The formalism of Fused Webs for multi-parton scattering amplitudes

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Resummation
REF
Evolution
Factorization
2022
workshop
October 31 - November 4, 2022

Resummation, Evolution, Factorization 2022 (Online)
31 Oct - 04 Nov 2022

Plan of the talk

- Webs in multi-parton amplitudes
- Properties of web mixing matrices
- Fused Cwebs formalism
- Summary

Agarwal, Pal, Srivastav, AT ;
JHEP 06 (2022) 020

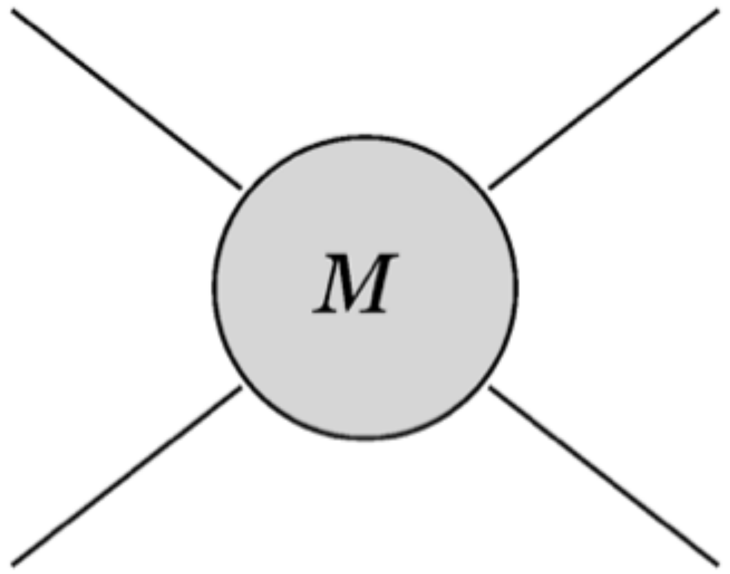
Agarwal, Magnea, Pal, AT ;
JHEP 03 (2021) 188

Agarwal, Danish, Magnea, Pal, AT ;
JHEP 05 (2020) 128

Time

Multi-parton Scattering Amplitude In IR limit

IR behaviour $\leftrightarrow$ Wilson line correlator



Soft matrix

$$\mathcal{S}_n\Big(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\Big) \equiv \langle 0 | \prod_{k=1}^{n} \Phi_{\beta_k}(\infty, 0) | 0 \rangle$$

Soft anomalous dimension

$$\mathcal{S}_n\Big(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\Big) = \mathcal{P} \exp\left[-\frac{1}{2}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\, \mathbf{\Gamma}_n\Big(\beta_i \cdot \beta_j, \alpha_s(\lambda^2), \epsilon\Big)\right]$$

Diagrammatic Exponentiation
(A complementary approach)

Kinematic factor $K(D)$

Color factor $C(D)$

$$S_n(\gamma_i) = \sum_D K(D)\, C(D)$$

$$S_n\left(\gamma_i\right) = \exp\left[\mathscr{W}_n\left(\gamma_i\right)\right]$$

Modified colour factors $\widetilde{C}(D)$

$$\mathscr{W}(\gamma_i) = \sum_D K(D)\, \widetilde{C}(D)$$

Mitov, Sterman, Sung; 2010

Gardi, Laenen, Stavenga, White; 2010

Gardi, Smillie, White; 2011

Gardi, White; 2011

Dukes, Gardi, Steingrimsson, White; 2013

Gardi, Smillie, White; 2013

Dukes, Gardi, McAslan, Scott, White; 2016

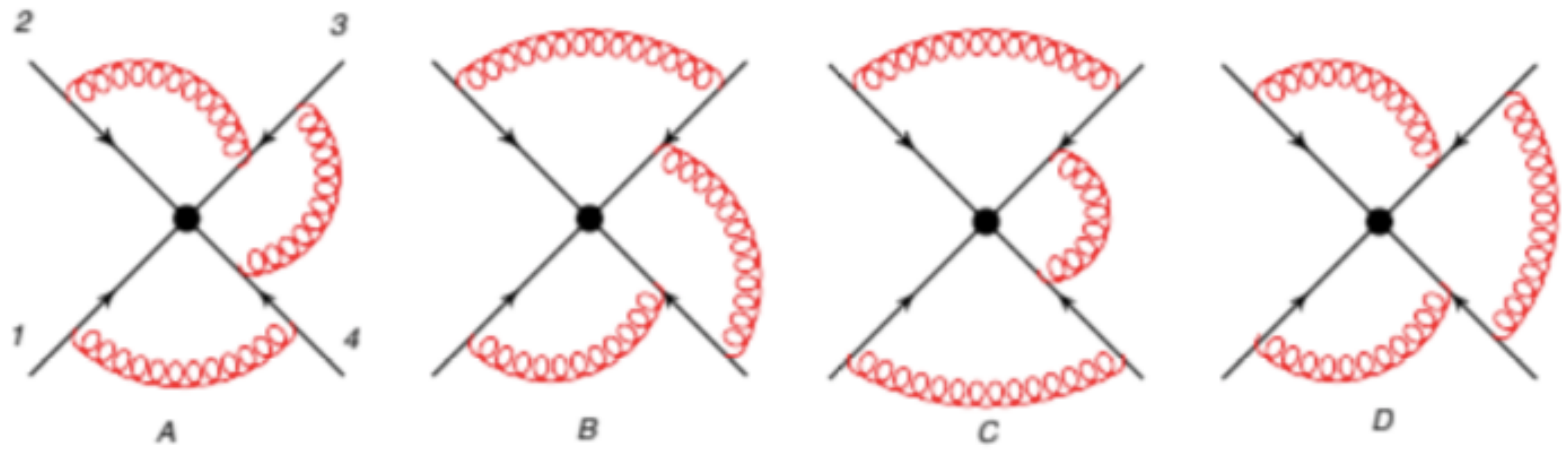
See also: Vladimirov, 2014-2017 for Alternative approach

For Eikonal Form factors these are called webs **Gatheral; Frenkel, Taylor; Sterman**

Multi-parton Webs

Web (w) : A set of diagrams closed under permutations of the gluon attachments on the Wilson lines.

(Gardi, Smillie, White, *et al*, *2010-2013*)



The exponent $W(\gamma_i)$
grouped into webs

$$S_n = \exp\Big(\sum w\Big)$$

$R_w(D, D')$
Web mixing matrix

$$S_n = \exp\left(\sum_{D,D'\in w} K(D)\, R_w(D, D')\, C(D)\right)$$

Properties of web mixing matrices

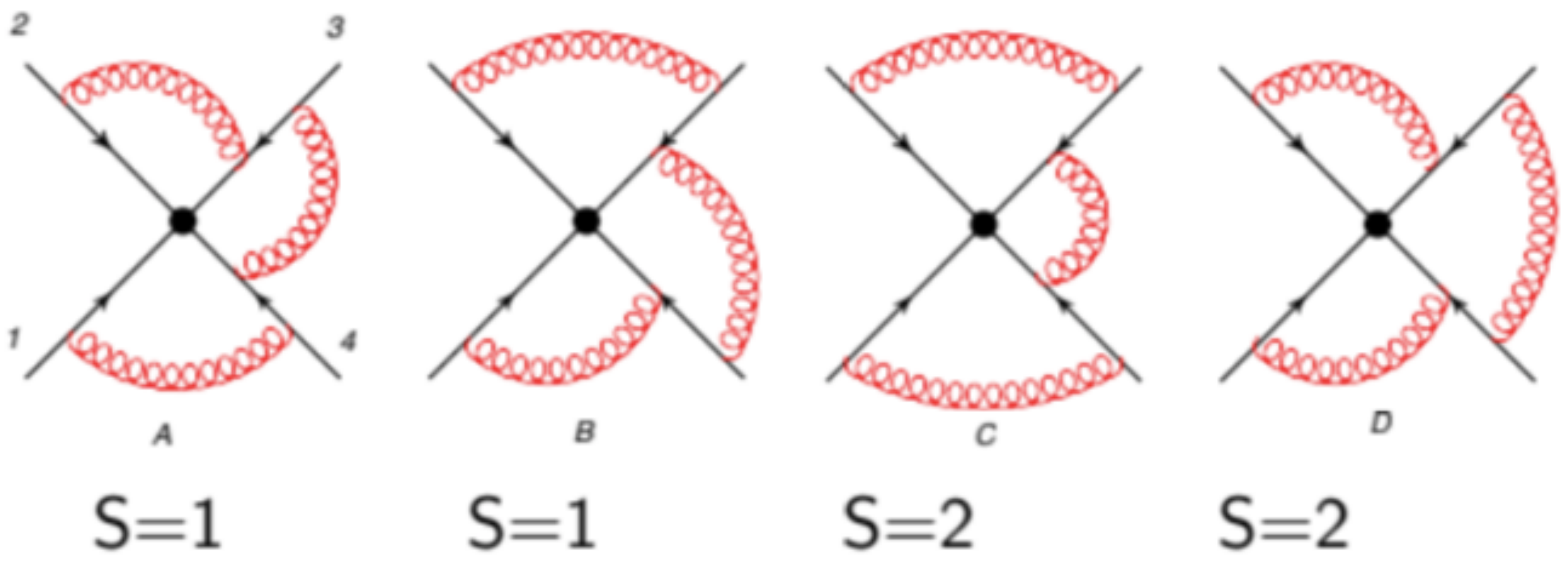
(Gardi, Smillie, White, et al 2010-2013)

Projector $$R_w^2 = R_w$$

(Gardi, Smillie, White, 2013)

Row sum rule $$\sum_{D'} R_w(D, D') = 0$$

Column sum rule (Conjecture) $$\sum_{D} s(D)\, R_w(D, D') = 0$$



$\Longrightarrow$

Exponentiated colour factor

Connection with Mathematical structures (Posets)

(Dukes, Gardi, McAslan, Scott, White)

Is there a pattern?

$$R = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$R = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$R = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$R = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

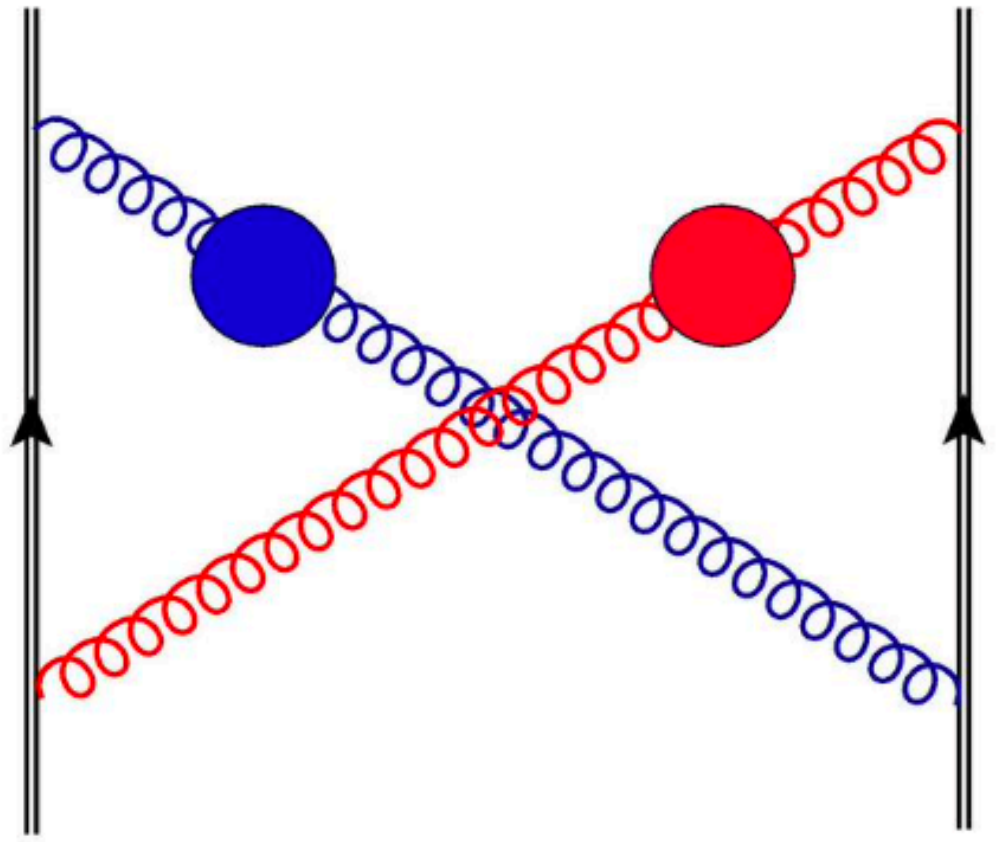
$$R = \begin{pmatrix} \frac{1}{6} & 0 & -\frac{1}{3} & 0 & \frac{1}{6} & 0 \\ -\frac{1}{3} & \frac{1}{2} & -\frac{1}{3} & 0 & \frac{2}{3} & -\frac{1}{2} \\ -\frac{1}{3} & 0 & \frac{2}{3} & 0 & -\frac{1}{3} & 0 \\ \frac{1}{6} & -\frac{1}{2} & -\frac{1}{3} & 1 & \frac{1}{6} & -\frac{1}{2} \\ \frac{1}{6} & 0 & -\frac{1}{3} & 0 & \frac{1}{6} & 0 \\ \frac{2}{3} & -\frac{1}{2} & -\frac{1}{3} & 0 & -\frac{1}{3} & \frac{1}{2} \end{pmatrix}$$

Cweb mixing matrices at 4 loops are reported in

Agarwal, Magnea, Pal, AT ;
JHEP 03 (2021) 188

Agarwal, Danish, Magnea, Pal, AT ;
JHEP 05 (2020) 128

Drawing the diagrams slightly differently
(Apologies for inconvenience!)



The tails of the Wilson lines are not visually meeting at the origin.

This makes drawing the diagrams easy.

Classification of diagrams

Irreducible

Partially Entangled

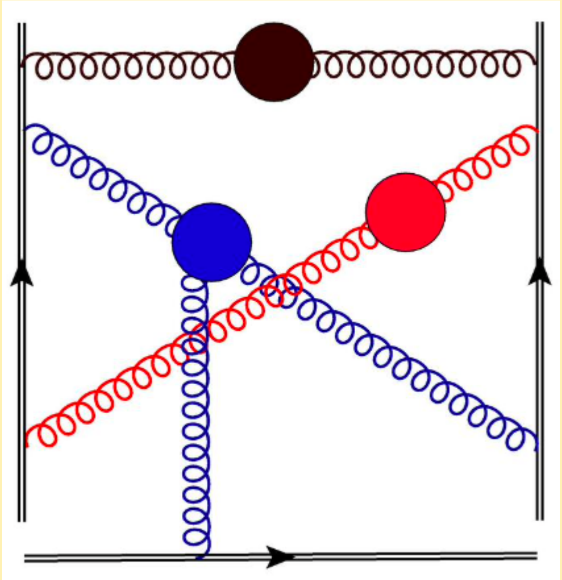
Completely Entangled

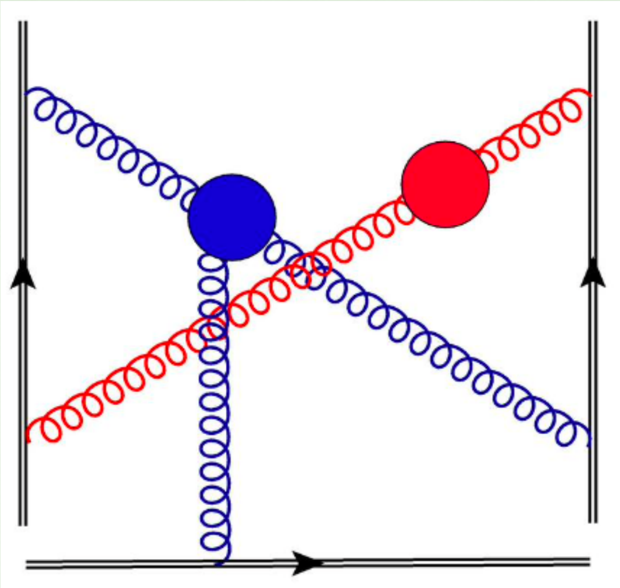
$s(d) = 0$

Reducible

$s(d) \neq 0$

Classification of diagrams





Irreducible

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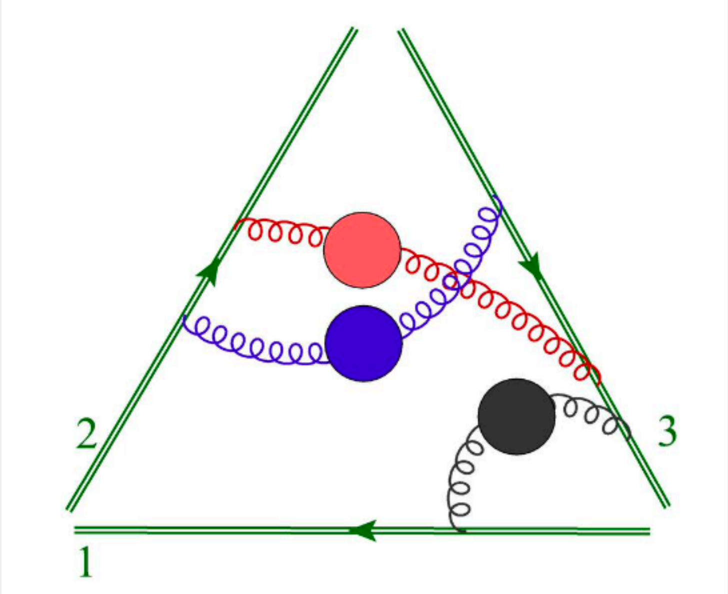

Webs containing only reducible diagrams
$\left(S(d_i) = 0, \quad \forall i\right)$

Uniqueness Theorem: For a given column weight vector

$S = \{s(d_1), \ldots, s(d_n)\}$

$s(d_i) \neq 0, \ \forall i$

the mixing matrix is unique.

Agarwal, Pal, Srivastav, AT ;
JHEP 06 (2022) 020

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Agarwal, Pal, Srivastav, AT ;
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	Web-1 at $\mathcal{O}(\alpha_s^N)$	Web-2 at $\mathcal{O}(\alpha_s^M)$
Identical	$S = \{s(d_1), \ldots, s(d_n)\}$	$S = \{s(d_1), \ldots, s(d_n)\}$
Identical	R	R

A general web containing both reducible & irreducible diagrams

Normal Ordering

d_1	$\cdots$	d_k	d_{k+1}	$\cdots$	d_l	d_{l+1}	$\cdots$	d_{l+m}

Completely Entangled | Partially Entangled | Reducible

$$R = \left(\begin{array}{cc|c} I_{k\times k} & (A_U)_{k\times(l-k)} & \\ O_{(l-k)\times k} & (A_L)_{(l-k)\times(l-k)} & B_{l\times m} \\ \hline \multicolumn{2}{c|}{O_{m\times l}} & D_{m\times m} \end{array}\right)$$

A and D diagonal blocks of mixing matrix R

$$R = \left(\begin{array}{cc|c} I_{k\times k} & (A_U)_{k\times(l-k)} & \\ O_{(l-k)\times k} & (A_L)_{(l-k)\times(l-k)} & B_{l\times m} \\ \hline & O_{m\times l} & D_{m\times m} \end{array}\right)$$

Agarwal, Pal, Srivastav, AT ;
JHEP 06 (2022) 020

The Block D satisfies the known properties of the mixing matrix!

$D^2 = D$ Satisfy Row Sum Rule Satisfy Column Sum Rule

Block D

Agarwal, Pal, Srivastav, AT ;
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$$R = \left(\begin{array}{cc|c} I_{k\times k} & (A_U)_{k\times(l-k)} & \multirow{2}{*}{$B_{l\times m}$} \\ O_{(l-k)\times k} & (A_L)_{(l-k)\times(l-k)} & \\ \hline \multicolumn{2}{c|}{O_{m\times l}} & D_{m\times m} \end{array}\right)$$

The Block D satisfies the known properties of the mixing matrix!

$D^2 = D$ Satisfy Row Sum Rule Satisfy Column Sum Rule

If $S = \{s_{l+1}, \ldots, s_{l+m}\}$ With all entries non vanishing

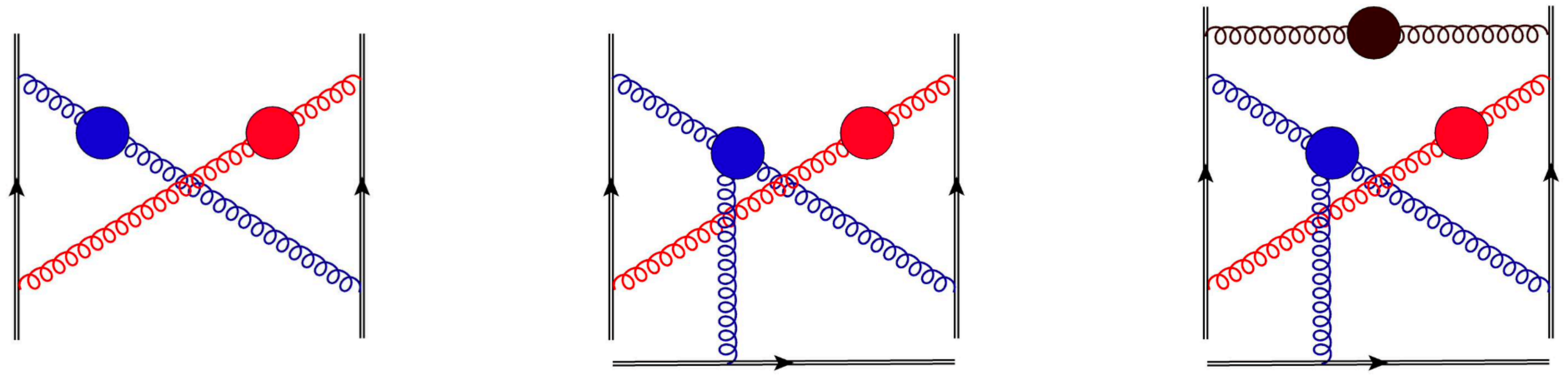
Using Uniqueness Theorem

D block is known if any web with same S has been calculated.

Block A

Coarse graining : The idea of Fused Webs

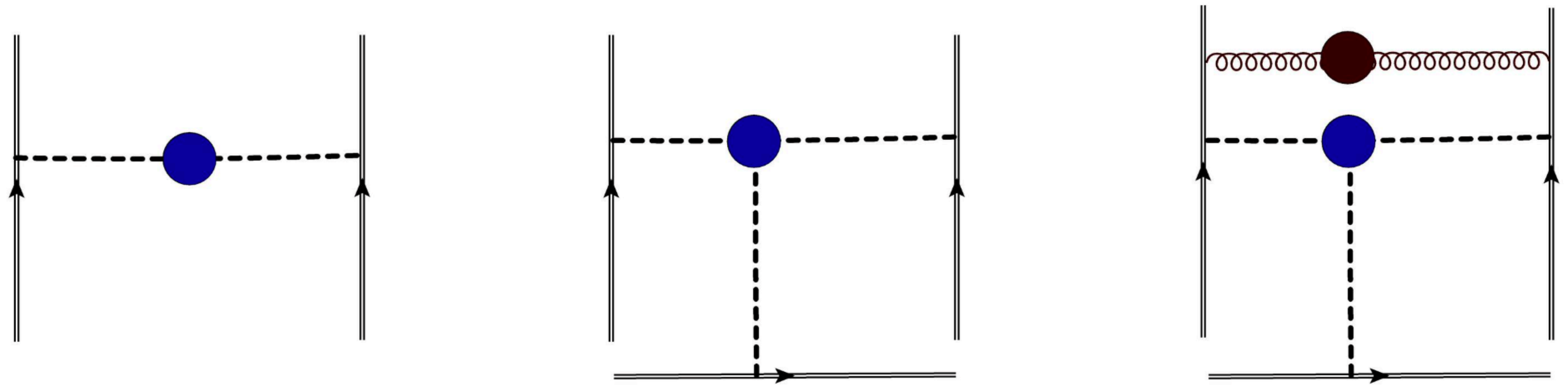
Original diagrams



(a) (b) (c)

$$R = \left(\begin{array}{cc|c} I_{k\times k} & (A_U)_{k\times(l-k)} & \multirow{2}{*}{$B_{l\times m}$} \\ O_{(l-k)\times k} & (A_L)_{(l-k)\times(l-k)} & \\ \hline \multicolumn{2}{c|}{O_{m\times l}} & D_{m\times m} \end{array}\right)$$

Fused diagrams



(d) (e) (f)

Colour factor of a Fused diagram = Colour factor of the original diagram

s-factors are defined in the usual way.

Application of fused web formalism

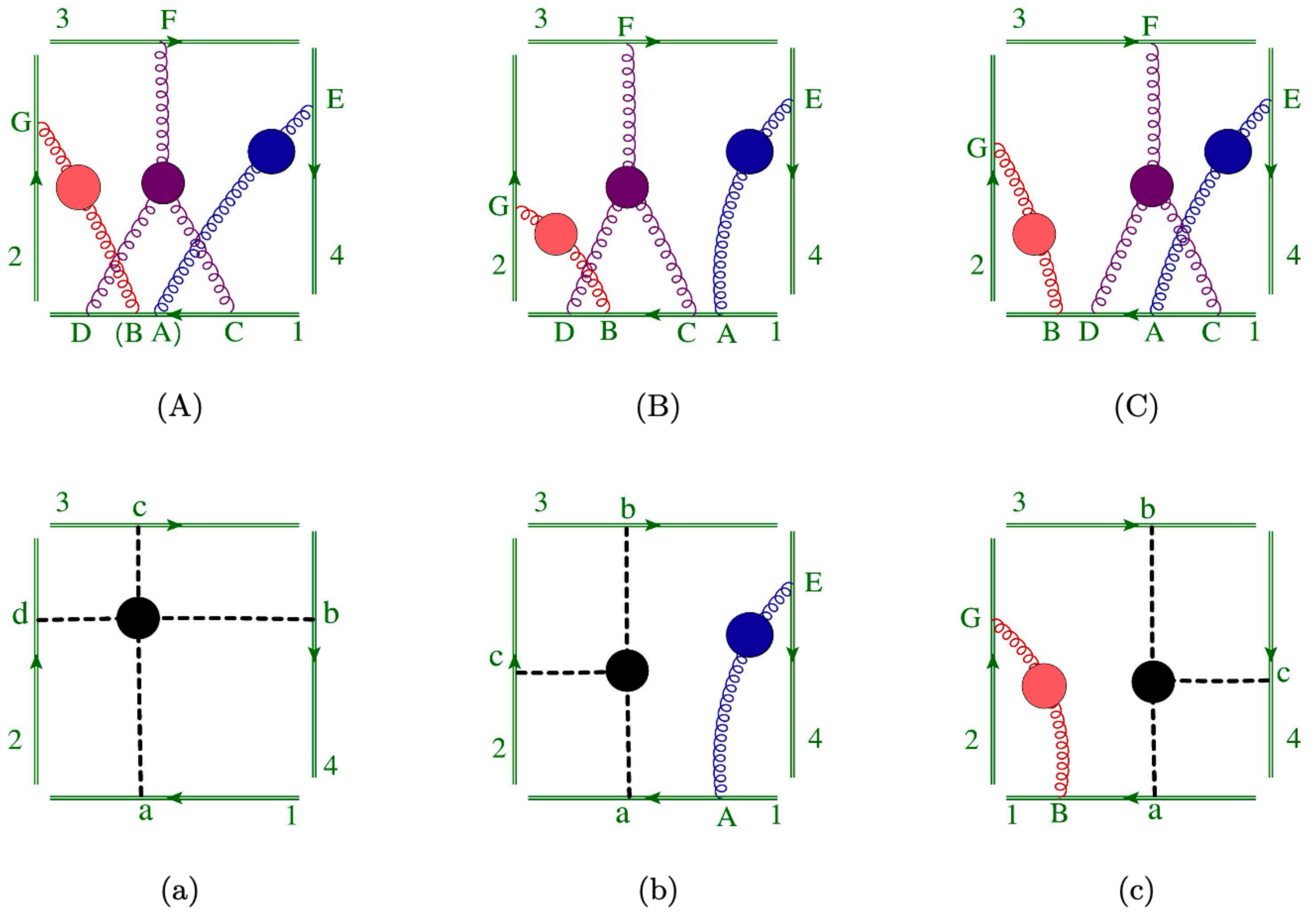
Cweb $W_4^{(2,1)}(1,1,1,4)$

12 diagrams

Completely Entangled: 2
Partially Entangled:4 → Block A

Reducible: 6 → Block D

$$R = \left(\begin{array}{c|cc|c} I_2 & \multicolumn{2}{c|}{A_U} & \\ \hline O_{4\times 2} & R(1_2) & X & B \\ & O_{2\times 2} & R(1_2) & \\ \hline \multicolumn{3}{c|}{O_{6\times 6}} & D \end{array}\right)$$



Cweb $W_4^{(2,1)}(1,1,1,4)$

(Its only completely and partially entangled diagrams)

Application of fused web formalism

Cweb $W_4^{(2,1)}(1,1,1,4)$;

12 diagrams

Completely Entangled: 2
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Reducible: 6 → Block D

$$R = \left(\begin{array}{c|cc|c} I_2 & \multicolumn{2}{c|}{A_U} & \\ \hline O_{4\times 2} & R(1_2) & X & B \\ & O_{2\times 2} & R(1_2) & \\ \hline \multicolumn{3}{c|}{O_{6\times 6}} & D \end{array}\right)$$

$$R = \frac{1}{6}\begin{pmatrix} 6 & 0 & -3 & -3 & -3 & -3 & -1 & 2 & 2 & -1 & 2 & 2 \\ 0 & 6 & -3 & -3 & -3 & -3 & 2 & -1 & 2 & 2 & 2 & -1 \\ 0 & 0 & 3 & -3 & 0 & 0 & -1 & -1 & -1 & -1 & 2 & 2 \\ 0 & 0 & -3 & 3 & 0 & 0 & 2 & -1 & 2 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 3 & -3 & -1 & -1 & 2 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -3 & 3 & -1 & 2 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & -1 & -1 & 2 \end{pmatrix};$$

Agarwal, Pal, Srivastav, AT ;
JHEP 06 (2022) 020

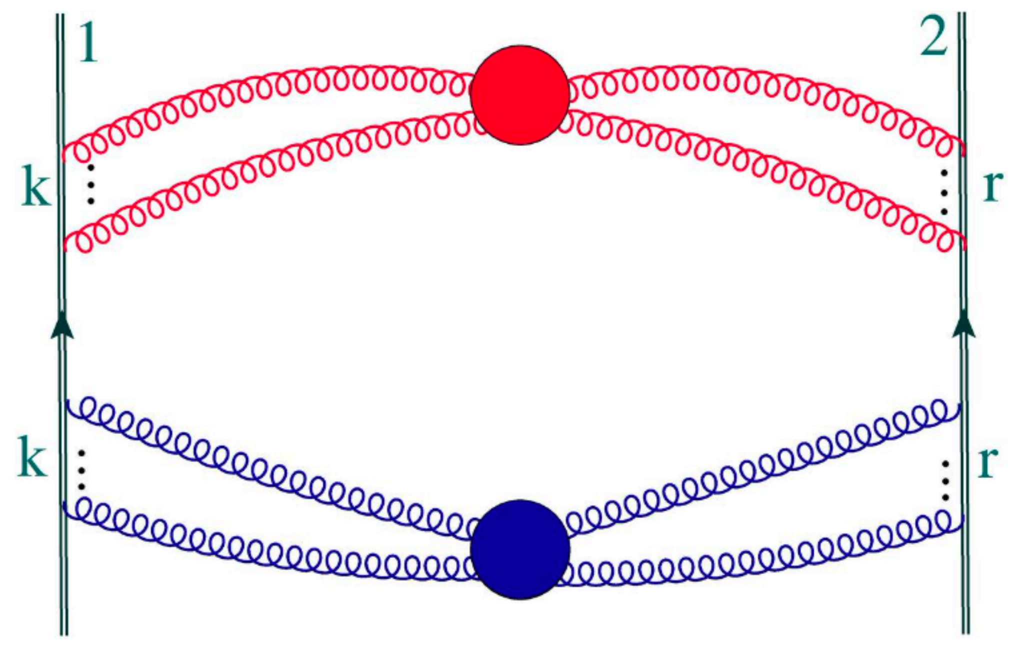
Application of fused web formalism

The sum of diagonal entries = Rank

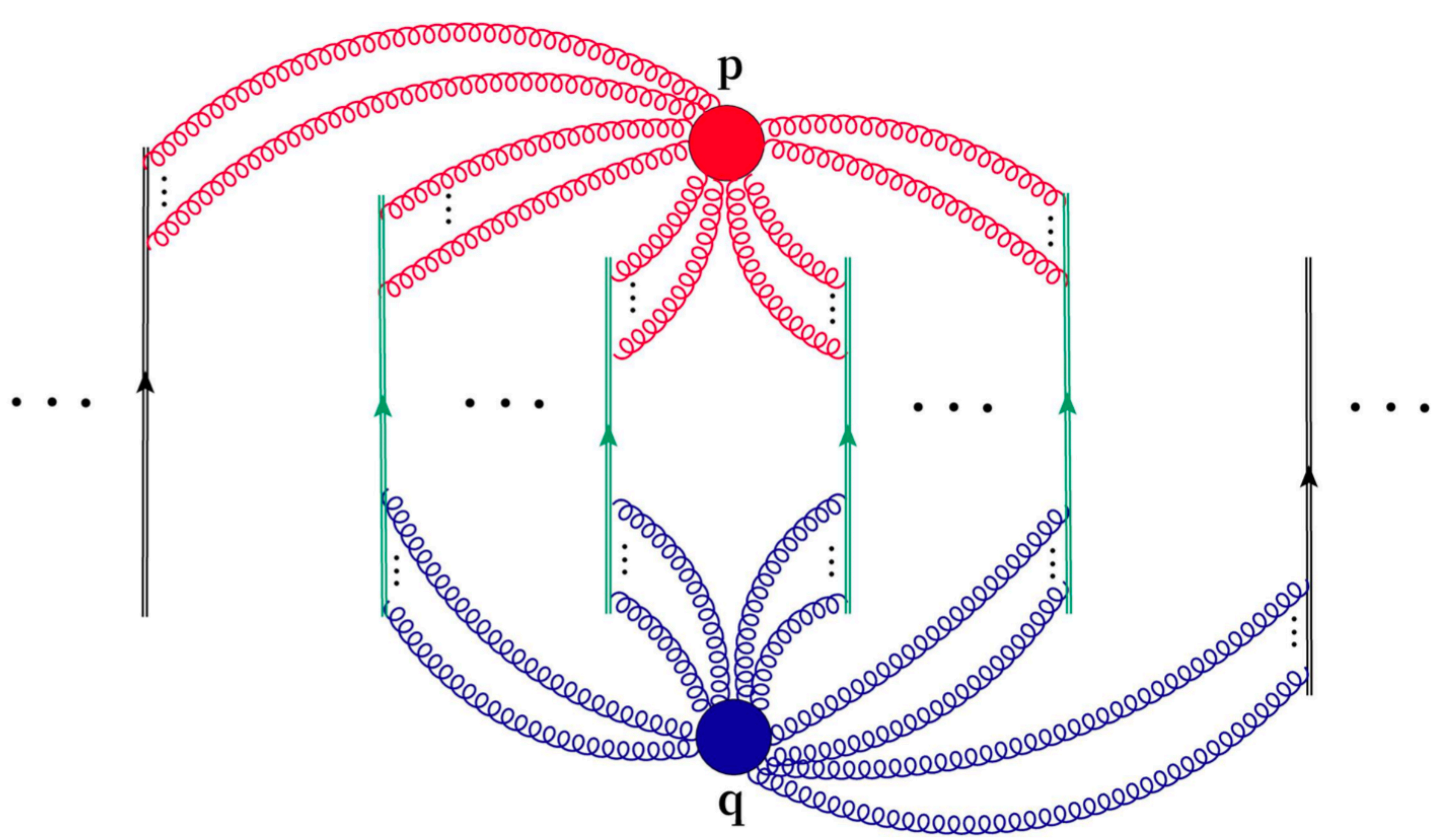
Rank = # of Exponentiated Colour factors

We can obtain the number of exponentiated colour factors
Using Fused Webs formalism

All order predictions for two special classes



$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



$$R = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & -1/2 & -1/2 \\ 0 & 1 & 0 & \ldots & 0 & -1/2 & -1/2 \\ 0 & 0 & 1 & \ldots & 0 & -1/2 & -1/2 \\ \vdots & \vdots & \vdots & \vdots & \vdots\,\vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -1/2 & -1/2 \\ 0 & 0 & 0 & \ldots & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & \ldots & 0 & -1/2 & 1/2 \end{pmatrix}.$$

Summary

- Using our Fused Web formalism we can obtain the diagonal blocks of R
- Diagonal Blocks are I or mixing matrices themselves
- # Exponentiated colour factors can be predicted using the diagonal blocks
- All order predictions can be made for special classes

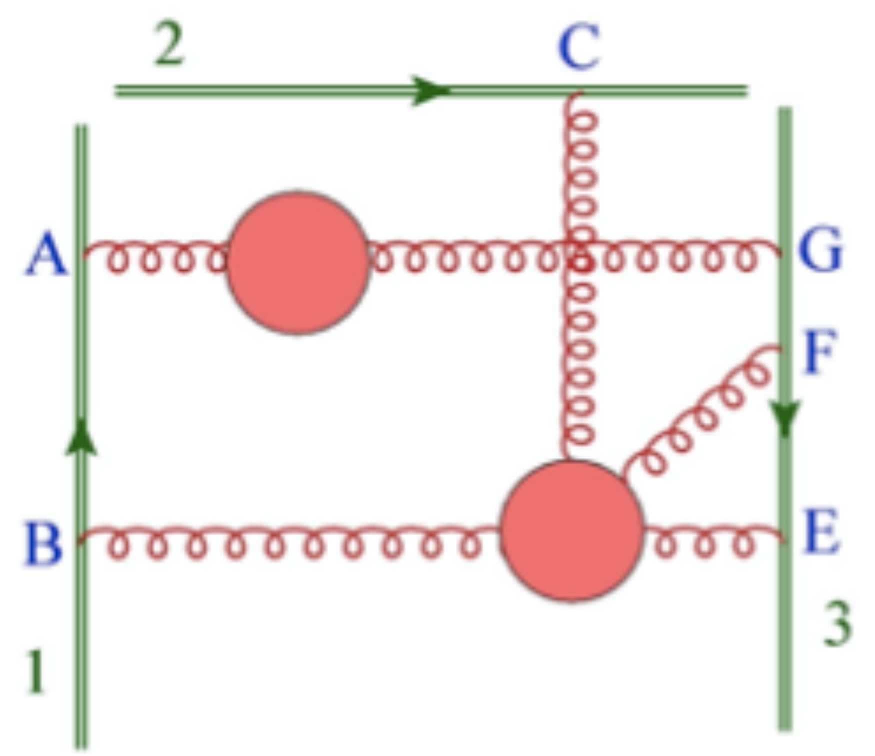
Thank You!

Backup Slides

New Results at 4 loops

(3 and 2-leg webs)

AT et al (to appear)



Diagrams	Sequences	S-factors
C_1	$\{\{BA\},\{GFE\}\}$	0
C_2	$\{\{BA\},\{FGE\}\}$	0
C_3	$\{\{BA\},\{FEG\}\}$	1
C_4	$\{\{AB\},\{GFE\}\}$	1
C_5	$\{\{AB\},\{FGE\}\}$	0
C_6	$\{\{AB\},\{FEG\}\}$	0

$$R = \begin{pmatrix} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \end{pmatrix} \qquad D = (\mathbf{1}_5, 0)$$

Exponentiated Colour Factors

$$\begin{aligned}(YC)_1 &= if^{afk}f^{bcg}f^{efg}\mathbf{T}_1^b\mathbf{T}_1^a\mathbf{T}_2^c\mathbf{T}_3^e\mathbf{T}_3^k + if^{aeh}f^{bcg}f^{efg}\mathbf{T}_1^b\mathbf{T}_1^a\mathbf{T}_2^c\mathbf{T}_3^h\mathbf{T}_3^f \\ &\quad - if^{abm}f^{bcg}f^{efg}\mathbf{T}_1^m\mathbf{T}_2^c\mathbf{T}_3^e\mathbf{T}_3^f\mathbf{T}_3^a\end{aligned}$$

$$(YC)_2 = if^{afk}f^{bcg}f^{efg}\mathbf{T}_1^b\mathbf{T}_1^a\mathbf{T}_2^c\mathbf{T}_3^e\mathbf{T}_3^k - if^{abm}f^{bcg}f^{efg}\mathbf{T}_1^m\mathbf{T}_2^c\mathbf{T}_3^e\mathbf{T}_3^f\mathbf{T}_3^a$$

$$(YC)_3 = -if^{abm}f^{bcg}f^{efg}\mathbf{T}_1^m\mathbf{T}_2^c\mathbf{T}_3^e\mathbf{T}_3^f\mathbf{T}_3^a$$

$$(YC)_4 = if^{afk}f^{bcg}f^{efg}\mathbf{T}_1^a\mathbf{T}_1^b\mathbf{T}_2^c\mathbf{T}_3^e\mathbf{T}_3^k + if^{aeh}f^{bcg}f^{efg}\mathbf{T}_1^a\mathbf{T}_1^b\mathbf{T}_2^c\mathbf{T}_3^h\mathbf{T}_3^f$$

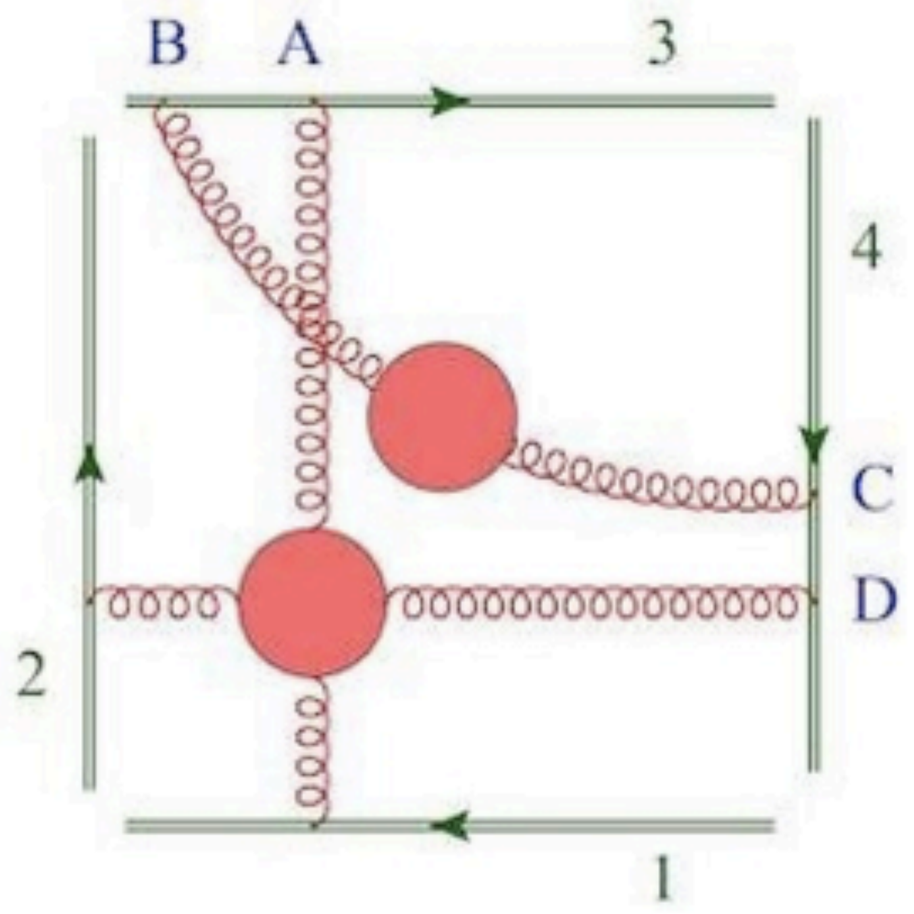
$$(YC)_5 = if^{afk}f^{bcg}f^{efg}\mathbf{T}_1^a\mathbf{T}_1^b\mathbf{T}_2^c\mathbf{T}_3^e\mathbf{T}_3^k$$

Results at 4 loops

(One of the 4-leg webs)

$\mathbf{W}_{4,\mathrm{I}}^{(1,0,1)}(1,1,2,2)$

Agarwal, Danish, Magnea, Pal, AT ; 2020



Diagrams	Sequences	S-factors
C_1	$\{\{BA\},\{CD\}\}$	1
C_2	$\{\{BA\},\{DC\}\}$	0
C_3	$\{\{AB\},\{CD\}\}$	0
C_4	$\{\{AB\},\{DC\}\}$	1

$$R = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \quad D = (\mathbf{1}_3, 0)$$

Exponentiated Color factors

$$(YC)_1 = i f^{abg} f^{cdg} f^{edh} \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^e \mathbf{T}_3^c \mathbf{T}_4^h - i f^{abg} f^{cdg} f^{cej} \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^j \mathbf{T}_4^d \mathbf{T}_4^e \,,$$

$$(YC)_2 = -i f^{abg} f^{cdg} f^{cej} \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^j \mathbf{T}_4^d \mathbf{T}_4^e \,,$$

$$(YC)_3 = i f^{abg} f^{cdg} f^{edh} \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^e \mathbf{T}_3^c \mathbf{T}_4^h - f^{abg} f^{cdg} f^{cej} f^{edh} \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^j \mathbf{T}_4^h \,.$$

Mixing matrices

Cwebs — Set of diagrams built out of gluon correlators

Agarwal, Danish, Magnea, Pal, AT ; 2020

Replica Trick — N_r identical copies of gauge fields are introduced, Wilson lines are replicated

Gardi, Laenen, Stavenga, White, 2010
See also: Vladimirov, 2014-2017

Replicated correlator

$$\mathcal{S}_n^{\text{repl.}}(\gamma_i) = \left[\mathcal{S}_n(\gamma_i)\right]^{N_r} = \exp\left[N_r \, \mathcal{W}_n(\gamma_i)\right]$$

Order N_r term

$$= \mathbf{1} + N_r \, \mathcal{W}_n(\gamma_i) + \mathcal{O}(N_r^2)$$

Combinatorics to extract ECF

- Assign replica number i to each connected gluon correlator
- Replica ordering operator to order colour generators $\mathbf{T}_k^i$ on each line
- # of hierarchies $h(m)$ between m replica numbers
- …
- Algorithm gives ECF

Inhouse Mathematica Code

The algorithm from generation of diagrams → computation ECF is implemented →Mixing matrices

Results at 4 loops

Agarwal, Danish, Magnea, Pal, AT ; 2020

Wilson line Correlators (**Cwebs**)	# of webs	Largest dimension of mixing matrix
5 legs	9	24
4 legs	21	24
3 legs	23	36
2 legs	8	36

Loop order (m)	Maximum number of hierarchies
1	**1**
2	3
3	13
4	75
5	541
6	**4683**

Fubini numbers
1,3,13,75,541,4683, …

Generating Function of Fubini numbers $h(m)$

$$\frac{1}{2-\exp(x)} - 1 \equiv \sum_{m=1}^{\infty} h(m)\,\frac{x^m}{m!}\,.$$

Soft Anomalous Dimension

IR behaviour of scattering amplitude $\leftrightarrow$ Wilson line correlator

Soft matrix

$$\mathcal{S}_n\Big(\beta_i\cdot\beta_j,\alpha_s(\mu^2),\epsilon\Big)\equiv\langle 0|\prod_{k=1}^{n}\Phi_{\beta_k}(\infty,0)\,|0\rangle$$

The Wilson line

$$\Phi_\beta(\infty,0)\equiv P\exp\left[ig\int_0^\infty d\lambda\,\beta\cdot\mathbf{A}(\lambda\beta)\right]$$

Soft anomalous dimension

$$\mathcal{S}_n\Big(\beta_i\cdot\beta_j,\alpha_s(\mu^2),\epsilon\Big)=\mathcal{P}\exp\left[-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\boldsymbol{\Gamma}_n\Big(\beta_i\cdot\beta_j,\alpha_s(\lambda^2),\epsilon\Big)\right]$$

Multi-parton Webs

Web (w) : A set of diagrams closed under permutations of the gluon attachments on the Wilson lines.

(Gardi, Smillie, White, *et al*)

The exponent $W(\gamma_i)$ **grouped into webs**

$$S_n(\gamma_i) = \exp\left(\sum w\right)$$

$R_w(D, D')$
Web mixing matrix

$$= \exp\left(\sum_{D,D' \in w} K(D)\, R_w(D, D')\, C(D)\right)$$

A 3 loop web
4×4 **mixing matrix**