# Abelian monopoles, charge quantization, and the Witten effect

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## Contents



# 1 Introduction

## 1.1 Why monopoles?

This being the first seminar in the series, let me start with a few words on why monopoles are interesting. Indeed, studying them may call for a justification given that their experimental relevance remains an open question. My subjective answers to this are:

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- including monopoles becomes a necessity when studying many field theories, among which the GUT or extra-dimensional extensions of the Standard model, the theories with extended supersymmetry, the EFTs of string theory, ...,
- including them has many direct physical consequences in cosmology, astrophysics, model building, etc, hence they are testable theoretical constructions,
- including them leads to many interesting effects, or in other words they are great excuses to study aspects of QFT.

## 1.2 References

For this first lecture, I drew from lectures of S. Coleman (*The magnetic monopole fifty years later* [1]), J. Preskill (*Magnetic monopoles* [2]) and from the book of Y. Shnir (*Magnetic monopoles* [3]). Classic papers will be mentioned along the flow, and are listed with very introductory comments in [4].

# 2 A new source of magnetic fields

Let us start slowly with two of the Maxwell equations,

$$\nabla \cdot \mathbf{E} = \rho_e \ , \quad \nabla \cdot \mathbf{B} = 0 \ , \tag{2.1}$$

and two of their consequences. First, the magnetic flux on a closed surface $S = \partial V$ vanishes, by Gauss' theorem,

$$\oiint_{\partial V} d\mathbf{S} \cdot \mathbf{B} = \iiint_V dV \, \nabla \cdot \mathbf{B} = 0 \ . \tag{2.2}$$

Second, putting a charged particle at rest at the origin of spacetime, i.e. turning on a charge density $\rho_e = e\delta^{(3)}(\mathbf{r})$, one generates a radial electric field

$$\mathbf{E} = \frac{e}{4\pi} \frac{\mathbf{r}}{r^3} \ , \tag{2.3}$$

whose flux on any closed surface surrounding the source is $e$.

Therefore, Maxwell equations do not allow for magnetic point-like sources which would generate a radial magnetic field. However, one may wonder whether those equations should be modified to allow for such sources, and if so, what are the physical consequences of and/or requirements on such a choice. Let us therefore place ourselves in the situation where the magnetic field reads

$$\boxed{\mathbf{B} = g \frac{\mathbf{r}}{r^3}} \ . \tag{2.4}$$

Such a field is generated by a non-zero magnetic charge density $\rho_m = 4\pi g \delta^{(3)}(\mathbf{r})$, corresponding to a point-like object called *magnetic monopole.*

In classical physics, one can work out the implications of such a background magnetic field on the trajectory of a particle of electric charges $e$ and velocity $\mathbf{v}$, which feels the Lorentz force

$$\mathbf{F} = e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \ . \tag{2.5}$$

This generates many intriguing phenomena, which will be encountered briefly in this lecture and more in subsequent ones. But the aspect of monopoles which we aim to discuss today has to do with their interplay with quantum mechanics.

# 3 Which gauge potential?

In classical mechanics, only electric and magnetic fields are physical and needed to compute the dynamics of charged particles. However, in quantum mechanics, the Schrödinger equation assumes the minimal replacement

$$\nabla \to \nabla - ie\mathbf{A} \ , \tag{3.1}$$

which explicitly features the vector gauge potential $\mathbf{A}$, usually defined such that $\mathbf{B} = \nabla \times \mathbf{A}$, consistently with $0 = \nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A})$. However, for the background magnetic field of a magnetic monopole, one does not have $\nabla \cdot \mathbf{B} = 0$ globally, but only locally, therefore one cannot define a regular $\mathbf{A}$ at each space point. Classically this is not a problem, as knowing the electric and magnetic field is all what one needs, but quantum mechanics effects exist which are directly sensitive to topological properties of $\mathbf{A}$, as we discuss now. This will motivate making sense of gauge potentials for a monopole field, which has non-trivial consequences.

## 3.1 Aharonov-Bohm effect

The archetypal example of a quantum mechanical effect which is directly sensitive to $\mathbf{A}$ is the Aharonov-Bohm effect. It involves a set-up where an electrically charged quantum particle is sent through a double-slit experiment, leading to interference fringes. In addition, an infinite solenoid is placed such that the classical trajectories circle it, see Figure 1 for a schematic representation.

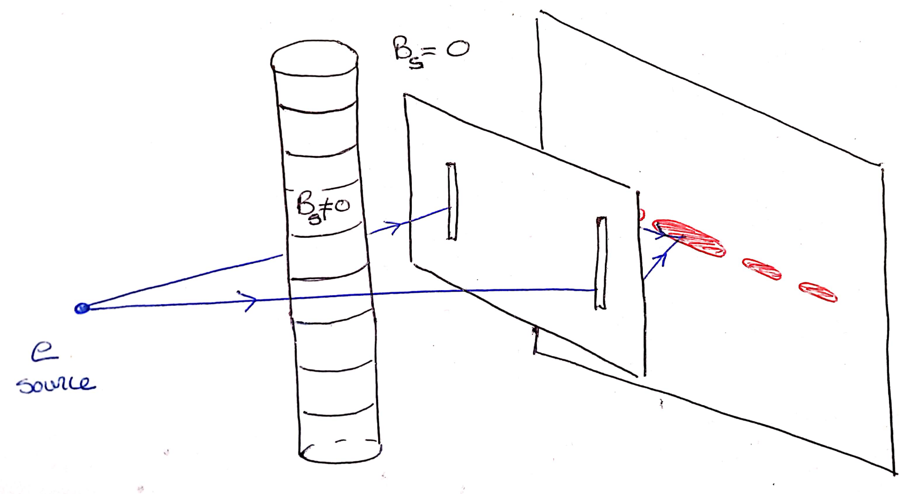


Figure 1: Setup for an Aharonov-Bohm experiment

The magnetic field $\mathbf{B}_s$ is constant in the solenoid and aligned with its axis, while it vanishes outside. Therefore the classical trajectory of the particle is not affected by the presence of the solenoid. This stops being true in the quantum case. For an electrically charged particle, such as the electron, of mass $m$ and charge $e$, Schrödinger equation for the wave-function $\psi$ reads

$$\left(i\frac{\partial}{\partial t} + \frac{1}{2m}\left[\nabla - ie\mathbf{A}\right]^2 - e\phi\right)\psi(\mathbf{r},t) = 0 \ . \tag{3.2}$$

On the static monopole background, the scalar gauge potential $\phi = 0$, and one can recycle a solution $\psi_0(\mathbf{r},t)$ to the free equation (i.e. when $\mathbf{A} = 0$) into a solution $\psi(\mathbf{r},t)$ of the full equation by forming

$$\psi(\mathbf{r},t) = \psi_0(\mathbf{r},t)e^{ie\int_0^r d\mathbf{r}'\cdot\mathbf{A}(\mathbf{r}')} \ . \tag{3.3}$$

Since the initial conditions of the two expressions match, if $\psi_0(\mathbf{r},t)$ describes the quantum propagation of a particle whose initial state was prepared in a given way, then $\psi(\mathbf{r},t)$ describes the quantum propagation of this particle when the initial state was prepared in the same way[1]. Let us now assume that one observes the interference pattern generated by the set-up in Figure 1 when the solenoid is absent. The fringes are captured by $\sin(\arg\psi_{0,1} - \arg\psi_{0,2})$ where $\psi_{0,1/2}$ describe the propagation of the particle through each slit respectively. Now, let us repeat the same experiment in presence of the solenoid. The fringes are now captured by $\sin(\arg\psi_1 - \arg\psi_2)$, where $\psi_{1/2}$ are connected to $\psi_{0,1/2}$ as in (3.3). This implies that

$$\arg\psi_1 - \arg\psi_2 = \arg\left(\psi_{0,1}e^{ie\int_{P_1} d\mathbf{r}'\cdot\mathbf{A}}\psi_{0,2^*}e^{-ie\int_{P_2} d\mathbf{r}'\cdot\mathbf{A}}\right) = \arg\psi_{0,1} - \arg\psi_{0,2} + e\oint_C d\mathbf{r}'\cdot\mathbf{A} \ . \tag{3.4}$$

[1](3.3) only applies for a propagation subject to the sole gauge potential. We treat the phenomenon of diffraction at the level of the slits in the usual way, namely we assume that the only effect of the slit is to act as a source coherent with the incoming wave.

(see Figure 2 for the definition of the integration contours).

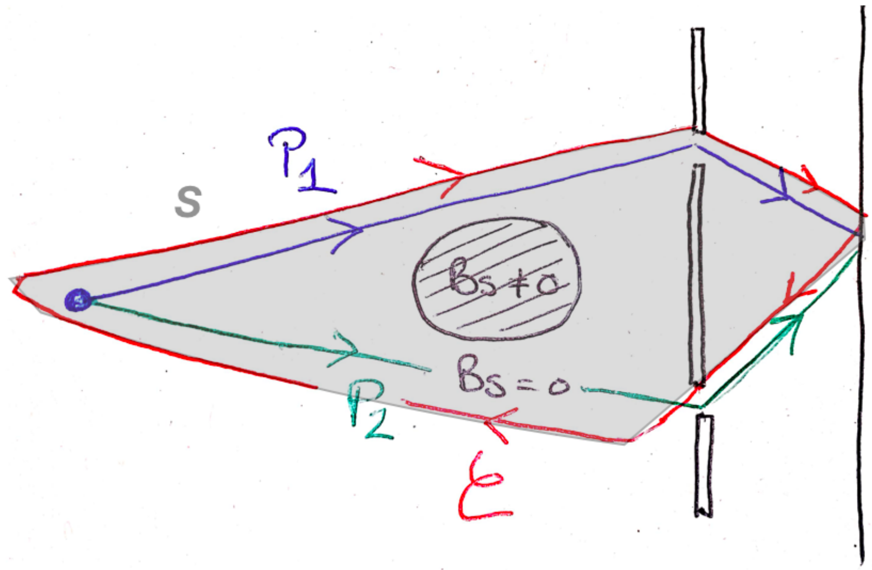


Figure 2: Integration contours for the Aharonov-Bohm experiment

Therefore, the fringes are shifted by an amount determined by $e\int_{\mathcal{C}} d\mathbf{r}'\cdot\mathbf{A}$. The surprising result is that the latter is non-zero, even when $\mathcal{C}$ lies outside of the solenoid, where the magnetic field is identically zero ! The reason is Stokes' theorem, which states that

$$\oint_{\mathcal{C}} d\mathbf{r}'\cdot\mathbf{A} = \int_S d\mathbf{S}\cdot(\nabla\times\mathbf{A}) = \int_S d\mathbf{S}\cdot\mathbf{B} \tag{3.5}$$

for any surface $S$ whose boundary is $\mathcal{C}$. We can use for instance $S$ as drawn on Figure 2, which means that

$$\int_S d\mathbf{S}\cdot\mathbf{B} = (\text{transverse solenoid area})\times B_s = \text{transverse magnetic flux in the solenoid} \equiv \Phi_s\ . \tag{3.6}$$

This effect illustrates the fact that a non-zero gauge potential affects quantum mechanical observables, even in regions of vanishing magnetic field. Hence, one needs to define gauge potentials in order to couple quantum-mechanically charged particles to electromagnetic fields. We now discuss how to make sense of a gauge potential for the field generated by a magnetic monopole.

## 3.2 Gauge potential of a monopole field

We argued previously that one cannot define a regular $\mathbf{A}$ at each space point to describe the field of a magnetic monopole. Historically, the first approach, due to Dirac [5], was to allow that $\mathbf{A}$ has a localized singular behavior along a half-line, dubbed *Dirac string*. For instance, allowing for a string on the positive $z$-axis, one finds the potential

$$\mathbf{A} = -\frac{g(1+\cos\theta)}{r\sin\theta}\mathbf{e}_\varphi = g\left(\frac{1+\cos\theta}{r\sin\theta}\sin\varphi, -\frac{1+\cos\theta}{r\sin\theta}\cos\varphi, 0\right) = \frac{g}{r}\frac{\mathbf{r}\times\mathbf{n}}{r-\mathbf{r}\cdot\mathbf{n}} \tag{3.7}$$

where in the last equality $\mathbf{n} = (0, 0, 1)$. This potential indeed verifies

$$\nabla \times \mathbf{A} = \mathbf{B} = g\frac{\mathbf{r}}{r^3} \ , \tag{3.8}$$

*except when it is singular*, where the computation is ill-defined. The direction of the string given by the vector $\mathbf{n}$ could be chosen arbitrarily, and that corresponds to a gauge transformation. For instance, considering a gauge transformation $U = e^{2ig\varphi}$, one finds that

$$\mathbf{A} \to \mathbf{A} - iU^{-1}\nabla U = \frac{g(1 - \cos\theta)}{r\sin\theta}\mathbf{e}_\varphi \ , \tag{3.9}$$

which now features a singular string on the negative $z$-axis. Note that the gauge transformation is not well-defined on the $z$-axis, which corresponds to the union of both singular Dirac strings. Suitable functions $U(\theta, \varphi)$ can be identified that move the string to any direction (see section 1.4 of [3]).

One may be annoyed by the fact that our description of the monopole involves a singular region of spacetime. It does not have to be the case, as was shown by Wu and Yang [6]: it suffices to separate space in several (slightly overlapping) patches, on each of which one defines a regular gauge potential which describes the field (2.4). The only constraint on such constructions comes from overlapping regions, where several gauge potentials are defined although they should correspond to the same physics. That is possible if they are related by gauge transformations on such overlapping regions. An example is found by considering two patches, corresponding to two half-spheres covering the north and south poles respectively, see Figure 3 in the limit where the overlap region reduces to the equator. We can now choose on the north sphere the potential whose string lies on the negative $z$-axis, and vice-versa on the south sphere. In addition, we know from the paragraph above that the two potentials are related by the gauge transformation $U = e^{2ig\varphi}$ everywhere except on the $z$-axis, in particular on the equator. Therefore, we have a fully regular description of the monopole.

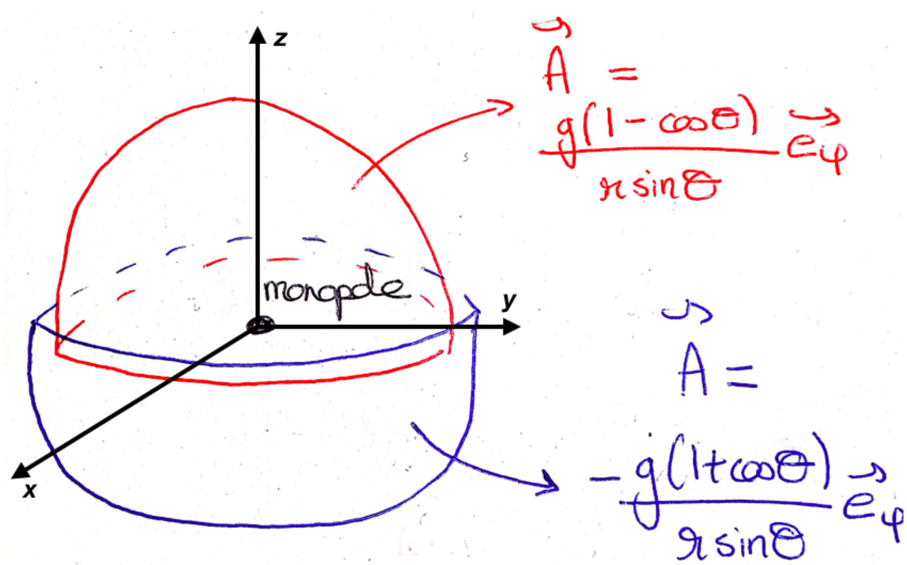


Figure 3: Wu-Yang patches

# 4 Quantized charges

Now that we found expressions for the gauge potential of a monopole field, let us see if it makes sense to couple it to quantum particles. The remarkable answer will be that *yes, but under certain conditions on the charge spectrum.*

## 4.1 Dirac quantization condition

The reason is actually rather easy to see given what we already discussed. Imagine performing double-slit measurements somewhere far down the $z$-axis (see Figure 4 for notations), with a magnetic monopole at the origin.

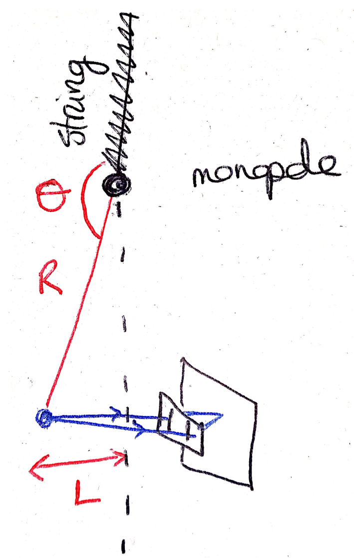


Figure 4: Aharonov-Bohm experiment with a monopole

The presence of the magnetic field will induce a phase shift for the wave-functions (hence a shift for the interference fringes), which can be computed following similar steps as in section 3.1. Approximating the path cut by the classical trajectories between the source, the slits and the fringes by a circle, one finds using (3.7) that

$$\int_C d\mathbf{r}' \cdot \mathbf{A} = -2\pi g\,(1 + \cos\theta) \approx -\pi g \left(\frac{L}{R}\right)^2 \tag{4.1}$$

where we assumed $L \ll R$. The impact on the fringes therefore vanishes when $R \to \infty$ for fixed $L$, consistently with the fact that the magnetic flux spanned by the surface of the experiment also vanishes. However, this computation was performed assuming (3.7), but as we discussed, we could have instead chosen that the Dirac string lies on the negative $z$-axis, passing right through the experiment although the monopole itself remains well separated (equivalently, we could imagine performing the same experiment in the symmetric location with respect to the origin, far up the $z$-axis, without changing the direction of the string). This has the effect of turning the apparent magnetic flux to

$$\int_{\mathcal{C}} d\mathbf{r}' \cdot \mathbf{A} = 2\pi g \left(1 - \cos\theta\right) \approx 4\pi g - \pi g \left(\frac{L}{R}\right)^2 \ , \tag{4.2}$$

which now does not vanish when $R \to \infty$ for fixed $L$. This is a shocking result ! As we argued above, the position of the string is gauge-dependent, so it ought to be an unphysical artefact. However, we are talking here about interference experiments, namely physical observables. The only way to reconcile the two observations is to demand that the phase shift induced by the string is unobservable, i.e. multiple of $2\pi$. We therefore find that

$$4\pi e g = 2\pi n \implies \boxed{eg = \frac{n}{2} \ , \ n \in \mathbb{Z}} \ . \tag{4.3}$$

This is the celebrated *Dirac quantization condition*. Let us pause and make a few comments.

First, the shift in the magnetic flux can be understood by realizing that the Dirac string is nothing but an infinitesimally-thin half-infinite solenoid with constant internal magnetic flux. Therefore, the magnetic flux in the string is opposite to that generated by the monopole over a given closed surface cutting through the string, since the total flux over this surface should vanish when one resolves the structure of the solenoid. This is also recovered when one regularizes the gauge potential, as done for instance in section 1.3 of [3]. This shows that one should not compute quantities close to the Dirac string.

Second, if you don't like the singularity in the Dirac string, this story has an analogue for the construction of Wu and Yang. Imagine this time performing Aharonov-Bohm experiments at the equator of Figure 3, with the two interference paths circling the equator. The experiment is therefore sensitive to $\int_{\text{equator}} d\mathbf{r}' \cdot \mathbf{A}$. One can compute $\int_{\text{equator}} d\mathbf{r}' \cdot \mathbf{A}$ on any of the two patches, and since this quantity is observable, it should yield the same result up to a multiple of $2\pi/e$:

$$\int_{\mathcal{C}} d\mathbf{r}' \cdot \mathbf{A}^{\text{north}} - \int_{\mathcal{C}} d\mathbf{r}' \cdot \mathbf{A}^{\text{south}} = \int_{\mathcal{C}} d\mathbf{r}' \cdot \nabla \left(2g\varphi\right) = 4\pi g \implies eg = \frac{n}{2} \ . \tag{4.4}$$

This argument is free from any singularity, and is equivalent to saying that the integrated phase of a quantum wave-function should be uniquely defined, since it is observable. In particular, it must be gauge-invariant. Under gauge transformations, the wave-function transforms as $\psi \to U^e \psi$. Therefore, acting with the transformation which allows to change patches at the equator in the Wu-Yang construction, its integrated phase is shifted by

$$\int_{\mathcal{C}} \frac{\log U^e}{i} = 4\pi e g \ , \tag{4.5}$$

i.e. the integrated gauge transformation $e^{i\int_{\mathcal{C}} 2eg\varphi}$ should also be single valued. Note that this is a quantum constraint: although $U = e^{2ig\varphi}$ is not single-valued in spacetime, the only thing necessary in the classical setup was that the gauge transformation $U^{-1}\nabla U = \frac{2ig}{r\sin\theta}\mathbf{e}_\varphi$ is single-valued.

Third, instead of using Aharonov-Bohm arguments, one can obtain the quantization condition from arguments related to angular momentum. Let us consider the (non-relativistic) equation of motion of a charge in the field of a monopole:

$$m\frac{d^2\mathbf{r}}{dt^2} = e\mathbf{v}\times\mathbf{B} = \frac{eg}{r^3}\frac{d\mathbf{r}}{dt}\times\mathbf{r}\ . \tag{4.6}$$

The usual angular momentum $m\mathbf{r}\times\mathbf{v}$ is not conserved during the induced motion, but there exists a conserved generalized angular momentum:

$$\mathbf{L} = m\mathbf{r}\times\mathbf{v} - \frac{eg\mathbf{r}}{r}\quad . \tag{4.7}$$

Very surprisingly, even a static charge-monopole system has angular momentum, given by $eg$ ! The latter can be identified with the angular momentum contained in the electric-field, which can be computed from the Maxwell action:

$$\int d^3r'\,(\mathbf{r}'\times[\mathbf{E}\times\mathbf{B}]) = -eg\hat{\mathbf{r}}\ , \tag{4.8}$$

where we used the explicit form of the magnetic field $\mathbf{B} = -\nabla\left(\frac{g}{r}\right)$, integration by parts (with vanishing gauge fields at infinity) and the Maxwell equation $\nabla\cdot\mathbf{E} = e\delta^{(3)}(\mathbf{r}-\mathbf{r}')$. Given the quantization of angular momentum in quantum mechanics, it becomes clear that one can obtain the Dirac quantization in this way. Indeed, one can check that the quantum operators associated to (4.7) commute with the Hamiltonian and verify angular momentum commutation relations. Upon looking for common eigenfunctions of the Hamiltonian and the squared angular momentum $\mathbf{L^2}$, and upon imposing the constraints on the eigenvalues derived from the angular momentum commutation algebra, one finds that $eg$ has to be integer or half integer (see section 2.3 of [3] for more details).

Finally, the quantization condition can be considered as an explanation of charge quantization different than gauge unification : if there exists a single magnetic monopole in the universe, all electrically charged particles must have a charge quantized in units of that monopole charge. This could explain why the ratios of the charges of known particles are not equal to $\pi, \sqrt{2}$, etc.

## 4.2 Dyons

One can generalize the quantization condition to the case of dyons, namely particles which carry both electric and magnetic charges. Given a static dyon of charges $e, g$ at the origin, the electromagnetic field it generates is simply given by eqs. (2.3)-(2.4). Studying a quantum particle of electric charge $\tilde{e}$ on the classical background of this dyon, one can run the same arguments as in the previous sections[2] and obtain that

$$\tilde{e}g = \frac{n}{2}\ . \tag{4.9}$$

[2]The scalar gauge potential is now non-zero on the background of the dyon. However, as it is regular everywhere (except at the origin: $\phi = \frac{e}{4\pi r}$) it does not affect the Schrödinger equation, or the interference experiments, at large distances.

This condition holds for an electrically charged particle coupled to a dyon of magnetic charge $g$. When both particles are dyons of charges $(e_1, g_1)$ and $(e_2, g_2)$, the quantization condition generalizes to

$$\boxed{e_1 g_2 - e_2 g_1 = \frac{n}{2}\ ,\ \ n \in \mathbb{Z}}\ . \tag{4.10}$$

It reduces to the aforementioned conditions when one of the particles has vanishing magnetic charge. This generalized condition, due to Schwinger and Zwanziger [7], can be derived by requiring that each of the dyons cannot access the Dirac string of the other, or by evaluating the angular momentum of the system. Instead of going through these derivations, let me instead note that it can be obtained from eq. (4.9) after using electric-magnetic duality. The latter, whose complete scope will be presented more thoroughly in subsequent lectures, is a symmetry of the free Maxwell equations, under which electric and magnetic fields are mixed

$$\mathbf{E} \to \cos\alpha \mathbf{E} + \sin\alpha \mathbf{B}\ , \quad \mathbf{B} \to -\sin\alpha \mathbf{E} + \cos\alpha \mathbf{B}\ , \tag{4.11}$$

a particular case of which is the more usual exchange symmetry under which $\mathbf{E} \to \mathbf{B}\ ,\ \mathbf{B} \to -\mathbf{E}$. For this symmetry to remain valid when charged matter is introduced, one needs to act on $(e, 4\pi g)$ as on $(\mathbf{E}, \mathbf{B})$. This symmetry is manifestly violated for our usual electromagnetism, as there is no magnetic state which could mix with the electron under this symmetry. However, it is an exact symmetry of certain theories with extended supersymmetry or string theories, and given that the kind of consistency conditions we are after here do not depend on the precise nature of the interactions, we can use this duality to infer the right relations. Any consistency condition or physical quantity cannot depend on the duality frame considered, therefore they must depend on duality-invariant quantities. Given two dyons 1 and 2, those are the inner and outer products [1]

$$e_1 e_2 + 16\pi^2 g_1 g_2 \text{ and } e_1 g_2 - g_1 e_2\ . \tag{4.12}$$

By writing the most general formula which depends on those expressions and asking that it reduces to the Dirac quantization condition when $e_1 = g_2 = 0$, one derives (4.10).

# 5 CP violation and the Witten effect

Let me finish this first lecture with an introductory discussion of the interplay between monopole properties and CP violation, and of the Witten effect [8].

One can wonder about the charge spectrum compatible with the quantization condition. Let us assume that there exist particles with unit magnetic charge $g = 1$. Then, the quantization condition implies that the electric charges $e_i$ of such particles are separated by an half-integer:

$$e_i - e_j = \frac{n}{2}\ . \tag{5.1}$$

However, their absolute values seem to be completely free, namely they could be $\pi$ and $\pi + \frac{1}{2}$. This freedom reduces when considering the implications of symmetry assumptions. In particular, it is known that under CP, the electric charge of a particle changes sign. Since under parity one has $(\mathbf{E},\mathbf{B})\to(-\mathbf{E},\mathbf{B})$, the magnetic charge instead remains fixed under the action of CP. Therefore, in a CP preserving theory, there must exist a dyon of charges $(-e,g)$ for each dyon of charge $(e,g)$. The quantization condition applied to those two related dyons then yields

$$eg=\frac{n}{4}\ . \tag{5.2}$$

For the aforementioned case of unit magnetic charge dyons, that implies that the electric charges $e$ must be multiple of $\frac{1}{4}$. In addition, if there exists a dyon of magnetic charge 1 such that $4e$ is even, the same holds for all other dyons of magnetic charge 1, in virtue of (5.1), and identically for even $4e$. Therefore, CP conservation introduces additional quantization conditions on the dyon charges.

Witten then computed the deviation from these new quantization constraints when a specific kind of CP-violation is introduced, the theta term:

$$\mathcal{L}\supset-\frac{\theta}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}\ , \tag{5.3}$$

where $\tilde{F}^{\mu\nu}\equiv\frac{\epsilon^{\mu\nu\rho\sigma}}{2}F_{\rho\sigma}$. The theta term violates CP, and sources a shift in the electric charges of dyons. To see this, let us consider the dynamics of the gauge field sourced by a static magnetic monopole at the origin (the argument is taken from section 5.2 of [1]). We expand scalar and vector potentials around the solution of a static monopole,

$$\mathbf{E}=\nabla\phi\ ,\quad \mathbf{B}=\frac{g}{r^3}\mathbf{r}+\nabla\times\mathbf{A}\ , \tag{5.4}$$

and we compute the part of the action which depends on $\theta$:

$$\mathcal{L}\supset-\frac{\theta}{4\pi^2}\mathbf{E}\cdot\mathbf{B}=\frac{\theta}{4\pi^2}\phi\nabla\cdot\mathbf{B}=\frac{\theta g}{\pi}\phi\,\delta^{(3)}(\mathbf{r})\ , \tag{5.5}$$

where we used integration by parts in the first equation. This is to be compared with the action of an electrically charged particle:

$$\mathcal{L}=\left(\frac{1}{2}m\mathbf{v}^2-e\phi+e\mathbf{v}\cdot\mathbf{A}\right)\delta^{(3)}(\text{wordline})\ , \tag{5.6}$$

which reduces to $-e\phi\delta^{(3)}(\mathbf{r})$ for a static particle at the origin. We found that the theta term grants the monopole an electric charge $e=-\frac{\theta g}{\pi}$ ! This is the so-called *Witten effect.* Since the theory violates CP, this charge can take any real value and is not constrained. If we had instead turned on the radial electric field of a static point source, we would have found that the theta term is a total derivative and does not induce any dynamics.

Although it is beyond the scope of my plan for this lecture, let me mention in passing that Witten did more than working out the shift in electric charge of the dyons, he also worked out the spectrum of dyons in theories which resolve the structure of the monopoles in terms of spontaneously broken non-abelian gauge theories (this topic will be discussed in the next lecture of this series). Such spectra have been recently used to compute the $\theta$-dependent vacuum energy induced by monopoles, which has applications in axion physics [9].

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