The Quantum Mechanics of a Perfect Fluid

Riccardo Rattazzi - EPFL

- S. Dubovsky, T. Gregoire, A. Nicolis, RR 2005

- S. Endlich, A. Nicolis, RR, J. Wang 2010

- A. Nicolis, R. Penco, F. Piazza, RR 2015

- S. Endlich, W. Goldberger, RR, I. Rothstein 2015 unpublished

- A. Dersy, A. Khmelnitsky, RR now

- What is cosmology?

- gravity & condensed matter playing together

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}}$$

- $g_{\mu\nu}$ & $T_{\mu\nu}$ homogeneous and isotropic

$T_{\mu\nu} \Longleftrightarrow$ finite density QFT with Poincaré group broken to $ISO(3)$

- $ISO(3,1) \Longrightarrow ISO(3)$ *spontaneously*

- long distance dynamics universally described by "hydrodynamic" modes ($\equiv$ Goldstones)

- gravity + hydro modes $\sim \begin{cases} \text{EM} + \text{Cooper pairs} \\ SU(2)\times U(1) + \text{Higgs} \end{cases}$

Ex: hot plasma $\Longrightarrow$ $L_{micro} \sim \frac{1}{T}$ $\quad L_{curv} = H^{-1} \sim \frac{M_P}{T^2}$

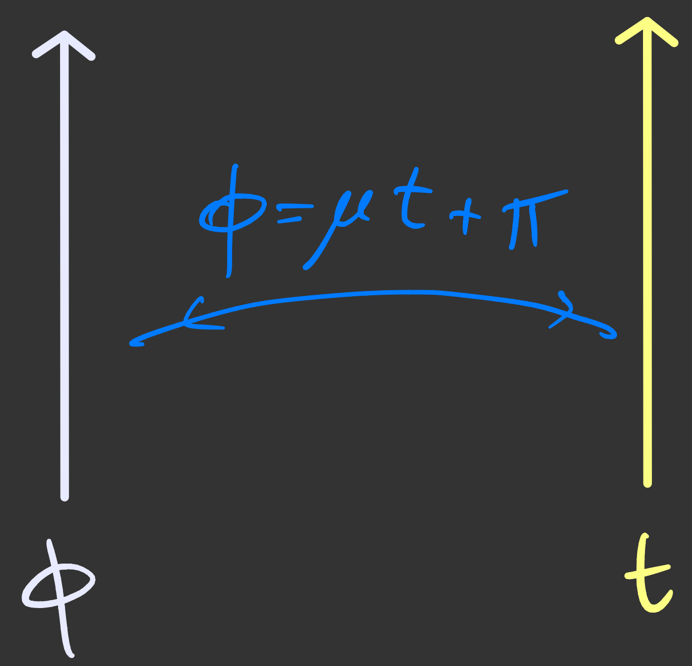
Which options for dynamics behind $T_{\mu\nu}$ are even possible?

$$\sim$$

What is the "space" of sensible EFTs realizing $ISO(3,1) \xrightarrow{\text{spontaneously}} ISO(3)$?

Ex: Superfluid

$\boxed{\text{Son } 2002}$

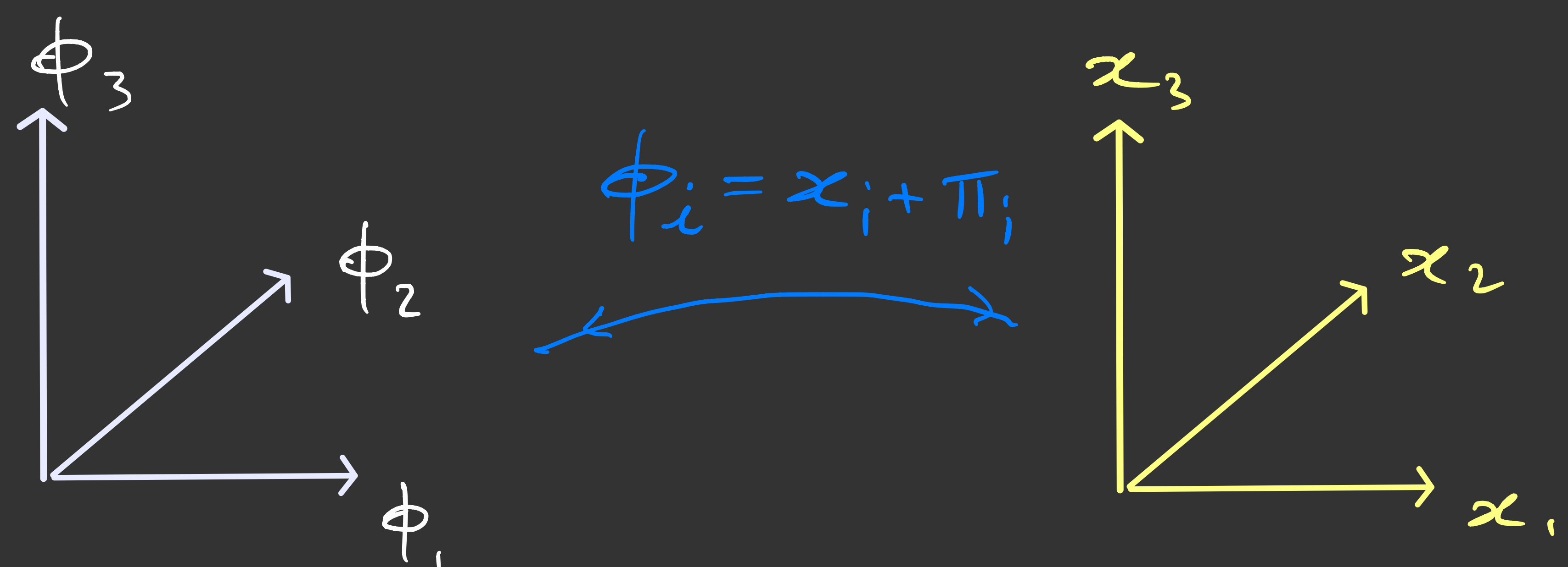


- $ISO(3,1) \times U(1)_\phi \to ISO(3) \times \overline{P}_0$

$$\overline{P}_0 = P_0 + \mu Q$$

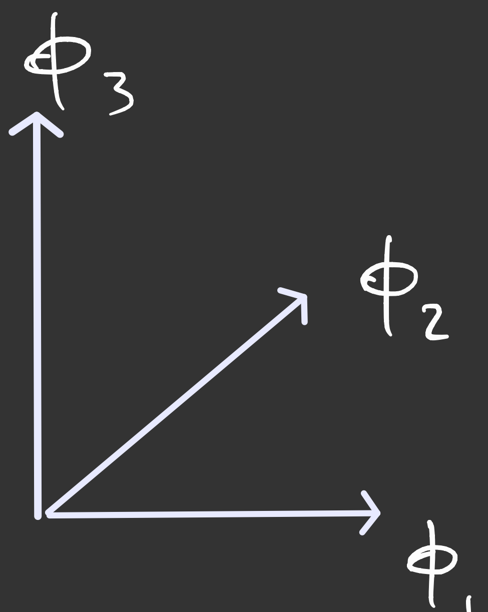
- $\mathcal{L} = F(\partial_\mu \phi \partial^\mu \phi)$

Ex: Solid

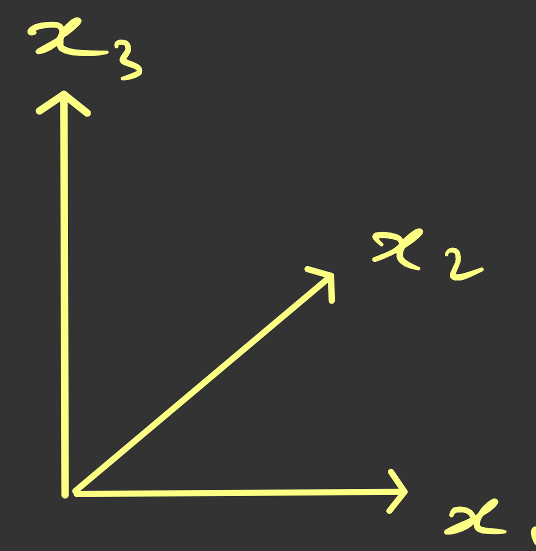


Ex Fluid

$\boxed{\text{Carter } 1973}$



$$\phi_i = x_i + \pi_i$$



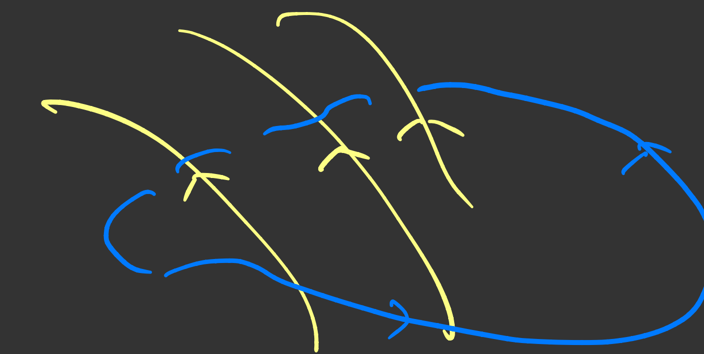
$$\begin{cases} \phi^I \longrightarrow \xi^I(\phi) \in \mathrm{Sdiff} \\[2ex] \left|\dfrac{\partial \xi}{\partial \phi}\right| = 1 \end{cases}$$

$$\left.\begin{array}{l} B^{IJ} \equiv \partial_\mu \phi^I \partial^\mu \phi^J \\[2ex] \mathcal{L} = F(\det B) \end{array}\right\} \Longrightarrow$$

Euler eqs. for relativistic perfect fluid

- Symmetries

$\longrightarrow$ Kelvin Theorem



$\oint_\gamma v \cdot dl$ = const on flow

- Quadratic action

$$\phi^i = x^i + \pi^i \longrightarrow \mathcal{L} \propto \left[\frac{1}{2} \dot{\underline{\pi}}^2 - c_s^2 (\underline{\nabla} \cdot \underline{\pi})^2 + O(\pi^3) \right]$$

- longitudinal $\vec{\pi}_L = \vec{\nabla} \varphi \implies \omega = c_s K$

$\boxed{\bullet \text{ transverse } \vec{\pi}_T = \vec{\nabla} \wedge \vec{w} \implies \omega = 0 \quad \forall K}$

$$\vec{\pi}_\perp \to \underbrace{\mathcal{L} \propto (\dot{\vec{\pi}}_\perp)^2 + O(\pi^3)}_{\text{Not a Fock space of states !!}}$$

- Does this system make sense quantum mechanically?
- Perhaps not (apparently no ordinary fluid at $T=0$)

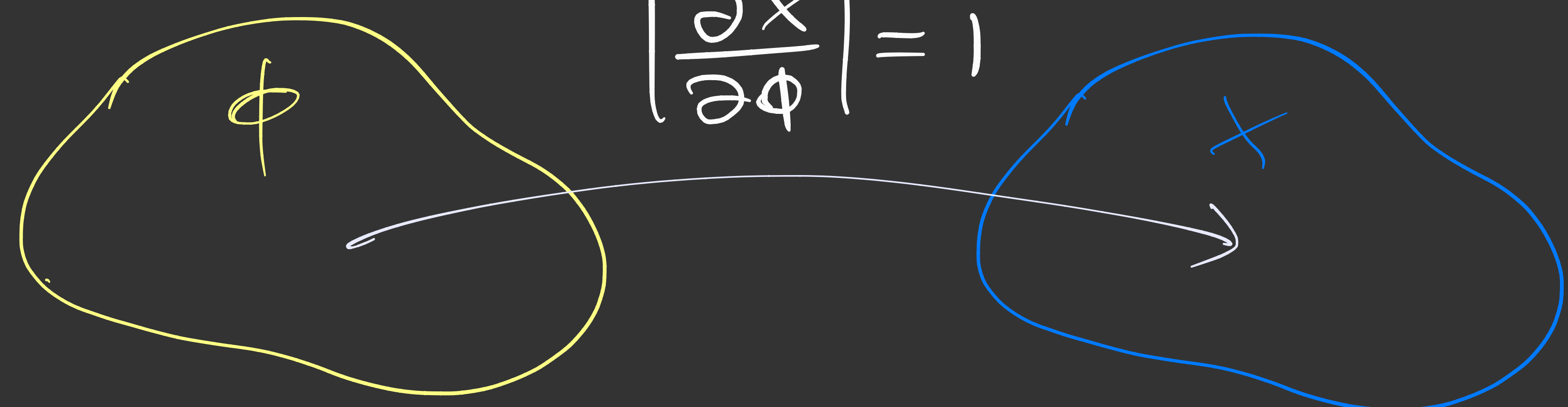
☆ Landau 1941 . . .

☆ . . . then Feynman 1953

Zoom on $\Pi_\perp$: incompressible limit

- $v \ll c_s \implies$ "integrate out sound"

$$\left|\frac{\partial x}{\partial \phi}\right| = 1$$



- $v = \dfrac{dx(\phi, t)}{dt}$
- $S = \int \left(\frac{1}{2} \rho v^2 + \dots \right) dt\, d^2\phi$

$$\implies \boxed{\left(\partial_t + \vec{v} \cdot \vec{\nabla}\right)\left(\vec{\nabla} \wedge \vec{v}\right) = 0}$$

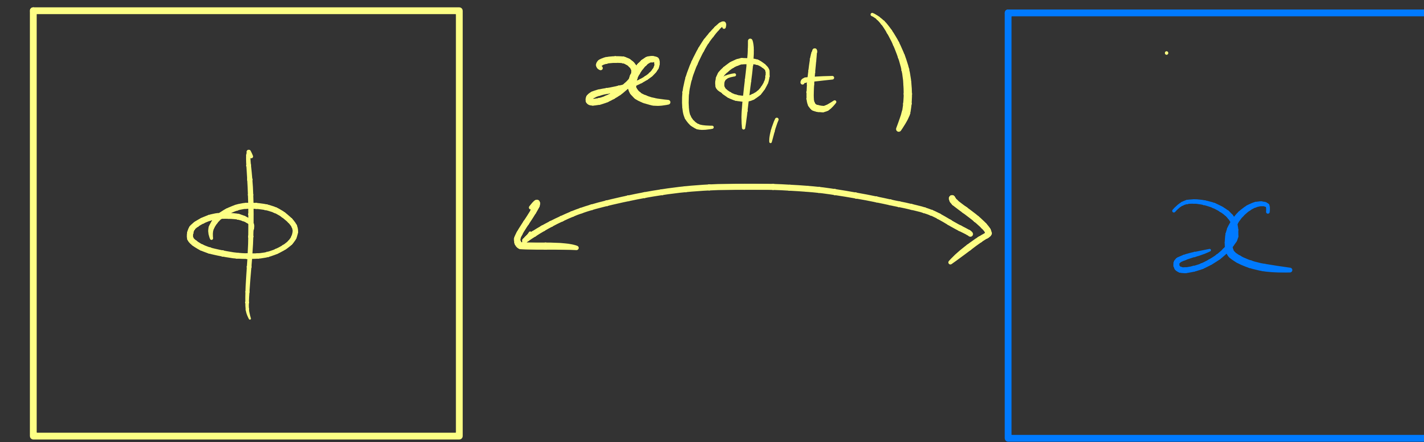
• Ex free particle $\mathcal{L} = \frac{m}{2} v^2 \longrightarrow H = \frac{p^2}{2m}$ $E_k = \frac{k^2}{2m}$

• fluid $\mathcal{L} = \frac{\rho}{2} \left[\dot{X}(\phi, t) \right]^2 \longrightarrow H = \frac{(\ldots)}{\rho}$ $\rho = \frac{M}{L^2}$

■ UV-independence $\Longrightarrow$ $\vec{k}$ only available scale: $E_k \sim \frac{k^4}{\rho}$

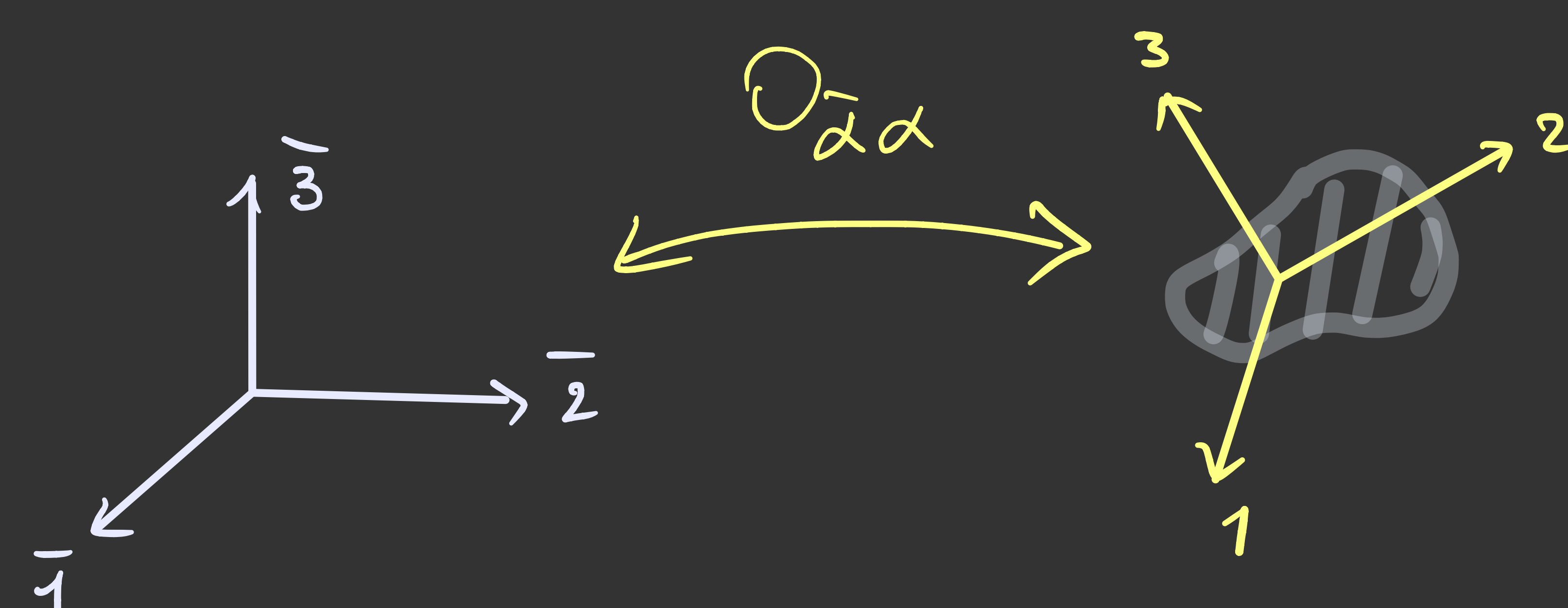
■ UV-dependence $\Longrightarrow \exists \Lambda$ (Landau 1941): $E_k \sim \frac{\Lambda^4}{\rho}$

Fluid on T^2



- $x(\phi, t) \in \mathrm{Sdiff}(T^2) \Rightarrow$ motion on group manifold

Rigid Body



- $O_{\bar{\alpha}\alpha}(t) \in SO(3)$

see V.I. Arnol'd "Mechanics" textbook

Group Action

- rigid body $\longrightarrow$ $O_{\bar{\alpha}\alpha} \longrightarrow U_L O U_R^{-1}$

$$SO(3)_L \times SO(3)_R$$

$SO(3)_L \downarrow$ symm, $SO(3)_R \not\downarrow$ not symm

- fluid $\longrightarrow$ $x(\phi) \longrightarrow f_R(x(f_L^{-1}(\phi)))$

$$\equiv f_R \circ x \circ f_L^{-1}$$

$$\Longrightarrow (\mathrm{Sdiff})_L \times (\mathrm{Sdiff})_R$$

$(\mathrm{Sdiff})_L \downarrow$ symm, $(\mathrm{Sdiff})_R \not\downarrow$ not symm

Hamiltonian

$$R_i = -I_{ij}\Omega_j$$

$$H = \frac{1}{2} I^{-1}_{ij} R_i R_j$$

$$[R_i, R_j] = i\,\varepsilon_{ijk} R_k$$

$$\Big\Downarrow \text{ Peter-Weyl}$$

$$\mathcal{H} = \bigoplus (\ell, \ell)$$

left right

$$R(x) = \rho\, \nabla \wedge v(x) \equiv \rho\, \omega(x)$$

$$H = \frac{1}{2\rho} \int d^2x \; R\, \nabla^{-2} R$$

$$[R_{\underline{n}}, R_{\underline{m}}] = i(\underline{n} \wedge \underline{m}) R_{\underline{n}+\underline{m}}$$

$$\Downarrow$$

$$\mathcal{H} = \bigoplus (q, q) = \, ?$$

$(q, q) \neq (0, 0) \Rightarrow \infty$-degeneracy

How to deal with infinite degeneracy?

A || gauge $(\mathrm{Sdiff})_L$ $\sim$ project on ground state $(0,0)$

$\Longrightarrow$ fluid $\longrightarrow$ superfluid   Feynman :)

B || drop (r,r) structure: just $(\mathrm{Sdiff})_R$ on <u>some</u> $\mathcal{H}$

... still not straightforward as Sdiff is ∞

A remarkable fact

learned from George Savvidy

$$\mathrm{Sdiff}(T^2) \sim \lim_{N \to \infty} SU(N)$$

- J. Hoppe, PhD thesis 1982
- Fairlie, Fletcher, Zachos 1988
- Pope, Stelle 1989

- Girvin, MacDonald, Platzman 1985
-
-
- Wiegmann 2013

A clever basis of SU(N) algebra ('t Hooft 1978)

- $\square = \mathbb{Z}_\alpha \qquad \alpha = -\frac{N-1}{2}, \dots, 0, \dots \frac{N-1}{2} \qquad \alpha \in \left[-\frac{N-1}{2}, \frac{N-1}{2}\right]$

- $\underline{n} \equiv (n_1, n_2) \in \left[-\frac{N-1}{2}, \frac{N-1}{2}\right] \times \left[-\frac{N-1}{2}, \frac{N-1}{2}\right] \longrightarrow N^2$

- $J_{\underline{n}}$

$$\boxed{\left[J_{\underline{n}}, J_{\underline{m}}\right] = -2i \sin\left(\frac{\pi\, \underline{n} \wedge \underline{m}}{N}\right) J_{\underline{n}+\underline{m}}}$$

$\underline{n} \neq (0,0)$

$$R_{\underline{n}} \equiv -\frac{N}{2\pi} J_{\underline{n}} \Rightarrow [R_{\underline{n}}, R_{\underline{m}}] = i\frac{N}{\pi}\sin\left(\frac{\pi\, \underline{n}\wedge\underline{m}}{N}\right) R_{\underline{n}+\underline{m}}$$

$$\boxed{|\underline{n}|, |\underline{m}| \ll \sqrt{N} \Longrightarrow [R_{\underline{n}}, R_{\underline{m}}] \simeq i(\underline{n}\wedge\underline{m}) R_{\underline{n}+\underline{m}}}$$

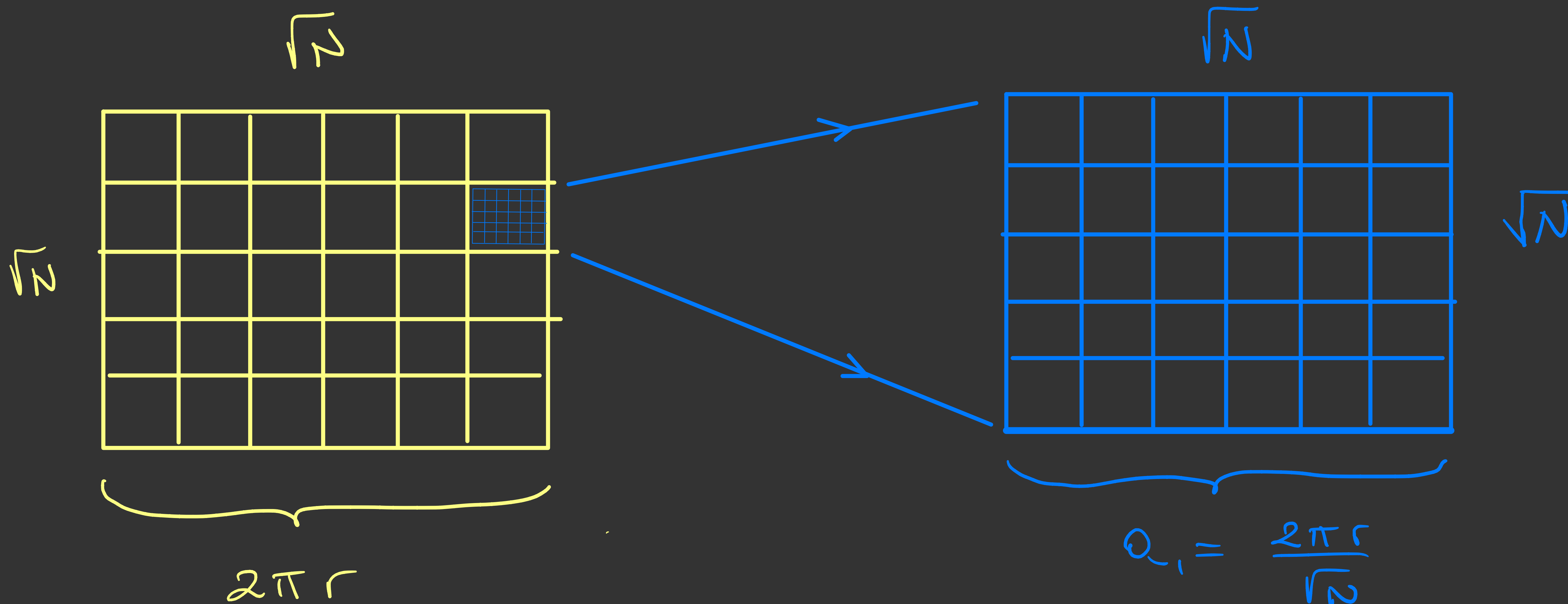
$\boxed{\text{SU(N) rigid body}}$ $\sim$ $\boxed{\text{UV completion of perfect fluid}}$

- $\underline{n} \in \left[-\frac{N-1}{2}, \frac{N-1}{2}\right]^2 \longrightarrow$ space latticed

- $\underline{n} \xleftrightarrow{\text{Fourier}} \underline{x}$ $\quad \underline{x} \in \left[-\frac{N-1}{2}, \frac{N-1}{2}\right]^2$

$$R(\underline{x}) \equiv \frac{1}{N} \sum_{\underline{n}} \omega^{-\underline{n}\cdot\underline{x}} R_n \qquad \omega = e^{\frac{2\pi i}{N}}$$

☐ $|\vec{n}| \lesssim \sqrt{N} \longleftrightarrow |\vec{\Delta x}| \gtrsim \sqrt{N}$



☐ fluid regime $|\vec{P}| \lesssim \frac{1}{a_1} \equiv \Lambda$

Hamiltonian

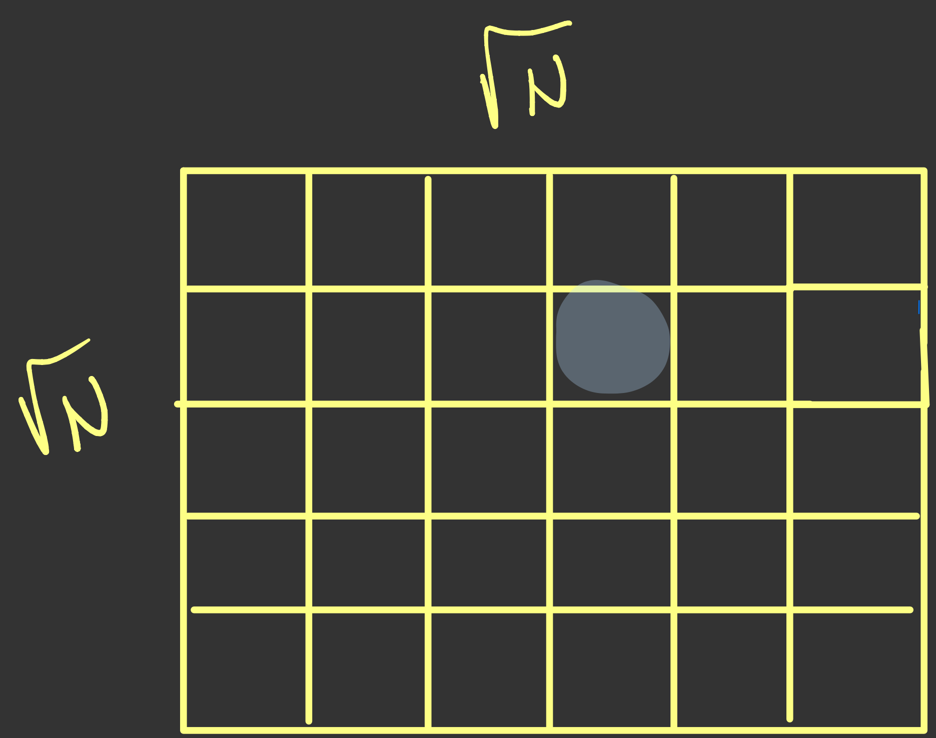
$$H = \frac{1}{2\rho}\left(\frac{1}{2\pi r^2}\right)^2 \sum_{\underline{n}}^{|\underline{n}|<\sqrt{N}} \frac{R_{\underline{n}} R_{-\underline{n}}}{\underline{n}^2}$$

$$= \frac{\Lambda^4}{32\pi^2\rho} \sum_{\underline{n}}^{|\underline{n}|<\sqrt{N}} \frac{J_{\underline{n}} J_{-\underline{n}}}{\underline{n}^2}$$

Spectrum

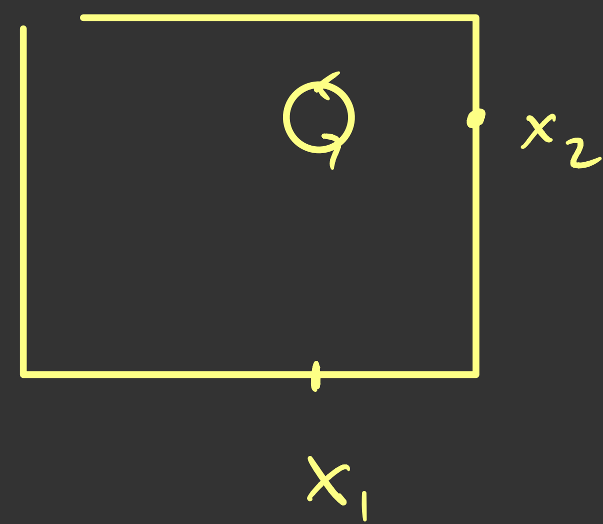
- trivial irrep $\bullet$ $\Longrightarrow$ $E = 0$ $\equiv$ ground state

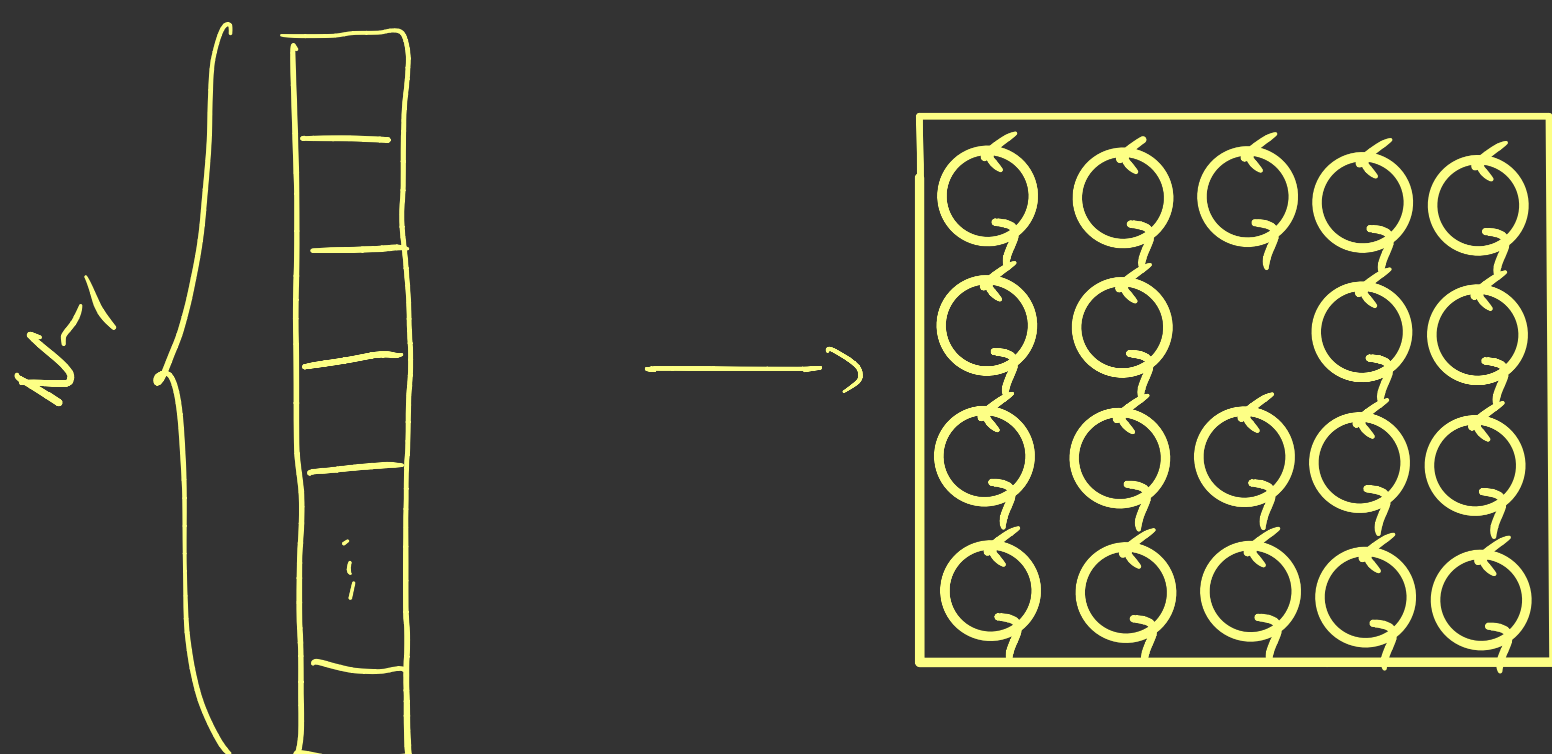
- fundamental $\Box$ $\Longrightarrow$ $E_{\Box} = \dfrac{\Lambda^4}{16\pi^2 \rho} \ln \Lambda r$ Landau!

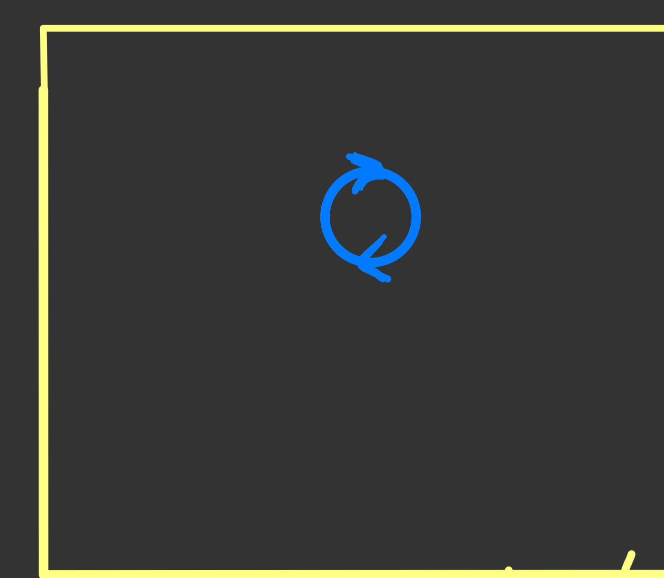


- $\sqrt{N} \times \sqrt{N} = N$ states
- $|\vec{x}\rangle$

- $\langle x | \nabla \wedge v(y) | x \rangle \sim \frac{\Lambda^2}{2\pi\rho}\left[\delta^2(x-y) - \frac{1}{V}\right]$

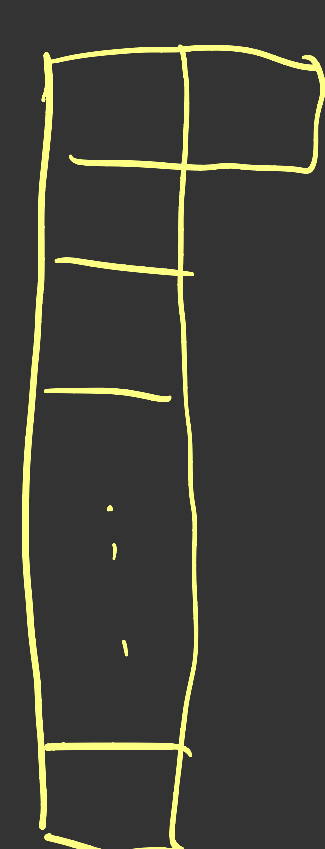
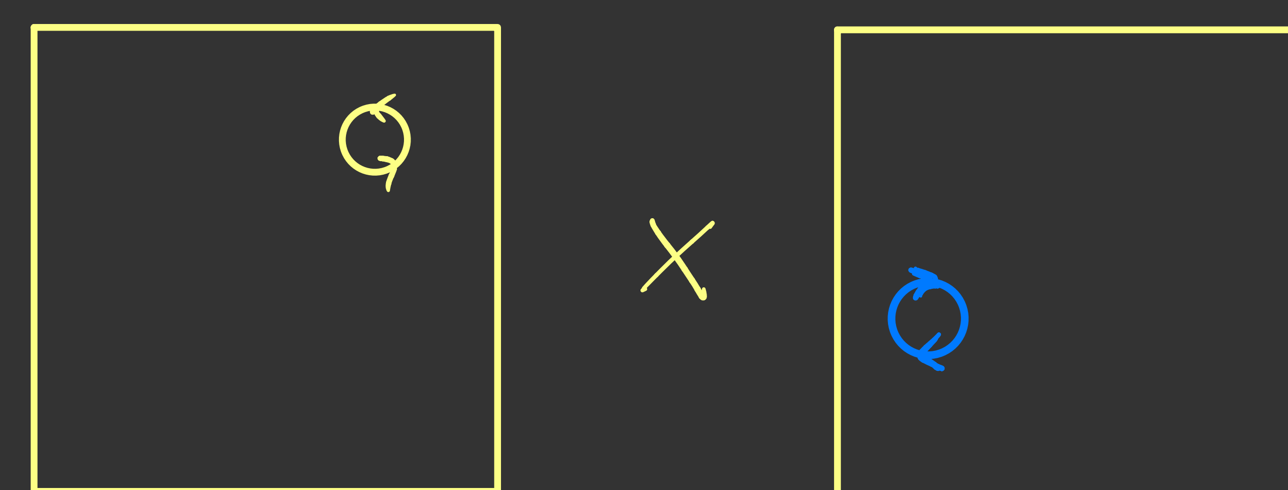
- $|x\rangle \longrightarrow$ 

- 

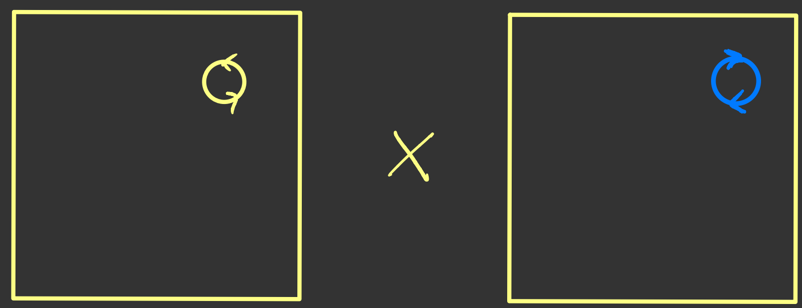


- Lowest irreps 

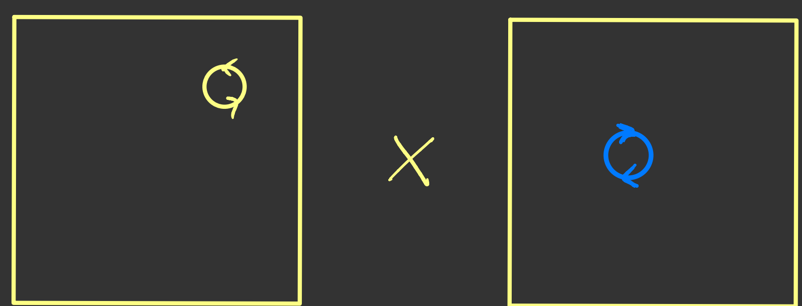
  all gapped

- Adjoint  $\sim$ 

  $N \times N$



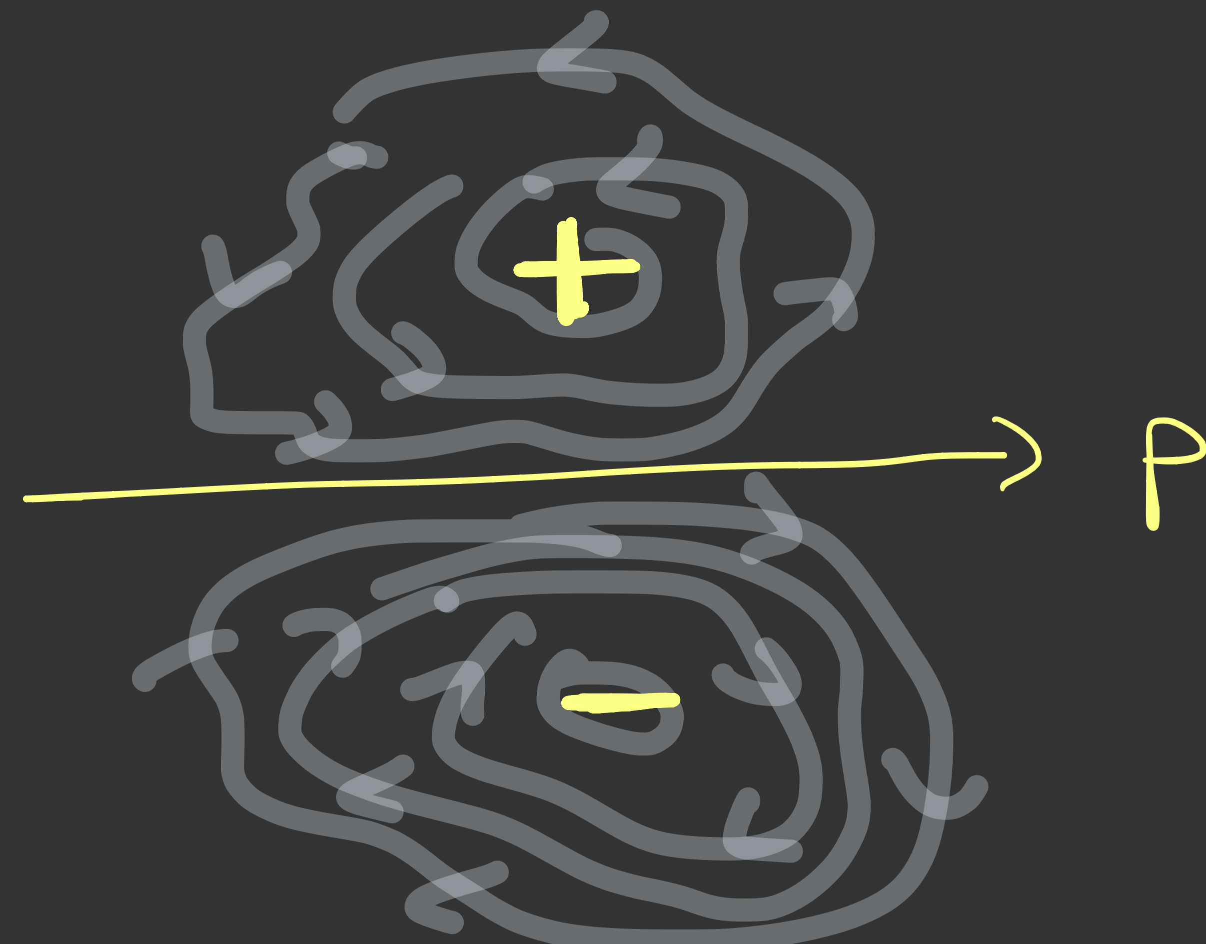
$\Longrightarrow$ ungapped $(\sim N)$



$\Longrightarrow$ gapped $\sim N^2$

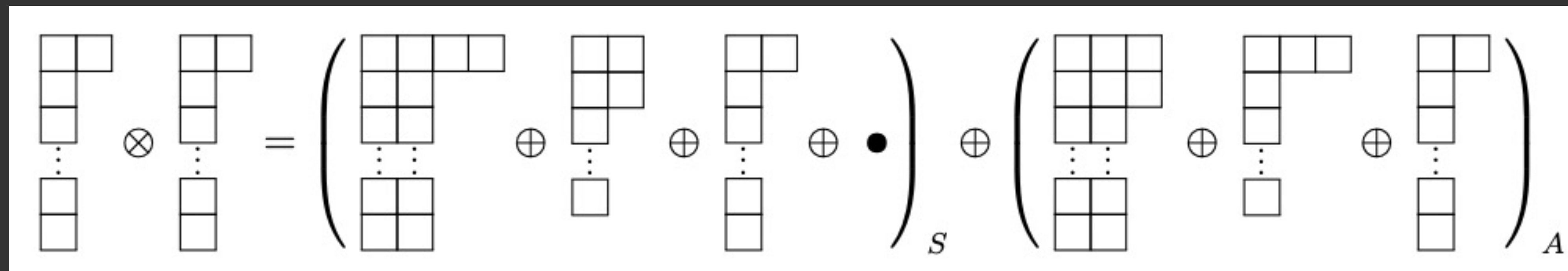
$$E_{adj}(P) = \begin{cases} \dfrac{P^2}{4\rho a_1^2} \equiv \dfrac{\Lambda^2 P^2}{16\pi^2 \rho} & P \ll \Lambda = \dfrac{1}{a_1} \\[2ex] \dfrac{\Lambda^4}{16\pi^3 \rho} \ln \dfrac{P^2}{\Lambda^2} & P \gtrsim \Lambda \end{cases}$$

- Vortons $\equiv$ quanta with vorticity dipole



$$d^I \equiv \int d^2x \, x^I \langle \vec{p} | \vec{\nabla} \wedge \vec{v} | \vec{p} \rangle = \frac{P^J \varepsilon^{JI}}{\rho}$$

Scattering



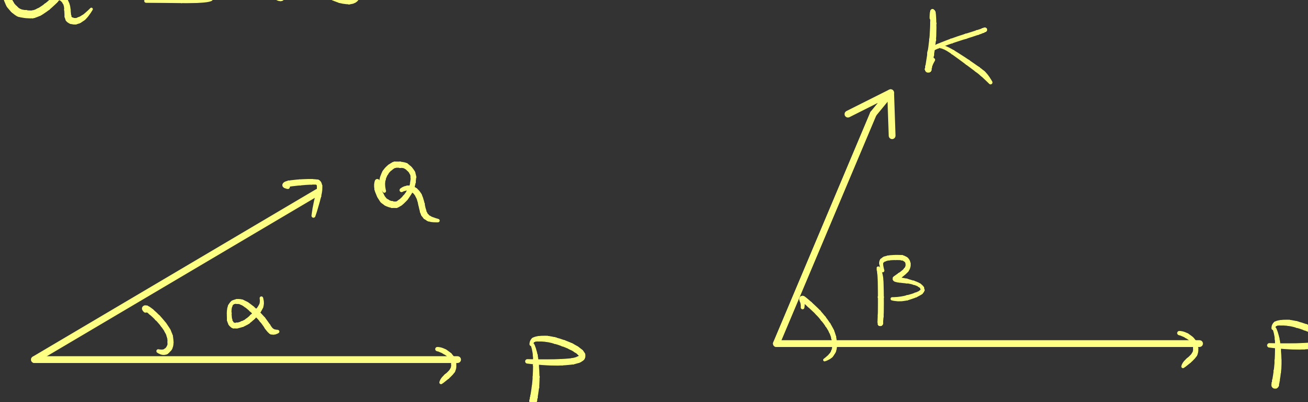
N^4 $\quad$ N^2 $\quad$ 1 $\quad$ N^4 $\quad$ N^2

bosonic vorton EFT

fermionic vorton EFT

$$\mathcal{L} = i\phi^\dagger\dot{\phi} - \frac{1}{2\rho}\left(\partial\phi^\dagger \wedge \partial\phi\right)\frac{1}{-\nabla^2}\left(\partial\phi^\dagger \wedge \partial\phi\right)$$

• $Q^2 = K^2$

• 

• $A_{boson} =$  $+$  $= \frac{1}{8\rho}\left(P^2 - Q^2\right)$

• $A_{fermion} =$  $-$  $= \frac{1}{8\rho}\left(P^2\cos^2(\alpha+\beta) - Q^2\cos^2(\alpha-\beta)\right)$

Summary

2D quantum perfect fluid $\sim$ SU(N) rigid body
$N \to \infty$

- $SU(N)_L \times SU(N)_R \longrightarrow$ infinite <u>hidden</u> degeneracy

- $\left.\begin{array}{l} \cancel{SU(N)_L} \times SU(N)_R \\ \equiv \text{vorticity dynamics} \end{array}\right\} \Longrightarrow$
  - vortices $\longrightarrow$ gapped
  - vortons $\longrightarrow$ ungapped

does this thing exist?