Status of electroweak computations w.r.t. numerical treatment of Feynman integrals

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Introduction

Many formal successfull studies are available on the market.

- Loop tree duality [Capatti, Hirschi, Pelloni, Ruijl, 2021]
- Unitarity cut techniques [Abreu, Ita, Page,Tschernow,2021]
- pySecDec approach [Long Chen, Heinrich, Jones, Kerner, Klappert, Schlenk, 2021]
- Auxiliary mass flow [Brønnum-Hansen, Melnikov, Quarroz, Chen-YuWang, 2021]
- Solving a system of differential equations numerically [Lee, Smirnov, Smirnov, 2018], [Mandal, Zhao, 2019], [Moriello, 2019], [Bonciani, Del Duca, Frellesvig, Henn, Hidding, Maestri, Moriello, Salvatori, Smirnov, 2019], [Hidding, 2020], [Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020]
- Many computational steps are automated. We are free to pick the individual computer programs and glue them together.
- The glue to connect the individuel tools for the full automation is still missing.

In this talk we demonstrate one possible engineering attempt to automate the differential equations approach.

Electroweak Precision Physics

	Experiment	Theory uncertainty	Main source
$M_W\,[\mathrm{MeV}]$	80385 ± 15	4	$N_f^2\alpha^3$, $N_f\alpha^2\alpha_\mathrm{s}$
$\sin^2\theta^\mathrm{l}_\mathrm{eff}\,[10^{-5}]$	23153 ± 16	4.5	$N_f^2\alpha^3$, $N_f\alpha^2\alpha_\mathrm{s}$
$\Gamma_Z\,[\mathrm{MeV}]$	2495.2 ± 2.3	0.4	$N_f^2\alpha^3$, $N_f\alpha^2\alpha_\mathrm{s}$, $\alpha\alpha_\mathrm{s}^2$
$\sigma^0_\mathrm{had}\,[\mathrm{pb}]$	41540 ± 37	6	$N_f^2\alpha^3$, $N_f\alpha^2\alpha_\mathrm{s}$
$R_b = \Gamma^b_Z/\Gamma^\mathrm{had}_Z\,[10^{-5}]$	21629 ± 66	15	$N_f^2\alpha^3$, $N_f\alpha^2\alpha_\mathrm{s}$

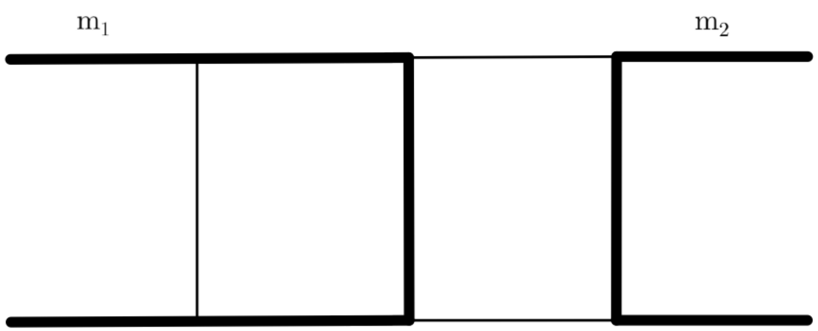
- The number of Z-bonos collected at LEP is 1.7×10^7
- Many pseudo observables are determined with high precision
- Present theoretical predictions are accurate enough to fullfill experimental demands

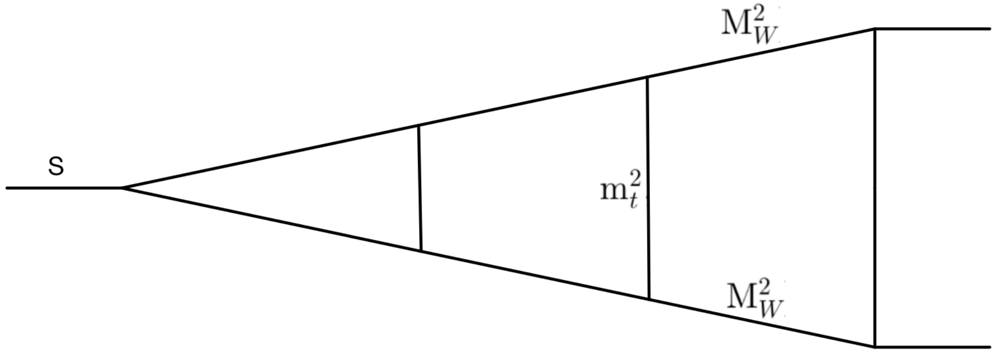
Overview Experiment Future

	Experiment uncertainty			Theory uncertainty	
	ILC	**CEPC**	**FCC-ee**	**Current**	**Future**
$M_W\,[\mathrm{MeV}]$	3-4	3	1	4	1
$\sin^2\theta^{\mathrm{l}}_{\mathrm{eff}}\,[10^{-5}]$	1	2.3	0.6	4.5	1.5
$\Gamma_Z\,[\mathrm{MeV}]$	0.8	0.5	0.1	0.4	0.2
$R_b[10^{-5}]$	14	17	6	15	7

- The concepts for the new experiments will have new demands to the theoreticle predictions
- **FCC-ee** will generate 5×10^{12} Z-bosons which is 10^5 more than during the LEP times
- The projection to the theory errors in the future assumes that the missing corrections $\alpha\alpha_{\mathrm{s}}^2$, $N_f^2\alpha^3$, $N_f\alpha^2\alpha_{\mathrm{s}}$ will become available

Samples of two-loop and three-loop Feynman integrals





- We project all Feynman integrals to scalar integrals
- We need to compute all Feynman integrals only up to the finite order in $\epsilon = (4-d)/2$, d the space time dimension
- At the end we want to be able to compute all three-loop Feynman integrals appearing in e.g. the $Z\bar{b}b$ vertex numerically with at least eight significant digits of accuracy in physical kinematic regions

Parameters

- The integrals depend on up to four dimensionless parameters

$$\left\{ \frac{M_H^2}{M_Z^2}, \frac{M_W^2}{M_Z^2}, \frac{m_t^2}{M_Z^2}, \frac{(s+i\delta)}{M_Z^2} \right\} \Big|_{s=M_Z^2} \tag{1}$$

- Many of them contain ultraviolet and infrared singularities, even though the divergences cancel in the final result
- Computations involve $\mathcal{O}(100)$ master integrals

Feynman integral

$$T(a_1, \ldots, a_N) = \int \left(\prod_{i=1}^{L} \mathrm{d}^d \ell_i \right) \frac{1}{P_1^{a_1} P_2^{a_2} \cdots P_N^{a_N}}, \quad N = \frac{L}{2}(L+1) + LE \tag{2}$$

- $P_j = q_j^2 - m_j^2$, $j = 1, \ldots, N$, are the inverse propagators
- The momenta q_j are linear combinations of the loop momenta ℓ_i, $i = 1, \ldots, L$ for an L-loop integral, and external momenta p_k, $k = 1, \ldots, E$ for $E + 1$ external legs
- The m_j are the propagator masses
- The a_j are the (integer) propagator powers

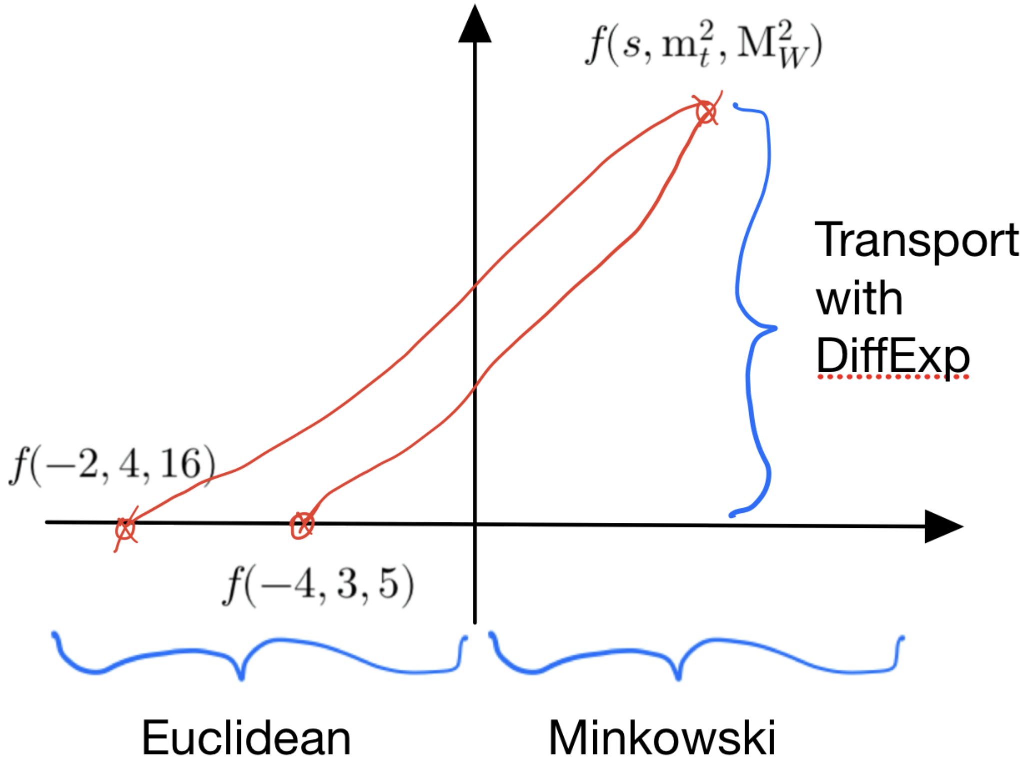
Differential Equations

- Each family of Feynman integrals $T(a_1, \ldots, a_N)$ may be charcterized through a system of differential equations [Kotikov, 1991], [Remiddi, 1997][Gehrmann, Remiddi, 2000]
$$\partial_{s_i} \vec{f} = M_{s_i}(s_i, \epsilon) \vec{f} \tag{3}$$
and a set of master integrals $\vec{f}$
- We take derivatives on kinematic invariants and masses denoted as s_i in $\vec{f}$
- We express these derivatives again as a linear combination in terms of the same master integrals with the help of integration-by-parts identities [Chetyrkin, Tkachov, 1981]

Caesar: Blueprint for Multi Loop Feynman Integral Computation

- Main developers of Caesar Martijn Hidding and J.U.
- Caesar has to interface Kira [Klappert, Lange, Maierhöfer, Usovitsch, 2020], Reduze 2 [Von Manteuffel, Studerus, 2012], pySecDec [Borowka et al., 2018] and DiffExp [Martijn Hidding, 2021].
- For all the programs we have prepared templates which are filled automatically for each problem individually
- Kira is the backbone / major bottleneck of the Caesar implementation - solves linear system of equations
- Reduze 2 - finds candidates for a finite basis of master integrals
- pySecDec computes the finite integrals in Euclidean regions - boundary terms
- DiffExp transports the Euclidean point to an arbitrary physical point with well understood propagation of errors
- Repeat the chain of tools for different Euclidean points to get an error estimate

One Possible Application of Caesar

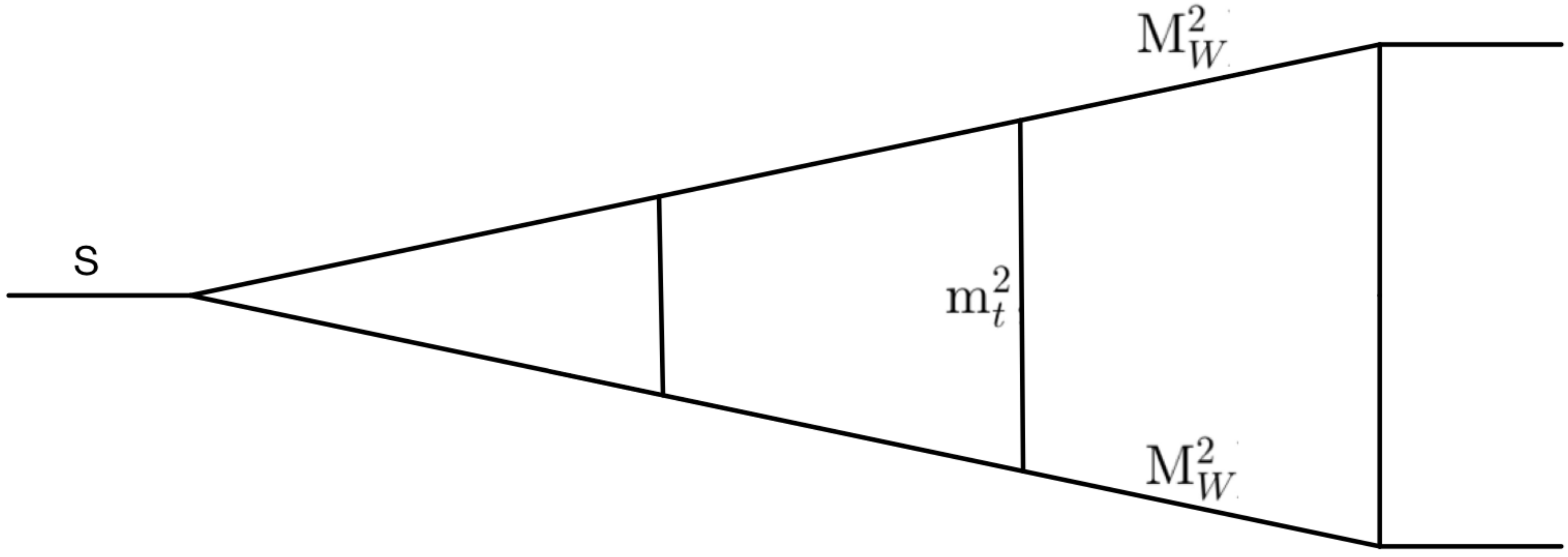


- All master integrals $f_i(\ldots)$ are finite integrals (Reduze)
- Master integrals $f_i(\ldots)$ are evaluated numerically in Euclidean regions (pySecDec)
- System of differential equations is generated with (Kira)
- Use series expansion of the system of differential equations to transport from the Eucliden points to Minkowskien physical regions (DiffExp)

Benefits of the Blueprint Caesar

- We may set all masses to physical values — reductions with Kira simplify enormously
- Finite integrals in Euclidean regions — avoid the contour deformation and the tedious resolving of the UV or IR divergences
- Proof of concept available in other projects [Frellesvig, Hidding, Maestri, Moriello, Salvatori, 2020], [Faela, Lange, Sch ̀lönwalda, Steinhauser, 2021]

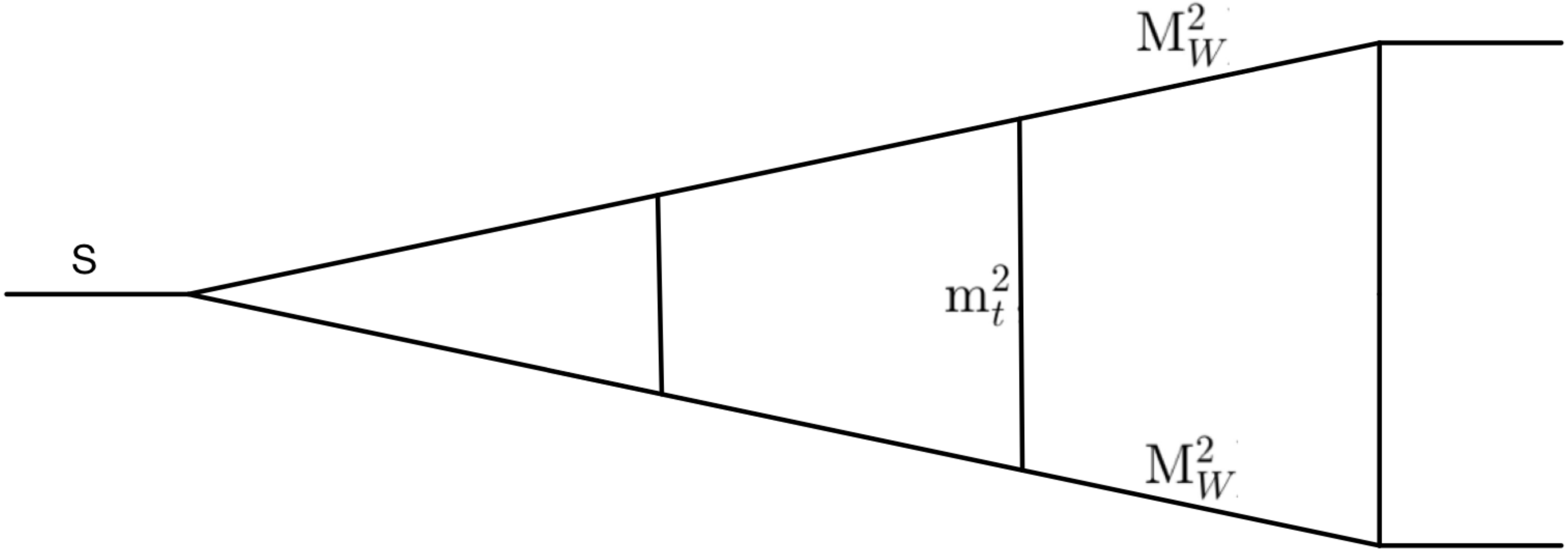
Caesar: Integralfamily v3t181



- In Euclidean regions (s, M_W^2, m_t^2)=(-2,4,16)

-> v3t181$^{d=4-2\epsilon}[1,1,1,1,0,1,1,1,1,0,0,0] =$ $0.1339526664441601839027 49812$ with 25 significant digits

Caesar: Integralfamily v3t181

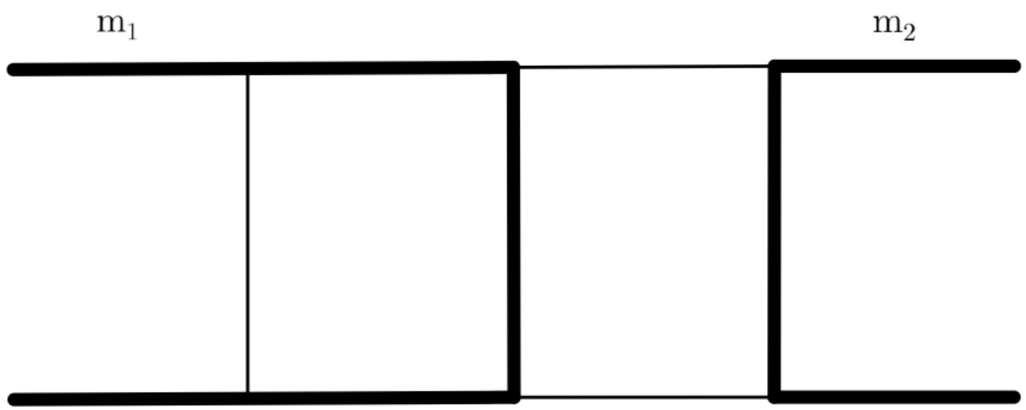


- In physical regions $(s, M_W^2, m_t^2)=(1,(\frac{401925}{4559382})^2, (\frac{433000}{227969})^2)$

-> v3t181$^{d=4-2\epsilon}$[1, 1, 1, 1, 1, 1, 1, 1, 1, -3, 0, 0] =

$$\frac{1.99999999981 + 8.18*10^{-12}\, i}{\epsilon^3} + \frac{9.87003934692 + 18.84955592198\, i}{\epsilon^2} - \frac{26.50733688118 - 41.19670709595\, i}{\epsilon} + (2.29574696253 + 201.06880202144\, i) + O(\epsilon)$$

Caesar: Integralfamily Bhabha



- In physical regions (s,t,m_1^2,m_2^2)=(2,5,4,16)

-> bhabha$^{d=6-2\epsilon}[1,2,1,2,1,1,1,0,0] =$
$(0.0002973066815\ +0.001542581913\ i)$
$-(0.002805345908\ -0.003106827180\ i)\ \epsilon+O\left(\epsilon^2\right)$

Outlook

- Get a basis where the matrix of the system of differential equations is linear in ϵ
  - -> DiffExp does order of magnitudes faster transport of the boundary terms

Conclusions

- The first physics goals are already in 6 month reach
- Important is the knowledge transfer and to get people motivated to engineer other methods for practical applications
- Strong computing resources are needed not only for the final product but also for the development of the tools.
- Without spending significant effort on simplification of the basis, we can numerically solve the differential equations of non-trivial 3-loop Feynman integrals.
- By choosing the basis representatives to be finite integrals, we can obtain precise numerical boundary conditions in the Euclidean region using pySecDec.
- We find that the precision of the boundary conditions in the Euclidean region carries over to the physical region.
- The process can be fully automated.