitrescu, Intriligator '16] show that there are no continuous 2-form symmetries in 5d or 6d SCFTs, while many examples of theories with discrete 2-form symmetries are known by now.

Anomalies

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The canonical example is the theory of a Weyl fermion in four dimensions charged under a $U(1)$ gauge symmetry

$$\mathcal{L} = \frac{1}{2g} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \psi^\dagger (i\partial_\mu - A_\mu) \sigma^\mu \psi$$

which looks fine classically, but is inconsistent quantum-mechanically.

A new approach to anomalies

One concise way to state the problem is that it might not be possible to define the phase of the partition function in a well defined way, as a function of the background fields modulo gauge invariance:

$$Z[A^g] = e^{i\mathcal{A}(A,g)} Z[A] \, .$$

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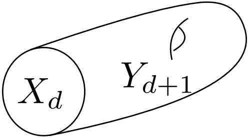
Recent developments [Dai, Freed '94], [Witten '15] have shed new light on this old topic.

These recent developments are geared towards condensed matter, but there are also interesting consequences for high energy physics.

The Dai-Freed viewpoint on anomalies

Consider the case that your space-time X_d is the boundary of some manifold Y_{d+1}, over which all the relevant structures on X_d extend.



We define the path integral of a fermion ψ on X_d as [Dai, Freed '04]

$$Z_\psi = |Z_\psi| e^{-2\pi i\, \eta(\mathcal{D}_{Y_{d+1}})}$$

with

$$\eta(\mathcal{D}_{Y_{d+1}}) = \frac{\dim\ker \mathcal{D}_{Y_{d+1}} + \sum_{\lambda\neq 0} \mathrm{sign}(\lambda)}{2}\,.$$

[*] For the experts, this is the same η that appears in the APS index theorem.

Why is this prescription useful

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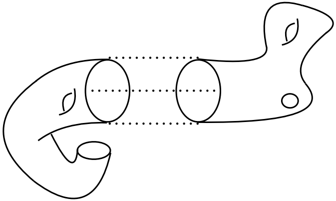
Nevertheless, it has very nice properties: if we change the orientation of the manifold the phase of the partition function changes sign:

$$e^{2\pi i\,\eta(\mathcal{D}_A)} = e^{-2\pi i\,\eta(\mathcal{D}_{\overline{A}})}$$

and it is "local", in the sense that η behaves nicely under gluing:

$$e^{2\pi i \eta(\mathcal{D}_A)} e^{2\pi i\,\eta(\mathcal{D}_B)} = e^{2\pi i\,\eta(\mathcal{D}_{A+B})}$$



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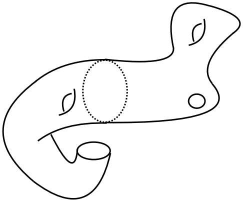

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Anomalies, in this language, come from situations in which the phase of the partition function depends on the choice of Y_{d+1}:

$$e^{-2\pi i\,\eta(\mathcal{D}_{Y_{d+1}})} \neq e^{-2\pi i\,\eta(\mathcal{D}_{Y'_{d+1}})} \tag{1}$$

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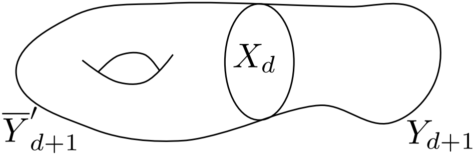
even if $\partial Y_{d+1} = \partial Y'_{d+1} = X_d$.

Gluing Y_{d+1} and $\overline{Y}'_{d+1}$ over X_d to form the closed manifold W_{d+1}, we find that the partition function is well defined as a function of the fields on X_d only if on every such W_{d+1}

$$e^{-2\pi i\,\eta(\mathcal{D}_{W_{d+1}})} = e^{-2\pi i\,\eta(\mathcal{D}_{Y_{d+1}})}/e^{-2\pi i\,\eta(\mathcal{D}_{Y'_{d+1}})} = 1 \tag{2}$$



The Dai-Freed viewpoint on anomalies

The theory with partition function

$$Z^{\mathcal{A}}(Y_{d+1}, A) = e^{2\pi i\, \eta(\mathcal{D}_A)}$$

is an example of a topological field theory in $(d+1)$-dimensions, known in this context as the *anomaly theory*.

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So when talking about anomalies, it is very natural to consider topological theories in one dimension higher. Later on I will give examples of anomaly theories for 1-form symmetries.

Higher form symmetries

Classifying $\mathcal{N} = 4$ theories

Known $\mathcal{N} = 4$ theories in four dimensions are classified by a choice of gauge group G (with algebra $\mathfrak{g}$), and some discrete θ angles. [Aharony, Seiberg, Tachikawa '13]

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One can distinguish the different global forms by studying the partition function on four-manifolds $\mathcal{M}_4$ with $H^2(\mathcal{M}_4, C) \neq 0$, or by studying the properties and correlators of extended operators.

Classifying $\mathcal{N} = 4$ theories

When computing the partition function of a $\mathcal{N} = 4$ theory on some closed manifold $\mathcal{M}_4$ we do:

$$Z_{\mathcal{N}=4}[\mathcal{M}_4, \cdot] = \sum_{[F]} \int [DA][D\lambda][D\Phi] e^{-S_{\mathfrak{g}}[\tau, A, \lambda, \Phi]}$$

where $[F]$ denotes the homotopy class of the bundle over $\mathcal{M}_4$. Which classes $[F]$ should we include in the sum?

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There is a genuine choice to be made here.

Classifying $\mathcal{N} = 4$ theories

$SU(2)$ vs. $SO(3)$

I will briefly illustrate this in the case $\mathfrak{g} = \mathfrak{su}(2)$. There are two Lie groups with algebra $\mathfrak{su}(2)$: $SU(2)$ and $SO(3) = PSU(2) = SU(2)/\mathbb{Z}_2$.

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Every $SU(2)$ bundle can be interpreted as a $SO(3)$ bundle, but in sufficiently complicated manifolds there are $SO(3)$ bundles that cannot be understood as $SU(2)$ bundles.

Classifying $\mathcal{N} = 4$ theories

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The obstruction to understanding $SO(3)$ bundles as $SU(2)$ bundles is encoded by elements $w_2 \in H^2(\mathcal{M}_4; \mathbb{Z}_2)$, known as *Stiefel-Whitney classes*. If a $SO(3)$ bundle E has $w_2(E) \neq 0$ then it cannot be lifted to $SU(2)$.

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In constructing the partition function, we sum over all $SO(3)$ bundles, including those with $w_2 \neq 0$ (the "$SO(3)$" theory), or only over those with $w_2 = 0$ (the "$SU(2)$" theory):

$$Z_{SO(3)}[\mathcal{M}_4, \cdot] = \sum_{[F]\in SO(3)} \int [DA][D\lambda][D\Phi] e^{-S_{\mathfrak{g}}[\tau, A, \lambda, \Phi]}$$

$$Z_{SU(2)}[\mathcal{M}_4, \cdot] = \sum_{[F]\in SU(2)} \int [DA][D\lambda][D\Phi] e^{-S_{\mathfrak{g}}[\tau, A, \lambda, \Phi]}$$

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- In the $SU(N)/\mathbb{Z}_N$ theory we gauge this electric 1-form symmetry by summing over all backgrounds for the symmetry. A magnetic 1-form symmetry counting 't Hooft loops emerges.

Holography and global structure

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Answered in [Witten '98]. The key insight is that we view the possible 4-dimensional theories as states in the Hilbert space of a 5-dimensional topological "bulk" theory, taking the radial direction as "time". [Friedan, Shenker '87], [Verlinde '88], [Moore, Seiberg '88], [Witten '89], ..., [Witten '98], ..., [Belov, Moore], ...

Quantization of the bulk TQFT

(Following [Witten '98])

The reduction of IIB on S^5 gives an effective action

$$L_{\mathsf{CS}} = \frac{N}{2\pi i} \int_{X_5} B_2 \wedge dC_2 \,.$$

The equations of motion are

$$dB_2 = dC_2 = 0$$

and B_2, C_2 are canonically conjugate ($B_2 = C_2 = 0$ is disallowed!):

$$[B_{ij}(x), C_{kl}(y)] = -\frac{2\pi i}{N} \epsilon_{ijkl} \delta^4(x-y) \,.$$

Quantization of the bulk TQFT

(Following [Witten '98])

In order to specify the boundary conditions, in addition to specifying the vevs of local gauge invariant operators, we need to specify

$$\alpha(R) = \exp\left(i\int_R B_2\right) \quad ; \quad \beta(S) = \exp\left(i\int_S C_2\right) \tag{3}$$

for any $R, S \subset \mathcal{M}_4$ near the boundary, $X_5 \approx \mathbb{R} \times \mathcal{M}_4$.

They do not commute:

$$\alpha(R)\beta(S) = \beta(S)\alpha(R)\exp\left(\frac{2\pi i}{N} R\cdot S\right) . \tag{4}$$

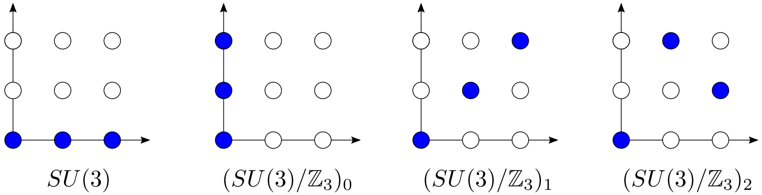
So a state cannot be a simultaneous eigenstate of both when $R \cdot S \neq 0$ mod N. In terms of boundary conditions, we cannot fix Dirichlet boundary conditions for both B_2 and C_2 simultaneously.

Quantization of the bulk TQFT

(Following [Witten '98])

The different global forms for $\mathfrak{su}(N)$ are then determined by the different boundary values of the B_2 and C_2 fields. In an appropriate duality frame:

- $\beta(R) = 1$ for all $R \mapsto SU(N)$.
- $\alpha(R) = 1$ for all $R \mapsto (SU(N)/\mathbb{Z}_N)_0$.
- $\alpha(R)\beta(R)^k = 1$ for all $R \mapsto (SU(N)/\mathbb{Z}_N)_k$.
- ...



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The symmetry theory

Back to geometric engineering

Consider, as an example of a theory that cannot be understood holographically, M-theory on $\mathbb{C}^2/\Gamma$. This gives rise to 7d SYM with gauge algebra $\mathfrak{g}_\Gamma$. The 1-form symmetry group of G_Γ (the simply connected form) is its centre:

$\Gamma \subset SU(2)$	$\mathfrak{g}_\Gamma$	G_Γ	$Z(G_\Gamma)$
$\mathbb{Z}_N$	$\mathfrak{su}(N)$	$SU(N)$	$\mathbb{Z}_N$
Binary dihedral $\mathrm{Dic}_{(2k-2)}$	$\mathfrak{so}(4k)$	$Spin(4k)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
Binary dihedral $\mathrm{Dic}_{(2k-1)}$	$\mathfrak{so}(4k+2)$	$Spin(4k+2)$	$\mathbb{Z}_4$
Binary tetrahedral $2T$	$\mathfrak{e}_6$	E_6	$\mathbb{Z}_3$
Binary octahedral $2O$	$\mathfrak{e}_7$	E_7	$\mathbb{Z}_2$
Binary icosahedral $2I$	$\mathfrak{e}_8$	E_8	1

Other global forms are possible, for instance $SU(N)/\mathbb{Z}_N$, which has a magnetic 4-form symmetry.

Where is the data for the global form?

The form of the singularity does not fully fix the global form of the gauge group, only the algebra. Either:

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In [IGE, Heidenreich, Regalado '19] we argued[1] that (like in holography) it is the second option that is realised: the choice of global form for the gauge group is encoded in a choice of boundary conditions (at infinity) for the supergravity fields, and all possible global forms can be obtained in this way. (Related work: [Del Zotto, Heckman, Park, Rudelius '15], [Morrison, Schäfer-Nameki, Willett '20], [Albertini, Del Zotto, IGE, Hosseini '20], [Closset, Schäfer-Nameki, Wang '20], [Del Zotto, IGE, Hosseini '20], . . .)

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Non-commutativity of fluxes in M-theory

Let us put M-theory on $\mathcal{M}_{11} = \mathcal{N}_{10} \times \mathbb{R}$. We will try to understand the Hilbert space $\mathcal{H}(\mathcal{N}_{10})$, or more precisely its grading by flux. This was done in [Freed, Moore, Segal '06].

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M-theory contains 3-form gauge fields C_3. The magnetic charge is measured by the topological class of C_3. To measure the electric charge, recall that in the Hamiltonian formulation of the theory the canonical momentum Π_{C_3} conjugate to C_3 is $\star G_4$. This is what we integrate to measure the electric charge.

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$$\psi(C_3 + \lambda) = e^{2\pi i \int_{\mathcal{N}_{10}} Q_e \lambda} \psi(C_3)$$

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So we cannot simultaneously measure electric and magnetic charges, if there are flat topologically non-trivial λ. This is the case iff $\mathrm{Tor}\, H^4(\mathcal{N}_{10}) \neq 0$.

Non-commutativity of fluxes in M-theory

This can be restated in terms of the flux operators, as follows: for every $\sigma \in \mathrm{Tor}\, H_6(\mathcal{N}_{10};\mathbb{Z}) = \mathrm{Tor}\, H^4(\mathcal{N}_{10};\mathbb{Z})$ there is a unitary flux operator Φ_σ. Similarly for any
$\sigma' \in \mathrm{Tor}(H_3(\mathcal{N}_{10};\mathbb{Z})) = \mathrm{Tor}\, H^7(\mathcal{N}_{10};\mathbb{Z})$.

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These operators in general do not commute:

$$\Phi_\sigma \Phi_{\sigma'} = e^{2\pi i\, \mathsf{L}(\sigma,\sigma')} \Phi_{\sigma'} \Phi_\sigma$$

where $\mathsf{L}(\sigma,\sigma')$ is the linking pairing on $\mathcal{N}_{10}$: choose $n \in \mathbb{Z}$ such that $n\sigma = \partial D$. Then

$$\mathsf{L}(\sigma,\sigma') = \frac{1}{n} D \cdot \sigma' \mod 1\,.$$

Non-commutativity of fluxes in M-theory

The pairing $\mathsf{L}(\cdot,\cdot)$ is *perfect*, which implies that if $\mathrm{Tor}(H_3(\mathcal{N}_{10};\mathbb{Z})) = \mathrm{Tor}(H_6(\mathcal{N}_{10};\mathbb{Z})) \neq 0$, then for each $\sigma \neq 0$ there is some σ' such that $\mathsf{L}(\sigma,\sigma') \neq 0$, and thus

$$\Phi_\sigma \Phi_{\sigma'} = e^{2\pi i\, \mathsf{L}(\sigma,\sigma')} \Phi_{\sigma'} \Phi_\sigma \neq \Phi_{\sigma'} \Phi_\sigma\,.$$

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$$\Phi_\sigma \Phi_{\sigma'} = e^{2\pi i\, \mathsf{L}(\sigma,\sigma')} \Phi_{\sigma'} \Phi_\sigma \neq \Phi_{\sigma'} \Phi_\sigma \,.$$

What this all implies, it that whenever $\mathrm{Tor}(H_3(\mathcal{N}_{10};\mathbb{Z})) \neq 0$ it is not possible to simultaneously diagonalize all Φ_σ. In particular, it is not consistent to take the simple "fluxless" choice $\Phi_\sigma = 1$ for all σ. We need to turn on *some* flux at infinity!

Maximal isotropic subspaces

Despite the perhaps unfamiliar setting, the final algebraic structure is the same as in holography: we have a Hilbert space, and a set of non-commuting operators acting on it.

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We can specify a state in the Hilbert space as usual: by choosing a maximal subspace $\mathcal{I} \subset \mathrm{Tor}(H_3(\mathcal{N}_{10});\mathbb{Z}) \times \mathrm{Tor}(H_6(\mathcal{N}_{10});\mathbb{Z})$ such that the corresponding group of operators $\{\Phi_x\}$ for $x \in \mathcal{I}$ is abelian, and imposing that

$$\Phi_x |0; L\rangle = |0; L\rangle \qquad \forall x \in \mathcal{I}$$

In our M-theory setting, this corresponds to setting to zero on the boundary as many fluxes as possible.

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In our M-theory setting, this corresponds to setting to zero on the boundary as many fluxes as possible.

It is an easy calculation that the set of choices agrees with the expectations from field theory ([Aharony, Seiberg, Tachikawa '13]) in all cases.

Back to M-theory on $\mathbb{C}^2/\Gamma$

We want to consider M-theory on a space $\mathcal{M}_{11} = \mathbb{C}^2/\Gamma \times \mathcal{M}_7$ with Γ a discrete subgroup of $SU(2)$. Let us apply our methods to classify the space of possible theories for a fixed $\mathfrak{g}$.

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We have that $\mathbb{C}^2/\Gamma$ is a cone over S^3/Γ, so in order to understand the boundary conditions at infinity we want to quantize the flux sector of M-theory on $\mathbb{R} \times S^3/\Gamma \times \mathcal{M}_7$.

Back to M-theory on $\mathbb{C}^2/\Gamma$

Γ acts freely on S^3, so $\pi_1(S^3/\Gamma) = \Gamma$. By Hurewicz's theorem

$$H_1(S^3/\Gamma) = \frac{\pi_1(S^3/\Gamma)}{[\pi_1(S^3/\Gamma), \pi_1(S^3/\Gamma)]} = \Gamma^{\mathsf{ab}}\,.$$

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The group Γ^{ab} is easy to determine:

$\Gamma \subset SU(2)$	$\mathfrak{g}_\Gamma$	Γ^{ab}
$\mathbb{Z}_N$	A_{N-1}	$\mathbb{Z}_N$
Binary dihedral $\mathrm{Dic}_{(2k-2)}$	D_{2k}	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
Binary dihedral $\mathrm{Dic}_{(2k-1)}$	D_{2k+1}	$\mathbb{Z}_4$
Binary tetrahedral $2T$	E_6	$\mathbb{Z}_3$
Binary octahedral $2O$	E_7	$\mathbb{Z}_2$
Binary icosahedral $2I$	E_8	1

(Notice that $\Gamma^{\mathsf{ab}} = Z(G_\Gamma)$, with G_Γ the simply connected Lie group with algebra g_Γ.)

Back to M-theory on $\mathbb{C}^2/\Gamma$

From here

$$H_*(S^3/\Gamma) = \{\mathbb{Z}, \Gamma^{\mathsf{ab}}, 0, \mathbb{Z}\}\,.$$

To make my life easier I will assume that $\mathcal{M}_7$ is closed and has no torsion in homology. Then Künneth's formula implies

$$\begin{aligned}\mathrm{Tor}(H_3(\mathcal{M}_7 \times S^3/\Gamma)) &= H_2(\mathcal{M}_7) \otimes H_1(S^3/\Gamma) = H_2(\mathcal{M}_7) \otimes \Gamma^{\mathsf{ab}} \\ &= H_2(\mathcal{M}_7; \Gamma^{\mathsf{ab}})\,.\end{aligned}$$

and similarly

$$\mathrm{Tor}(H_6(\mathcal{M}_7 \times S^3/\Gamma)) = H_5(\mathcal{M}_7; \Gamma^{\mathsf{ab}})\,.$$

Given elements $\sigma_a = a \otimes \ell_a$, $\sigma_b = b \otimes \ell_b$, we have the linking form

$$\mathsf{L}(\sigma_a, \sigma_b) = (a \cdot b)\, \mathsf{L}_{S^3/\Gamma}(\ell_a, \ell_b)\,.$$

Back to M-theory on $\mathbb{C}^2/\Gamma$

It is not difficult to compute the linking form on S^3/Γ, we find:

Γ	G_Γ	Γ^{ab}	L_Γ
$\mathbb{Z}_N$	$SU(N)$	$\mathbb{Z}_N$	$\frac{1}{N}$
$\mathrm{Dic}_{(4N-2)}$	$\mathrm{Spin}(8N)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathrm{Dic}_{(4N-1)}$	$\mathrm{Spin}(8N+2)$	$\mathbb{Z}_4$	$\frac{3}{4}$
$\mathrm{Dic}_{(4N)}$	$\mathrm{Spin}(8N+4)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\mathrm{Dic}_{(4N+1)}$	$\mathrm{Spin}(8N+6)$	$\mathbb{Z}_4$	$\frac{1}{4}$
$2T$	E_6	$\mathbb{Z}_3$	$\frac{2}{3}$
$2O$	E_7	$\mathbb{Z}_2$	$\frac{1}{2}$
$2I$	E_8	0	0

Back to M-theory on $\mathbb{C}^2/\Gamma$

Classification

The possible global forms of the $d = 7$ theories on $\mathcal{M}_7$ are given by maximal commuting subspaces of $H_2(\mathcal{M}_7; \Gamma^{\mathsf{ab}}) \times H_5(\mathcal{M}_7; \Gamma^{\mathsf{ab}})$, with commutators as above.

This result agrees with what one obtains from applying the ideas in [Gaiotto, Moore, Neitzke '10], [Aharony, Seiberg, Tachikawa '13].

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This result agrees with what one obtains from applying the ideas in [Gaiotto, Moore, Neitzke '10], [Aharony, Seiberg, Tachikawa '13].

An alternative derivation of this result can be obtained by thinking about screening of line operators, closely following [Aharony, Seiberg, Tachikawa '13]. This was done in geometric language in [Del Zotto, Heckman, Park, Rudelius '15], where they introduce the *defect group*, which in this case is

$$\mathbb{D} = \frac{H_2(\mathbb{C}^2/\Gamma, S^3/\Gamma)}{H_2(\mathbb{C}^2/\Gamma)} \times \frac{H_2(\mathbb{C}^2/\Gamma, S^3/\Gamma)}{H_2(\mathbb{C}^2/\Gamma)}$$

It is easy to show that $H_2(\mathbb{C}^2/\Gamma, S^3/\Gamma)/H_2(\mathbb{C}^2/\Gamma) = H_1(S^3/\Gamma) = \Gamma^{\mathsf{ab}}$.

The symmetry theory

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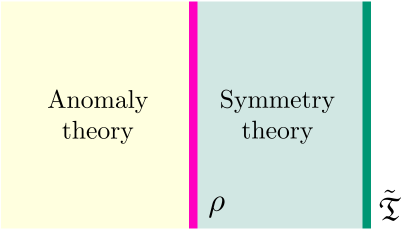
- We cannot speak of anomalies until we have chosen the symmetries.

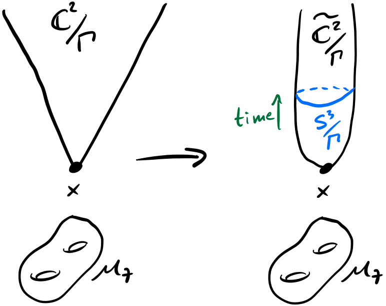
A picture (suggested by D. S. Freed) makes this precise



where $\tilde{\mathfrak{T}}$ encodes the (relative [Freed, Teleman '12]) theory of local dynamics, and ρ is a gapped interface encoding the choice of global form.

How symmetry theories appear in string theory

The derivation in [IGE, Heidenreich, Regalado '19] uses a modified asymptotic structure.



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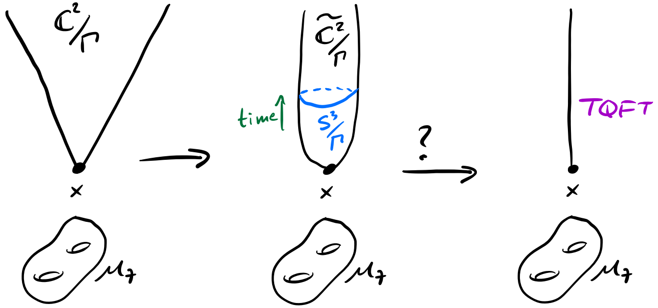

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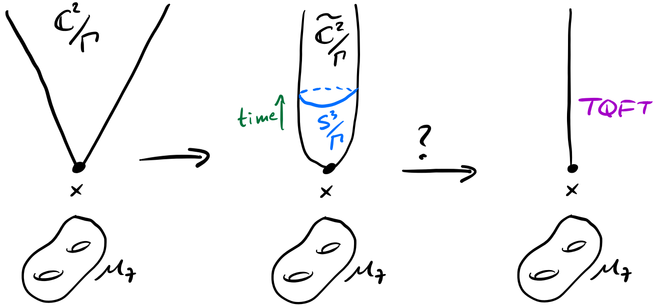
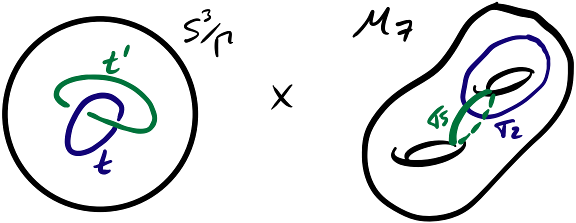


In this picture the boundary conditions at infinity that we need to specify in string theory correspond to ρ, so the anomaly theory itself is not visible.

The BF theory

In the full theory on $S^3/\Gamma \times X^8$ there are non-commuting flux operators [Freed, Moore, Segal '06] wrapping $t \times \sigma_2$ and $t' \times \sigma_5$, with $t, t' \in H_1(S^3/\Gamma) = \Gamma^{\text{ab}}$ and $\sigma_i \in H_i(X^8)$. Their commutation relations (on a spatial slice $\mathcal{M}_7$ of X^8) are

$$\Phi(t \times \sigma_2)\Phi(t' \times \sigma_5) = e^{2\pi i L(t,t')\sigma_2 \cdot \sigma_5}\Phi(t' \times \sigma_5)\Phi(t \times \sigma_2)\,.$$



The BF theory (continued)

Fix $\Gamma = \mathbb{Z}_N$ for concreteness. Then from the point of view of X_8 we have $\mathbb{Z}_N$ 2-surface operators and 5-surface operators whose relative phase goes with the intersection number divided by N. This can be represented as a

$$S_{\text{top}} = N \int_{X_8} B_2 \wedge dC_5$$

topological action (as in [Witten '98]).

Mixed anomalies

(2112.02092, with F. Apruzzi, F. Bonetti, S. Hosseini and S. Schäfer-Nameki)

The 7d theory, in addition to the 1-form and/or 4-form symmetries acting on Wilson lines / 't Hooft surfaces, has a $U(1)_I$ continuous 2-form symmetry acting on instanton surfaces.

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$$S_{\text{anomaly}} = \int_{X_8} dC_I^{(3)} \wedge r_{\mathfrak{g}} \frac{\mathcal{P}(B_2)}{2}$$

with $r_{\mathfrak{g}}\mathcal{P}(B_2)/2$ the fractional instanton number in the presence of a background for the 1-form symmetry, and $C_I^{(3)}$ the background for the instanton 1-form symmetry.

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This can be derived by "reducing" $\int_{\mathcal{M}_{11}} C_3 G_4 G_4 + C_3 X_8$ on S^3/Γ, *keeping track of the torsion sector*. (See also recent work by [Cvetič, Dierigl, Lin, Zhang '21].)

Differential cohomology

KK reductions beyond de Rham

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We can make sense of this by using **differential cohomology** (aka Cheeger-Simons cohomology or Deligne cohomology), a way of packing differential forms and cohomology classes together.

Differential cohomology

The degree d differential cohomology group $\breve{H}^d(\mathcal{M})$ fits into:

$$
\begin{tikzcd}
 & & \mathrm{Tor}\,H^p(\mathcal{M};\mathbb{Z}) \arrow[dr] & & \\
 & H^{p-1}(\mathcal{M};\mathbb{R}/\mathbb{Z}) \arrow[ur] \arrow[rr, "-\beta"] \arrow[dr, hook, "i"] & & H^p(\mathcal{M};\mathbb{Z}) \arrow[dr, "\varrho"] & \\
\frac{H^{p-1}(\mathcal{M};\mathbb{R})}{H^{p-1}_{\mathrm{Free}}(\mathcal{M};\mathbb{Z})} \arrow[ur] \arrow[dr] & & \breve{H}^p(\mathcal{M}) \arrow[ur, two heads, "I"] \arrow[dr, two heads, "R"] & & H^p_{\mathrm{Free}}(\mathcal{M};\mathbb{Z}) \\
 & \frac{\Omega^{p-1}(\mathcal{M})}{\Omega^{p-1}_{\mathbb{Z}}(\mathcal{M})} \arrow[ur, hook, "\tau"] \arrow[rr, "d_{\mathbb{Z}}"] \arrow[dr, "d"] & & \Omega^p_{\mathbb{Z}}(\mathcal{M}) \arrow[ur, "r"] & \\
 & & d\Omega^{p-1}(\mathcal{M}) \arrow[ur] & &
\end{tikzcd}
$$

and enjoys a product:

$$\breve{H}^p(\mathcal{M}) \star \breve{H}^q(\mathcal{M}) \to \breve{H}^{p+q}(\mathcal{M})\,.$$

Chern-Simons terms

The differential cohomology formulation of the M-theory Chern-Simons term "$C_3 \wedge G_4 \wedge G_4$" is

$$S_{\mathsf{CS}} = -\frac{1}{6}\int_{\mathcal{M}^{11}} \breve{G}_4 \star \breve{G}_4 \star \breve{G}_4 \,.$$

In differential cohomology, for $\breve{x} \in \breve{H}^p(\mathcal{M}^d)$ we have

$$\int_{\mathcal{M}^d} \breve{x} \qquad \in \breve{H}^1(\mathrm{pt}) = \mathbb{R}/\mathbb{Z}\,.$$

So the integral above is not well defined by itself, but it is well known that the whole M-theory action is. [Witten '96]

The differential KK reduction

On $\mathcal{M}^8 \times S^3/\Gamma$ we can expand

$$\breve{G}_4 = \breve{\gamma}_4 \star \breve{1} + \breve{B}_2 \star \breve{t}_2 + \dots$$

with $t_2 \in H^2(S^3/\Gamma) = \Gamma^{\mathsf{ab}}$ and $\breve{t}_2$ a flat representative of t_2.

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Then the reduction contains a term

$$S_{\mathsf{symm}} = \ldots + \left(-\frac{1}{2}\int_{S^3/\Gamma} \breve{t} \star \breve{t}\right) \int_{\mathcal{M}^8} \breve{\gamma}^4 \breve{B}_2^2 \,.$$

The term in parenthesis is the fractional instanton number for the generator of 1-form symmetry background. It is given by the level $-\frac{1}{2}$ spin-Chern-Simons invariant of S^3/Γ evaluated on a flat connection:

$$n_{\mathsf{inst}} = -\frac{1}{2}\int_{S^3/\Gamma} \breve{t} \star \breve{t} \,.$$

This geometrizes field theory results in [Witten '00], [Córdova, Freed, Lam, Seiberg '19], and is easy to generalise to compute anomalies in the space of coupling constants for non-Lagrangian theories.

How to compute

Extending results of [Gordon,Litherland '78], if $\mathcal{M}^d = \partial\mathcal{N}^{d+1}$ and x is torsional, then we can compute

$$\int_{\mathcal{M}^d} \breve{x} \star \breve{y} \star \dots$$

in terms of an intersection product in $\mathcal{N}^{d+1}$. The extra factor of $\frac{1}{2}$ requires the introduction of a quadratic refinement, which in our case amount to choosing $\mathcal{N}^{d+1}$ to be Calabi-Yau, and dividing by 2.

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By means of this formalism we can compute n_{inst} in the previous slide and (for example) the much more subtle anomaly theory in 5d for $SU(p)_q$ [Gukov, Pei, Hsin '20]

$$S^{(5d)}_{\text{anomaly}} = \int_{X_6} dC_I^{(1)} \wedge \frac{p(p-1)}{2\,\text{gcd}(p,q)}\mathcal{P}(B_2) + \frac{qp(p-1)(p-2)}{6\,\text{gcd}(p,q)^3} B_2^3 \,.$$

Applications

Generalised symmetries can be used to understand the dynamics of field theory [Gaiotto, Kapustin, Komargodski, Seiberg '17], . . . , [Komargodski, Ohmori, Roumpedakis, Seifnashri '20], . . .

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$$\text{geometry} \rightarrow \text{Symmetry theory} \rightarrow \text{SCFT}$$

Applications

Generalised symmetries can be used to understand the dynamics of field theory [Gaiotto, Kapustin, Komargodski, Seiberg '17], . . . , [Komargodski, Ohmori, Roumpedakis, Seifnashri '20], . . .

Developing tools to extra symmetry theories from the geometry of string theory might allow us to analyse cases where no Lagrangian in known.

It also suggests a way to refine the SCFT classification problem:

$$\text{geometry} \to \text{Symmetry theory} \to \text{SCFT}$$

We can also try to understand what this philosophy says about the "No global symmetries in Quantum Gravity" expectation in the literature.

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Life becomes more interesting when the QG is non-susy, then we expect bubbles of nothing to start popping up. [IGE, Montero, Sousa, Valenzuela '20]

Conclusions

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String theory provides a beautiful geometrisation of these developments. In some simple examples in 7d and 5d we could derive systematically the symmetry theory from doing dimensional reduction of the M-theory Chern-Simons sector on the link of the singularity. We did not need any Lagrangian information about the theory, only the geometry!

Future directions

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  - We can use it to learn about field theory, but also about string theory itself: the computations are very sensitive to details.
- Gravity is the big question mark. How does it fit in the usual Atiyah-Segal axiomatic framework? [Banerjee, Moore '22]