Leading isospin breaking effects in the HVP contribution to a_μ and to the running of α_{em}

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Standard model of particle physics

- The standard model is incomplete!
  - Unification with general relativity
  - Neutrino masses and oscillations
  - Baryogenesis
  - Dark matter
  - Mass hierarchies in fermion families
  - ...
- How to test the standard model (and find new physics)?

  Direct searches: Production of new particles

  - LEP: $e^+ + e^- \to X$
  - Tevatron: $p + \overline{p} \to X$
  - LHC: $p/Pb + p/Pb \to X$
  - ...

  Indirect searches: Precision experiments

  - Anomalous magnetic moment of the muon a_μ
  - Momentum dependence ("running") of $\alpha_{\rm em}(Q^2)$, $\sin^2(\theta_{\rm W})(Q^2)$, etc.
  - ...

The muon anomalous magnetic moment a_μ

- Magnetic moment of the muon $\vec{M}$:

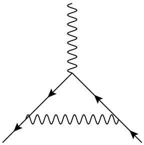
$$\vec{M} = g_\mu \frac{e}{2m_\mu} \vec{S},$$

  g_μ gyromagnetic ratio

- Relativistic QM (Dirac equation): $g_\mu = 2$

- Anomalous magnetic moment (QFT effects):

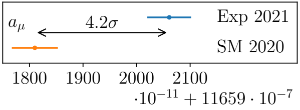
$$a_\mu = \frac{g_\mu - 2}{2}$$



LO-QED

- Experimental value: BNL + FNAL 2021[1] (0.35 ppm)
  vs. Standard Model prediction: Muon g-2 Theory Initiative 2020[2]

$$a_\mu^{\text{Exp}} = 116592061(41) \cdot 10^{-11}$$
$$a_\mu^{\text{SM}} = 116591810(43) \cdot 10^{-11}$$



$\Rightarrow$ New physics?

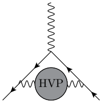
[1] Abi et al. 2021.
[2] Aoyama et al. 2020.

The muon anomalous magnetic moment a_μ

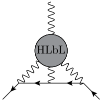
- Decomposition into $a_\mu^{\mathrm{SM}} = a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{QCD}} + a_\mu^{\mathrm{EW}}$

Contribution	$a_\mu \times 10^{11}$
QED (bis Ordnung $O(\alpha^5)$)	$116\ 584\ 718.93 \pm 0.10$
electroweak	153.6 ± 1.0
QCD	
HVP (LO)	$6\ 931 \pm 40$
HVP (NLO)	-98.3 ± 0.7
HVP (NNLO)	12.4 ± 0.1
HLbL	94 ± 19
Sum (theory)	$116\ 591\ 810 \pm 43$

Standard Model contributions to a_μ[3]



LO-HVP



HLbL

Error of LO-HVP contribution is dominant!

- LO-HVP via dispersion relation[4]:

$$a_\mu^{\mathrm{HVP}} = \frac{\alpha^2}{3\pi^2} \int_{4m_\pi^2}^{\infty} \mathrm{d}s \, \frac{K(s)}{s} R(s) \qquad R(s) = \frac{\sigma(e^+e^- \to \text{hadrons}(+\gamma))}{(4\pi\alpha^2)/(3s)}$$

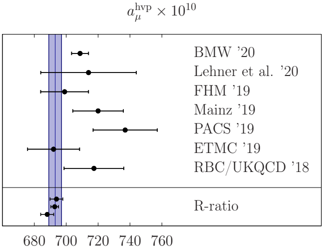
- Ab-initio prediction via Lattice QCD desirable

[3] Gérardin 2020.

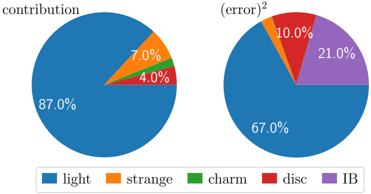
[4] Aoyama et al. 2020.

The muon anomalous magnetic moment a_μ

- Dispersive method gives smaller errors in comparison to Lattice QCD

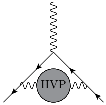


LO-HVP contribution to a_μ[5]



Relative contributions to a_μ^{HVP} from Mainz 2019[6].

- Precise determination via first principle computation in Lattice QCD (with sub-percent error)[7]:
  - QCD_{iso}: $m_{\text{u}} = m_{\text{d}}$, $\alpha_{\text{em}} = 0$
  - $(m_{\text{u}} - m_{\text{d}})/\Lambda \Rightarrow O(1\%)$-effect
  - $\alpha_{\text{em}} \approx 0.007 \Rightarrow O(1\%)$-effect

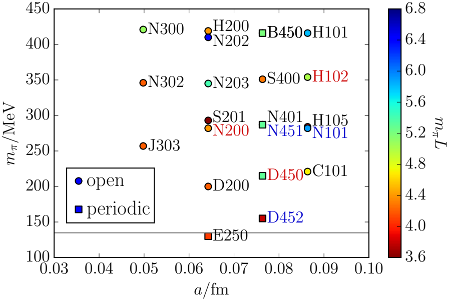


$\Rightarrow$ Isospin breaking effects become relevant!

[5] Gérardin 2020.
[6] Gérardin et al. 2019.
[7] Portelli 2013.

QCD+QED on QCD_{iso} gauge ensembles

- QCD_{iso} ensembles (CLS)[8]:
  - Tree-level improved Lüscher-Weisz gauge action
  - $N_f = 2 + 1$ $O(a)$-improved Wilson fermion action
  - Periodic/open temporal boundary conditions
  - $\text{tr}(M) = \textit{const.}$



	$(\frac{L}{a})^3 \times \frac{T}{a}$	a [fm]	m_π [MeV]	m_K [MeV]	$m_\pi L$	L [fm]
N200	$48^3 \times 128$	0.06426(76)	282(3)	463(5)	4.4	3.1
D452	$64^3 \times 128$	0.07634(97)	155	481	3.8	4.9
D450	$64^3 \times 128$	0.07634(97)	217(3)	476(6)	5.4	4.9
N451	$48^3 \times 128$	0.07634(97)	287(4)	462(5)	5.3	3.7
H102	$32^3 \times 96$	0.08636(10)	354(5)	438(4)	5.0	2.8
N101	$48^3 \times 128$	0.08636(10)	282(4)	460(4)	5.9	4.1

processed, work in progress

[8] Bruno et al. 2015; Bruno et al. 2017.

- QCD+QED action:

$$S[U, A, \Psi, \overline{\Psi}] = S_{\mathrm{g}}[U] + S_{\gamma}[A] + S_{\mathrm{q}}[U, A, \Psi, \overline{\Psi}]$$

  parametrised by

$$\varepsilon = (m_{\mathrm{u}}, m_{\mathrm{d}}, m_{\mathrm{s}}, \beta, e^2)$$

- $\mathrm{QCD_{iso}}$ + free photon field:

$$\varepsilon^{(0)} = (m_{\mathrm{u}}^{(0)}, m_{\mathrm{d}}^{(0)}, m_{\mathrm{s}}^{(0)}, \beta^{(0)}, 0) \qquad m_{\mathrm{u}}^{(0)} = m_{\mathrm{d}}^{(0)}$$

  m_{s} and β also renormalise under isospin breaking!

- $S_{\mathrm{g}}[U]$: like $\mathrm{QCD_{iso}}$-gauge action, $\beta^{(0)} \to \beta$

- $S_{\gamma}[A]$: Non-compact lattice QED, $\mathrm{QED_L}$ prescription[9] for IR-regularisation
  - No net charge on periodic volume $\mathbb{T}^3$:

$$Q = \int_{\mathbb{T}^3} \mathrm{d}^3x\, \rho = \int_{\mathbb{T}^3} \mathrm{d}^3x\, \partial^i E^i = \int_{\partial\mathbb{T}^3} \mathrm{d}S^i E^i = 0$$

[9] Hayakawa and Uno 2008.

QCD+QED action

- $S_\gamma[A]$: Non-compact lattice QED, $\mathrm{QED_L}$ prescription[10] for IR-regularisation
  - QED on $\mathbb{T}^4$: gauge symmetries with $\alpha^x = \alpha^x_{\mathrm{per}} + \frac{2\pi n^\mu}{eX^\mu} x^\mu$ and $n \in \mathbb{Z}^4$

    $$A^{x\mu} \mapsto A^{x\mu} - (\partial^\mu \alpha)^x \qquad \Psi^x \mapsto e^{\mathrm{i}e\alpha^x}\Psi^x \qquad \overline{\Psi}^x \mapsto \overline{\Psi}^x e^{-\mathrm{i}e\alpha^x}$$

    $\Rightarrow$ Photon differential operator is non-invertible (zero modes)
  - $n = 0$: local gauge fixing: Coulomb gauge (Feynman gauge as cross check)
  - $n \neq 0$: shift symmetry of A (large gauge transformations) broken by $\mathrm{QED_L}$ constraint:

    $$\sum_{\vec{x}} A^{x\mu} = 0 \quad \forall\mu, \forall x^0$$

- $S_\mathrm{q}[U, A, \Psi, \overline{\Psi}]$: like $\mathrm{QCD_{iso}}$ action, $(m_\mathrm{u}^{(0)}, m_\mathrm{d}^{(0)}, m_\mathrm{s}^{(0)}) \to (m_\mathrm{u}, m_\mathrm{d}, m_\mathrm{s})$
  Introduce electromagnetic interaction by QCD+QED gauge links

  $$U^{x\mu} \to W^{x\mu} = U^{x\mu} e^{\mathrm{i}aeQA^{x\mu}} \qquad Q = \mathrm{diag}\Big(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\Big)$$

- $\mathrm{QCD_{iso}}$ $O(a)$-improved, isospin breaking however introduces $O(a)$ lattice artefacts

[10] Hayakawa and Uno 2008.

QCD+QED on QCD_{iso} gauge ensembles

- Generation of new QCD+QED gauge ensembles is expensive!
  $\Rightarrow$ Monte-Carlo reweighting[11] on existing QCD_{iso} gauge ensembles

- QCD_{iso} ensembles generated with $S_{\text{eff}}^{(0)}[U] = S_{\text{g}}^{(0)}[U] - \log(Z_{\text{q}}^{(0)}[U])$:

$$\langle \mathcal{O}[U] \rangle_{\text{eff}}^{(0)} = \frac{\int DU\, e^{-S_{\text{eff}}^{(0)}[U]}\, \mathcal{O}[U]}{\int DU\, e^{-S_{\text{eff}}^{(0)}[U]}} \qquad Z_{\text{q}}^{(0)}[U] = \int D\Psi D\overline{\Psi}\, e^{-S_{\text{q}}^{(0)}[U,\Psi,\overline{\Psi}]}$$

- Define QED on fixed QCD_{iso} gauge field background U:

$$\langle \mathcal{O}[U,A,\Psi,\overline{\Psi}] \rangle_{\text{q}\gamma} = \frac{1}{Z_{\text{q}\gamma}[U]} \int DAD\Psi D\overline{\Psi}\, e^{-S_\gamma[A] - S_{\text{q}}[U,A,\Psi,\overline{\Psi}]}\, \mathcal{O}[U,A,\Psi,\overline{\Psi}]$$

$$Z_{\text{q}\gamma}[U] = \int DAD\Psi D\overline{\Psi}\, e^{-S_\gamma[A] - S_{\text{q}}[U,A,\Psi,\overline{\Psi}]}$$

- Expectation value with reweighting factor $R[U]$:

$$\langle \mathcal{O}[U,A,\Psi,\overline{\Psi}] \rangle = \frac{\langle R[U]\, \langle \mathcal{O}[U,A,\Psi,\overline{\Psi}] \rangle_{\text{q}\gamma} \rangle_{\text{eff}}^{(0)}}{\langle R[U] \rangle_{\text{eff}}^{(0)}} \qquad R[U] = \frac{e^{-S_{\text{g}}[U]}\, Z_{\text{q}\gamma}[U]}{e^{-S_{\text{g}}^{(0)}[U]}\, Z_{\text{q}}^{(0)}[U]}$$

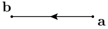
- Evaluation of $\langle \mathcal{O}[U,A,\Psi,\overline{\Psi}] \rangle_{\text{q}\gamma}$ and $R[U]$:
  Compute $q\gamma$-path integral stochastically/via lattice perturbation theory around QCD_{iso}

[11] Ferrenberg and Swendsen 1988; Duncan et al. 2005; Divitiis et al. 2013.

- Evaluate $\langle \mathcal{O}[U, A, \Psi, \overline{\Psi}] \rangle_{\mathrm{q}\gamma}$ and $R[U]$ using a perturbative expansion[12] in $\Delta\varepsilon = \varepsilon - \varepsilon^{(0)}$ around $\varepsilon^{(0)}$ with $\varepsilon = (m_\mathrm{u}, m_\mathrm{d}, m_\mathrm{s}, \beta, e^2)$ (RM123)

- Free isosymmetric quark propagator on $\mathrm{QCD_{iso}}$ gauge field background U $(\mathbf{a}, \mathbf{b} \equiv (xfcs))$:

$$S_\mathrm{q}^{(0)}[U, \Psi, \overline{\Psi}] = \overline{\Psi}_\mathbf{a} D^{(0)}[U]^\mathbf{a}{}_\mathbf{b} \Psi^\mathbf{b}$$



$$= (D^{(0)}[U]^{-1})^\mathbf{b}{}_\mathbf{a}$$

- Free photon-propagator $(\mathbf{c} \equiv (x\mu))$:

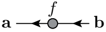
$$S_\gamma[A] = \frac{1}{2} A_{\mathbf{c_2}} \Delta^{\mathbf{c_2}}{}_{\mathbf{c_1}} A^{\mathbf{c_1}}$$

$$= (\Delta^{-1})^{\mathbf{c_2}}{}_{\mathbf{c_1}}$$

- Quark- and quark-photon-vertices:

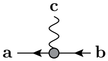
$$S_\mathrm{q}[U, A, \Psi, \overline{\Psi}] - S_\mathrm{q}^{(0)}[U, \Psi, \overline{\Psi}] = \overline{\Psi}_\mathbf{a} (D[U, A]^\mathbf{a}{}_\mathbf{b} - D^{(0)}[U]^\mathbf{a}{}_\mathbf{b}) \Psi^\mathbf{b}$$

$$= -\sum_f \Delta m_f \overline{\Psi}_\mathbf{a}$$

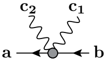


$$\Psi^\mathbf{b}$$

$$- e\, \overline{\Psi}_\mathbf{a}$$



$$\Psi^\mathbf{b} A^\mathbf{c} - \frac{1}{2} e^2\, \overline{\Psi}_\mathbf{a}$$



$$\Psi^\mathbf{b} A^{\mathbf{c_2}} A^{\mathbf{c_1}} + O(e^3)$$

$$e^{\mathrm{i}aeQA^{x\mu}} = \mathbb{1} + \mathrm{i}aeQA^{x\mu} - \frac{1}{2} a^2 e^2 Q^2 (A^{x\mu})^2 + O(e^3)$$

[12] Divitiis et al. 2013.

Perturbative expansion of QCD+QED around $\mathrm{QCD_{iso}}$

- $\Delta\beta$-vertex:

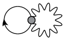
$$\frac{e^{-S_\mathrm{g}[U]}}{e^{-S_\mathrm{g}^{(0)}[U]}} = 1 + \Delta\beta$$ 

$$+ O(\Delta\beta^2)$$

- Reweighting factor $R[U] = \frac{e^{-S_\mathrm{g}[U]}\, Z_{\mathrm{q}\gamma}[U]}{e^{-S_\mathrm{g}^{(0)}[U]}\, Z_\mathrm{q}^{(0)}[U]}$:

$$R[U] = Z_\gamma^{(0)} \Big(1 + \sum_f \Delta m_f$$ 

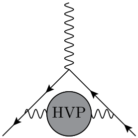
$$+ \Delta\beta$$ 

$$+ e^2 \Big($$  $$+$$  $$+$$  $$\Big) + O(\Delta\varepsilon^2) \Big)$$

$$Z_\gamma^{(0)} = \int DA\, e^{-S_\gamma[A]}$$

Observables in QCD+QED

- Pseudo-scalar meson masses m_{π^+}, m_{π^0}, m_{K^+}, m_{K^0} from $\langle \mathcal{M}\mathcal{M}^\dagger \rangle$
  $\Rightarrow$ For hadronic renormalisation scheme
- Baryon masses $m_p, m_n, m_\Omega, m_\Lambda, m_{\Sigma^0}$ from $\langle \mathcal{B}\mathcal{B}^\dagger \rangle$
  $\Rightarrow$ Search for scale setting candidate
- Renormalised HVP funtion $\hat{\Pi}_{\mathcal{V}_\mathrm{R}^\gamma \mathcal{V}_\mathrm{R}^\gamma}$ from $\langle \mathcal{V}_\mathrm{R}^\gamma \mathcal{V}_\mathrm{R}^\gamma \rangle$
  $\Rightarrow$ Renormalisation of vector currents required
- Renormalisation factors $Z_{\mathcal{V}_\mathrm{R}\mathcal{V}}$ for $\mathcal{V}_\mathrm{R} = Z_{\mathcal{V}_\mathrm{R}\mathcal{V}}\mathcal{V}$ (including mixing)
- LO-HVP contribution to a_μ from $\hat{\Pi}_{\mathcal{V}_\mathrm{R}^\gamma \mathcal{V}_\mathrm{R}^\gamma}$ / $\langle V_\mathrm{R}^\gamma V_\mathrm{R}^\gamma \rangle$
- Hadronic contribution to the running of α_em from $\hat{\Pi}_{\mathcal{V}_\mathrm{R}^\gamma \mathcal{V}_\mathrm{R}^\gamma}$ / $\langle V_\mathrm{R}^\gamma V_\mathrm{R}^\gamma \rangle$



Hadronic renormalisation scheme for QCD+QED

- Fix bare parameters am_u, am_d, am_s, β, e^2 by renormalisation scheme

- Physical point of $\mathrm{QCD_{iso}}$ is ambiguous. Convention:

$$\left(m_\pi^2\right)^{\mathrm{QCD_{iso}}} = \left(m_{\pi^0}^2\right)^{\mathrm{phys}}$$
$$\left(m_K^2\right)^{\mathrm{QCD_{iso}}} = \frac{1}{2}\left(m_{K^+}^2 + m_{K^0}^2 - m_{\pi^+}^2 + m_{\pi^0}^2\right)^{\mathrm{phys}}$$

(suggested at muon $g-2$ theory initiative workshop 2021[13])

- Goal: Extend this scheme to a full scheme for QCD+QED

- Pseudo-scalar meson masses in χPT + QED[14]:

$$\hat{m} = \frac{1}{2}(m_\mathrm{u} + m_\mathrm{d}) \qquad \varepsilon = \frac{\sqrt{3}}{4}\frac{m_\mathrm{d} - m_\mathrm{u}}{m_\mathrm{s} - \hat{m}} \quad \pi^0\text{-}\eta \text{ mixing angle}$$

At $O(e^2 p^0)$ and $O(\varepsilon)$:

$$m_{\pi^0}^2 = 2B\hat{m} \qquad m_{K^+}^2 = B\left((m_\mathrm{s} + \hat{m}) - \frac{2\varepsilon}{\sqrt{3}}(m_\mathrm{s} - \hat{m})\right) + 2e^2 Z F^2$$

$$m_{\pi^+}^2 = 2B\hat{m} + 2e^2 Z F^2 \qquad m_{K^0}^2 = B\left((m_\mathrm{s} + \hat{m}) + \frac{2\varepsilon}{\sqrt{3}}(m_\mathrm{s} - \hat{m})\right)$$

In chiral limit: F pion decay constant, B vacuum condensate parameter, Z dimensionless coupling constant

[13] Blum et al. 2021.

[14] Neufeld and Rupertsberger 1996.

Hadronic renormalisation scheme for QCD+QED

- Proxies for bare parameters $m_\mathrm{u}+m_\mathrm{d}$, m_s, $m_\mathrm{u}-m_\mathrm{d}$ and e^2:

$$m^2_{\pi^0} = B(m_\mathrm{u}+m_\mathrm{d}) \qquad m^2_{K^+}+m^2_{K^0}-m^2_{\pi^+} = 2Bm_\mathrm{s}$$
$$m^2_{K^+}-m^2_{K^0}-m^2_{\pi^+}+m^2_{\pi^0} = B(m_\mathrm{u}-m_\mathrm{d}) \qquad m^2_{\pi^+}-m^2_{\pi^0} = 2e^2ZF^2$$

$\Rightarrow$ Equate proxies in theory and experiment

- Scheme for $\mathrm{QCD_{iso}}$ ($e^2=0$, $m_\mathrm{u}=m_\mathrm{d} \Rightarrow m^2_{K^+}=m^2_{K^0}$, $m^2_{\pi^+}=m^2_{\pi^0}$):

$$m^2_{\pi^0} \qquad m^2_{K^+}+m^2_{K^0}-m^2_{\pi^+}$$

- Scheme for QCD+QED:

$$m^2_{\pi^0} \quad m^2_{K^+}+m^2_{K^0}-m^2_{\pi^+} \quad m^2_{K^+}-m^2_{K^0}-m^2_{\pi^+}+m^2_{\pi^0} \quad \alpha_\mathrm{em}$$
$$\Leftrightarrow \qquad m^2_{\pi^0} \qquad m^2_{K^0} \qquad m^2_{K^+}-m^2_{\pi^+} \qquad \alpha_\mathrm{em}$$

Alternatively:

$$m^2_{\pi^0} \quad m^2_{K^+}+m^2_{K^0}-m^2_{\pi^+} \quad m^2_{K^+}-m^2_{K^0}-m^2_{\pi^+}+m^2_{\pi^0} \quad m^2_{\pi^+}-m^2_{\pi^0}$$
$$\Leftrightarrow \qquad m^2_{\pi^0} \qquad m^2_{\pi^+} \qquad m^2_{K^0} \qquad m^2_{K^+}$$

Hadronic renormalisation scheme for QCD+QED

- α_{em} does not renormalise at leading order $\Rightarrow e^2 = 4\pi\alpha_{\mathrm{em}} = \frac{4\pi}{137.035\ldots}$
- Neglect IB effects in lattice spacing $a \Rightarrow \Delta\beta = 0$
- Match proxies for $m_{\mathrm{u}} + m_{\mathrm{d}}$ and m_{s} on each ensemble and set proxy for $m_{\mathrm{u}} - m_{\mathrm{d}}$ to physical value
- Determination of $\Delta\varepsilon = (a\Delta m_{\mathrm{u}}, a\Delta m_{\mathrm{d}}, a\Delta m_{\mathrm{s}}, \Delta\beta, e^2)$ at LO: $am_H = (am_H)^{(0)} + \sum_l \Delta\varepsilon_l (am_H)_l^{(1)} + O(\Delta\varepsilon^2)$ for $H = \pi^0, \pi^+, K^0, K^+$ $\Rightarrow$ scheme translates into a system of linear equations (simplified):

$$\sum_l \Delta\varepsilon_l \Big((am_{\pi^0})^{(0)} (am_{\pi^0})_l^{(1)} \Big) = 0$$
$$\sum_l \Delta\varepsilon_l \Big((am_{K^0})^{(0)} (am_{K^0})_l^{(1)} \Big) = 0$$
$$\sum_l \Delta\varepsilon_l \Big((am_{K^+})^{(0)} (am_{K^+})_l^{(1)} - (am_{\pi^+})^{(0)} (am_{\pi^+})_l^{(1)} \Big) = \frac{1}{2} a^{(0)} (m_{K^+}^2 - m_{K^0}^2 - m_{\pi^+}^2 + m_{\pi^0}^2)^{\mathrm{phys}}$$
$$\Delta\varepsilon_{\Delta\beta} = \Delta\beta = 0$$
$$\Delta\varepsilon_{e^2} = e^2 = 4\pi\alpha_{\mathrm{em}}$$

- Need to compute $(am_H)^{(0)}$ and $(am_H)_l^{(1)}$ for $H = \pi^0, \pi^+, K^0, K^+$ and $l = a\Delta m_{\mathrm{u}}, a\Delta m_{\mathrm{d}}, a\Delta m_{\mathrm{s}}, \Delta\beta, e^2$ using hadron spectroscopy methods

Hadron spectroscopy from two-point functions

- Time-ordered two-point function $t_2 > t_1$:
  $$C(t_2, t_1) = \langle \mathcal{O}_2(t_2) \mathcal{O}_1(t_1) \rangle$$
  $\mathcal{O}_1$ creates hadronic state, $\mathcal{O}_2$ annihilates hadronic state

- Projection on vacuum state for $T \gg t_2 > t_1 \gg 0$:
  $$C(t_2, t_1) \to \sum_n \langle 0|\mathcal{O}_2|n\rangle \langle n|\mathcal{O}_1|0\rangle \, e^{-(E_n - E_0)(t_2 - t_1)}$$
  complete set of states $|n\rangle$ with $H|n\rangle = E_n|n\rangle$

- Projection on ground state: Asymptotic behaviour for $t_2 \gg t_1$:
  $$C(t_2, t_1) \to \langle 0|\mathcal{O}_2|1\rangle \langle 1|\mathcal{O}_1|0\rangle \, e^{-(E_1 - E_0)(t_2 - t_1)} = c \, e^{-m(t_2 - t_1)}$$

- Perturbative expansion of $C(t_2, t_1)$, c and m in $\Delta\varepsilon_l$ with $l = a\Delta m_{\mathrm{u}}, a\Delta m_{\mathrm{d}}, a\Delta m_{\mathrm{s}}, \Delta\beta, e^2$:
  $$X = (X)^{(0)} + \sum_l \Delta\varepsilon_l (X)^{(1)}_l + O(\Delta\varepsilon^2)$$

- Fit asymptotic behaviour to correlation function order by order:
  $$C^{(0)}(t_2, t_1) \to c^{(0)} e^{-m^{(0)}(t_2 - t_1)}$$
  $$C^{(1)}_l(t_2, t_1) \to \left(c^{(1)}_l - c^{(0)} m^{(1)}_l (t_2 - t_1)\right) e^{-m^{(0)}(t_2 - t_1)}$$

- Similarly: Effective mass[15]:

$$(am_{\text{eff}}(t_2,t_1))^{(0)} = \log\left(\frac{(C(t_2,t_1))^{(0)}}{(C(t_2+a,t_1))^{(0)}}\right) \to am^{(0)}$$

$$(am_{\text{eff}}(t_2,t_1))_l^{(1)} = \frac{(C(t_2,t_1))_l^{(1)}}{(C(t_2,t_1))^{(0)}} - \frac{(C(t_2+a,t_1))_l^{(1)}}{(C(t_2+a,t_1))^{(0)}} \to am_l^{(1)}$$

- Extract masses from zero-momentum projected two-point functions:

$$C(x_2^0,x_1^0) = \langle \mathcal{O}_2^{x_2^0}\mathcal{O}_1^{x_1^0}\rangle \qquad \mathcal{O}_i^{x^0} = \frac{a^3}{\sqrt{|\Lambda_{123}|}}\sum_{\vec{x}} \mathcal{O}_i^x$$

- Operators have to be expanded:

$$\mathcal{O} = \mathcal{O}^{(0)} + e\mathcal{O}^{(\frac{1}{2})} + \frac{1}{2}e^2\mathcal{O}^{(1)} + O(e^3)$$

QED gauge link: $e^{\mathrm{i}aeQA^{x\mu}} = \mathbb{1} + \mathrm{i}aeQA^{x\mu} - \frac{1}{2}a^2e^2Q^2(A^{x\mu})^2 + O(e^3)$

- Pseudo-scalar masses from $\langle \mathcal{P}^{x_2^0 i_2}\mathcal{P}^{x_1^0 i_1}\rangle$:

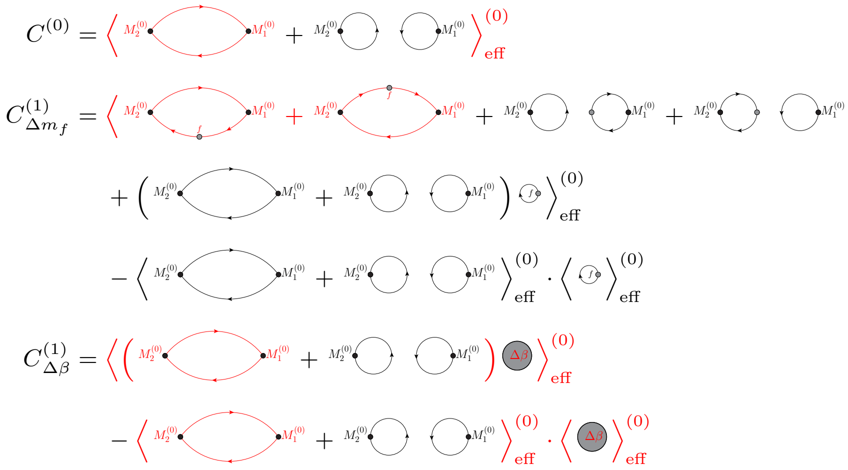
$\mathcal{P}^{xi} = \overline{\Psi}^x\Lambda^i\gamma^5\Psi^x$ $\qquad$ Λ^i determines flavour content

$\Rightarrow \mathcal{P}^{(0)} = \mathcal{P}$, $\mathcal{P}^{(\frac{1}{2})} = 0$, $\mathcal{P}^{(1)} = 0$

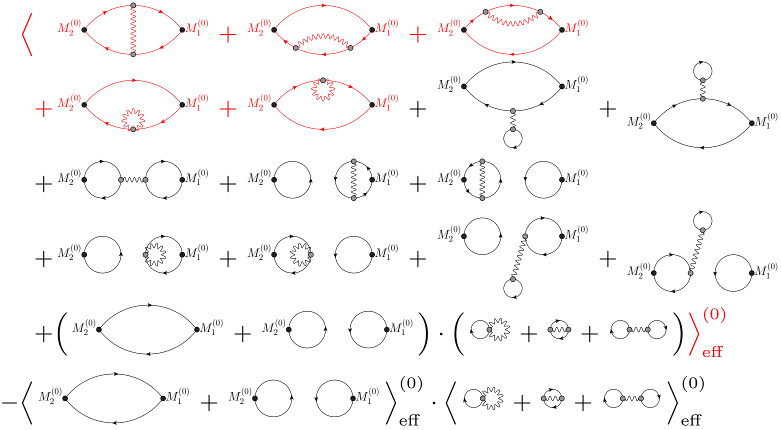
[15] Boyle et al. 2017.

Mesonic two-point functions

- Diagrammatic expansion of $C = \langle \mathcal{M}_2 \mathcal{M}_1 \rangle$ (quark-connected contributions):



$$C^{(1)}_{e^2} =$$



- Evaluated with stochastic spin-explicit $U(1)$ timeslice quark sources, generalised one-end-trick and Z_2 photon sources
- Noise reduction: covariant approximation averaging[16], truncated solver method[17]

[16] Shintani et al. 2015.
[17] Bali et al. 2010.

- Photon action for non-compact lattice QED:

$$F^{\mu_2\mu_1} = \overrightarrow{\partial}_{\mathrm{F}}^{\mu_2} A^{\mu_1} - \overrightarrow{\partial}_{\mathrm{F}}^{\mu_1} A^{\mu_2} \qquad \overrightarrow{\partial}_{\mathrm{F}}^{\mu\, x_2}{}_{x_1} = \frac{1}{a}(\delta^{x_2+a\hat{\mu}}_{x_1} - \delta^{x_2}_{x_1})$$

$$S_\gamma[A] = \frac{1}{4}\sum_x \sum_{\mu_1,\mu_2} F^{x\mu_1\mu_2} F^{x\mu_1\mu_2} + S_{\gamma\mathrm{gf}}[A] = \frac{1}{2} A_{\mathbf{c_2}} \Delta^{\mathbf{c_2}}{}_{\mathbf{c_1}} A^{\mathbf{c_1}}$$

- Quark and gauge fields U boundary conditions (bc) determined by $\mathrm{QCD_{iso}}$ setup $\Rightarrow$ Choose bc for A consistently with U

- Periodic bc: obvious

- Open temporal bc: Electric components of F vanish:

$$F^{x0\mu}|_{x^0=-a} = 0 \qquad F^{x0\mu}|_{X^0-a} = 0 \qquad \text{for } \mu = 1,2,3$$

Implemented by imposing suitable bc on A:

$$\mu = 0: \qquad A^{x0}|_{x^0=-a} = 0 \qquad A^{x0}|_{x^0=X^0-a} = 0 \qquad \text{Dirichlet bc}$$

$$\mu = 1,2,3: \qquad (\overrightarrow{\partial}_{\mathrm{F}}^{0} A^{\mu})^x|_{x^0=-a} = 0 \qquad (\overrightarrow{\partial}_{\mathrm{F}}^{0} A^{\mu})^x|_{x^0=X^0-a} = 0 \qquad \text{Neumann bc}$$

- Goal: Derive analytic expression for photon propagator Σ and $\sqrt{\Sigma}$:

$$\Sigma^{\mathbf{c_2}\mathbf{c_1}} = (\Delta^{-1})^{\mathbf{c_2}}{}_{\mathbf{c_1}} =$$ 

$$\sqrt{\Sigma}^{\mathbf{c_2}}{}_{\mathbf{c_1}} =$$ 

Photon field

- Construct basis change transformations $A^{p_2\mu_2} = \mathfrak{B}^{p_2\mu_2}{}_{x_1\mu_1} A^{x_1\mu_1}$ to block-diagonalise Δ in p-space consistent with boundary conditions:

$$\Delta^{p_4\mu_4}{}_{p_1\mu_1} = \mathfrak{B}^{p_4\mu_4}{}_{x_3\mu_3} \Delta^{x_3\mu_3}{}_{x_2\mu_2} (\mathfrak{B}^{-1})^{x_2\mu_2}{}_{p_1\mu_1}$$

  $\Rightarrow$ Use eigenfunctions of elementary difference operators $\overrightarrow{\partial}{}^{\mu}_{\mathrm{F}}$, $\overleftarrow{\partial}{}^{\mu}_{\mathrm{F}}$

- Periodic bc in μ-direction: discrete Fourier transform $\mathfrak{F}_\mu$:
  $x^\mu \in \{0, \ldots, X^\mu - a\}$, $p^\mu \in \frac{2\pi}{aX^\mu}\{0, \ldots, X^\mu - a\}$

$$\mathfrak{F}_\mu{}^{p^\mu}{}_{x^\mu} = \sqrt{\frac{a}{X^\mu}} \exp(-\mathrm{i} p^\mu x^\mu) \qquad (\mathfrak{F}_\mu^{-1})^{x^\mu}{}_{p^\mu} = \sqrt{\frac{a}{X^\mu}} \exp(\mathrm{i} p^\mu x^\mu)$$

- Temporal Dirichlet bc: discrete sine transformation $\mathfrak{S}_0$:
  $x^0 \in \{0, \ldots, X^0 - 2a\}$, $p^0 \in \frac{\pi}{aX^0}\{a, \ldots, X^0 - a\}$

$$\mathfrak{S}_0{}^{p^0}{}_{x^0} = (\mathfrak{S}_0^{-1})^{x^0}{}_{p^0} = \sqrt{\frac{2a}{X^0}} \sin(p^0(x^0 + a))$$

- Temporal Neumann bc: discrete cosine transformation $\mathfrak{C}_0$:
  $x^0 \in \{0, \ldots, X^0 - a\}$, $p^0 \in \frac{\pi}{aX^0}\{0, \ldots, X^0 - a\}$

$$\mathfrak{C}_0{}^{p^0}{}_{x^0} = (\mathfrak{C}_0^{-1})^{x^0}{}_{p^0} = \begin{cases} \sqrt{\frac{a}{X^0}} & p^0 = 0 \\ \sqrt{\frac{2a}{X^0}} \cos(p^0(x^0 + \frac{a}{2})) & p^0 \neq 0 \end{cases}$$

Photon field

- Δ block-diagonal in p-space representation $\Rightarrow$ obtain Σ in p-space representation performing algebraic 4×4 matrix inversion

- Example: Block-diagonalised photon propagator in Coulomb gauge on periodic lattice:

$$\Sigma^{p\mu_2}{}_{p\mu_1} = \frac{1}{(\sum_\mu p_{\mathrm{B}}^\mu p_{\mathrm{F}}^\mu)(\sum_{\mu\neq 0} p_{\mathrm{B}}^\mu p_{\mathrm{F}}^\mu)} \cdot \begin{pmatrix} \sum_\mu p_{\mathrm{B}}^\mu p_{\mathrm{F}}^\mu & & & \\ & p_{\mathrm{B}}^2 p_{\mathrm{F}}^2 + p_{\mathrm{B}}^3 p_{\mathrm{F}}^3 & -p_{\mathrm{B}}^2 p_{\mathrm{F}}^1 & -p_{\mathrm{B}}^3 p_{\mathrm{F}}^1 \\ & -p_{\mathrm{B}}^1 p_{\mathrm{F}}^2 & p_{\mathrm{B}}^1 p_{\mathrm{F}}^1 + p_{\mathrm{B}}^3 p_{\mathrm{F}}^3 & -p_{\mathrm{B}}^3 p_{\mathrm{F}}^2 \\ & -p_{\mathrm{B}}^1 p_{\mathrm{F}}^3 & -p_{\mathrm{B}}^2 p_{\mathrm{F}}^3 & p_{\mathrm{B}}^1 p_{\mathrm{F}}^1 + p_{\mathrm{B}}^2 p_{\mathrm{F}}^2 \end{pmatrix}^{\mu_2}_{\mu_1}$$

$$\Sigma^{(p\mu)_2}{}_{(p\mu)_1} = 0 \qquad p_2 \neq p_1.$$

Lattice momenta:

$$p_{\mathrm{F}}^\mu = -\frac{\mathrm{i}}{a}(\exp(\mathrm{i}ap^\mu) - 1) \qquad p_{\mathrm{B}}^\mu = -\frac{\mathrm{i}}{a}(1 - \exp(-\mathrm{i}ap^\mu))$$

- Construct $\sqrt{\Sigma}$ from Σ by analytical diagonalisation in μ-coordinates:

$$\sqrt{\Sigma}^p{}_p = U(p)\,\mathrm{diag}\left(\frac{1}{\sqrt{\sum_{\mu\neq 0} p_{\mathrm{B}}^\mu p_{\mathrm{F}}^\mu}}, \frac{1}{\sqrt{\sum_\mu p_{\mathrm{B}}^\mu p_{\mathrm{F}}^\mu}}, \frac{1}{\sqrt{\sum_\mu p_{\mathrm{B}}^\mu p_{\mathrm{F}}^\mu}}, 0\right)(U(p))^{-1}$$

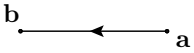
Mesonic two-point functions

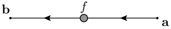
- Stochastic estimation of photon all-to-all propagator:

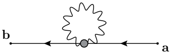
  Introduce stochastic source J with $\langle J_{\mathbf{c_2}} J^{\mathbf{c_1}} \rangle_J = \delta^{\mathbf{c_1}}_{\mathbf{c_2}}$, $J^{x\mu} \in \{+1, -1\}$

  $A[J]^{\mathbf{c_2}} = \sqrt{\Sigma}^{\mathbf{c_2}}{}_{\mathbf{c_1}} J^{\mathbf{c_1}} =$  $J^{\mathbf{c_1}}$ $\quad \langle A[J]^{\mathbf{c_2}} A[J]^{\mathbf{c_1}} \rangle_J = \Sigma^{\mathbf{c_2 c_1}} =$ 

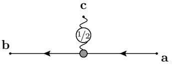
- Elementary building blocks for diagrams:

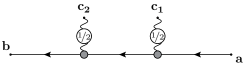
  $\Psi[\eta]^{\mathbf{b}} =$  $\eta^{\mathbf{a}}$

  $\Psi_{V_{\bar{q}qf}}[\eta]^{\mathbf{b}} =$  $\eta^{\mathbf{a}}$

  $\Psi_{V_{\bar{q}q\gamma\gamma}\Sigma}[\eta]^{\mathbf{b}} =$  $\eta^{\mathbf{a}}$

  depends on $\Sigma^{x\mu x\mu}$, direct computation using translational symmetries (boundary conditions!) $\Rightarrow$ no stochstic estimate required

  $\Psi_{V_{\bar{q}q\gamma}\sqrt{\Sigma}}[J, \eta]^{\mathbf{b}} =$  $J^{\mathbf{c}} \eta^{\mathbf{a}}$

  $\Psi_{V_{\bar{q}q\gamma}\sqrt{\Sigma} V_{\bar{q}q\gamma}\sqrt{\Sigma}}[J_2, J_1, \eta]^{\mathbf{b}} =$  $J^{\mathbf{c}_2} J^{\mathbf{c}_1} \eta^{\mathbf{a}}$

  Combine building blocks $\Rightarrow$ reduce number of inversions: $(3 + n_J) \cdot n_\eta$

- Quark-propagator and vertices are γ^5-hermitian for real A (for real J)
  $\Rightarrow \gamma^5 \Psi^\dagger \gamma^5$ gives reversed quark line

- Add spin/flavour structures of interpolation operators for contractions

Masses of pseudo-scalar mesons

- Pion and kaon masses:

	m_{π^+} [MeV]	m_{π^0} [MeV]	$m_{\pi^+} - m_{\pi^0}$ [MeV]
N200	$284.1(9)_{\mathrm{st}}(3.3)_{\mathrm{a}}[3.4]$	$281.8(9)_{\mathrm{st}}(3.3)_{\mathrm{a}}[3.4]$	$2.232(107)_{\mathrm{st}}(26)_{\mathrm{a}}[109]$
D450	$220.0(6)_{\mathrm{st}}(2.7)_{\mathrm{a}}[2.8]$	$216.7(6)_{\mathrm{st}}(2.7)_{\mathrm{a}}[2.8]$	$3.31(8)_{\mathrm{st}}(4)_{\mathrm{a}}[9]$
H102	$355.8(9)_{\mathrm{st}}(4.3)_{\mathrm{a}}[4.4]$	$353.8(9)_{\mathrm{st}}(4.3)_{\mathrm{a}}[4.3]$	$1.968(70)_{\mathrm{st}}(24)_{\mathrm{a}}[74]$

Pion masses

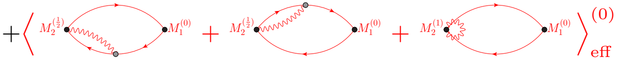
	m_{K^+} [MeV]	m_{K^0} [MeV]
N200	$460.7(5)_{\mathrm{st}}(5.3)_{\mathrm{a}}[5.4]$	$464.9(5)_{\mathrm{st}}(5.4)_{\mathrm{a}}[5.4]$
D450	$474.35(25)_{\mathrm{st}}(5.90)_{\mathrm{a}}[5.97]$	$478.27(25)_{\mathrm{st}}(5.95)_{\mathrm{a}}[5.88]$
H102	$437.0(8)_{\mathrm{st}}(5.3)_{\mathrm{a}}[5.4]$	$441.3(7)_{\mathrm{st}}(5.3)_{\mathrm{a}}[5.3]$

Kaon masses

- Leading QED finite volume corrections not included
- Scale setting error dominates error of meson masses
- Isospin partner possess compatible masses within errors
- Pion mass splitting is significant

Renormalisation of the local vector current

- Flavour-diagonal vector currents $\mathcal{V} = (\mathcal{V}^0, \mathcal{V}^3, \mathcal{V}^8)$ with $\Lambda^0 = \frac{1}{\sqrt{6}}\mathbb{1}$, $\Lambda^3 = \frac{1}{2}\lambda^3$ and $\Lambda^8 = \frac{1}{2}\lambda^8$
- Electromagnetic current $\mathcal{V}^\gamma = \mathcal{V}^3 + \frac{1}{\sqrt{3}}\mathcal{V}^8$
- Use two lattice discretisations of vector current:
  Local discretisation:
  $$\mathcal{V}_\mathrm{l}^{x\mu i} = \overline{\Psi}^x \Lambda^i \gamma^\mu \Psi^x$$
  Conserved discretisation (fulfils vector Ward identity in QCD+QED):
  $$\mathcal{V}_\mathrm{c}^{x\mu i} = \frac{1}{2}\left(\overline{\Psi}^{x+a\hat{\mu}}(W^{x\mu})^\dagger \Lambda^i(\gamma^\mu + \mathbb{1})\Psi^x + \overline{\Psi}^x \Lambda^i (\gamma^\mu - \mathbb{1}) W^{x\mu}\Psi^{x+a\hat{\mu}}\right)$$
  $$W^{x\mu} = U^{x\mu} e^{\mathrm{i}aeQA^{x\mu}} \quad \Rightarrow \quad \mathcal{V}_\mathrm{c} = \mathcal{V}_\mathrm{c}^{(0)} + e\mathcal{V}_\mathrm{c}^{(\frac{1}{2})} + \frac{1}{2}e^2\mathcal{V}_\mathrm{c}^{(1)} + O(e^3)$$
- Compute $\langle \mathcal{V}_\mathrm{l}^{x_2^0} \mathcal{V}_\mathrm{l}^{x_1^0} \rangle$ and $\langle \mathcal{V}_\mathrm{c}^{x_2^0} \mathcal{V}_\mathrm{l}^{x_1^0} \rangle$
- Additional diagrams for $\langle \mathcal{V}_\mathrm{c} \mathcal{V}_\mathrm{l} \rangle$:
  $$C_{e^2}^{(1)} = \quad \left(\langle \mathcal{M}_2^{(0)} \mathcal{M}_1^{(0)} \rangle\right)_{e^2}^{(1)}$$



$$+\left(\text{quark-disconnected contributions}\right)$$

Renormalisation of the local vector current

- Flavour-diagonal vector currents $\mathcal{V} = (\mathcal{V}^0, \mathcal{V}^3, \mathcal{V}^8)$ with $\Lambda^0 = \frac{1}{\sqrt{6}}\mathbb{1}$, $\Lambda^3 = \frac{1}{2}\lambda^3$ and $\Lambda^8 = \frac{1}{2}\lambda^8$ may mix under renormalisation:

$$\mathcal{V}_{\mathrm{l,R}} = Z_{\mathcal{V}_{\mathrm{l,R}}\mathcal{V}_{\mathrm{l}}}\mathcal{V}_{\mathrm{l}} \qquad\qquad \mathcal{V}_{\mathrm{c,R}} = Z_{\mathcal{V}_{\mathrm{c,R}}\mathcal{V}_{\mathrm{c}}}\mathcal{V}_{\mathrm{c}}$$

  Assumption: $Z_{\mathcal{V}_{\mathrm{c,R}}\mathcal{V}_{\mathrm{c}}} = \mathbb{1}$ (lattice vector Ward identity)

- Renormalisation condition[18] $\langle 0|\mathcal{V}_{\mathrm{c,R}}|V\rangle = \langle 0|\mathcal{V}_{\mathrm{l,R}}|V\rangle$:

$$\langle \mathcal{V}^{x_2^0}_{\mathrm{c,R}}\mathcal{V}^{x_1^0}_{\mathrm{l,R}}\rangle \to \langle \mathcal{V}^{x_2^0}_{\mathrm{l,R}}\mathcal{V}^{x_1^0}_{\mathrm{l,R}}\rangle \quad \text{for} \quad T \gg x_2^0 \gg x_1^0 \gg 0$$

- Expressed in terms of bare correlation functions

$$\langle \mathcal{V}^{x_2^0}_{\mathrm{c}}\mathcal{V}^{x_1^0}_{\mathrm{l}}\rangle \to Z_{\mathcal{V}_{\mathrm{l,R}}\mathcal{V}_{\mathrm{l}}}\,\langle \mathcal{V}^{x_2^0}_{\mathrm{l}}\mathcal{V}^{x_1^0}_{\mathrm{l}}\rangle \quad \text{for} \quad T \gg x_2^0 \gg x_1^0 \gg 0$$

- Define effective renormalisation factor:

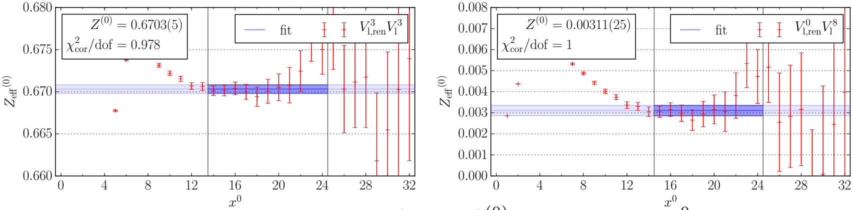
$$Z_{\mathrm{eff},\mathcal{V}_{\mathrm{l,R}}\mathcal{V}_{\mathrm{l}}}(x_2^0, x_1^0) = \left(\frac{1}{3}\sum_{\mu=1}^{3}\langle \mathcal{V}^{x_2^0\mu}_{\mathrm{c}}\mathcal{V}^{x_1^0\mu}_{\mathrm{l}}\rangle\right)\left(\frac{1}{3}\sum_{\mu=1}^{3}\langle \mathcal{V}^{x_2^0\mu}_{\mathrm{l}}\mathcal{V}^{x_1^0\mu}_{\mathrm{l}}\rangle\right)^{-1}$$
$$\to Z_{\mathcal{V}_{\mathrm{l,R}}\mathcal{V}_{\mathrm{l}}} \quad \text{for} \quad T \gg x_2^0 \gg x_1^0 \gg 0$$

- Perturb. expansion: $Z_{\mathcal{V}_{\mathrm{R}}\mathcal{V}} = (Z_{\mathcal{V}_{\mathrm{R}}\mathcal{V}})^{(0)} + \sum_l \Delta\varepsilon_l\,(Z_{\mathcal{V}_{\mathrm{R}}\mathcal{V}})^{(1)}_l + O(\Delta\varepsilon^2)$

[18] Maiani and Martinelli 1986.

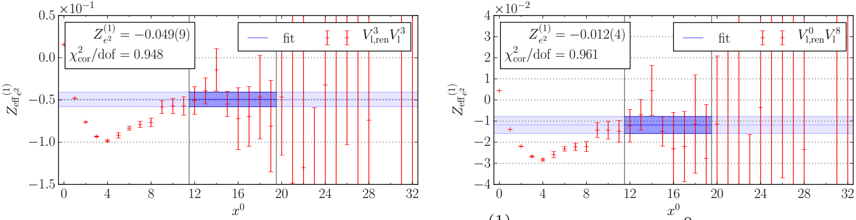
Renormalisation of the local vector current

- $(Z_{\mathcal{V}_{\mathrm{l,R}}\mathcal{V}_{\mathrm{l}}})^{(0)}$:



Effective renormalisation factors $(Z_{\mathcal{V}_{\mathrm{l,R}}\mathcal{V}_{\mathrm{l}}})^{(0)}$ as a function of x^0 on N200.

- $(Z_{\mathcal{V}_{\mathrm{l,R}}\mathcal{V}_{\mathrm{l}}})^{(1)}_l$ with $l = a\Delta m_{\mathrm{u}}, a\Delta m_{\mathrm{d}}, a\Delta m_{\mathrm{s}}, \Delta\beta, e^2$:



Effective renormalisation factors $(Z_{\mathcal{V}_{\mathrm{l,R}}\mathcal{V}_{\mathrm{l}}})^{(1)}_{e^2}$ as a function of x^0 on N200.

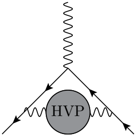
Renormalisation of the local vector current

- Results for $Z_{\text{eff},\mathcal{V}_{\text{l,R}}\mathcal{V}_\text{l}}$:

$(Z_{\text{eff},\mathcal{V}_{\text{l,R}}\mathcal{V}_\text{l}})^{(0)}$	$\begin{pmatrix} 0.6681(4) & 0.0 & 0.00311(23) \\ 0.0 & 0.6703(5) & 0.0 \\ 0.00311(23) & 0.0 & 0.66598(22) \end{pmatrix}$
$(Z_{\text{eff},\mathcal{V}_{\text{l,R}}\mathcal{V}_\text{l}})^{(1)}$	$\begin{pmatrix} -0.0016593(16) & -0.000919(34) & -0.0005236(14) \\ -0.000919(34) & -0.0020295(6) & -0.000650(24) \\ -0.0005236(14) & -0.000650(24) & -0.0012894(27) \end{pmatrix}$
$Z_{\text{eff},\mathcal{V}_{\text{l,R}}\mathcal{V}_\text{l}}$	$\begin{pmatrix} 0.6664(4) & -0.000919(34) & 0.00258(23) \\ -0.000919(34) & 0.6683(5) & -0.000650(24) \\ 0.00258(23) & -0.000650(24) & 0.66469(22) \end{pmatrix}$

Renormalisation factors $Z_{\mathcal{V}_{\text{l,R}}\mathcal{V}_\text{l}}$ for the local vector current $\mathcal{V}_\text{l}$ on N200

- $\mathcal{V}_\text{l}^3$ mixes with $\mathcal{V}_\text{l}^0$ and $\mathcal{V}_\text{l}^8$ in QCD+QED
- $(Z_{\text{eff},\mathcal{V}_{\text{l,R}}\mathcal{V}_\text{l}})^{(1)}$ is $O(1\%)$ of $(Z_{\text{eff},\mathcal{V}_{\text{l,R}}\mathcal{V}_\text{l}})^{(0)}$ for diagonal entries
- $(Z_{\text{eff},\mathcal{V}_{\text{l,R}}\mathcal{V}_\text{l}})^{(1)}$ is $O(10\%)$ of $(Z_{\text{eff},\mathcal{V}_{\text{l,R}}\mathcal{V}_\text{l}})^{(0)}$ for off-diagonal entries
- 1st order corrections are significant



- Compute a_μ^{HVP} in time-momentum representation[19]:

$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \mathrm{d}x^0\, \tilde{K}(x^0, m_\mu) \int \mathrm{d}x^3 \langle \mathcal{V}_\mathrm{R}^{\gamma x \mu_2} \mathcal{V}_\mathrm{R}^{\gamma 0 \mu_1} \rangle$$

- Renormalised electromagnetic current:

$$\mathcal{V}_\mathrm{R}^\gamma = \mathcal{V}_\mathrm{R}^3 + \frac{1}{\sqrt{3}} \mathcal{V}_\mathrm{R}^8 = \sum_{i=0,3,8} \left(Z_{\mathcal{V}_\mathrm{R}^3 \mathcal{V}^i} + \frac{1}{\sqrt{3}} Z_{\mathcal{V}_\mathrm{R}^8 \mathcal{V}^i} \right) \mathcal{V}^i$$

- Replace integration by finite sum up to x^0_{cut}: $\int_0^\infty \mathrm{d}x^0 \to \sum_{x^0=0}^{x^0_{\mathrm{cut}}}$
- $\langle \mathcal{V}_\mathrm{R}^{\gamma t_2} \mathcal{V}_\mathrm{R}^{\gamma t_1} \rangle$ exhibits noise problem for large $x^0 = t_2 - t_1$:

  $\Rightarrow$ Single state reconstruction via fit (effective description)

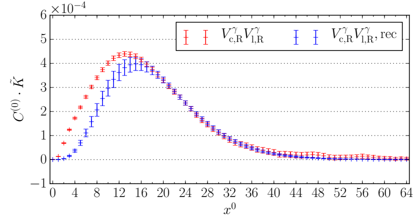
$$\langle \mathcal{V}_\mathrm{R}^{\gamma t_2} \mathcal{V}_\mathrm{R}^{\gamma t_1} \rangle_{\mathrm{rec}} = c\, e^{-m(t_2 - t_1)}$$

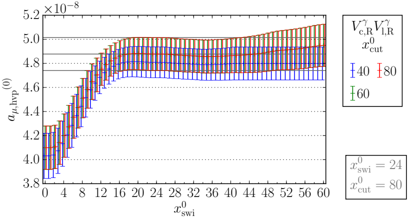
- Switch between $\langle \mathcal{V}_\mathrm{R}^{\gamma t_2} \mathcal{V}_\mathrm{R}^{\gamma t_1} \rangle$ and reconstruction $\langle \mathcal{V}_\mathrm{R}^{\gamma t_2} \mathcal{V}_\mathrm{R}^{\gamma t_1} \rangle_{\mathrm{rec}}$ at x^0_{swi}
- Perturbative expansion, neglect IB in the scale a for am_μ^{phys}

[19] Bernecker and Meyer 2011; Francis et al. 2013; Della Morte et al. 2017.

LO-HVP contribution to the muon anomalous magnetic moment a_μ

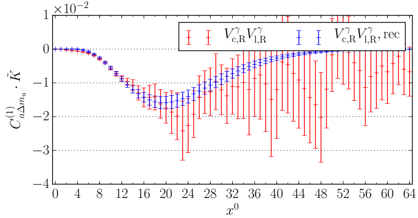
- $(a_\mu^{\mathrm{HVP}})^{(0)}$:

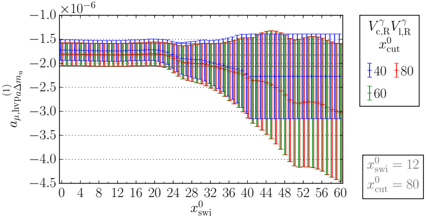




Integrand $\langle \mathcal{V}^\gamma_{\mathrm{c,R}} \mathcal{V}^\gamma_{\mathrm{l,R}} \rangle^{(0)} \cdot \tilde{K}$ and $(a_{\mu,\mathrm{HVP}})^{(0)}$ on N200. $1\,\mathrm{fm} = 15.5(1)\,a$.

- $(a_\mu^{\mathrm{HVP}})^{(1)}_l$:





Integrand $\langle \mathcal{V}^\gamma_{\mathrm{c,R}} \mathcal{V}^\gamma_{\mathrm{l,R}} \rangle^{(1)}_{a\Delta mu} \cdot \tilde{K}$ and $(a_{\mu,\mathrm{HVP}})^{(1)}_{a\Delta mu}$ on N200. $1\,\mathrm{fm} = 15.5(1)\,a$.

LO-HVP contribution to the muon anomalous magnetic moment a_μ

- Results for $a_\mu^{\rm HVP}$:

$a_\mu^{\rm HVP}$ from $\langle \mathcal{V}^\gamma_{\rm c,R} \mathcal{V}^\gamma_{\rm l,R} \rangle$

	$(a_\mu^{\rm HVP})^{(0)}$ $[10^{10}]$	$(a_\mu^{\rm HVP})^{(1)}$ $[10^{10}]$	$a_\mu^{\rm HVP}$ $[10^{10}]$	$\frac{(a_\mu^{\rm HVP})^{(1)}}{(a_\mu^{\rm HVP})^{(0)}}$ [%]
N200	$488(9)_{\rm st}(10)_{\rm a}[14]$	$-0.6[7]$	$487(9)_{\rm st}(10)_{\rm a}[13]$	$-0.12[15]$
D450	$541(8)_{\rm st}(12)_{\rm a}[15]$	$0.97[99]$	$542(9)_{\rm st}(12)_{\rm a}[15]$	$0.18[18]$
H102	$440(4)_{\rm st}(10)_{\rm a}[10]$	$1.7[4]$	$441(4)_{\rm st}(10)_{\rm a}[11]$	$0.38[8]$

$a_\mu^{\rm HVP}$ from $\langle \mathcal{V}^\gamma_{\rm l,R} \mathcal{V}^\gamma_{\rm l,R} \rangle$

	$(a_\mu^{\rm HVP})^{(0)}$ $[10^{10}]$	$(a_\mu^{\rm HVP})^{(1)}$ $[10^{10}]$	$a_\mu^{\rm HVP}$ $[10^{10}]$	$\frac{(a_\mu^{\rm HVP})^{(1)}}{(a_\mu^{\rm HVP})^{(0)}}$ [%]
N200	$491(8)_{\rm st}(11)_{\rm a}[13]$	$-0.8[7]$	$490(8)_{\rm st}(11)_{\rm a}[13]$	$-0.16[14]$
D450	$546(8)_{\rm st}(12)_{\rm a}[15]$	$1.49[99]$	$548(8)_{\rm st}(13)_{\rm a}[15]$	$0.27[18]$
H102	$445(4)_{\rm st}(10)_{\rm a}[10]$	$1.6[4]$	$447(4)_{\rm st}(10)_{\rm a}[11]$	$0.36[8]$

- Compatible results for both lattice discretisations
- Scale setting uncertainty dominates error of $(a_\mu^{\rm HVP})^{(0)}$
- $(a_\mu^{\rm HVP})^{(1)}$ is a $O(0.5\%)$ correction to $(a_\mu^{\rm HVP})^{(0)}$
- $(a_\mu^{\rm HVP})^{(1)}$ smaller than error of $(a_\mu^{\rm HVP})^{(0)}$

Hadronic contribution to the running of α_{em}

- Running of α_{em}:

$$\alpha_{\text{em}}(p^2) = \frac{\alpha_{\text{em}}}{1 - \Delta\alpha_{\text{em}}(p^2)}$$

- Hadronic contributions:

$$\Delta\alpha_{\text{em}}^{\text{had}}(-p^2) = 4\pi\alpha\, \hat{\Pi}_{\mathcal{V}_{\text{R}}^{\gamma}\mathcal{V}_{\text{R}}^{\gamma}}(p^2)$$

  with subtracted vacuum polarisation function $\hat{\Pi}(p^2) = \Pi(p^2) - \Pi(0)$

- Time-momentum representation of $\hat{\Pi}_{\mathcal{V}_{\text{R}}^{\gamma}\mathcal{V}_{\text{R}}^{\gamma}}(p^2)$[20]:

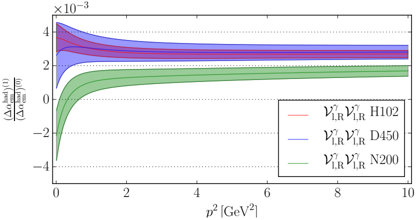
$$\hat{\Pi}_{\mathcal{V}_{\text{R}}^{\gamma}\mathcal{V}_{\text{R}}^{\gamma}}(p^2)\,\delta^{\mu_2\mu_1} = \int_0^\infty \mathrm{d}x^0\, K(p^2, x^0) \int \mathrm{d}x^3\, \langle \mathcal{V}_{\text{R}}^{\gamma x \mu_2} \mathcal{V}_{\text{R}}^{\gamma 0 \mu_1} \rangle$$

$$K(\omega^2, t) = -\frac{1}{\omega^2}\left(\omega^2 t^2 - 4\sin^2\left(\frac{\omega t}{2}\right)\right)$$

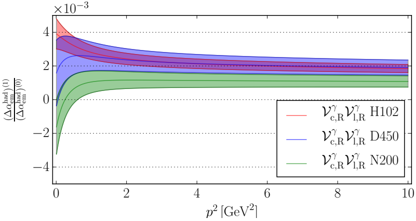
- Reconstruction of $\langle \mathcal{V}_{\text{R}}^{\gamma t_2} \mathcal{V}_{\text{R}}^{\gamma t_1} \rangle$ for large $x^0 = t_2 - t_1$ as for a_μ^{HVP}
- Perturbative expansion

[20] Bernecker and Meyer 2011.

- Relative isospin breaking correction to $\Delta\alpha_{\text{em}}^{\text{had}}(p^2)$:



from $\langle \mathcal{V}^{\gamma}_{\text{l,R}}\mathcal{V}^{\gamma}_{\text{l,R}} \rangle$



from $\langle \mathcal{V}^{\gamma}_{\text{c,R}}\mathcal{V}^{\gamma}_{\text{l,R}} \rangle$

- At $p^2 = 1\,\text{GeV}^2$: $(\Delta\alpha_{\text{em}}^{\text{had}})^{(1)}$ is a $O(0.5\%)$ correction to $(\Delta\alpha_{\text{em}}^{\text{had}})^{(0)}$
- Correction less relevant for larger p^2
- Scale setting uncertainty dominates error of $(\Delta\alpha_{\text{em}}^{\text{had}})^{(0)}$
- $(\Delta\alpha_{\text{em}}^{\text{had}})^{(1)}$ smaller than error of $(\Delta\alpha_{\text{em}}^{\text{had}})^{(0)}$

In collaboration with Alexander M. Segner, Andrew D. Hanlon and Hartmut Wittig:

- In QCD_{iso} CLS scale setting based on $\frac{2}{3}(f_K + \frac{1}{2} f_\pi)$[21]
- Computation of decay constants f_K and f_π in QCD+QED is demanding[22]:
  - infrared divergences in intermediate stages
  - cancel taking virtual photons exchanged between quarks and charged decay products as well as emitted real final state photons into account

  $\Rightarrow$ Hadron masses for scale setting preferred, e.g. Ω^-, Σ^0, Λ
- Interpolating operators for baryons based on Clebsch-Gordan construction[23]
- QCD covariant QED non-covariant smearing, Wuppertal quark field smearing on APE-smeared QCD gauge field
  (for QED covariant smearing $(\mathcal{B})^{\frac{1}{2}} \neq 0$ and $\mathcal{B}^{(1)} \neq 0 \Rightarrow$ additional diagrams)
- GEVP analysis (operator basis and Σ^0-Λ mixing)

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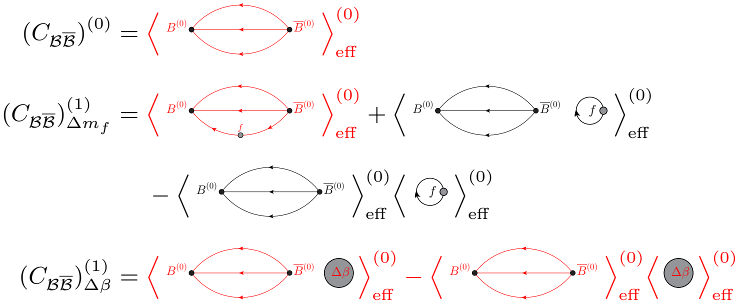
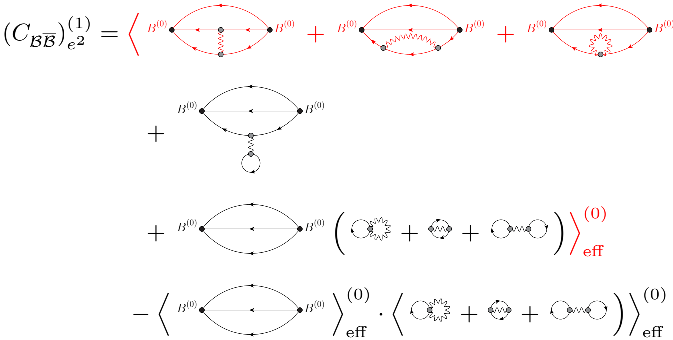
[21] Bruno et al. 2017.
[22] Carrasco et al. 2015; Giusti et al. 2018; Di Carlo et al. 2019; Desiderio 2020.
[23] Basak et al. 2005.

Baryonic two-point functions

- Diagrammatic expansion of $C = \langle \mathcal{B}\overline{\mathcal{B}} \rangle$ (quark-connected contributions):



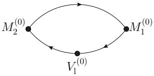
Baryonic two-point functions

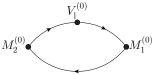


- Evaluated with quark point sources and stochastic Z_2 photon sources
- Noise reduction: covariant approximation averaging[24], truncated solver method[25]

[24] Shintani et al. 2015.
[25] Bali et al. 2010.

- Include LO QED finite volume corrections for pseudo-scalar masses[26]
- Determination of a_μ^{HVP} using bounding method[27]
- Reduce noise of $\langle VV \rangle$ using exact low-mode averaging (at small m_π)
- Determine isospin-breaking effects in octet and decouplet baryons[28]
  $\Rightarrow$ find QCD+QED scale setting candidate
- Determination of $Z_{\mathcal{V}_{\mathrm{l,R}}\mathcal{V}_{\mathrm{l}}}$ by means of vector Ward identity including IB effects[29]





+ first order diagrams

  $\Rightarrow$ Enables combination of this effort with Mainz $O(a)$-improved $\mathrm{QCD_{iso}}$ computation[30]

- Include quark-disconnected and sea-quark contributions

[26] Borsanyi et al. 2015.
[27] Borsanyi et al. 2017; Blum et al. 2018.
[28] with A. Segner, A. Hanlon and H.Wittig
[29] with M. Padmanath and H.Wittig
[30] Gérardin et al. 2019.