Constrained Kinematic Fits - Exercises

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First copy the material for these exercises. In the shell type
```
$> cp /statistics-school/ConstrainedFits/ConstrainedFits_2011.tgz .
```
Then unpack the archive, change to the new directory and source the setup script.
```
$> tar xvzf ConstrainedFits_2011.tgz
$> cd ConstrainedFits
$> .  setup.sh
```
The macros will be used in the exercises as described below.

Exercise 1: Jet Energies in Z_0 Decay

In this exercise a simple and somehow artificial example of a kinematic fit is studied. First the case of a linear constraint on the energies of two jets is considered, later the case of a constraint on their invariant mass, which is not linear any more.

As a physics scenario we consider the decay of a Z_0 into two quarks, giving a signature of two jets in the detector. We make some assumptions in order to keep this example as simple as possible:

1. The decaying Z_0 is produced at rest, therefore the momenta of the two jets have opposite directions.

2. There is no uncertainty on the direction of the jets. Only their energy measurement is subject to uncertainties. Both jets have the same energy uncertainty $\sigma_{E_1} = \sigma_{E_2}$.

3. There is no correlation between the jet energies.

That means the energies of the two jets (E_1, E_2) remain the only parameters.

(a) Energy Constraint

It follows from conservation of four-momentum that in this case $E_{Z_0} = M_{Z_0} = E_1 + E_2$. We will use this constraint in the first example:

(i) Write down the function L for this problem containing a χ^2 -term for the jet energies and the constraint of the jet energies on the Z_0 energy. (Remember that $\sigma_{E_1} = \sigma_{E_2}$)

(ii) Find the stationary point of this function, i.e. calculate the derivatives of L with respect to the parameters (jet energies E_1, E_2 and lagrange multiplier λ).

You should obtain a set of three linear equations which can be solved analytically. Try to solve the system of equations in order to find the expressions for the fitted energies E_1 and E_2 at the extremum of L. Check if fulfill the constraint. (**Do not spend more than 10 minutes on the calculation.** You can find the solution on the next pages.)

(iii) The solution to this fitting problem is implemented in the ROOT macro `exercise1a.C`. Take a look at it in a text editor:

```
$> emacs exercise1a.C
```

Read quickly through the macro and make a list of the quantities which will be plotted when running it. Sketch what you would expect for the

- Individual jet energies before and after the fit.
- Distribution in E_1 - E_2 plane before and after the fit
- Dijet energy $(E_1 + E_2)$ before and after the fit
- χ^2-distribution (A plot of χ^2-curves for different ndf can be found in the lecture on p.5)

Then start ROOT and run the macro:

```
$> root
root [0] .x exercise1a.C
```

Compare the plots to your expectations. Then inspect the plots related to the fit quality (χ^2, χ^2-Probability, Pulls). Would you trust the results of this fit? Why (not)?

Note that in this case the pulls are defined as normalized corrections w.r.t to the measured values, not the true ones.

$$p_i = \frac{\Delta E_i}{\sigma_{\Delta E_i}}$$

with

$$\sigma_{\Delta E_i} = \sqrt{\sigma^2_{E_i} - \sigma^2_{E_i^{fit}}}$$

(iv) Now edit the macro and change the input parameters of the 'jets'. What is the fit result in case of systematically shifted input energies? What happens if the uncertainties are not the same?

(b) Invariant Mass Constraint

Now we try to apply a constraint on the invariant mass of the two jets. As they are back-to-back their fourvectors can be written $p_i = (E_i, 0, 0, \pm E_i)$. The invariant mass then becomes:

$$M_{\text{inv}} = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 = 4 \cdot E_1 \cdot E_2$$

This constraint is not linear anymore and the iterative solution is implemented in the ROOT macro `exercise1b.C`. The corresponding formulae can be found on the next page.

(i) Which distributions do you expect to change with respect to the first part? What will the distribution of the dijet invariant mass look like? Check it by running the macro.

```
$> root
root [0] .x exercise1b.C
```

Solution Exercise 1a:

(a) The function to be minimized is

$$L = \frac{(\hat{E}_1 - E_1)^2}{\sigma_{E_1}^2} + \frac{(\hat{E}_2 - E_2)^2}{\sigma_{E_2}^2} + \lambda \cdot (E_1 + E_2 - M_Z)$$

where $\hat{E}_{1/2}$ are the input energies while $E_{1/2}$ denote the energies obtained in the fit.

(b) For finding the stationary point one needs to calculate the derivatives w.r.t. the three parameters (E_1, E_2, λ):

$$\begin{aligned}
\frac{\partial L}{\partial E_1} &= -2\frac{(\hat{E}_1 - E_1)}{\sigma_{E_1}^2} + 2\lambda \stackrel{!}{=} 0 \\
\frac{\partial L}{\partial E_2} &= -2\frac{(\hat{E}_2 - E_2)}{\sigma_{E_2}^2} + 2\lambda \stackrel{!}{=} 0 \\
\frac{\partial L}{\partial \lambda} &= E_1 + E_2 - M_Z \stackrel{!}{=} 0
\end{aligned}$$

As solution of this system of equations one finds:

$$E_1 = \frac{\hat{E}_1 \cdot \sigma_{E_2}^2 + (M_Z - \hat{E}_2) \cdot \sigma_{E_1}^2}{(\sigma_{E_1}^2 + \sigma_{E_2}^2)} \tag{1}$$

$$E_2 = \frac{\hat{E}_2 \cdot \sigma_{E_1}^2 + (M_Z - \hat{E}_1) \cdot \sigma_{E_2}^2}{(\sigma_{E_1}^2 + \sigma_{E_2}^2)} \tag{2}$$

$$\lambda = \frac{(\hat{E}_1 + \hat{E}_2 - M_Z)}{(\sigma_{E_1}^2 + \sigma_{E_2}^2)}$$

As could be expected the solution is symmetric in E_1, E_2.

In the case of a common uncertainty $\sigma_{E_1} = \sigma_{E_2} = \sigma$ the result is

$$\begin{aligned}
E_1 &= \frac{M_Z}{2} + \frac{1}{2} \cdot (\hat{E}_1 - \hat{E}_2) \\
E_2 &= \frac{M_Z}{2} + \frac{1}{2} \cdot (\hat{E}_2 - \hat{E}_1) \\
\lambda &= \frac{(\hat{E}_1 + \hat{E}_2 - M_Z)}{2\sigma^2}
\end{aligned}$$

For the calculation of the pulls we need the variance of the fitted energies. It can be

obtained using the coordinate transformations given by equations (1) and (2).

$$\begin{aligned}
\sigma^2_{E_1^{fit}} &= \left|\frac{\partial E_1}{\partial \hat{E}_1}\right|^2 \cdot \sigma^2_{E_1} + \left|\frac{\partial E_1}{\partial \hat{E}_2}\right|^2 \cdot \sigma^2_{E_2} \\
&= \left|\frac{\sigma^2_{E_2}}{(\sigma^2_{E_1} + \sigma^2_{E_2})}\right|^2 \cdot \sigma^2_{E_1} + \left|\frac{-\sigma^2_{E_1}}{(\sigma^2_{E_1} + \sigma^2_{E_2})}\right|^2 \cdot \sigma^2_{E_2}
\end{aligned}$$

And in the same way

$$\sigma^2_{E_2^{fit}} = \left|\frac{-\sigma^2_{E_2}}{(\sigma^2_{E_1} + \sigma^2_{E_2})}\right|^2 \cdot \sigma^2_{E_1} + \left|\frac{\sigma^2_{E_1}}{(\sigma^2_{E_1} + \sigma^2_{E_2})}\right|^2 \cdot \sigma^2_{E_2}$$

Again this simplifies for common uncertainties to

$$\sigma^2_{fit} = \frac{1}{2}\sigma^2$$

Using these new variances one can calculate the normalized corrections to the parameters (pulls).

$$p_i = \frac{\Delta E_i}{\sigma_{\Delta E_i}}$$

with

$$\sigma_{\Delta E_i} = \sqrt{\sigma^2_{E_i} - \sigma^2_{E_i^{fit}}}$$

Solution Exercise 1b:

In case of non-linear constraints the solution can be found iteratively, solving the following matrix equation in each iteration.

$$\begin{pmatrix} 1/\sigma_1^2 & 0 & 4E_2 \\ 0 & 1/\sigma_2^2 & 4E_1 \\ 4E_2 & 4E_1 & 0 \end{pmatrix} \begin{pmatrix} \Delta E_1 \\ \Delta E_2 \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$$

where $c = B \cdot \Delta y^* + f^* = \begin{pmatrix} 4E_2^* & 4E_1^* \end{pmatrix} \begin{pmatrix} \Delta E_1^* \\ \Delta E_2^* \end{pmatrix} + (4E_1^* E_2^* - M_{Z_0}^2)$ from the previous iteration.

By matrix inversion one obtains the solution (omitting the *s here):

$$\begin{aligned} \Delta E_1 &= \frac{E_2 \sigma_1^2}{4(E_1^2 \sigma_2^2 + E_2^2 \sigma_1^2)} \cdot c \\ \Delta E_2 &= \frac{E_1 \sigma_2^2}{4(E_1^2 \sigma_2^2 + E_2^2 \sigma_1^2)} \cdot c \\ \lambda &= -\frac{1}{16(E_1^2 \sigma_2^2 + E_2^2 \sigma_1^2)} \cdot c \end{aligned}$$

The covariance matrix for the fitted energies then reads:

$$V_{fit} = \frac{\sigma_1^2 \sigma_2^2}{E_1^2 \sigma_2^2 + E_2^2 \sigma_1^2} \begin{pmatrix} E_1^2 & -E_1 E_2 \\ -E_1 E_2 & E_2^2 \end{pmatrix}$$

Exercise 2: Jets in Top Quark Decay

The physics case considered in the following exercises are top-quark pair decays as they will be measured at the LHC. We use a sample of selected Monte Carlo events generated with Pythia for the LHC center-of-mass energy of $\sqrt{s} = 14$ TeV.

Furthermore we now consider all of the jets' parameters in the laboratory frame. The jet fourvectors are parametrized with transverse energy (E_T), pseudorapidity (η) and the azimuthal angle (ϕ). A C++ tool[1] is used in the macros for the actual fitting of this multi-parameter problems. As the problems become more complex, these exercises will mainly serve for illustration.

First the hadronic decay of a top quark will be studied. We only consider the case with a correct association of the jets to the W/top.

(a) Fit of two jets to the W Mass

W bosons are produced in the decay $t \to W + b$, implying that the invariant mass of the W boson and the b quark jet yields the top-quark mass. The two jets from the W decay are being constrained to the W mass: $(p_1 + p_2)^2 - M_W^2 = 0$. As extension of the previous exercise one can write:

$$L = \sum_{i=1}^{2} \left(\frac{\hat{E}_i - E_i}{\sigma_{E_i}^2} + \frac{\hat{\eta}_i - \eta_i}{\sigma_{\eta_i}^2} + \frac{\hat{\phi}_i - \phi_i}{\sigma_{\phi_i}^2} \right) + \lambda \cdot ((p_1 + p_2)^2 - M_W^2)$$

The iterative determination of the stationary point will be done by the fitting tool.

(i) Sketch your expectation for the following distributions before and after the fit:

- Invariant mass of the W jets system
- Invariant mass of all three jets (top mass)
- Resolution ($\frac{y^{\text{rec}} - y^{\text{gen}}}{y^{\text{gen}}}$) for the jets' E_T , W and top mass.

(ii) The fit is implemented in the ROOT macro `exercise2a.C`. Start root, load the library of the fitting tool and compile the macro before running: (Did you run the `setup.sh` script before?)

```
$> root
root [0] gSystem->Load("libkinfitter.so")
root [1] .L exercise2a.C+
root [2] run2a()
```

Compare the plots to your expectations.

(b) Fit of three jets to the top-mass

In addition to part (a) the b-jet is included and all three jets are constrained to the top-mass in a second constraint: $(p_1 + p_2 + p_b)^2 - M_{\text{top}}^2 = 0$.
In principle such a setup could be used for jet calibration. Here we study only the impact of the additional constraint.
(i) What will be the effect of the constraint for the resolution of the b-jet? How many degrees of freedom does the fit have? How should the χ^2-distribution change w.r.t. to part (a)?

(ii) This is implemented in the ROOT macro `exercise2b.C`. Start root, load the library of the fitting tool and compile the macro before running:

```
$> root
root [0] gSystem->Load("libkinfitter.so")
root [1] .L exercise2b.C+
root [2] run2b()
```

Compare the distributions to those from part (a). Does the χ^2 meet your expectations?

Exercise 3: Fitting Top-Quark Pair Decays

Now the full event with its two decaying top pairs is included in the fit.

(a) Fit of hadronic top-pair event with an equal top mass constraint

The W mass constraint is now applied on both decay branches. In addition we require the two top masses, i.e. the invariant masses of $W + b$ jets to be the same.
(i) This is implemented in the ROOT macro `exercise3a.C`. Start root, load the library of the fitting tool and compile the macro before running:

```
$> root
root [0] gSystem->Load("libkinfitter.so")
root [1] .L exercise3a.C+
root [2] run3a()
```

How does this improve the top mass reconstruction w.r.t. exercise 2a? Can you trust these fit results?

(b) Fit of semi-leptonic top pair event

If one top quark decays leptonically the neutrino momentum enters the fit as unmeasured parameters.

(i) Which and how many parameters enter the fit? How many constraints are needed for an overconstrained kinematic fit of the event? Which constraints can be used in this example? How many degrees of freedom does the fit then have?
(ii) Here we use the two constraints on the W mass, two constraints from p_T-balance and the additional constraint of an equal top mass on both branches. This is implemented in the ROOT macro `exercise3b.C`. Start root, load the library of the fitting tool and compile the macro before running:

```
$> root
root [0] gSystem->Load("libkinfitter.so")
root [1] .L exercise3b.C+
root [2] run3b()
```

How does the fit change w.r.t. to the fully hadronic channel? Is the neutrino momentum being correctly reconstructed? Does the χ^2 correspond to the counted degrees of freedom?

(iii) The fit setup can be changed to use other constraints. Edit the macro `exercise3b.C` (lines 172-179) to check what happens if one

- excludes the equal top mass constraint.
- adds the constraints on the top mass.
- ...

References

[1] J. D'Hondt et al., Fitting of Event Topologies with External Kinematic Constraints in CMS, **CMS AN 2005/025**