# Two-scale evolution from rapidity-ordered BFKL cascade[1]

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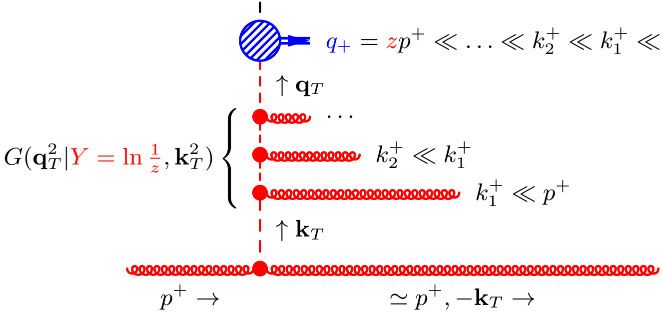
# Motivation

1. NLO corrections to the BFKL/BK/JIMWLK kernel and Impact-factors as well as many NLO corrections in CGC/Saturation studies contain double logs $\alpha_s \ln^2(\mu^2/\mathbf{q}_T^2)$ for $\mu^2 \gg \mathbf{q}_T^2$. All-order structure of these very large ($\mathbf{q}_T$ is often integrated down to zero!) corrections remains unknown despite vast body of literature devoted to this problem.
2. In several recent studies [Müller *et.al.* 13'; M.N. 20'; Hentschinski *et.al.* 21'; Taels *et.al.* 22 ] these "Sudakov" terms where found **with different coefficients and even signs!** The coefficient of this term strongly depends on the procedure of "double-counting subtraction" between evolution and NLO correction. It means, that just adding Sudakov formfactor on top of small-x UPDF could not be always correct.
3. In TMD factorization the resummation of Sudakov logs is based on the structure of **rapidity divergences** in TMDs and soft-factors. In Lipatov's EFT, the BFKL kernel is also a coefficient of the rapidity-divergence in 4-Reggeon Green's function. Are these RDs the same or different? Is there an overlap and could it be exploited?

# Standard HEF – resummation of $\ln 1/z$

The setup of standard High-Energy Factorization [Collins, Ellis, 91'; Catani, Ciafaloni, Hautmann, 91',94'] in the **LLA** $(\sum_n \alpha_s^n \ln^{n-1}\frac{1}{z},\ z=\frac{q^+}{p^+})$ and in the **LP** w.r.t. $z$, treatment like in [Kirschner, Segond, 10']:

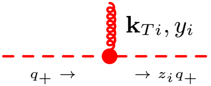



**Notice, that $k^+$-conservation is taken care of by the MRK!**

# Reminder: Building blocks of BFKL Green's function

**For the squared amplitude:**

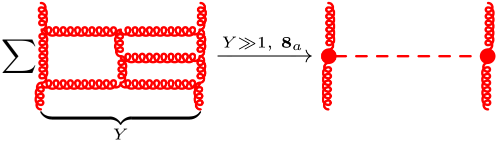
- Real emisssion – squared *Lipatov's vertex:*




$$= \hat{\alpha}_s \frac{(2\pi)^{2\epsilon}}{\pi \mathbf{k}_{Ti}^2} d^2\mathbf{k}_{Ti} dy_i, \quad dy_i = \frac{dz_i}{z_i(1-z_i)} = dz_i \underbrace{\left(\frac{1}{z_i} + \frac{1}{1-z_i}\right)}_{\text{CCFM kernel}}$$

where $\boxed{\hat{\alpha}_s = \alpha_s C_A/\pi}$

- Virtual corrections – *Regge factors*:




$$\propto \exp\left[2\omega_g(\mathbf{p}_T^2) Y\right],$$

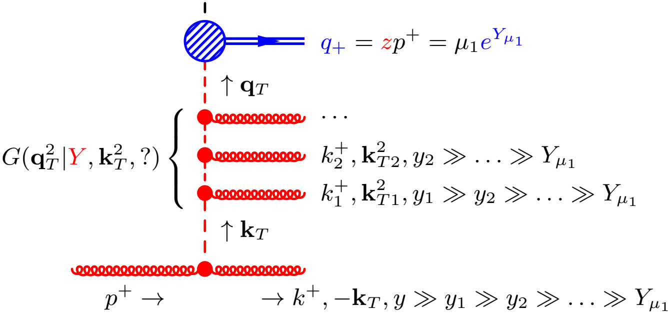
where $\omega_g(\mathbf{p}_T^2)$ – **one-loop gluon Regge trajectory**:

$$\omega_g(\mathbf{p}_T^2) = -\frac{\hat{\alpha}_s}{4} \int \frac{d^{2-2\epsilon}\mathbf{k}_T}{\pi(2\pi)^{-2\epsilon}} \frac{\mathbf{p}_T^2}{\mathbf{k}_T^2(\mathbf{p}_T - \mathbf{k}_T)^2} = \frac{\hat{\alpha}_s}{2\epsilon}(\mathbf{p}_T^2)^{-\epsilon} \frac{(4\pi)^{\epsilon}\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \text{ (in DR)}$$

$$= -\hat{\alpha}_s \ln \frac{\mathbf{p}_T^2}{\lambda^2} \text{ (for } \mathbf{k}_{Ti}^2 > \lambda^2 \text{ regularization)}$$

# New setup: resummation of rapidity logs

The same tools should allow the resummation of corrections enhanced by **difference in physical rapidity** ($y = \frac{1}{2}\ln\frac{k^+}{k^-}$) between rebounded gluon and the hard process (*like with Müller-Navelet dijets*, also motivated by studies of [Balitsky, Tarasov, 15'; Balitsky, Chirilli 20']): $Y = y_0 - Y_{\mu_1} = \ln\frac{\mu_1}{|\mathbf{k}_T|}\frac{k_+}{q_+}$.




**Two possible solutions:**

- *Kinematic constraint* (yet unexplored...):

$$k_{i-1}^+ > k_i^+,$$

- or just **restore $k^+$-conservation**?

**Problem in the DGLAP region:** if $k_T$-ordering

$$\mathbf{k}_T^2 \ll \mathbf{k}_{T1}^2 \ll \mathbf{k}_{T2}^2 \ll \ldots \ll \mu_1^2,$$

happens to be "stronger" than $y$-ordering $y \gg y_1 \gg y_2 \gg \ldots \gg Y_{\mu_1}$, then it is possible that:

$$|\mathbf{k}_T|e^{y} \ll |\mathbf{k}_{T1}|e^{y_1} \ll \ldots \ll \mu_1 e^{y_{\mu_1}} \Rightarrow k^+ \ll k_1^+ \ll k_2^+ \ll \ldots \ll q^+,$$

and **the MRK treatment of $k^+$-conservation breaks-down.**

# Resummation factor

Collinearly un-subtracted resummation factor:

$$\tilde{\mathcal{C}}(z, \mathbf{q}_T^2, \mu_1^2) = \hat{\alpha}_s \int\limits_{Y_{\mu_1}}^{+\infty} dy \int \frac{d^2\mathbf{k}_T}{\pi \mathbf{k}_T^2}\, G\Big(\mathbf{q}_T^2, zp_+ \Big|\, y - Y_{\mu_1}, \mathbf{k}_T^2, p_+ - k_+\Big),$$

where $y$ and $\mathbf{k}_T$ – rapidity and transverse momentum of the *rebounded gluon*, $k^+ = |\mathbf{k}_T| e^y$, $G$ – (modified) BFKL Green's function with longitudinal-momentum dependence.

The $k^+$-conservation $\delta$-function can be factorised using Fourier transform in $x^-$:

$$\delta\left(p_+ - k_+ - k_1^+ - \ldots - k_n^+ - zp_+\right) = \int\limits_{-\infty}^{+\infty} \frac{dx_-}{2\pi} e^{ix_-(p_+(1-z)-k_+)} \prod_{i=1}^{n} e^{-ix_- k_i^+},$$

where $k_i^+ = |\mathbf{k}_{Ti}| e^{y_i}$. So we introduce: $G\Big(\mathbf{q}_T^2 \Big|\, Y, \mathbf{p}_T^2, x_-\Big)$

# Evolution for modified BFKL Green's function

$$\frac{\partial G\left(\mathbf{q}_T^2 \middle| \, Y, \mathbf{p}_T^2, x_-\right)}{\partial Y} = \hat{\alpha}_s \int d^{2-2\epsilon}\mathbf{k}_T \, K(\mathbf{k}_T^2, \mathbf{p}_T^2, x_-, Y) G\left(\mathbf{q}_T^2 \middle| \, Y, (\mathbf{p}_T - \mathbf{k}_T)^2, x_-\right),$$

with

$$K(\mathbf{k}_T^2, \mathbf{p}_T^2, x_-, Y) = \delta^{(2-2\epsilon)}(\mathbf{k}_T) \, \frac{(\mathbf{p}_T^2)^{-\epsilon}}{\epsilon} \, \frac{(4\pi)^\epsilon \Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} + \frac{\exp[-ix_-|\mathbf{k}_T|e^Y]}{\pi(2\pi)^{-2\epsilon}\mathbf{k}_T^2}.$$

Asymptotic solution for $x_- \gg |\mathbf{q}_T|^{-1}$ [M.N., 21']:

$$\boxed{G_{\text{asy.}}\left(\mathbf{q}_T^2 \middle| \, Y, \mathbf{p}_T^2, x_-\right) = \delta(\mathbf{q}_T^2 - \mathbf{p}_T^2)\left(\mathbf{q}_T^2 x_-^2\right)^{-\hat{\alpha}_s Y} \exp\left[-\hat{\alpha}_s Y \left(2\gamma_E + i\pi\right) - \hat{\alpha}_s Y^2\right],}$$

in the previous work only the effects of the highlighted term where studied.
**Note that:**

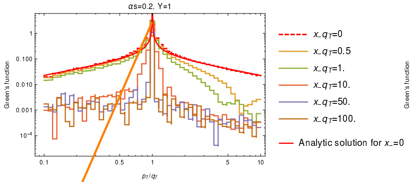
$$G\left(\mathbf{q}_T^2 \middle| \, Y, \mathbf{p}_T^2, x_- \gg |\mathbf{q}_T|^{-1}\right) \sim \left(|\mathbf{q}_T| x_-\right)^{-2\hat{\alpha}_s Y}.$$

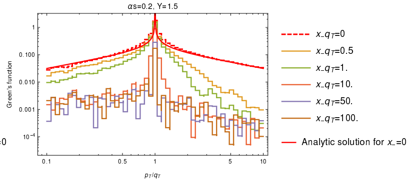
**The "heavy" soft-gluon tail of the shockwave?** Doesn't look like a boosted distribution $f(x_- e^{-Y}) \underset{Y\to\infty}{\longrightarrow} \delta(x_-)$.

# (Preliminary) Monte-Carlo results for $G$

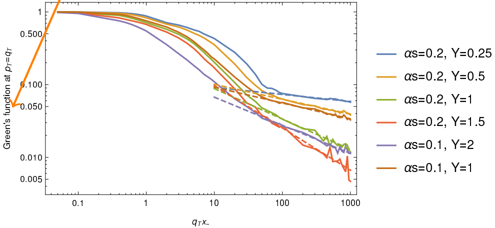
The LL BFKL equation can be solved using efficient Monte-Carlo algorithm [C. Schmidt, 96'] (also implemented in the BFKLex MC [G. Chachamis et.al.]) which gives $(y_i, \mathbf{k}_{Ti})$ for all emissions with $|\mathbf{k}_{Ti}| > \lambda$, then the modified Green's function can be calculated as:

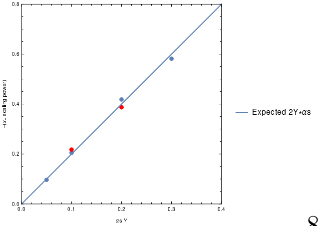
$$G\left(\mathbf{q}_T^2 \middle| Y, \mathbf{p}_T^2, x_-\right) = \left\langle \prod_{\text{emissions}} \exp\left[-ix_-|\mathbf{k}_{Ti}|e^{y_i}\right] \right\rangle_{\text{events}}$$



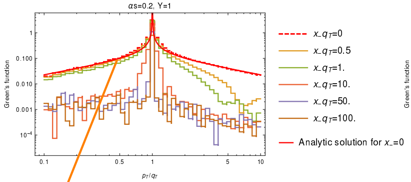



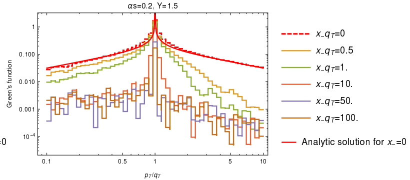

$G_{\text{MC}}(\mathbf{p}_T^2 = \mathbf{q}_T^2) \sim (\mathbf{q}_T^2 x_-^2)^{-2\hat{\alpha}_s Y}$ at $x_- \gg |\mathbf{q}_T|$:



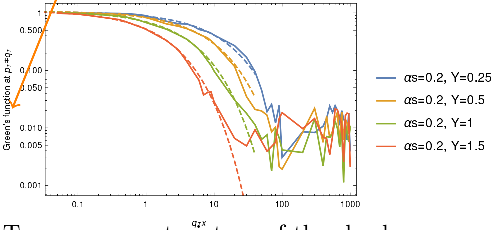



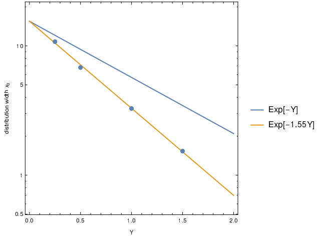
# (Preliminary) Monte-Carlo results for $G$




$G_{\rm MC}(\mathbf{p}_T^2 \neq \mathbf{q}_T^2) \sim e^{-(x/x_0)^{0.7}}$ at small $x_-|\mathbf{q}_T|$ with $x_0$ decreasing with $Y$:




Two-component picture of the shock-wave emerges:

- *The quasi-classical component*, which is "narrow" in $x_-$ and Lorentz-contracts as $e^{-Y}$
- *The soft-gluon component*, has "heavy" power-law tail in $x_-$ and shrinks with increasing $Y$ differently

## From $G_{\text{asy.}}$ to $\tilde{\mathcal{C}}$

Substituting $G_{\text{asy.}}$ into:

$$\tilde{\mathcal{C}} = \hat{\alpha}_s \int_0^{\infty} dY \int_0^{\infty} \frac{d\mathbf{k}_T^2}{\mathbf{k}_T^2} \int_{-\infty}^{+\infty} \frac{p^+ dx_-}{2\pi} e^{ix_-\left(p^+(1-z)-k^+\right)} G\left(\mathbf{q}_T^2 \middle|\, Y, \mathbf{k}_T^2, x_-\right),$$

with $k^+ = zp^+|\mathbf{k}_T|e^Y/\mu_1$, we obtain:

$$\tilde{\mathcal{C}} \simeq \frac{\hat{\alpha}_s}{\mathbf{q}_T^2(1-z)} \int_0^{\infty} dY \left(\frac{p_+(1-z)}{|\mathbf{q}_T|}\right)^{2\hat{\alpha}_s Y} e^{-\hat{\alpha}_s Y^2 - 2\gamma_E \hat{\alpha}_s Y} f\left(1 - \frac{|\mathbf{q}_T|}{\mu_1}\frac{z}{1-z}e^Y, \hat{\alpha}_s Y\right),$$

where

$$f(\kappa, \alpha) = \int_{-\infty}^{+\infty} \frac{dx}{2\pi} x^{-2\alpha} e^{i(\kappa x - \pi\alpha)} = \frac{\kappa^{-1+2\alpha}\theta(\kappa)}{\Gamma(2\alpha)},$$

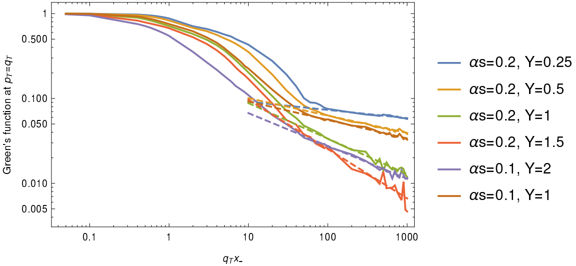
so finally we obtain a resummation factor depending on **two scales**:

$$\boxed{\tilde{\mathcal{C}} \simeq \frac{\hat{\alpha}_s}{\mathbf{q}_T^2(1-z)} \int_0^{Y_1} dY \exp\left[-\hat{\alpha}_s\left(Y^2 - 2Y(Y_2 - \gamma_E)\right)\right] \frac{\left(1 - e^{Y - Y_1}\right)^{-1+2\hat{\alpha}_s Y}}{\Gamma(2\hat{\alpha}_s Y)},}$$

with $Y_1 = \ln\left(\frac{\mu_1}{|\mathbf{q}_T|}\frac{1-z}{z}\right)$ and $Y_2 = \ln\left(\frac{\mu_2}{|\mathbf{q}_T|}\frac{1-z}{z}\right)$ where $\mu_2 = q_+$.

# Region of applicability

The obtained solution for $\tilde{\mathcal{C}}$ is applicable only if the integral over $x_-$ is dominated by $|\mathbf{q}_T|x_- \gg 1$ tail of the Green's function:




From the derivation, this is true if at least $p_+(1-z)/|\mathbf{q}_T| \lesssim 1$, i.e. $\alpha_s Y_2 \ll 1$ still.

The hierarchy

$$q_+ \frac{1-z}{z} \lesssim |\mathbf{q}_T| \ll \mu_1,$$

can be realised e.g. in production of heavy particle (e.g. Higgs, pseudoscalar quarkonium) in the direction of the projectile.

# Two-scale exponent

$$\tilde{\mathcal{C}} \simeq \frac{\hat{\alpha}_s}{\mathbf{q}_T^2(1-z)} \int_0^{Y_1} dY \exp\left[-\hat{\alpha}_s \left(Y^2 - 2Y(Y_2 - \gamma_E)\right)\right] \frac{\left(1-e^{Y-Y_1}\right)^{-1+2\hat{\alpha}_s Y}}{\Gamma(2\hat{\alpha}_s Y)},$$

In the limit $\alpha_s Y_1 \ll 1$, $\alpha_s Y_1^2 \sim 1$ the singularity at $Y = Y_1$ can be replaced by $\delta(Y - Y_1)$ and the integral can be calculated to be:

$$\tilde{\mathcal{C}} \propto \exp\left[-\hat{\alpha}_s \left(\ln^2 \frac{\mu_1}{|\mathbf{q}_T|} - 2\ln\frac{\mu_1}{|\mathbf{q}_T|}\left(\ln\frac{\mu_2}{|\mathbf{q}_T|} - \gamma_E\right)\right)\right].$$

Let's compare the scale-dependent exponent with the solution of CSS equations for $\Gamma_{\rm c}(\mu) = \text{const.}$:

$$\exp\left[\Gamma_{\rm c}\ln^2(\mu|\mathbf{x}_T|) - 2\Gamma_{\rm c}\ln(\sqrt{\zeta}|\mathbf{x}_T|)\ln(\mu|\mathbf{x}_T|) - \gamma_V\ln(\mu|\mathbf{x}_T|)\right],$$

which leads to identification:

$$\mu_1 \to \mu,\ \ \mu_2 \to \sqrt{\zeta} \text{ and } \Gamma_{\rm c} = -\hat{\alpha}_s,$$

the negative cusp anomalous dimension is weird...

Notation from [Vladimirov, Scimemi, 18']:

$$\frac{d}{d\ln\mu}\ln F(x,\mathbf{x}_T,\mu,\sqrt{\zeta}) = \gamma_F(\mu,\sqrt{\zeta}),\quad \frac{d}{d\ln\sqrt{\zeta}}\ln F(x,\mathbf{x}_T,\mu,\sqrt{\zeta}) = -\mathcal{D}_F(\mu,|\mathbf{x}_T|),$$

$$\frac{d}{d\ln\sqrt{\zeta}}\gamma_F(\mu,\sqrt{\zeta}) = -\frac{d}{d\ln\mu}\mathcal{D}_F(\mu,|\mathbf{x}_T|) = \Gamma_{\rm c}(\mu).$$

## Conclusions and outlook

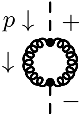
- Resummation factor depending on **two scales** arises from the asymptotic solution for $x_- \to \infty$, some analogy with solution of CSS equation can be made
- However the region of applicability of asymptotic solution is very narrow (**not** applicable at $z \ll 1$!), better understanding of $x_-$-dependence is needed
- Two-component picture of the shockwave produced by the energetic parton is emerging from the model, with "quasi-classical" component which Lorentz-contracts with increasing $Y$ and "soft-gluon" component which shrinks significantly slower

**Thank you for your attention!**

# Digression: Regge trajectory and RDs

Due to the presence of the $1/q^{\pm}$-factors in the induced vertices, loop integrals in Lipatov's High-energy EFT contain the light-cone (Rapidity) divergences:

$$\Pi^{(1)}_{ab} = q\downarrow$$




$$= g_s^2 C_A \delta_{ab} \int \frac{d^d q}{(2\pi)^D} \frac{\left(\mathbf{p}_T^2 (n_+ n_-)\right)^2}{q^2 (p-q)^2 q^+ q^-}$$

The regularization by explicit cutoff in rapidity was proposed by Lipatov [Lipatov, 1995] ($q^{\pm} = \sqrt{q^2 + \mathbf{q}_T^2} e^{\pm y}$, $p^+ = p^- = 0$):

$$\int \frac{dq^+ dq^-}{q^+ q^-} = \int\limits_{y_1}^{y_2} dy \int \frac{dq^2}{q^2 + \mathbf{q}_T^2},$$

then

$$\Pi^{(1)}_{ab} \sim \delta_{ab} \mathbf{p}_T^2 \times \boxed{\underbrace{\frac{C_A g_s^2}{2(2\pi)^3} \int \frac{\mathbf{p}_T^2 d^{D-2}\mathbf{q}_T}{\mathbf{q}_T^2 (\mathbf{p}_T - \mathbf{q}_T)^2}}_{\omega_g(\mathbf{p}_T^2)}} \times (y_2 - y_1) + \text{finite terms}$$