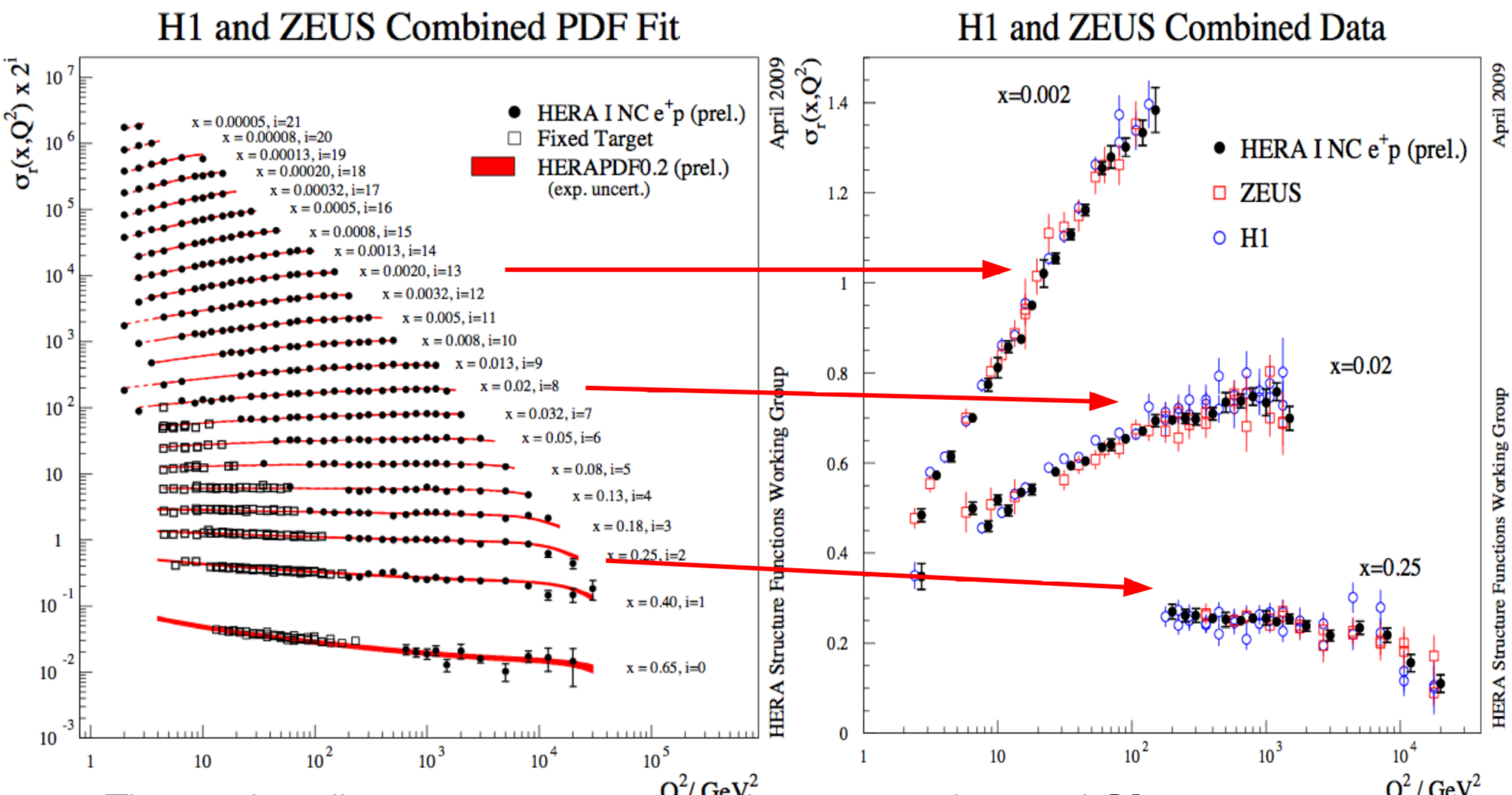
QCD and Monte Carlo techniques

- Hope that you are all ok !

How to apply these results ?

Applying DGLAP to DIS data ...

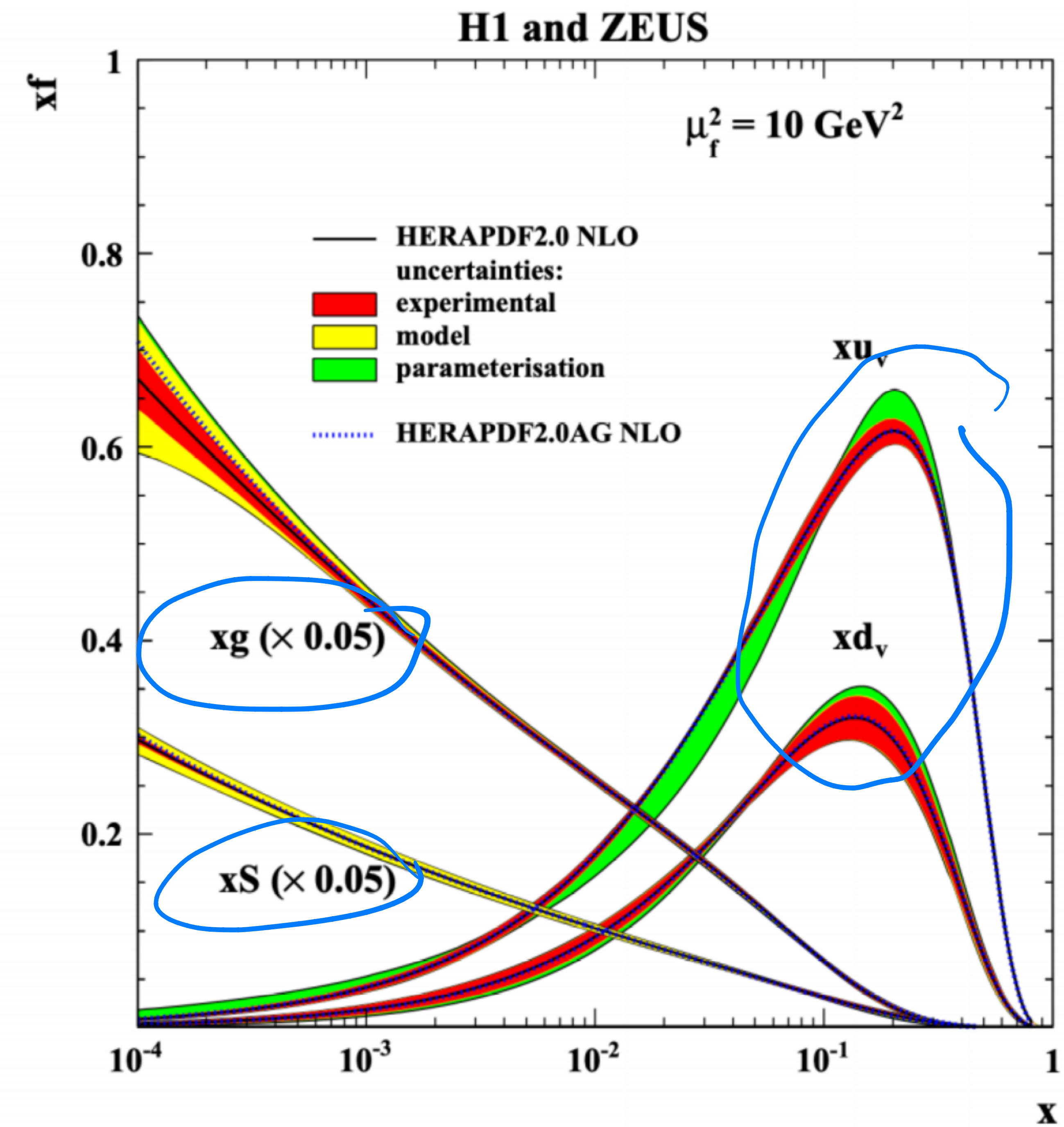


- Theory describes measurement over huge range in x and Q^2
- Success of theory (DGLAP)

Extraction of PDFs from DGLAP fits

- Sum rules are essential to constrain starting distributions

- Solve DGLAP equations
- adjust input parameters (starting distributions) such that F2 is best described
- extract PDFs as fct of x
- then DGLAP gives PDFs at any Q^2



Solving DGLAP equations ...

- Different methods to solve integro-differential equations
  - brute-force (BF) method (M. Miyama, S. Kumano CPC 94 (1996) 185)

$$\frac{df(x)}{dx} = \frac{f(x)_{m+1} - f(x)_m}{\Delta x_m} \qquad \int f(x)dx = \sum f(x)_m \Delta x_m$$

  - Laguerre method (S. Kumano J.T. Londergan CPC 69 (1992) 373, and L. Schoeffel Nucl.Instrum.Meth.A423:439-445,1999)
  - Mellin transforms (M. Glueck, E. Reya, A. Vogt Z. Phys. C48 (1990) 471)
  - QCDNUM: calculation in a grid in x,Q2 space (M. Botje Eur.Phys.J. C14 (2000) 285-297)
  - CTEQ evolution program in x,Q2 space: http://www.phys.psu.edu/~cteq/
  - QCDFIT program in x,Q2 space (C. Pascaud, F. Zomer, LAL preprint LAL/94-02, H1-09/94-404,H1-09/94-376)
  - MC method using Markov chains (S. Jadach, M. Skrzypek hep-ph/0504205)
  - Parton Branching Method (JHEP, 01(2018), 070, Phys. Lett. B, 772(2017), 446)
  - Monte Carlo method from iterative procedure
- brute-force method and MC method are best suited for detailed studies of branching processes !!!

Evolution code in LHAPDF

lhapdf is hosted by Hepforge, IPPP Durham

LHAPDF 6.4.0

Main page | PDF sets | Class hierarchy | Functions | Examples | More... | Search

LHAPDF Documentation

Introduction

LHAPDF is a general purpose C++ interpolator, used for evaluating PDFs from discretised data files. Previous versions of **LHAPDF** were written in Fortran 77/90 and are documented at http://lhapdf.hepforge.org/lhapdf5/.

LHAPDF6 vastly reduces the memory overhead of the Fortran **LHAPDF** (from gigabytes to megabytes!), entirely removes restrictions on numbers of concurrent PDFs, allows access to single PDF members without needing to load whole sets, and separates a new standardised PDF data format from the code library so that new PDF sets may be created and released easier and faster. The C++ LHAPDF6 also permits arbitrary parton contents via the standard PDG ID code scheme, is computationally more efficient (particularly if only one or two flavours are required at each phase space point, as in PDF reweighting), and uses a flexible metadata system which fixes many fundamental metadata and concurrency bugs in LHAPDF5.

Table of Contents

Compatibility routines are provided as standard for existing C++ and Fortran codes using the LHAPDF5 and PDFLIB legacy interfaces, so you can keep using your existing codes. But the new interface is much more powerful and pleasant to work with, so we think you'll want to switch once you've used it!

LHAPDF6 is documented in more detail in http://arxiv.org/abs/1412.7420

Can use LHAPDF to evolve starting distribution to any Q^2 with

- CTEQ, QCDNUM, and other evolution packages...

Graphical Plotter interface: TMDplotter & TMDlib

- TMDlib proposed in 2014 as part of REF workshop and developed since
- combine and collect different ansaetze and approaches:

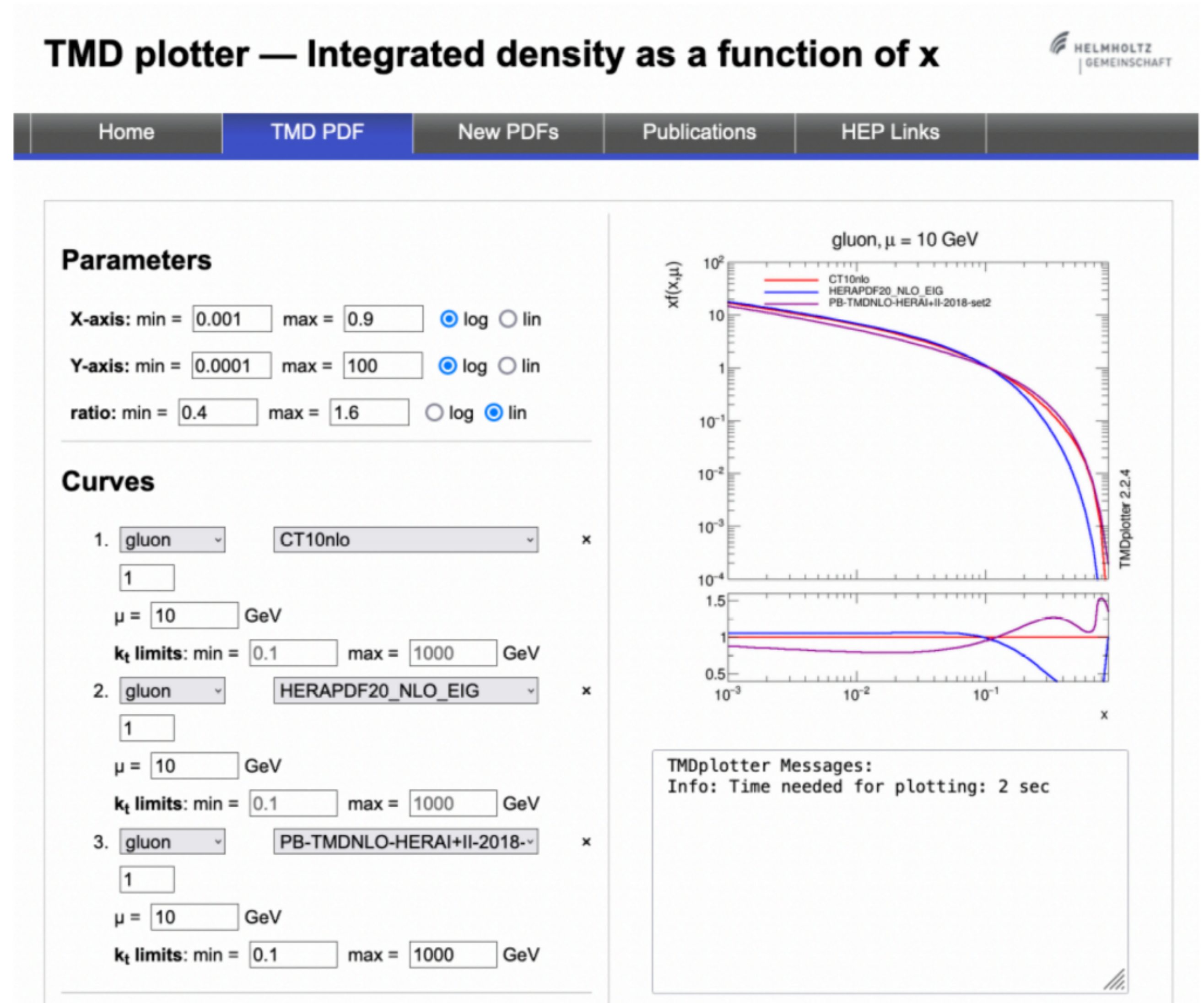
  http://tmd.hepforge.org/ and http://tmdplotter.desy.de

→ TMDlib: a library of parametrization of different TMDs and uPDFs (similar to LHApdf)

TMDlib and TMDplotter: library and plotting tools for transverse-momentum-dependent parton distributions, *F. Hautmann et al.* arXiv 1408.3015, Eur. Phys. J., C 74(12):3220, 2014.

Abdulov, N. A. and others TMDlib2 and TMDplotter: a platform for 3D hadron structure studies, Eur. Phys. J. C, 81(2021), 752

→ Also integrated pdfs (including photon pdf are available via LHAPDF)



Solution of DGLAP equation:

What happens at small x ?

What is happening at small x ?

- For $x \to 0$ only gluon splitting function matters:

$$P_{gg} = 6\left(\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right) = 6\left(\frac{1}{z} - 2 + z(1-z) + \frac{1}{1-z}\right)$$

$$P_{gg} \sim 6\frac{1}{z} \text{ for } z \to 0$$

- evolution equation is then:

$$\frac{dg(x,\mu^2)}{d\log\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi} g(\xi,\mu^2) P_{gg}\left(\frac{x}{\xi}\right)$$

$$xg(x,t) = xg(x,\mu_0^2) + \frac{3\alpha_s}{\pi}\int_{t_0}^{t} d\log t' \int_x^1 \frac{d\xi}{\xi}\xi g(\xi,t')$$ with $t = \mu^2$

$$x g(x,\mu^2) = x g(x,\mu_0^2) + \frac{3\alpha_s}{\pi} \int_{\mu_0^2}^{\mu^2} d\log\mu'^2 \int_x^1 d\log\xi \; \xi g(\xi,\mu'^2)$$

Fredholm type integral equation

$$x g_0 = C$$

$$x g_1 = C + \frac{3\alpha_s}{\pi} \int_{\mu_0}^{\mu^2} d\log\mu'^2 \int_x^1 d\log\xi \cdot C$$

$$= C + \frac{3\alpha_s}{\pi} \log \mu^2/\mu_0^2 \; \log 1/x \cdot C$$

$$x g_2 = C + \frac{3\alpha_s}{\pi} \log\mu^2/\mu_0^2 \log 1/x \cdot C + \left(\frac{3\alpha_s}{\pi}\right)^2 \int d\log\mu'^2 \, d\log\xi \; \log\frac{\mu'^2}{\mu_0^2} \log 1/\xi \; C$$

$$= \left(\frac{3\alpha_s}{\pi}\right)^2 \cdot \frac{1}{2} \log^2 \frac{\mu^2}{\mu_0^2} \frac{1}{2} \log^2 1/\xi \cdot C$$

$$x g_i = \sum_i \left(\frac{3\alpha_s}{\pi}\right)^i \left(\frac{1}{i!}\right)^2 \left(\log \mu^2/\mu_0^2\right)^i \cdot \left(\log 1/\xi\right)^i \cdot C$$

modified Bessel function!

$$x g(x,t) = C \cdot \exp\sqrt{\log\mu^2/\mu_0^2 \cdot \log 1/x}$$

Solving integral equations

Weisstein, Eric W. "Integral Equation Neumann Series." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/IntegralEquationNeumannSeries.htm

- Integral equation of Fredholm type:

$$\phi(x) = f(x) + \lambda \int_a^b K(x,y)\phi(y)dy$$

- solve it by iteration (Neumann series):

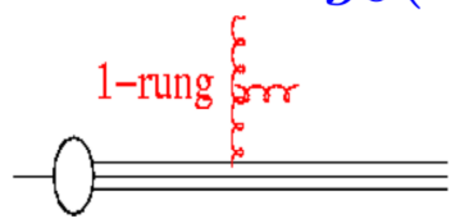
$$\begin{aligned}
\phi_0(x) &= f(x) \\
\phi_1(x) &= f(x) + \lambda \int_a^b K(x,y) f(y) dy \\
\phi_2(x) &= f(x) + \lambda \int_a^b K(x,y_1) f(y_1) dy_1 + \lambda^2 \int_a^b \int_a^b K(x,y_1) K(y_1,y_2) f(y_2) dy_2 dy_1 \\
\phi_n(x) &= \sum_{i=0}^{n} \lambda^i u_i(x) \\
u_0(x) &= f(x) \\
u_1(x) &= \int_a^b K(x,y) f(y) dy \\
u_n(x) &= \int_a^b \cdots \int_a^b K(x,y_1) K(y_1,y_2) \cdots K(y_{n-1},y_n) f(y_n) dy_1 \cdots dy_n
\end{aligned}$$

with the solution: $$\phi(x) = \lim_{n\to\infty} q_n(x) = \lim_{n\to\infty} \sum_{i=0}^{n} \lambda^i u_i(x)$$

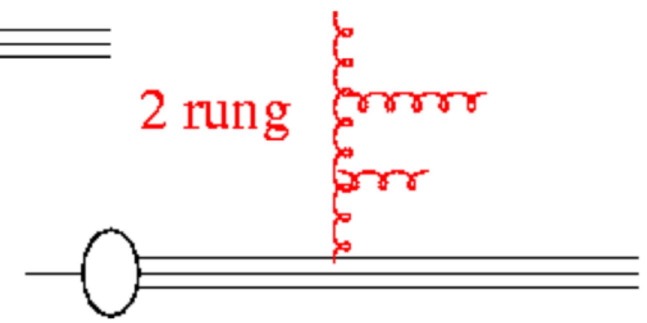
Estimates at small x: DLL

$$xg(x,t) = xg(x,\mu_0^2) + \frac{3\alpha_s}{\pi}\int_{t_0}^{t} d\log t' \int_x^1 \frac{d\xi}{\xi}\xi g(\xi,t') \quad \text{with} \quad t = \mu^2$$

- use constant starting distribution at small t: $xg_0(x) = C$

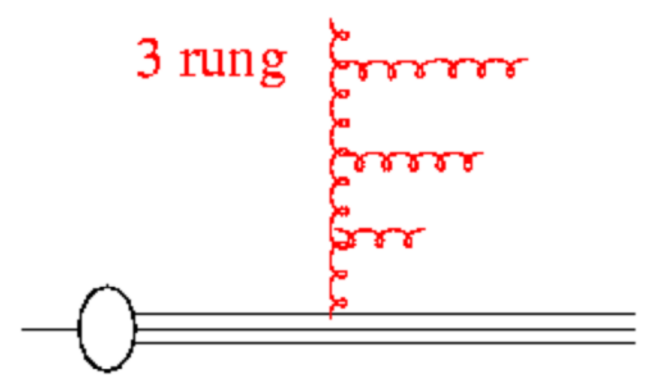


$$xg_1(x,t) = C + \frac{3\alpha_s}{\pi}\log\frac{t}{t_0}\log\frac{1}{x}C$$



$$xg_2(x,t) = C + \frac{1}{2}\frac{1}{2}\left(\frac{3\alpha_s}{\pi}\log\frac{t}{t_0}\log\frac{1}{x}\right)^2 C$$

$$\vdots$$



$$xg_n(x,t) = C + \frac{1}{n!}\frac{1}{n!}\left(\frac{3\alpha_s}{\pi}\log\frac{t}{t_0}\log\frac{1}{x}\right)^n C$$

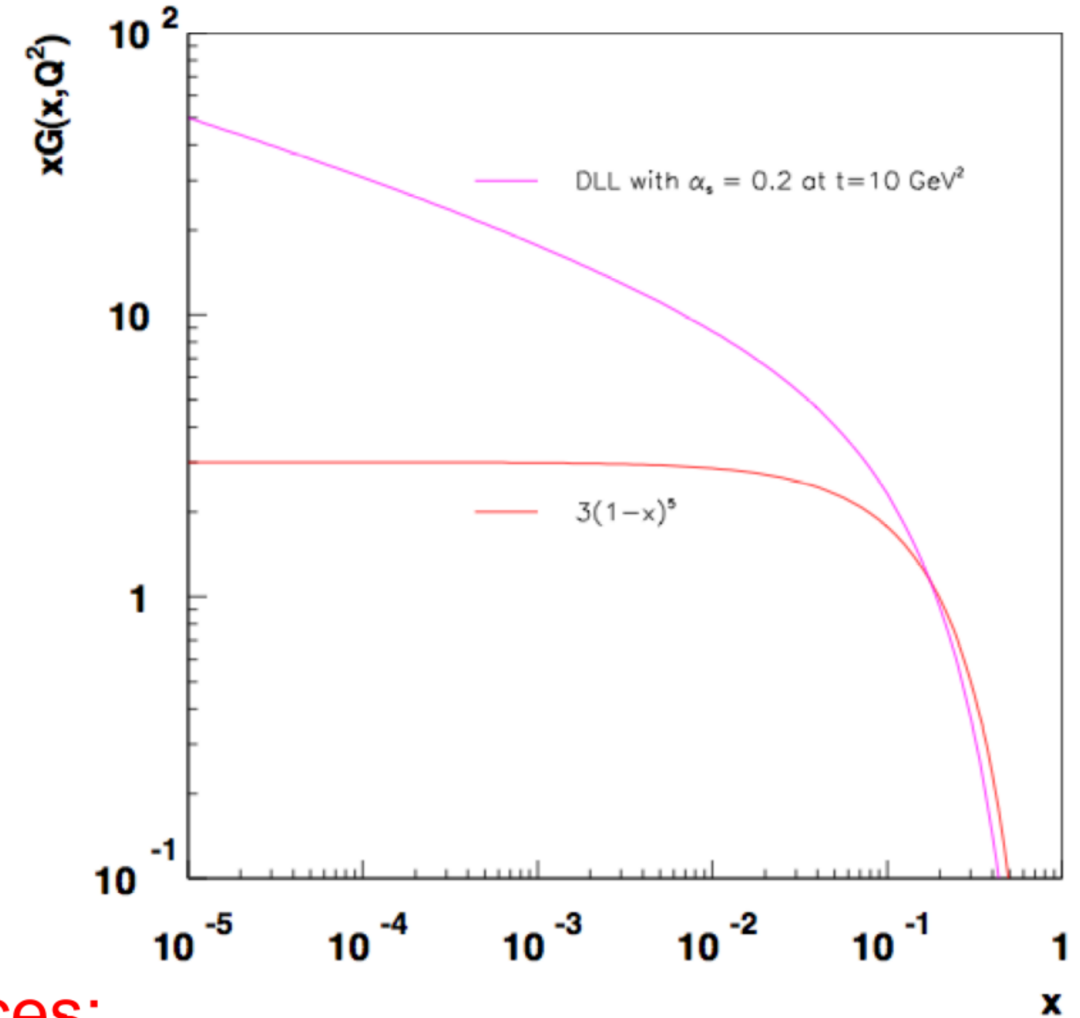
$$xg(x,t) = \sum_n \frac{1}{n!}\frac{1}{n!}\left(\frac{3\alpha_s}{\pi}\log\frac{t}{t_0}\log\frac{1}{x}\right)^n C$$

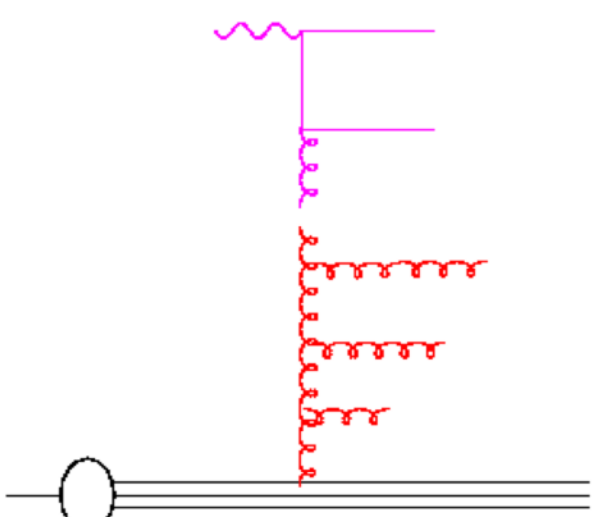
$$xg(x,t) \sim C\exp\left(2\sqrt{\frac{3\alpha_s}{\pi}\log\frac{t}{t_0}\log\frac{1}{x}}\right)$$

double leading log approximation (DLL)

Results from DLL approximation

- DLL arise from taking small x limit of splitting fct: $xg(x,t) \sim C \exp\left(2\sqrt{\frac{3\alpha_s}{\pi}\log\frac{\mu^2}{\mu_0^2}\log\frac{1}{x}}\right)$
  - $\log 1/x$ from small *x* limit of splitting fct
    $\log \mu^2/\mu_0^2$ from μ integration
  - strong ordering in *x* from small *x* limit
  - strong ordering in *t* from small *t* limit of ME...
- DLL gives rapid increase of gluon density from a flat starting distribution
- gluons are coupled to F_2... strong rise of F_2 at small x:



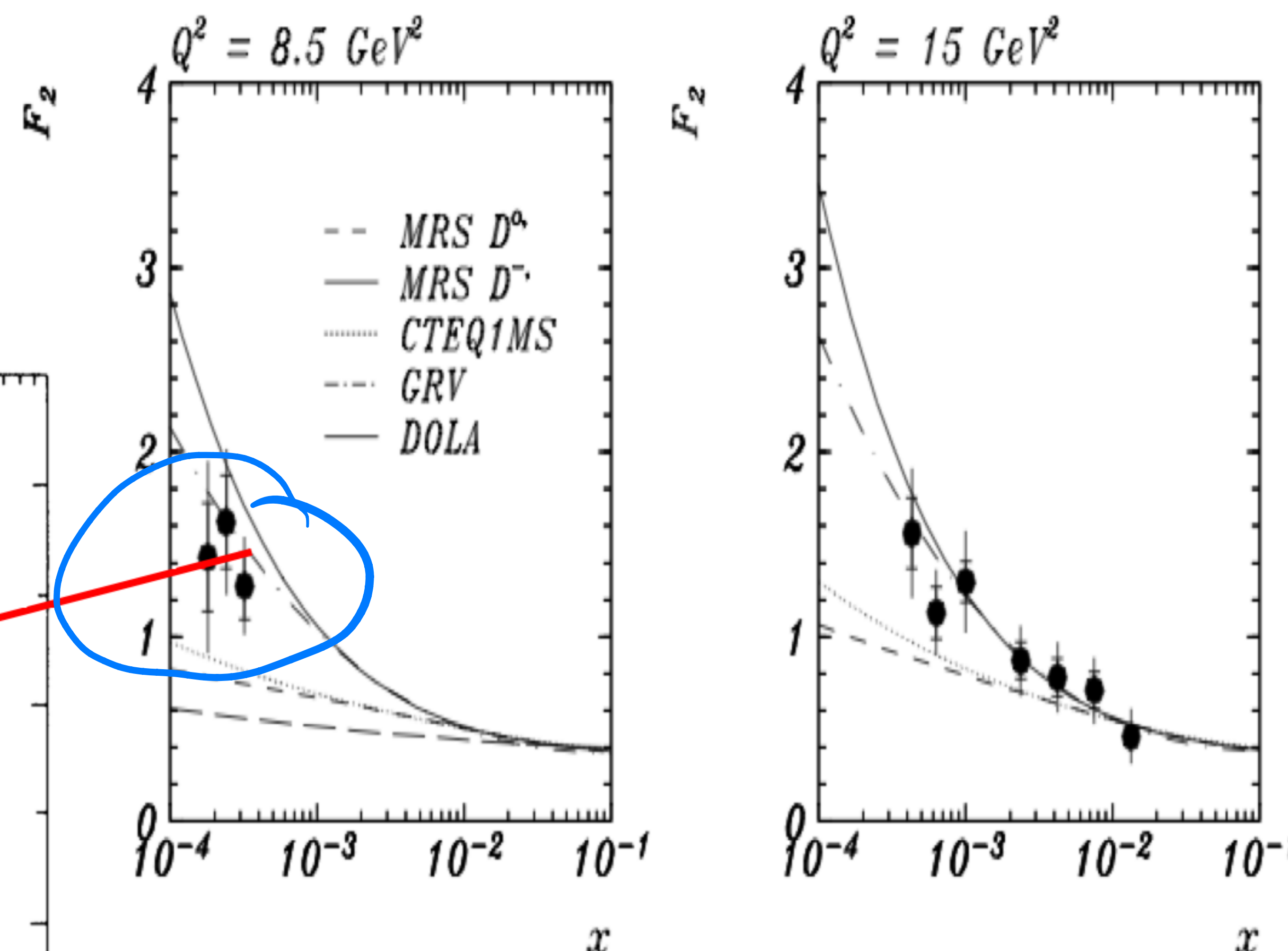


- consequences:
- rise continues forever ???
- what happens when too high gluon density ?

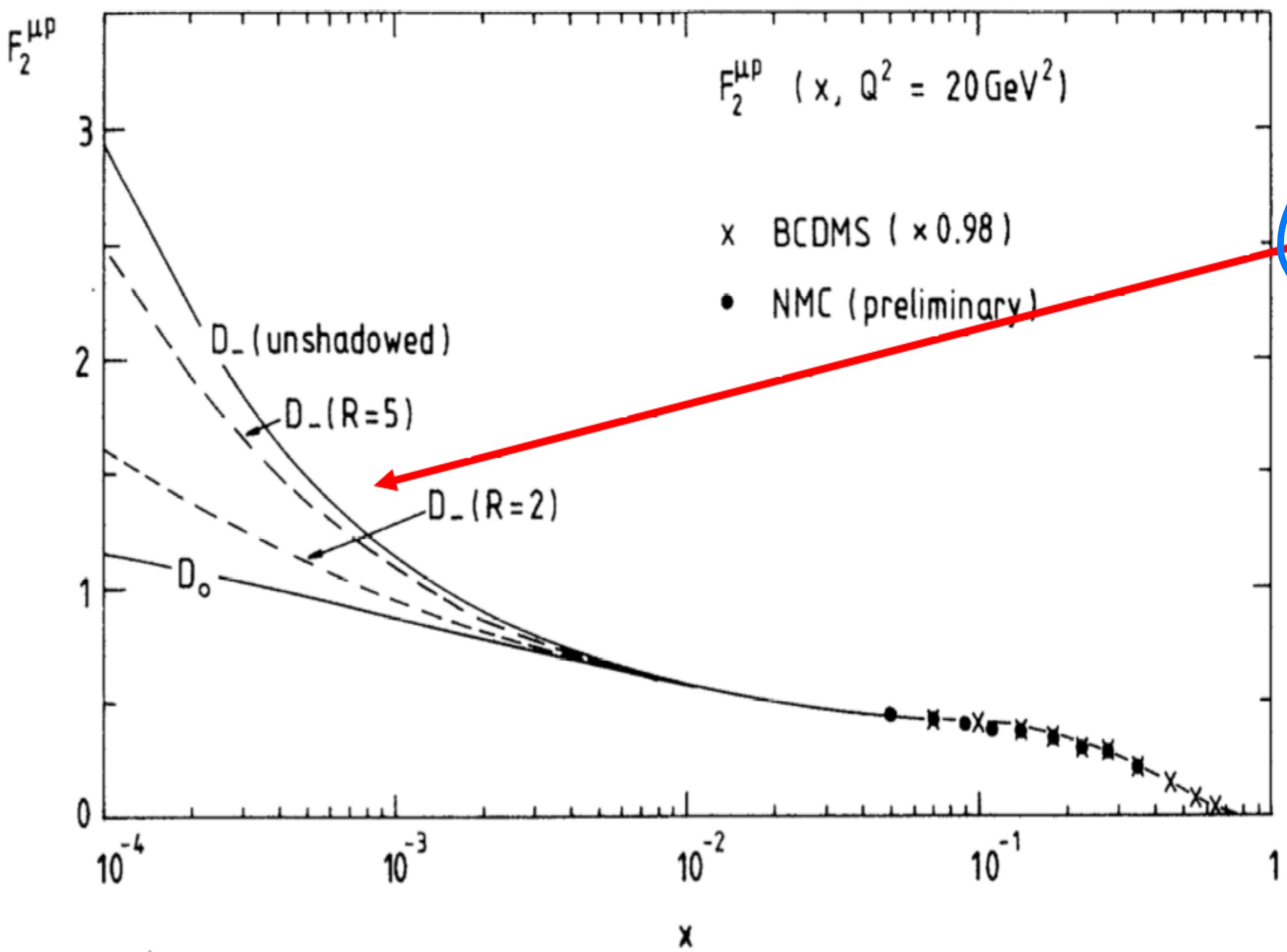
Remember the pre-HERA times

- Just before HERA started in 1992, new PDF fits (NLO DGLAP) were released, using all existing high precision data
- 1[st] HERA data 1992

H1 Nucl. Phys. B407 (1993) 515



Martin, Stirling, Roberts Apr 1992.
Phys.Rev.D47:867-882,1993.



From evolution equation to parton branching ...

How ?

Divergencies again...

- collinear divergencies factored into renormalized parton distributions
- what about soft divergencies ?

treated with "plus" prescription

with

$$\frac{1}{1-z} \to \frac{1}{1-z}_+ \qquad \int_0^1 dz \frac{f(z)}{(1-z)_+} = \int_0^1 dz \frac{f(z)-f(1)}{(1-z)}$$

- soft divergency treated with Sudakov form factor:

$$\Delta(t) = \exp\left[-\int_{t_0}^{t} \frac{dt'}{t'} \int^{z_{max}} dz \frac{\alpha_s}{2\pi} \tilde{P}(z)\right]$$

Sudakov form factor

$$\mu^2 \frac{\partial f(x,\mu^2)}{\partial \mu^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} P_+(z)\, f\left(\frac{x}{z}, \mu^2\right)$$

$$\int \frac{f(x)}{(1-x)_+} dx = \int \frac{f(x) - f(1)}{1-x} dx$$

$$= \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} P(z)\, f\left(x/z, \mu^2\right) - f(x,\mu^2) \int P(z)\, dz$$

$$\Delta_s = \exp\left\{ - \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \int dz \frac{\alpha_s}{2\pi} P(z) \right\}$$

$$\frac{\partial e^{-ax}}{\partial x} = -e^{-ax} \frac{\partial a(x)}{\partial x}$$

$$\frac{\partial \Delta_s}{\partial \mu^2} = -\Delta_s \left\{ \frac{1}{\mu^2} \int dz \frac{\alpha_s}{2\pi} P(z) \right\}$$

$$\mu^2 \frac{\partial f}{\partial \mu^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} P(z)\, f\left(\frac{x}{z}, \mu^2\right) + f(x,\mu^2) \frac{\mu^2}{\Delta_s} \frac{\partial \Delta_s}{\partial \mu^2}$$

$$\mu^2 \frac{\partial (f/\Delta)}{\partial \mu^2} = \frac{\alpha_s}{2\pi} \int \frac{dz}{z} P(z) \frac{f(x/z, \mu^2)}{\Delta_s}$$

DGLAP evolution again....

- differential form: $$t\frac{\partial}{\partial t} f(x,t) = \int \frac{dz}{z}\frac{\alpha_s}{2\pi} P_+(z)\, f\left(\frac{x}{z}, t\right)$$

$$\Delta_s(t) = \exp\left(-\int_x^{z_{max}} dz \int_{t_0}^{t} \frac{\alpha_s}{2\pi}\frac{dt'}{t'}\tilde{P}(z)\right)$$

- differential form using f/Δ_s with

$$t\frac{\partial}{\partial t}\frac{f(x,t)}{\Delta_s(t)} = \int \frac{dz}{z}\frac{\alpha_s}{2\pi}\frac{\tilde{P}(z)}{\Delta_s(t)} f\left(\frac{x}{z}, t\right)$$

- integral form

$$f(x,t) = f(x,t_0)\Delta_s(t) + \int \frac{dz}{z}\int \frac{dt'}{t'}\cdot\frac{\Delta_s(t)}{\Delta_s(t')}\tilde{P}(z) f\left(\frac{x}{z}, t'\right)$$

no – branching probability from t_0 to t

Sudakov and Poisson statistics

- Poisson statistics $$P(n,p) = \frac{p^n e^{-p}}{n!}$$

- for $n=0$, no-branch probability $P(0,p) = \exp(-p)$

- if exponent in Sudakov form factor represents the integrated splitting probability, then Sudakov gives no-branching probability:

$$\Delta_s(t) = \exp\left(-\int \frac{dt'}{t'} \int dz \frac{\alpha_s}{2\pi} P(z)\right)$$

- one-branch probability: $P(1,p) = p\exp(-p) = \Delta_s \int dz P(z)$

- see full DGLAP with Sudakov:

$$\begin{aligned} f(x,t) &= f(x,t_0)\Delta_s(t) + \int dz \int dx' \int \frac{dt'}{t'} \cdot \frac{\Delta_s(t)}{\Delta_s(t')} \frac{\alpha_s}{2\pi} \tilde{P}(z) \times \\ & \times f(x',t_0)\,\Delta_s(t')\delta(x - zx') \end{aligned}$$

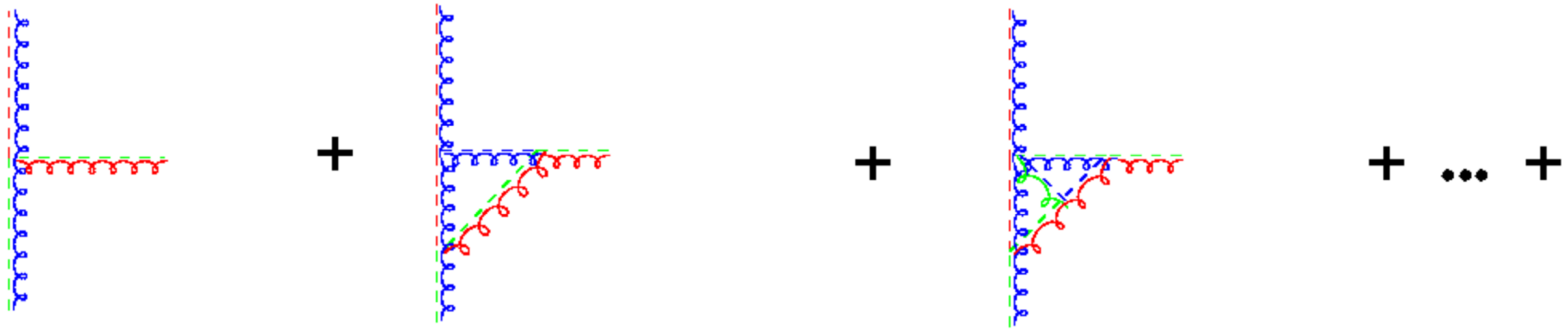
Sudakov form factor: all loop resum...

$g \to gg$ Splitting Fct $\tilde{P}(z) = \frac{\bar{\alpha}_s}{1-z} + \frac{\bar{\alpha}_s}{z} + ...$

- Sudakov form factor .... all loop resummation

$$\Delta_S = \exp\left(-\int dz \int \frac{dq'}{q'} \frac{\alpha_s}{2\pi} \tilde{P}(z)\right)$$

$$\Delta_S = 1 + \left(-\int dz \int \frac{dq}{q} \frac{\alpha_s}{2\pi} \tilde{P}(z)\right)^1 + \frac{1}{2!}\left(-\int dz \int \frac{dq}{q} \frac{\alpha_s}{2\pi} \tilde{P}(z)\right)^2 ...$$



$$\tilde{P}(z)\left[1 - \int\int dz \frac{dq}{q} \frac{\alpha_s}{2\pi} \tilde{P}(z) + \frac{1}{2!}\left(-\int\int dz \frac{dq}{q} \frac{\alpha_s}{2\pi} \tilde{P}(z)\right)^2 - ... -\right]$$

DGLAP re-sums leading logs...

$$f(x,t) = f(x,t_0)\Delta_s(t) + \int \frac{dz}{z} \int \frac{dt'}{t'} \cdot \frac{\Delta_s(t)}{\Delta_s(t')} \tilde{P}(z) f\left(\frac{x}{z}, t'\right)$$

- solve integral equation via iteration:

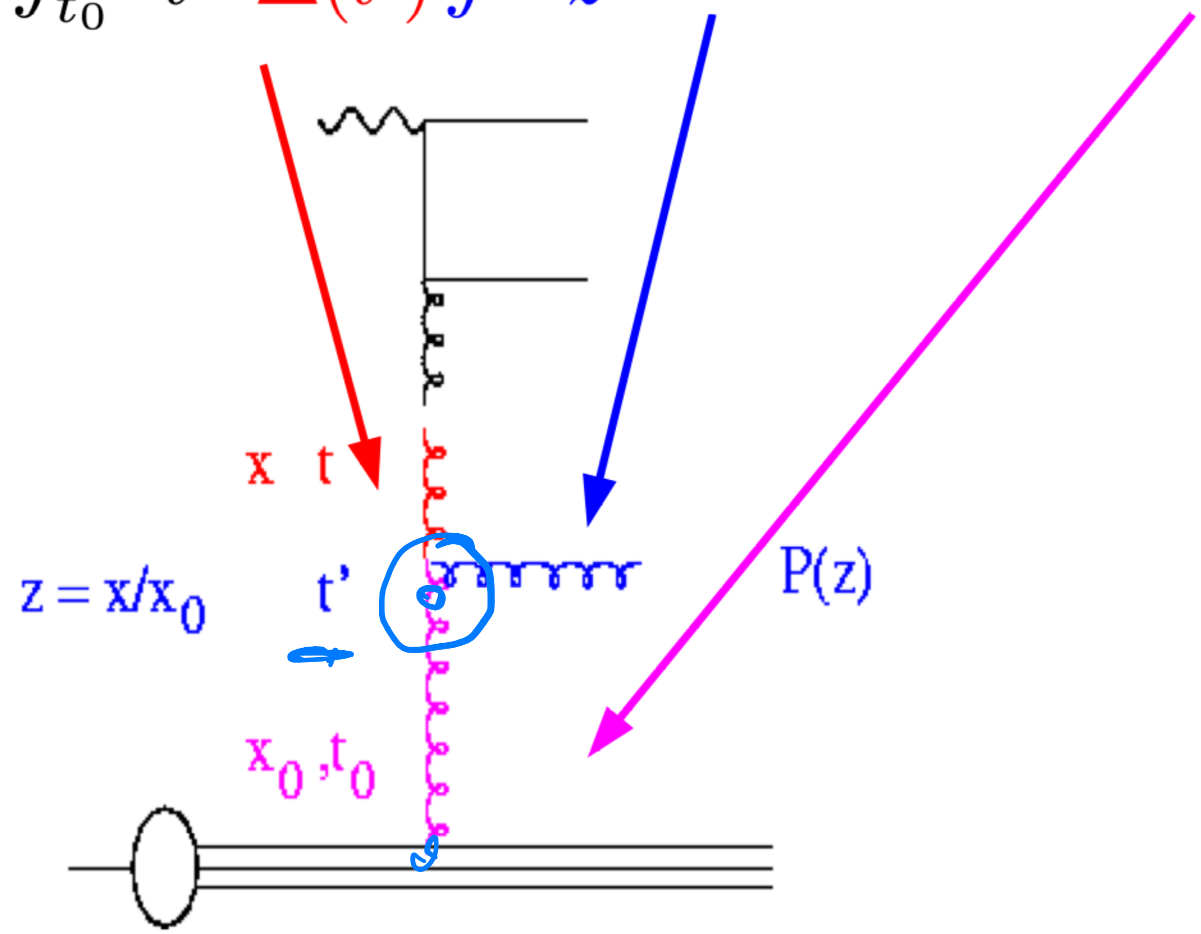
$$f_0(x,t) = f(x,t_0)\Delta(t)$$

$$f_1(x,t) = f(x,t_0)\Delta(t) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\Delta(t)}{\Delta(t')} \int \frac{dz}{z} \tilde{P}(z) f(x/z, t_0)\Delta(t')$$

from t' to t w/o branching

branching at t'

from t_0 to t' w/o branching



DGLAP re-sums leading logs...

$$f(x,t) = f(x,t_0)\Delta_s(t) + \int \frac{dz}{z}\int \frac{dt'}{t'}\cdot \frac{\Delta_s(t)}{\Delta_s(t')}\tilde{P}(z) f\left(\frac{x}{z}, t'\right)$$

- solve integral equation via iteration:

from t' to t w/o branching

branching at t'

from t_0 to t' w/o branching

$$f_0(x,t) = f(x,t_0)\Delta(t)$$

$$f_1(x,t) = f(x,t_0)\Delta(t) + \int_{t_0}^{t} \frac{dt'}{t'}\frac{\Delta(t)}{\Delta(t')}\int \frac{dz}{z}\tilde{P}(z) f(x/z,t_0)\Delta(t')$$

$$= f(x,t_0)\Delta(t) + \log\frac{t}{t_0} A \quad \Delta(t) f(x/z,t_0)$$

$$f_2(x,t) = f(x,t_0)\Delta(t) + \log\frac{t}{t_0} A \quad \Delta(t) f(x/z,t_0) +$$

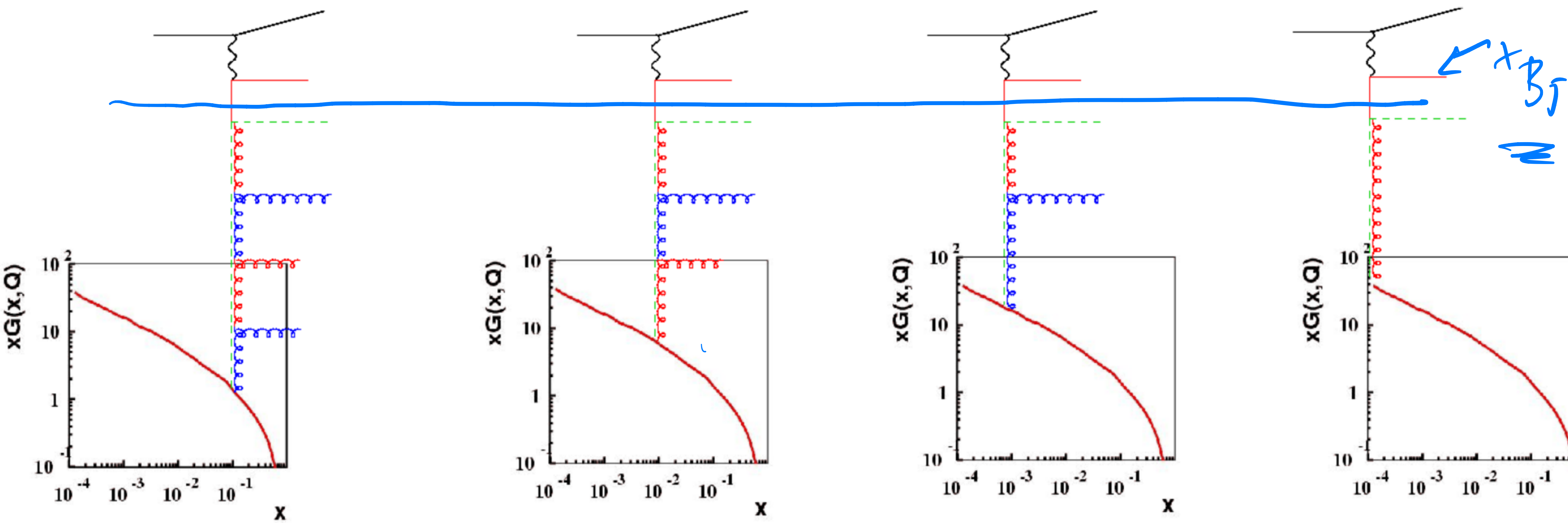
$$\frac{1}{2}\log^2\frac{t}{t_0} A \quad A \quad \Delta(t) f(x/z,t_0)$$

$$f(x,t) = \lim_{n\to\infty} f_n(x,t) = \lim_{n\to\infty}\sum_n \frac{1}{n!}\log^n\left(\frac{t}{t_0}\right) A^n \quad \Delta(t) f(x/z,t_0)$$

DGLAP re-sums $\log t$ to all orders !!!!!!!!!!!!!!!!!

DGLAP evolution equation... again...

- for fixed x and Q^2 chains with different branchings contribute
- iterative procedure, spacelike parton showering



$$f(x,t) = f_0(x,t_0)\Delta_s(t) + \sum_{k=1}^{\infty} f_k(x_k,t_k)$$

Evolution equation and parton branching method

- use momentum weighted PDFs: $xf(x,t)$

$$
\begin{aligned}
xf_a(x,\mu^2) \quad &= \quad \Delta_a(\mu^2)\; xf_a(x,\mu_0^2) \\
&+ \sum_b \int_{\mu_0}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \frac{\Delta_a(\mu^2)}{\Delta_a(\mu'^2)} \int_x^{z_M} dz P_{ab}^{(R)}(\alpha_s, z)\; \frac{x}{z} f_b\left(\frac{x}{z}, \mu'^2\right)
\end{aligned}
$$

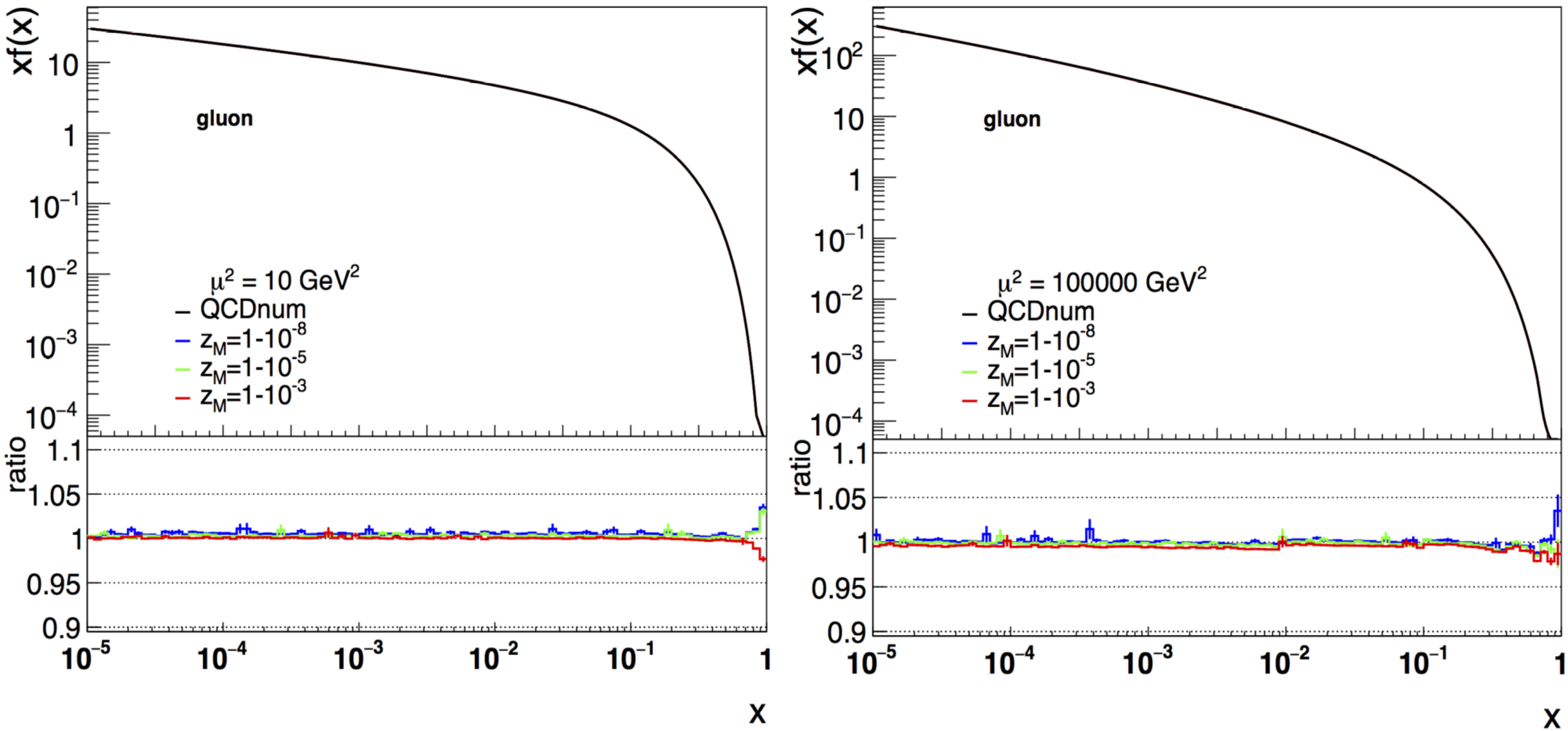
- with $P_{ab}{}^{(R)}\,(\alpha_s(t'),z)$ real emission probability (without virtual terms)
  - z_M introduced to separate real from virtual and non-emission probability
  - reproduces DGLAP up to $\mathcal{O}(1 - z_M)$

- make use of momentum sum rule to treat virtual corrections
  - use Sudakov form factor to treat non-resolvable and virtual corrections

$$
\Delta_a(z_M,\mu^2,\mu_0^2) = \exp\left(- \sum_b \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \int_0^{z_M} dz\; z\; P_{ba}^{(R)}(\alpha_s), z) \right)
$$

Validation of method with QCDnum at NLO

Hautmann, F., Jung, H., Lelek, A., Radescu, V., and Zlebcik, R.
Soft-gluon resolution scale in QCD evolution equations, Phys. Lett. B, 772(2017), 446



- Very good agreement with NLO - QCDnum if z_M is large enough:
  - approximation is of $\mathcal{O}(1 - z_M)$

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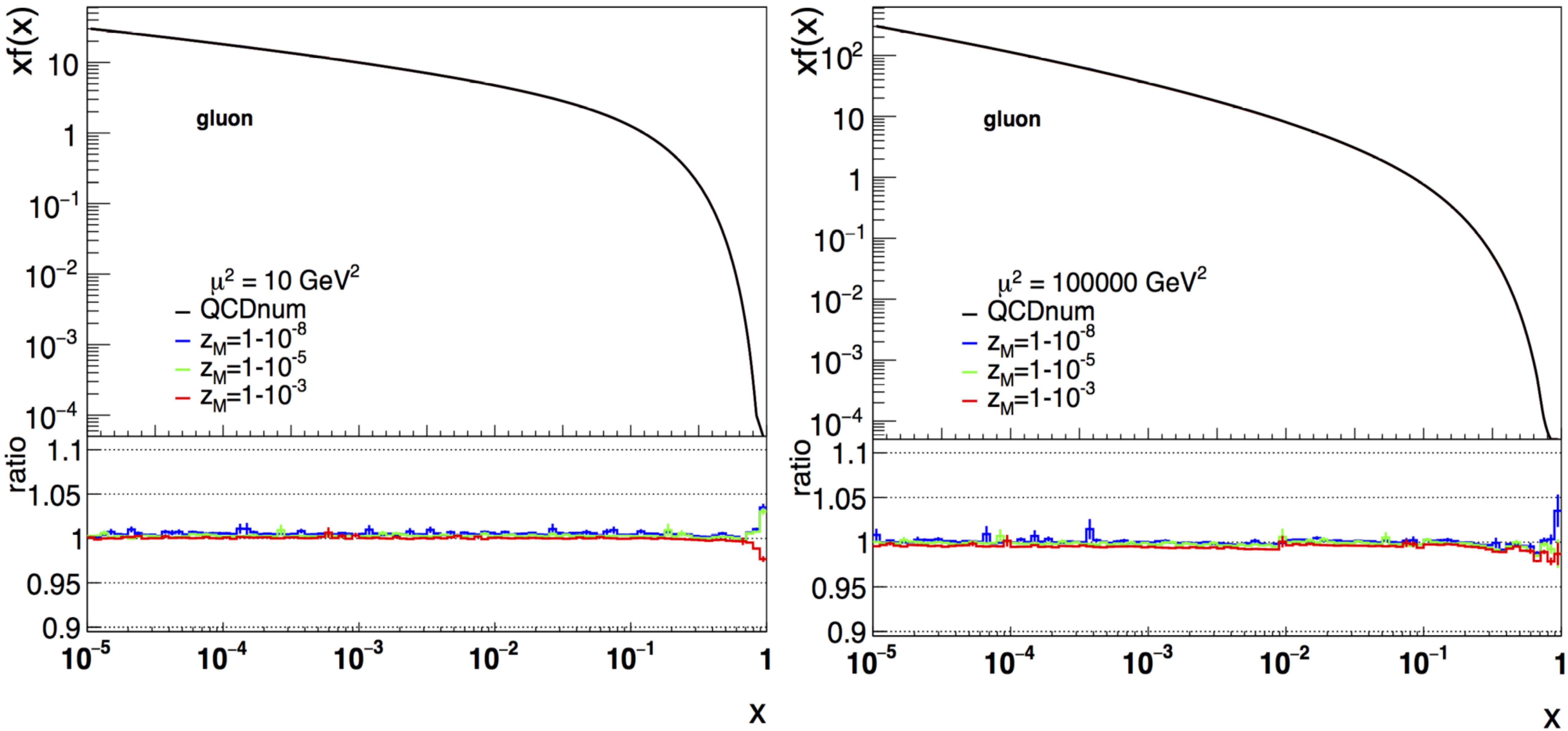

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