Andrea Simonelli,
INFN Torino

Istituto Nazionale di Fisica Nucleare
SEZIONE DI TORINO

In collaboration with M. Boglione

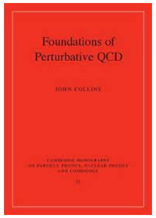
Latest results on the factorization of single-inclusive e^+e^- annihilation

Resummation
Evolution
Factorization
REF 2022 workshop
October 31 - November 4, 2022

STRONG 2020

Extraction of TMD Fragmentation Functions

Access to the 3D-dynamics of confinement

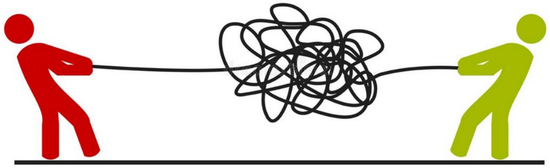

Foundations of Perturbative QCD

JOHN COLLINS

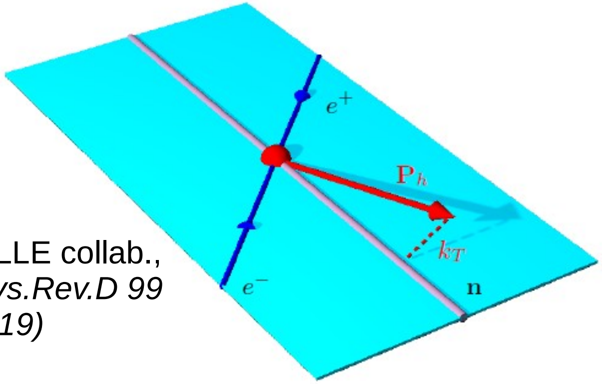
Standard TMD factorization
(Collins factorization formalism)

- SIDIS $d\sigma \sim H_{\text{SIDIS}}\, F\, D$
- DIA $d\sigma \sim H_{\text{DIA}}\, D_1\, D_2$

Always two TMDs that have to be extracted simultaneously



SIA would provide a much cleaner access to TMD FFs! $d\sigma \overset{??}{\propto} D$



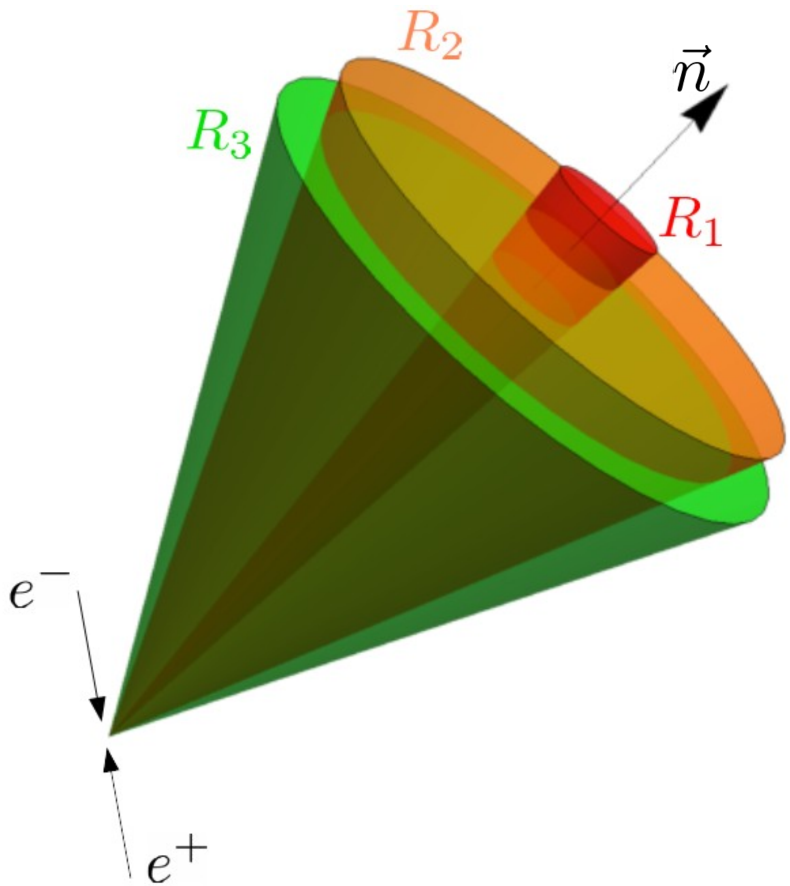
BELLE collab., *Phys.Rev.D 99 (2019)*

We will consider the process $e^+e^- \to h\,X$
The transverse momentum of the detected hadron is measured w.r.t. the thrust axis

$$z_h = \frac{E}{Q/2}, \quad T = \frac{\sum_i |\vec{P}_{(\text{c.m.}),\,i} \cdot \hat{n}|}{\sum_i |\vec{P}_{(\text{c.m.}),\,i}|}, \quad P_T \text{ w.r.t } \vec{n}$$

Physical intuition picture

Depending on where the hadron is located within the jet the underlying kinematics can be remarkably different, resulting in different factorization theorems



M. Boglione, A. Simonelli, JHEP 02 (2022) 013

The hadron is detected very close to the axis of the jet:

- Extremely small P_T
- Soft radiation affects significantly the transverse deflection of the hadron from the thrust axis

TMD FF + non-pert. SOFT contribution

The hadron is detected in the central region of the jet:

- Most common scenario
- Majority of experimental data fall into this case

TMD FF

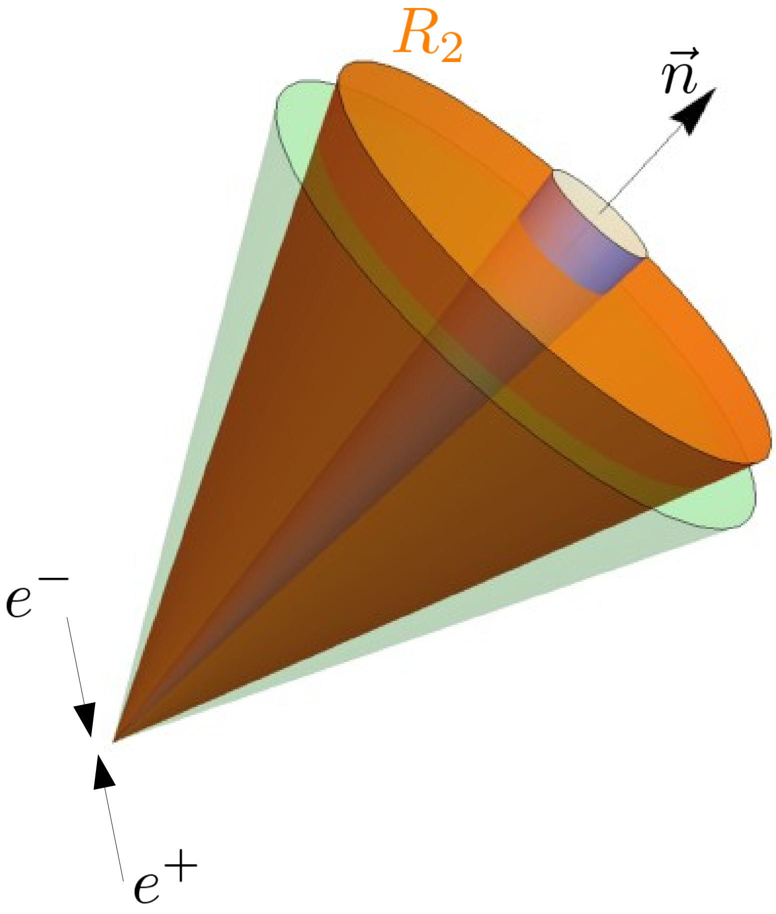
The hadron is detected near the boundary of the jet:

- Moderately small P_T
- The hadron P_T causes the spread of the jet affecting the topology of the final state (i.e. the value of thrust)

Generalized FJF

Physical intuition picture

Depending on where the hadron is located within the jet the underlying kinematics can be remarkably different, resulting in different factorization theorems



The hadron is detected very close to the axis of the jet:
- Extremely small P_T
- Soft radiation affects significantly the transverse deflection of the hadron from the thrust axis

TMD FF + non-pert. SOFT contribution

The hadron is detected in the central region of the jet:
- Most common scenario
- Majority of experimental data fall into this case

TMD FF

The hadron is detected near the boundary of the jet:
- Moderately small P_T
- The hadron P_T causes the spread of the jet affecting the topology of the final state (i.e. the value of thrust)

Generalized FJF

M. Boglione, A. Simonelli, JHEP 02 (2022) 013

Factorization theorem in Region 2

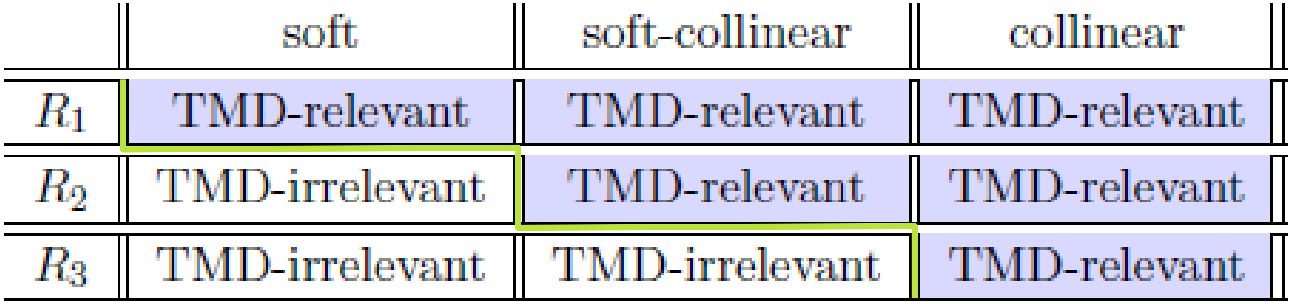
Forward soft-collinear radiation is **relevant** for TMD effects

Matches intuition!

The hadronization process is described by a **TMD FF**

$$d\sigma \sim H \times J(\tau) \times \frac{\mathcal{S}(\tau)}{Y_L(\tau)} \times D_{h/j}(z, P_T)$$

Matches intuition!



	soft	soft-collinear	collinear
R_1	TMD-relevant	TMD-relevant	TMD-relevant
R_2	TMD-irrelevant	TMD-relevant	TMD-relevant
R_3	TMD-irrelevant	TMD-irrelevant	TMD-relevant

Totally "symmetric" structure among the three regions

Region 2 is a truly independent kinematic region!

Other choices are possible, but they do not lead to independent kinematic configurations.

TMD UNIVERSALITY

The TMD FF appearing in the Region 2 factorized cross section is not defined as in SIDIS and DIA (standard TMD factorization).

The differences concern the non-perturbative region (large bT) as they are due to the impact of long-distance soft radiation in standard TMD factorization theorems:

$$D^{\text{usual}}(z, b_T) = D^{R_2}(z, b_T)\,\sqrt{M_S(b_T)}$$

Boglione, Simonelli, *Eur.Phys.J.C 81 (2021)*

Factorization Theorem of **Region 2**

TMD UNIVERSALITY

Not addressed in this talk

The TMD FF appearing in the Re[illegible] factorized cross section is not defined as in SIDIS and DIA (standard TMD fact[illegible]).

The differences concern the non-perturbative [illegible] (large bT) as they are due to the impact of long-distance soft radiation in standa[illegible] factorization theorems:

$$D^{\text{usual}}(z, b_T) = D^{R_2}(z, b_T)\,\sqrt{M_S(b_T)}$$

Boglione, Simonelli, *Eur.Phys.J.C 81 (2021)*

Factorization Theorem of **Region 2**

TMD UNIVERSALITY

Not addressed in this talk

The TMD FF appearing in the Re factorized cross section is not defined as in SIDIS and DIA (standard TMD fact n).

The differences concern the non-perturbative (large bT) as they are due to the impact of long-distance soft radiation in standa factorization theorems:

$$D^{\text{usual}}(z, b_T) = D^{R_2}(z, b_T)\,\sqrt{M_S(b_T)}$$

Boglione, Simonelli, *Eur.Phys.J.C 81 (2021)*

Factorization Theorem of **Region 2**

RAPIDITY DIVERGENCES TREATMENT

Standard treatment of rapidity divergences cannot be applied to Region 2.

$$\frac{\partial}{\partial y_1} \ldots \mathcal{S}(\tau, y_1, \ldots)\, D(z, b_T, y_1) \neq 0$$

Boglione, Simonelli, *JHEP 02 (2021) 076;*
JHEP 02 (2022) 013

SIA^{thr} has a **double nature**:

Thrust dependent observable

TMD observable

The thrust τ *naturally* regularizes the rapidity divergences.
The 2-jet limit $\tau \to 0$ corresponds to remove the regulator and to expose the rapidity divergences in fixed order calculations.

So the final result depends on a regulator? Yes, but...

1) The thrust is *measured*.
2) When the regulator is removed the (factorized) cross section vanishes, as showed by resummation.

The rapidity cut-offs $y_{1,2}$ *artificially* regularize the rapidity divergences.
The limits $y_{1,2} \to \pm\infty$ correspond to remove the regulator and to expose the rapidity divergences in fixed order calculations.

So the final result depends on a regulator? Yes (in principle), but…

1) The rapidity cut-offs are just mathematical tools.
2) In standard TMD factorization they cancel among themselves before the limit $y_{1,2} \to \pm\infty$ is taken and the final cross section is automatically rapidity cut-offs independent.

SIA$^{\text{thr}}$ has a **double nature**:

Thrust dependent observable

TMD observable

Both kind of regularization coexists in SIA$^{\text{thr}}$.
Therefore, it should not be surprising that the two mechanisms intertwine themselves and hence that thrust and rapidity regulators are strictly related.

There are cases where only the thrust survives…

$$\frac{\mathcal{S}(\tau, y_1, y_2)}{\mathcal{Y}_L(\tau, y_2)\,\mathcal{Y}_R(\tau, y_1)} = S_{\text{thr}}(\tau)$$

...and cases where only the rapidity cut-off…

$$\frac{\mathcal{G}^{\text{asy}}_{h/j}(\tau, z, P_T)}{\mathcal{C}_R(\tau, P_T, y_1)} = D_{h/j}(z, P_T, y_1)$$

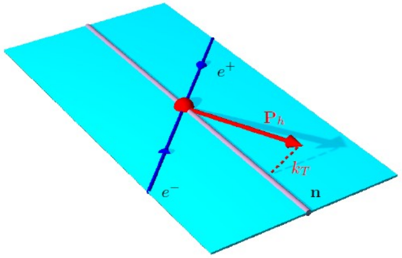
...but only in Region 2 the factorized cross section depends both on τ and on y_1 !

This signals a *redundancy* of regulators: one can be expressed in terms of the other.
In particular, the rapidity cut-off y_1 should be a function of thrust, such that when it is removed, also τ is removed. In other words:

$$\tau \to 0 \Longleftrightarrow y_1 \to +\infty$$

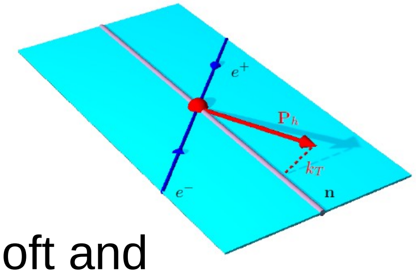
- Kinematic argument: $y_h \geq -\log\sqrt{\tau}$ ➡ $y_1 \propto -\log\sqrt{\tau}$

Rapidity of detected hadron



- Kinematic argument: $y_h \geq -\log\sqrt{\tau}$ ⇒ $y_1 \propto -\log\sqrt{\tau}$

Rapidity of detected hadron



- Formally: The double counting due to the overlap between soft and collinear (forward) radiation is cancelled only if the rapidity cut-off is fixed to a function of thrust and transverse momentum

SOFT-COLLINEAR

COLLINEAR

$$\boxed{\begin{aligned} k_T &\lesssim c_1/b_T \\ k_T &\lesssim Q\,e^{y_1}/u_E \end{aligned}}$$

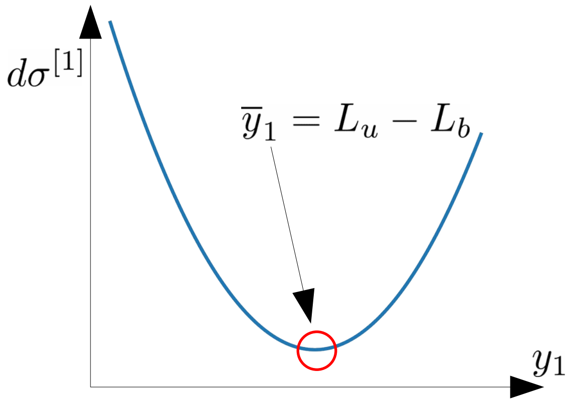
SOFT

$$\boxed{y_1 = L_u - L_b}$$

$$u_E = u\,e^{\gamma_E};\; c_1 = 2e^{-\gamma_E}$$

$$L_u = \log u_E$$

$$L_b = \log\left(b_T Q/c_1\right)$$



This is also the **minimum** of the factorized cross section as a function of y_1

The value of the rapidity cut-off that factorizes the cross section of Region 2 is the solution of Collins-Soper evolution equation

$$\frac{\partial}{\partial y_1} d\sigma_{R_2} = 0$$

$$\Updownarrow$$

$$\widehat{G}_R(u, y_1) = \widetilde{K}(b_T)$$

The solution that we have found is valid ONLY at perturbative level

$$\overline{y}_1 = L_u - L_b$$

Correct behavior
as $u \to +\infty$, i.e. $\tau \to 0$

Improper behavior (even divergent!)
as $b_T \to +\infty$, i.e. $P_T \to 0$

The rapidity cut-off must be large and positive for the factorization theorem to be valid

The solution that we have found is valid ONLY at perturbative level

$$\overline{y}_1 = L_u - L_b$$

Correct behavior as $u \to +\infty$, i.e. $\tau \to 0$

Improper behavior (even divergent!) as $b_T \to +\infty$, i.e. $P_T \to 0$

The rapidity cut-off must be large and positive for the factorization theorem to be valid

Non-perturbative corr.

$b^\star$ prescription:

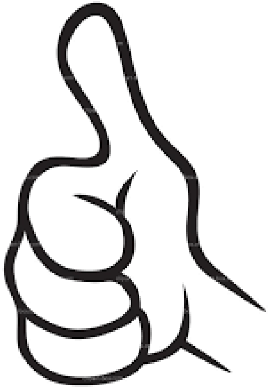
$$b_T \to \frac{b_T}{\sqrt{1 + \frac{b_T^2}{b_{\mathrm{MAX}}^2}}}$$

Crucial and central role of g_K in correlating thrust and transverse momentum

$$\mu_b^\star = c_1 / b_T^\star$$
$$\lambda_b^\star = 2\beta_0\, a_S(\mu_b^\star) L_b^\star$$

$$\overline{y}_1 = L_u - L_b^\star \left(1 + \frac{1 - e^{-\frac{2\beta_0}{\gamma_K^{[1]}}\left(g_K - \widetilde{K}^\star\right)}}{\lambda_b^\star} \right)$$

$$\begin{cases} \to L_b, & b_T \to 0 \\ \to \mathrm{const}, & b_T \to \infty \end{cases}$$



Also consistent with kinematics:

$$\widehat{y}_1 = -\log\sqrt{\tau} + b_T\text{-logs}$$

Factorization theorem in Region 2

$$d\sigma_{R_2} \sim H \; J(u) \, \frac{\mathcal{S}(u,\overline{y}_1,y_2)}{\mathcal{Y}_L(u,y_2)} \, \widetilde{D}_{h/j}(z,b_T,\overline{y}_1)$$

$$= \; H \, J \, \frac{\mathcal{S}}{\mathcal{Y}_L}\bigg|_{\text{ref. scale}} \exp\left\{ \int_{\mu_J}^{Q} \frac{d\mu'}{\mu'} \gamma_J + \frac{1}{2} \int_{\mu_S}^{Q} \frac{d\mu'}{\mu'} \gamma_S \right\} \times \; \widetilde{D}_{h/j}(z,b_T)\bigg|_{y_1=0}$$

$$\times \exp\left\{ \frac{1}{2} \int_{\mu_S}^{\mu_S e^{\overline{y}_1}} \frac{d\mu'}{\mu'} \left[\widehat{g} - \gamma_K \, \log\left(\frac{\mu'}{\mu_S}\right)\right] - \overline{y}_1 \;\; \widetilde{K}\bigg|_{\mu_S} \right\}$$

Factorization theorem in Region 2

Genuinely thrust. Exponent is *half* of standard thrust distributionin e+e- annihilation

$$
\begin{aligned}
d\sigma_{R_2} &\sim H\ J(u)\ \frac{\mathcal{S}(u,\overline{y}_1,y_2)}{\mathcal{Y}_L(u,y_2)}\ \widetilde{D}_{h/j}(z,b_T,\overline{y}_1) \\
&= H\ J\,\frac{\mathcal{S}}{\mathcal{Y}_L}\bigg|_{\text{ref. scale}} \exp\left\{\int_{\mu_J}^{Q}\frac{d\mu'}{\mu'}\gamma_J+\frac{1}{2}\int_{\mu_S}^{Q}\frac{d\mu'}{\mu'}\gamma_S\right\}\times\ \widetilde{D}_{h/j}(z,b_T)\Big|_{y_1=0} \\
&\times\exp\left\{\frac{1}{2}\int_{\mu_S}^{\mu_S e^{\overline{y}_1}}\frac{d\mu'}{\mu'}\left[\widehat{g}-\gamma_K\ \log\left(\frac{\mu'}{\mu_S}\right)\right]-\overline{y}_1\ \widetilde{K}\Big|_{\mu_S}\right\}
\end{aligned}
$$

Factorization theorem in Region 2

$$
\begin{aligned}
d\sigma_{R_2} &\sim H\ J(u)\ \frac{\mathcal{S}(u,\overline{y}_1,y_2)}{\mathcal{Y}_L(u,y_2)}\ \widetilde{D}_{h/j}(z,b_T,\overline{y}_1) \\
&= H\ J\ \frac{\mathcal{S}}{\mathcal{Y}_L}\bigg|_{\text{ref. scale}} \exp\left\{\int_{\mu_J}^{Q}\frac{d\mu'}{\mu'}\gamma_J + \frac{1}{2}\int_{\mu_S}^{Q}\frac{d\mu'}{\mu'}\gamma_S\right\} \times \widetilde{D}_{h/j}(z,b_T)\Big|_{y_1=0} \\
&\times \exp\left\{\frac{1}{2}\int_{\mu_S}^{\mu_S e^{\overline{y}_1}}\frac{d\mu'}{\mu'}\left[\widehat{g}-\gamma_K\ \log\left(\frac{\mu'}{\mu_S}\right)\right] - \overline{y}_1\ \widetilde{K}\Big|_{\mu_S}\right\}
\end{aligned}
$$

Genuinely TMD.
Reference scales as* in
standard TMD factorization

Factorization theorem in Region 2

$$d\sigma_{R_2} \sim H\ J(u)\ \frac{\mathcal{S}(u,\overline{y}_1,y_2)}{\mathcal{Y}_L(u,y_2)}\ \widetilde{D}_{h/j}(z,b_T,\overline{y}_1)$$

$$= H\ J\ \frac{\mathcal{S}}{\mathcal{Y}_L}\bigg|_{\text{ref. scale}} \exp\left\{\int_{\mu_J}^{Q}\frac{d\mu'}{\mu'}\gamma_J + \frac{1}{2}\int_{\mu_S}^{Q}\frac{d\mu'}{\mu'}\gamma_S\right\}\times \widetilde{D}_{h/j}(z,b_T)\bigg|_{y_1=0}$$

$$\times \exp\left\{\frac{1}{2}\int_{\mu_S}^{\mu_S e^{\overline{y}_1}}\frac{d\mu'}{\mu'}\left[\widehat{g}-\gamma_K\ \log\left(\frac{\mu'}{\mu_S}\right)\right]-\overline{y}_1\ \widetilde{K}\bigg|_{\mu_S}\right\}$$

Correlation part. It encodes the correlations between the measured variables

Factorization theorem in Region 2

$$d\sigma_{R_2} \sim H\ J(u)\ \frac{\mathcal{S}(u,\overline{y}_1,y_2)}{\mathcal{Y}_L(u,y_2)}\ \widetilde{D}_{h/j}(z,b_T,\overline{y}_1)$$

$$= H\ J\ \frac{\mathcal{S}}{\mathcal{Y}_L}\bigg|_{\text{ref. scale}} \exp\left\{\int_{\mu_J}^{Q}\frac{d\mu'}{\mu'}\gamma_J + \frac{1}{2}\int_{\mu_S}^{Q}\frac{d\mu'}{\mu'}\gamma_S\right\} \times \widetilde{D}_{h/j}(z,b_T)\Big|_{y_1=0}$$

$$\times \exp\left\{\frac{1}{2}\int_{\mu_S}^{\mu_S e^{\overline{y}_1}}\frac{d\mu'}{\mu'}\left[\widehat{g}-\gamma_K\ \log\left(\frac{\mu'}{\mu_S}\right)\right] - \overline{y}_1\ \widetilde{K}\Big|_{\mu_S}\right\}$$

The function g_K is in both terms and not only into the TMD FF

Resummation

$$\frac{d\sigma_{R_2}}{dz\,dT\,dP_T} \overset{\text{LL}}{=} -\frac{\sigma_B}{1-T}\,N_C \int \frac{d^2\vec{b}_T}{(2\pi)^2}\ \widetilde{D}^{\text{LL}}_{h/j}(z,b_T)\Big|_{y_1=0}$$

$$\times \exp\{-\log(1-T) f_1(\bullet)\}\ \gamma(\bullet)$$

- $\bullet = \{-a_S\,\beta_0\,\log(1-T), 2\,a_S\,\beta_0\,L_b^\star, g_K(b_T)\}$

Factorization theorem in Region 2

$$d\sigma_{R_2} \sim H\ J(u)\, \frac{\mathcal{S}(u,\overline{y}_1,y_2)}{\mathcal{Y}_L(u,y_2)}\, \widetilde{D}_{h/j}(z,b_T,\overline{y}_1)$$

$$= H\ J\, \frac{\mathcal{S}}{\mathcal{Y}_L}\bigg|_{\text{ref. scale}} \exp\left\{\int_{\mu_J}^{Q}\frac{d\mu'}{\mu'}\gamma_J + \frac{1}{2}\int_{\mu_S}^{Q}\frac{d\mu'}{\mu'}\gamma_S\right\} \times \widetilde{D}_{h/j}(z,b_T)\Big|_{y_1=0}$$

$$\times \exp\left\{\frac{1}{2}\int_{\mu_S}^{\mu_S e^{\overline{y}_1}}\frac{d\mu'}{\mu'}\left[\widehat{g} - \gamma_K \log\left(\frac{\mu'}{\mu_S}\right)\right] - \overline{y}_1\ \widetilde{K}\Big|_{\mu_S}\right\}$$

The function g_K is in both terms and not only into the TMD FF

Resummation

$$\frac{d\sigma_{R_2}}{dz\, dT\, dP_T} \overset{\text{NLL}}{=} -\frac{\sigma_B}{1-T} N_C \int \frac{d^2\vec{b}_T}{(2\pi)^2}\ \widetilde{D}^{\text{NLL}}_{h/j}(z,b_T)\Big|_{y_1=0}$$

$$\times \exp\left\{-\log(1-T) f_1(\bullet) + f_2(\bullet)\right\}\ \gamma(\bullet)$$

$$\bullet = \{-a_S\,\beta_0\,\log(1-T),\, 2\,a_S\,\beta_0\,L_b^{\star},\, g_K(b_T)\}$$

Factorization theorem in Region 2

$$d\sigma_{R_2} \sim H\ J(u)\ \frac{\mathcal{S}(u,\overline{y}_1,y_2)}{\mathcal{Y}_L(u,y_2)}\ \widetilde{D}_{h/j}(z,b_T,\overline{y}_1)$$

$$= H\ J\ \frac{\mathcal{S}}{\mathcal{Y}_L}\Big|_{\text{ref. scale}} \exp\left\{\int_{\mu_J}^{Q}\frac{d\mu'}{\mu'}\gamma_J + \frac{1}{2}\int_{\mu_S}^{Q}\frac{d\mu'}{\mu'}\gamma_S\right\} \times \widetilde{D}_{h/j}(z,b_T)\Big|_{y_1=0}$$

$$\times \exp\left\{\frac{1}{2}\int_{\mu_S}^{\mu_S e^{\overline{y}_1}}\frac{d\mu'}{\mu'}\left[\widehat{g} - \gamma_K \log\left(\frac{\mu'}{\mu_S}\right)\right] - \overline{y}_1\ \widetilde{K}\Big|_{\mu_S}\right\}$$

The function g_K is in both terms and not only into the TMD FF

Resummation

$$\frac{d\sigma_{R_2}}{dz\,dT\,dP_T} \overset{\text{NNLL}}{=} -\frac{\sigma_B}{1-T}\,N_C \int \frac{d^2\vec{b}_T}{(2\pi)^2}\ \widetilde{D}^{\text{NNLL}}_{h/j}(z,b_T)\Big|_{y_1=0}\ (1+a_S\,C_1(b_T))$$

$$\times \exp\left\{-\log(1-T)\,f_1(\bullet) + f_2(\bullet) - \frac{1}{\log(1-T)}f_3(\bullet)\right\}\left(\gamma(\bullet) - \frac{1}{\log(1-T)}\rho(\bullet)\right)$$

$\bullet = \{-a_S\,\beta_0\,\log(1-T),\, 2\,a_S\,\beta_0\,L_b^{\star},\, g_K(b_T)\}$

PHENOMENOLOGY

A *strongly simplified* version of this theorem is possible

1) $$\overline{y}_1 = L_u - \xcancel{L_b^\star \left(1 + \frac{1 - e^{-\frac{2\beta_0}{\gamma_K^{[1]}}\left(g_K - \widetilde{K}^\star\right)}}{\lambda_b^\star} \right)} \Longrightarrow$$

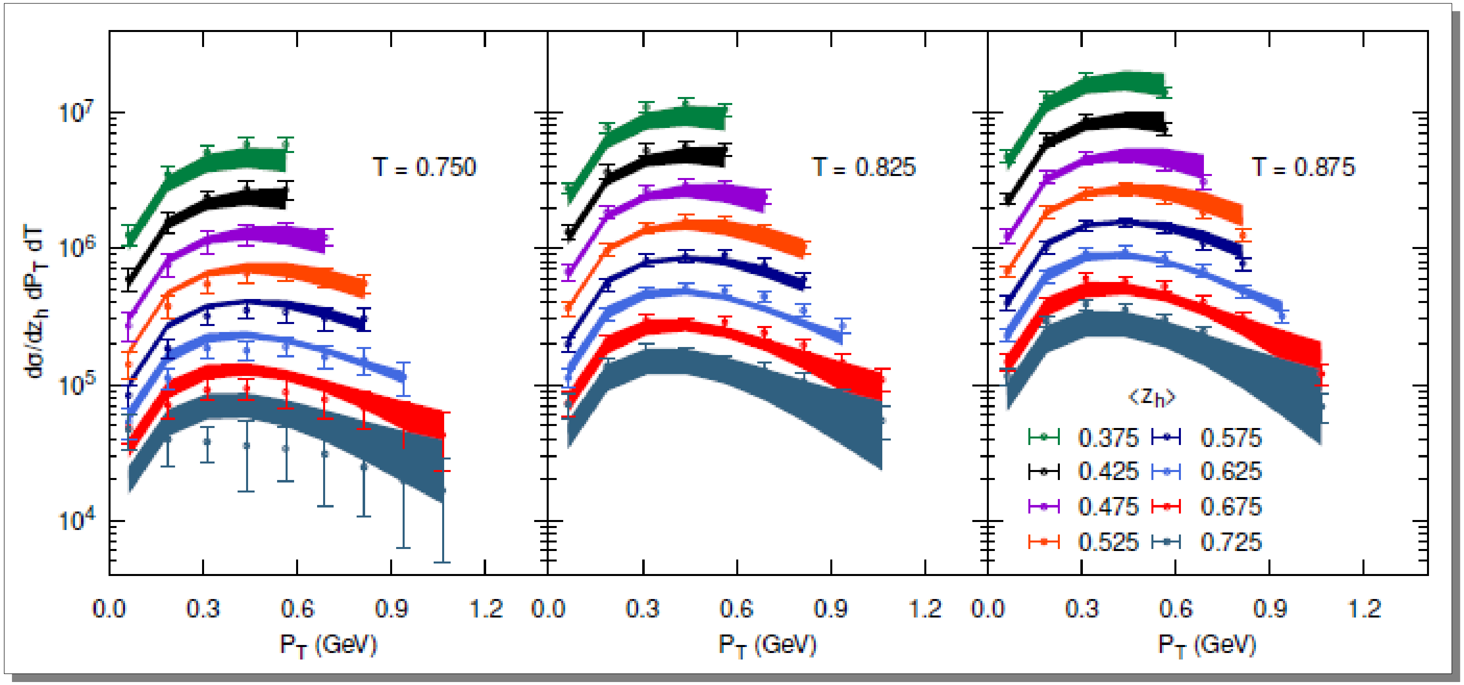
In thrust space:

$$\widehat{y}_1 = -\log\sqrt{\tau}$$

Just kinematics!

These approximations lead to the result of Boglione, Simonelli, *JHEP 02 (2021) 076*

2) Thrust is not resummed



PHENOMENOLOGY:
Boglione, Gonzalez-Hernandez, Simonelli, Phys.Rev.D 106 (2022) 7, 074024

DATA:
BELLE collab., *Phys.Rev.D 99 (2019)*

TMD Fragmentation Function

TMD effects

- The TMD FF model (POWER LAW):

$$M_D(z_h,\, P_T) \propto \left(M^2 + \frac{P_T^2}{z_h^2} \right)^{-p}$$

$$p,\ M \to p = \frac{1}{2}\left(\frac{3}{1-R} - 1 \right),\quad M = \frac{W}{z_h}\sqrt{\frac{3}{1-R}}$$

With:

$$R = 1 - c\,\frac{f(z)}{f(z_0)},\quad W = \frac{M_\pi}{R}$$

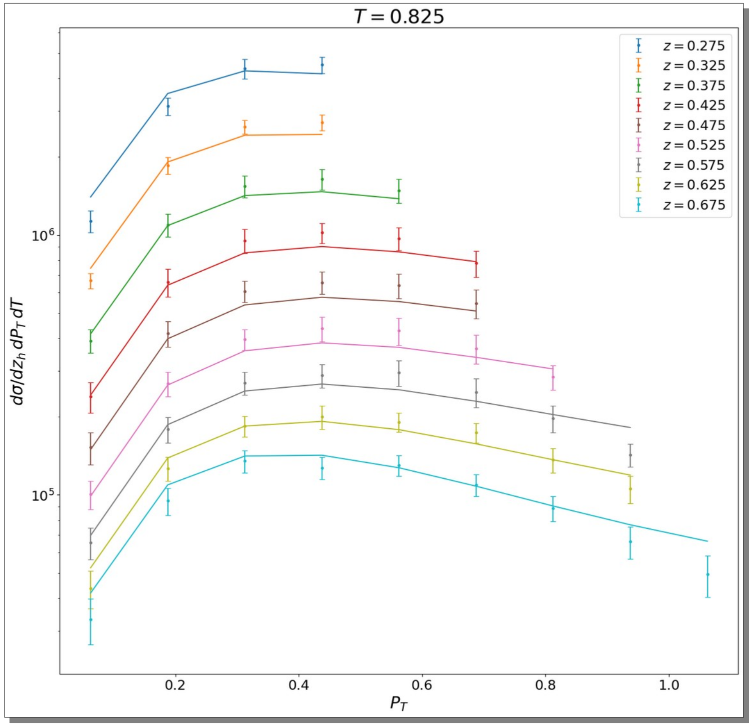
$$f(z) = 1 - (1-z)^{z_0/1-z_0}\, z$$

- The g_K-function:

$$g_K(b_T) = g_0 \tanh\left(a\,\frac{b_T^2}{b_{MAX}^2} \right)$$

Preliminary fit: $T = 0.825$

- Far from matching effects (higher topologies contributions)
- Far from NP effects genuinely due to thrust

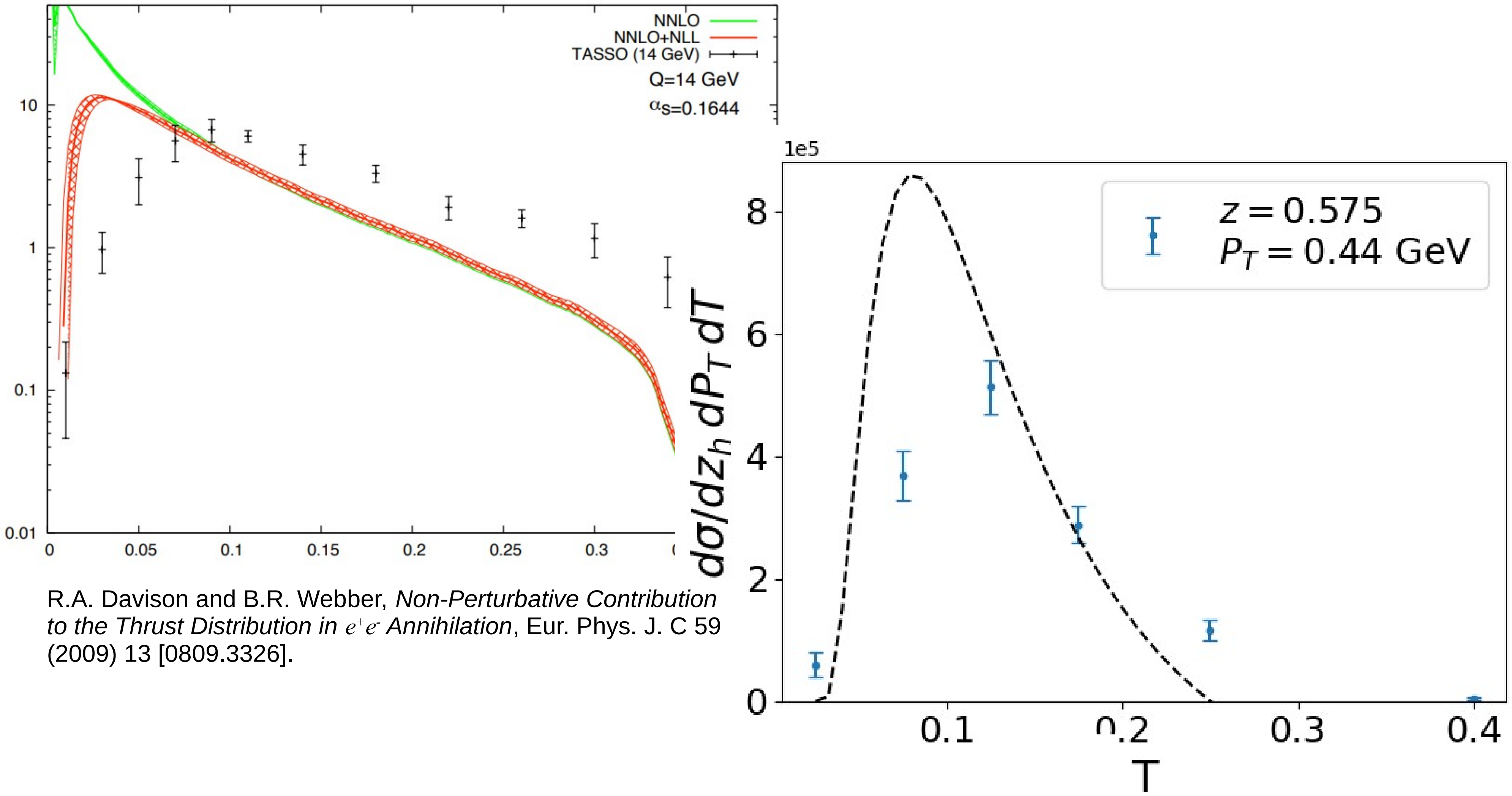


Thrust Non-Perturbative effects

- Shaping function:

$$f_{NP}(\tau) = \frac{1 - e^{-(\alpha\,\tau)^2}}{1 - e^{-\alpha^2}}$$

- Very simple model
- Cross-section level



R.A. Davison and B.R. Webber, *Non-Perturbative Contribution to the Thrust Distribution in e^+e^- Annihilation*, Eur. Phys. J. C 59 (2009) 13 [0809.3326].

Thrust Non-Perturbative effects

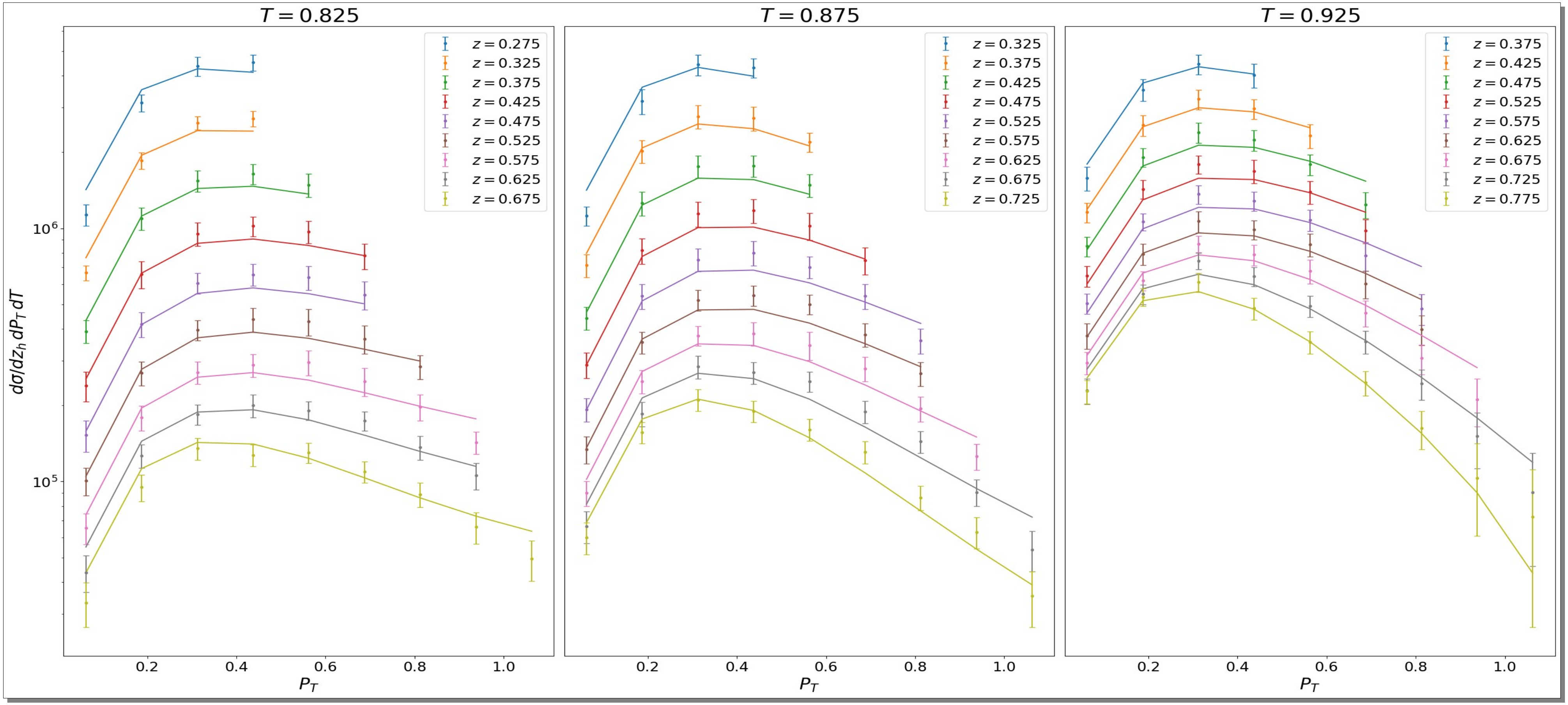
$\chi^2 = 1.12$

$\mathrm{M_D}: \quad z_0 = 0.556 \pm 0.005 \quad c = 0.529 \pm 0.004$

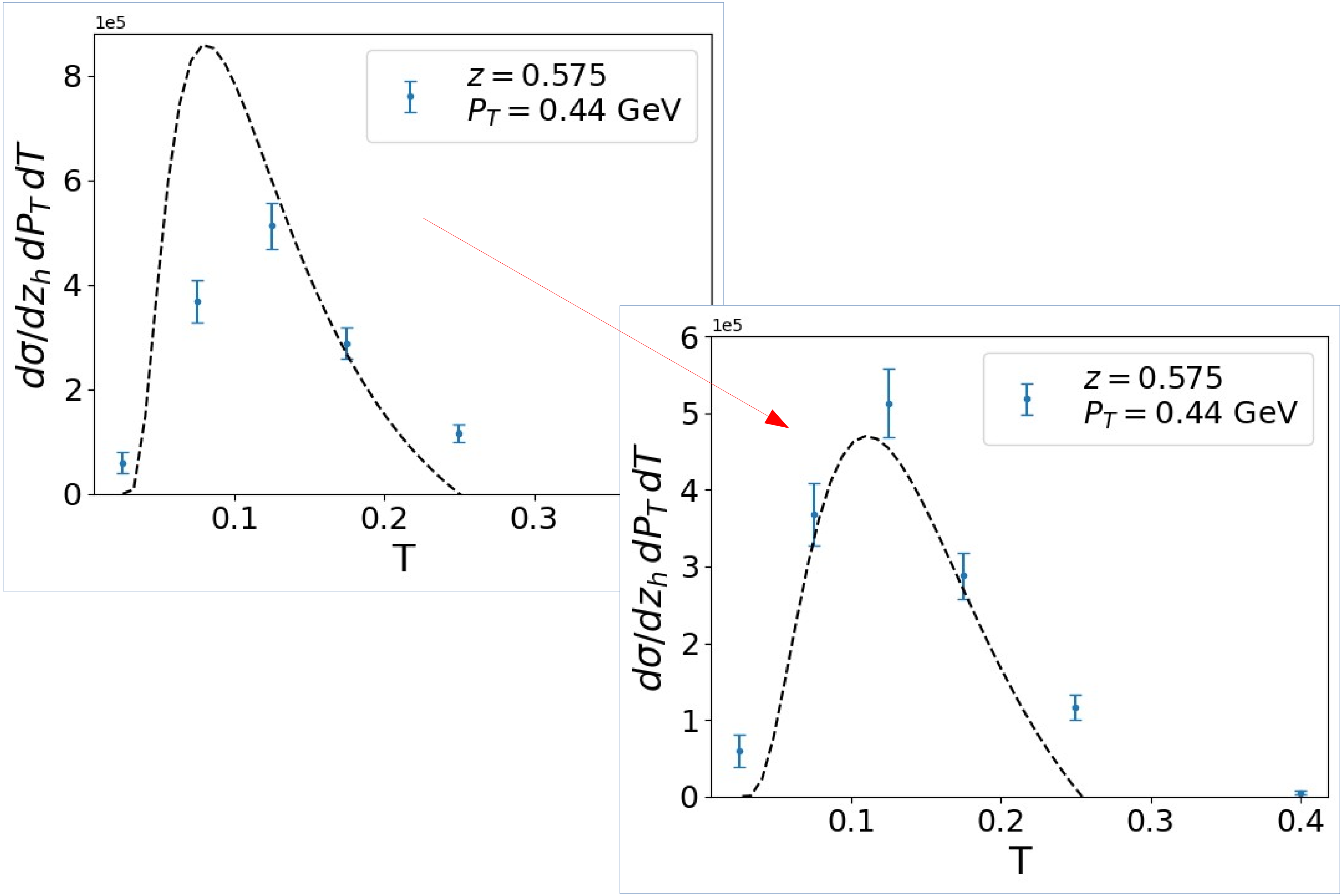
$\mathrm{g_K}: \quad g_0 = 0.294 \pm 0.007 \quad a = 4.176 \pm 0.904$

178 data points

$\mathrm{f_{NP}}: \ \alpha = 8.979 \pm 0.110$



Thrust Non-Perturbative effects



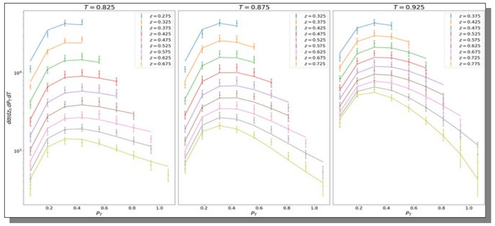
Conclusions

- The factorization properties of SIA$^{\text{thr}}$ have been deeply investigated.

- Factorization in Region 2 exposes the double nature of SIA$^{\text{thr}}$, an observable which is both thrust-dependent and TMD. This is made manifest in the relation linking the TMD rapidity regulator with the thrust.

$$\overline{y}_1 = L_u - L_b^\star \left(1 + \frac{1 - e^{-\frac{2\beta_0}{\gamma_K^{[1]}}\left(g_K - \widetilde{K}^\star\right)}}{\lambda_b^\star} \right)$$

- First phenomenological results



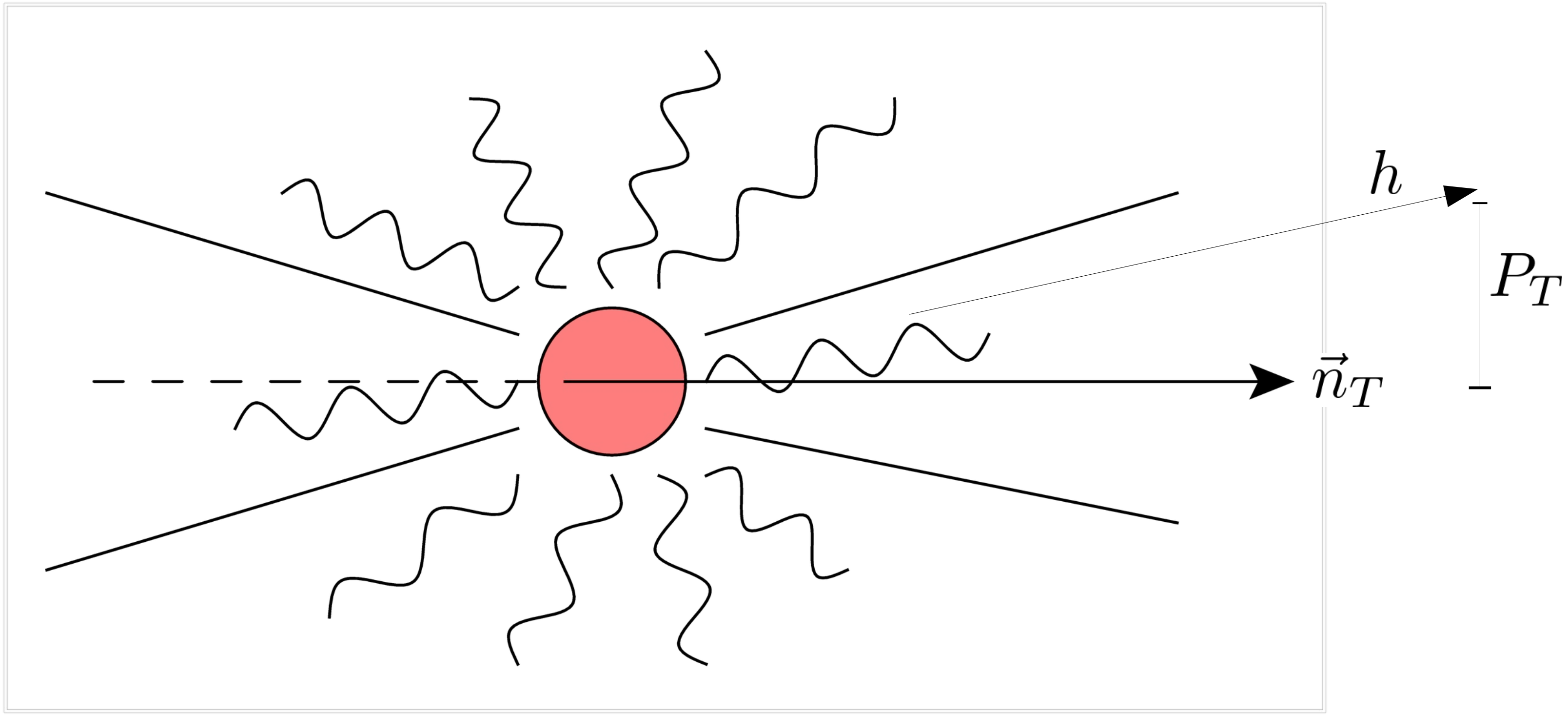
- The next step is to perform phenomenology on BELLE data without introducing simplifications to the formalism and provide the **first cleanest extraction of a TMD FF**.

BACK-UP SLIDES

Radiation decomposition

$$d\sigma \sim H$$

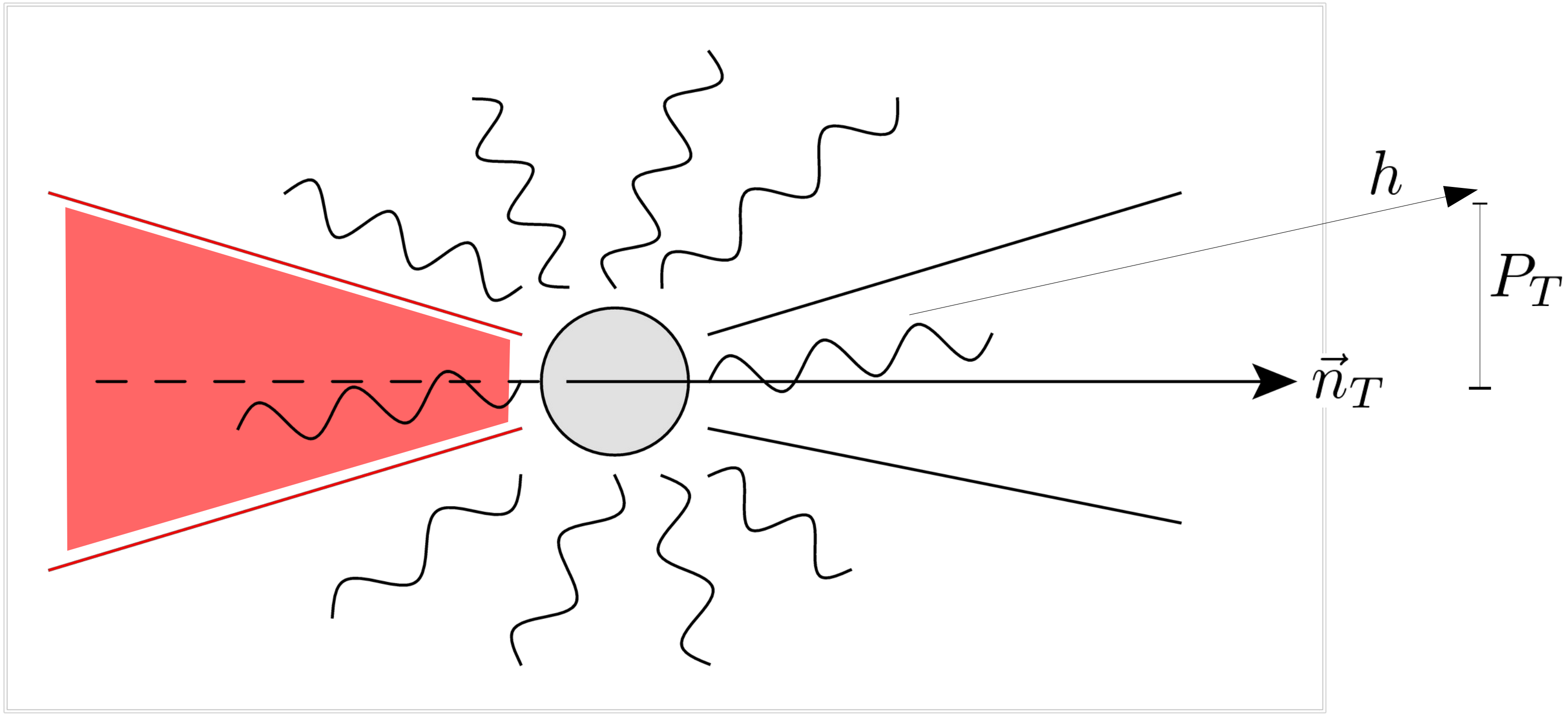
- Hard (far off-shell) radiation dresses $\gamma^\star \to q\bar{q}$ vertex



Radiation decomposition

$$d\sigma \sim H \times J(\tau)$$

- Hard (far off-shell) radiation dresses $\gamma^\star \to q\bar{q}$ vertex
- Backward radiation irrelevant for the transverse motion of the detected hadron

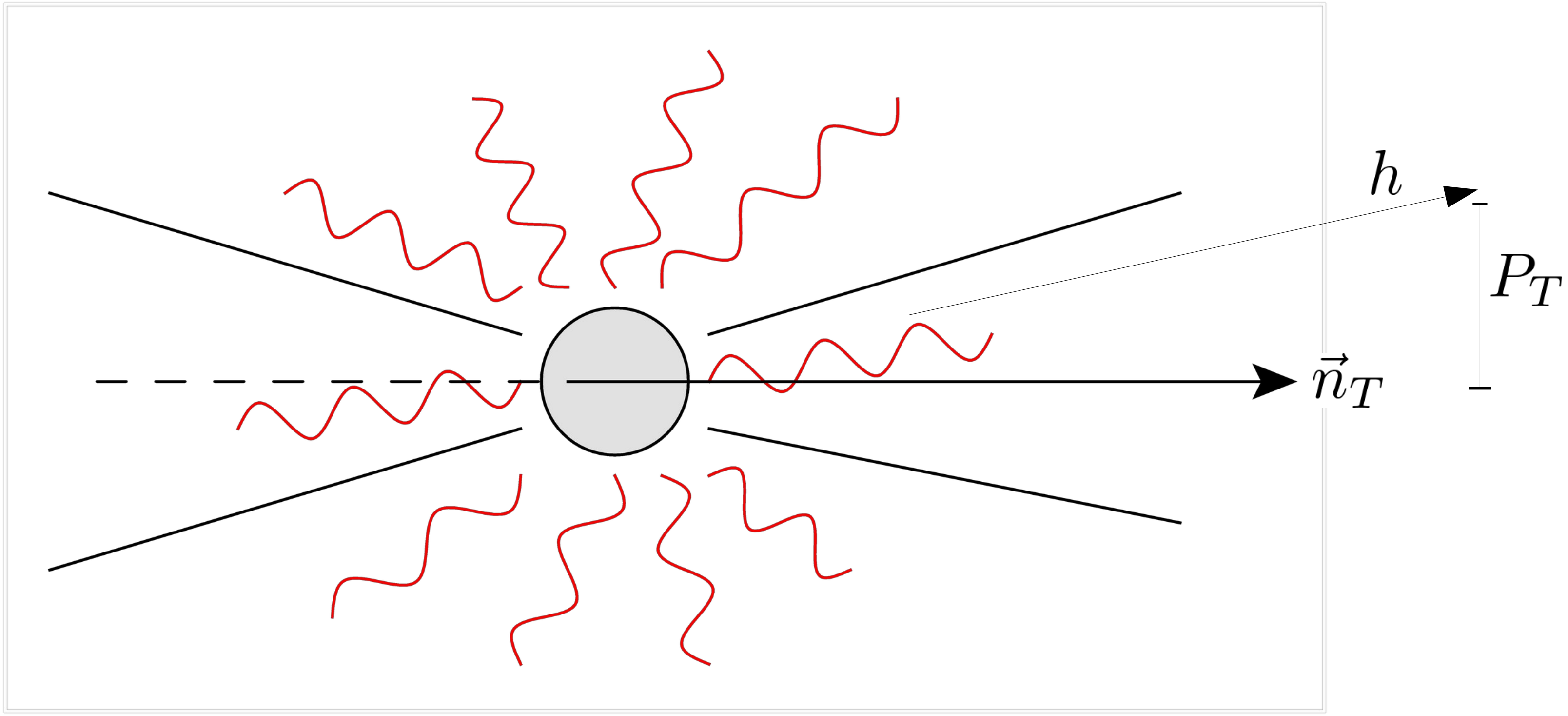


Radiation decomposition

Hallmark of R_2 and R_3

$$d\sigma \sim H \times J(\tau) \times \mathcal{S}(\tau)$$

- Hard (far off-shell) radiation dresses $\gamma^\star \to q\bar{q}$ vertex
- Backward radiation irrelevant for the transverse motion of the detected hadron
- Soft radiation does not affect the transverse motion of the detected hadron



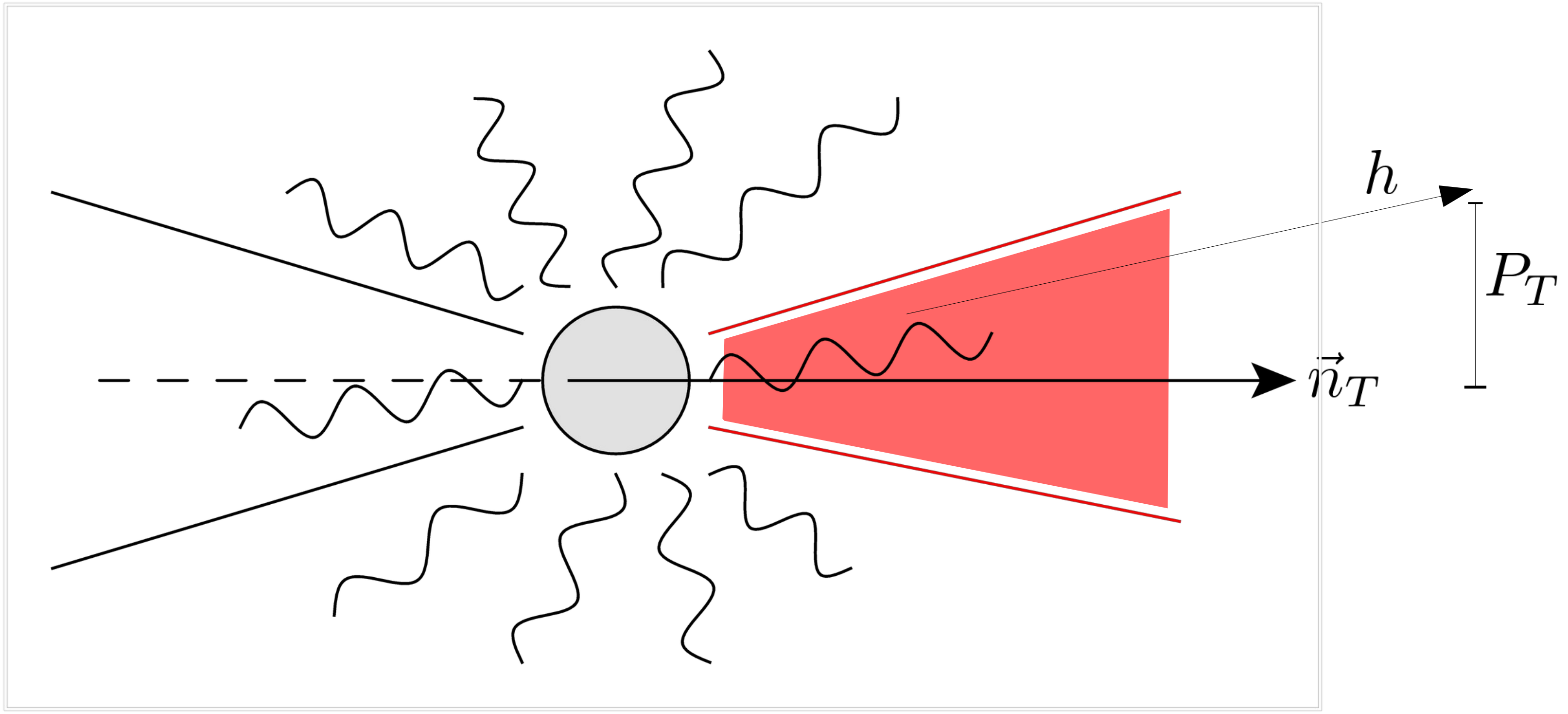
Radiation decomposition

Hallmark of R_2 and R_3

$$d\sigma \sim H \times J(\tau) \times \mathcal{S}(\tau) \times \mathcal{G}^{\text{asy}}_{h/j}(\tau, z, P_T)$$

Hallmark of R_1 and R_2

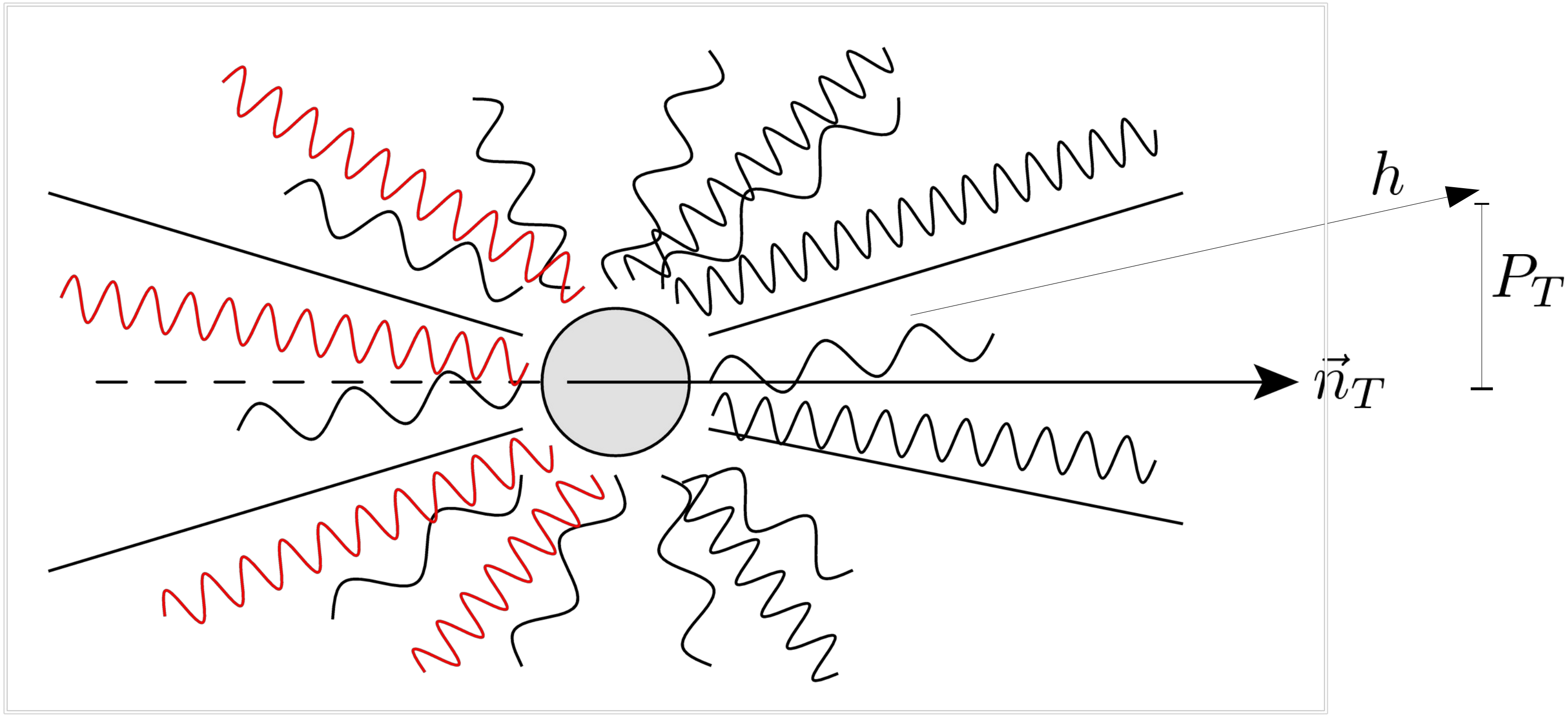
- Hard (far off-shell) radiation dresses $\gamma^\star \to q\bar{q}$ vertex
- Backward radiation irrelevant for the transverse motion of the detected hadron
- Soft radiation does not affect the transverse motion of the detected hadron
- Collinear (forward) radiation necessarily contributes to TMD effects



Subtractions

$$d\sigma \sim H \times J(\tau) \times \frac{\mathcal{S}(\tau)}{Y_L(\tau)} \times \mathcal{G}^{\text{asy}}_{h/j}(\tau, z, P_T)$$

- Hard (far off-shell) radiation dresses $\gamma^\star \to q\bar{q}$ vertex
- Backward radiation irrelevant for the transverse motion of the detected hadron
- Soft radiation does not affect the transverse motion of the detected hadron
- Collinear (forward) radiation necessarily contributes to TMD effects

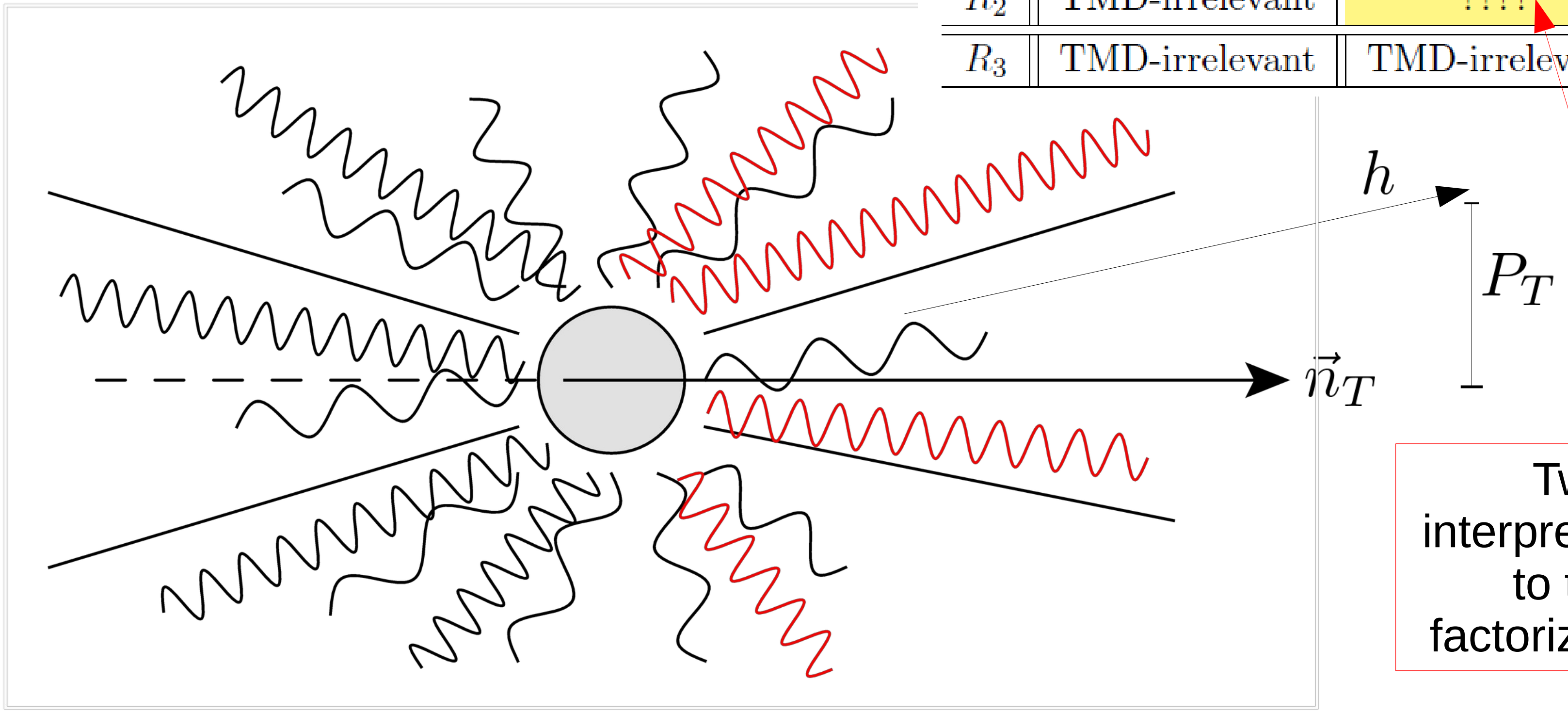


Subtractions

$$d\sigma \sim H \times J(\tau) \times \frac{\mathcal{S}(\tau)}{Y_L(\tau) \times ???} \times \mathcal{G}^{\text{asy}}_{h/j}(\tau, z, P_T)$$

- Hard (far off-shell) radiation dresses $\gamma^\star \to q\bar{q}$ vertex
- Backward radiation irrelevant for the transverse motion of the detected hadron
- Soft radiation does not affect the transverse motion of the detected hadron
- Collinear (forward) radiation necessarily contributes to TMD effects

	soft	soft-collinear	collinear
R_1	TMD-relevant	TMD-relevant	TMD-relevant
R_2	TMD-irrelevant	????	TMD-relevant
R_3	TMD-irrelevant	TMD-irrelevant	TMD-relevant



Two different interpretations that lead to two different factorization theorems!

Subtractions: approach I

1) Forward soft-collinear radiation is **TMD-irrelevant**

$$d\sigma \sim H \times J(\tau) \times \boxed{\begin{array}{c}\dfrac{\mathcal{S}(\tau)}{Y_L(\tau) \times Y_R(\tau)} \\ = S_{\text{thr}}(\tau)\end{array}} \times \mathcal{G}^{\text{asy}}_{h/j}(\tau, z, P_T)$$

The hadronization process is not described by a TMD FF

	soft	soft-collinear	collinear
R_1	TMD-relevant	TMD-relevant	TMD-relevant
R_2	TMD-irrelevant	TMD-irrelevant	TMD-relevant
R_3	TMD-irrelevant	TMD-irrelevant	TMD-relevant

The underlying physics of R_2 and R_3 is almost indistinguishable

The above scheme is much more close to a representation of just two kinematic regions, R_1 and R_3, and the “bulk” of the phase space exists just as a limit of its boundaries.

This limit is indeed a **matching region**, M, and not an independent kinematic configuration: the label “R_2” does not fit it anymore.

Matching between *what*??

$$\mathrm{R}_3 \qquad \Longrightarrow \qquad \mathrm{M} \qquad \dots \qquad \Longleftarrow \qquad \mathrm{R}_1$$

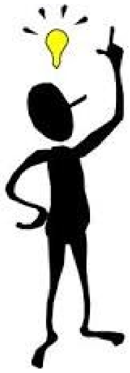
Low-P_T approximation
of the generalized FJF

$$\mathcal{G}_{h/j}(\tau, z, P_T) \to \mathcal{G}^{\mathrm{asy}}_{h/j}(\tau, z, P_T)$$

????
R1 seems to be too far

The previous forward-radiation scheme is *incomplete* and there must be (at least) another kinematic region between M and R_1

The “bulk” of the phase space deserves its own independent kinematic region R_2



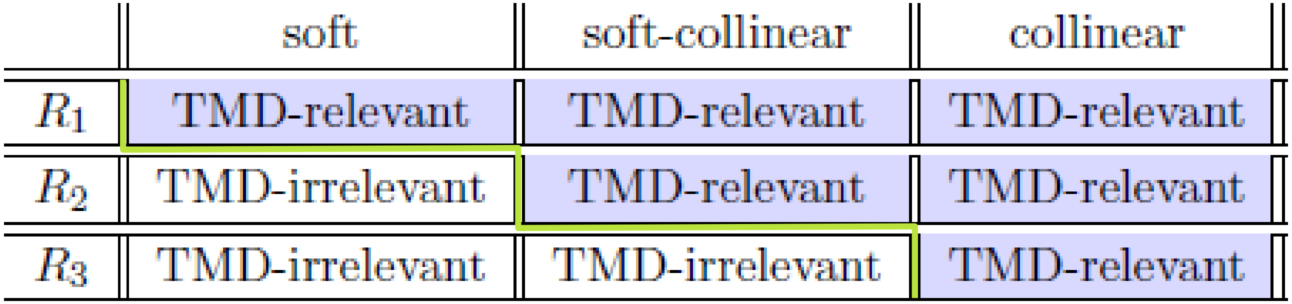
Subtractions: approach II

2) Forward soft-collinear radiation is **TMD-relevant**

Matches intuition!

$$d\sigma \sim H \times J(\tau) \times \frac{\mathcal{S}(\tau)}{Y_L(\tau)} \times \boxed{\begin{array}{l}\dfrac{\mathcal{G}^{\text{asy}}_{h/j}(\tau, z, P_T)}{\mathcal{C}_R(\tau, P_T)} \\ = D_{h/j}(z, P_T)\end{array}}$$

The hadronization process is described by a **TMD FF**



	soft	soft-collinear	collinear
R_1	TMD-relevant	TMD-relevant	TMD-relevant
R_2	TMD-irrelevant	TMD-relevant	TMD-relevant
R_3	TMD-irrelevant	TMD-irrelevant	TMD-relevant

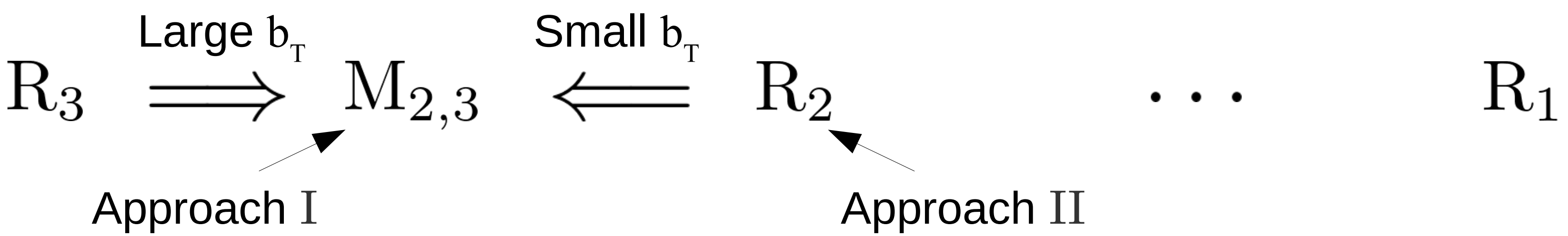
Matches intuition!

Totally “symmetric” structure among the three regions

Now **Region 2** is a truly independent kinematic region!

Kinematic structure of SIA^{thr}

i) Approach I leads to a *hybrid* btw R2 and R3 and indeed **matches** the two regions



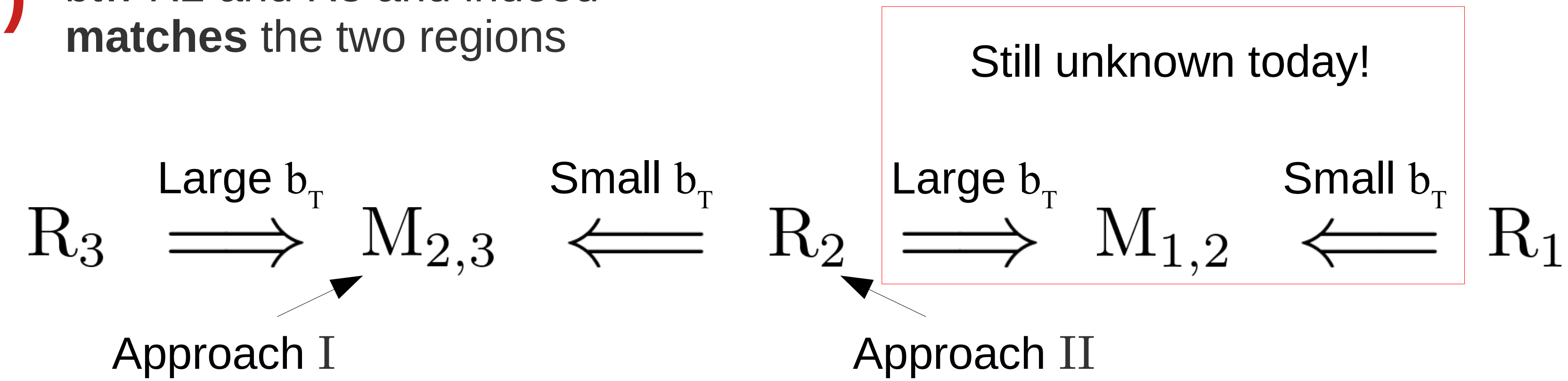
$$\mathrm{R}_3 \xRightarrow{\text{Large } b_T} \mathrm{M}_{2,3} \xLeftarrow{\text{Small } b_T} \mathrm{R}_2 \quad \cdots \quad \mathrm{R}_1$$

Having a well-defined factorization theorem in a matching region is a unusual and remarkable fact!

- Very helpful for phenomenological description of experimental data
- Extremely useful for constraining the non-perturbative behavior of generalized FJFs

Kinematic structure of SIA$^{\text{thr}}$

i) Approach I leads to a *hybrid* btw R2 and R3 and indeed **matches** the two regions



Still unknown today!

$$R_3 \xRightarrow{\text{Large } b_T} M_{2,3} \xLeftarrow{\text{Small } b_T} R_2 \xRightarrow{\text{Large } b_T} M_{1,2} \xLeftarrow{\text{Small } b_T} R_1$$

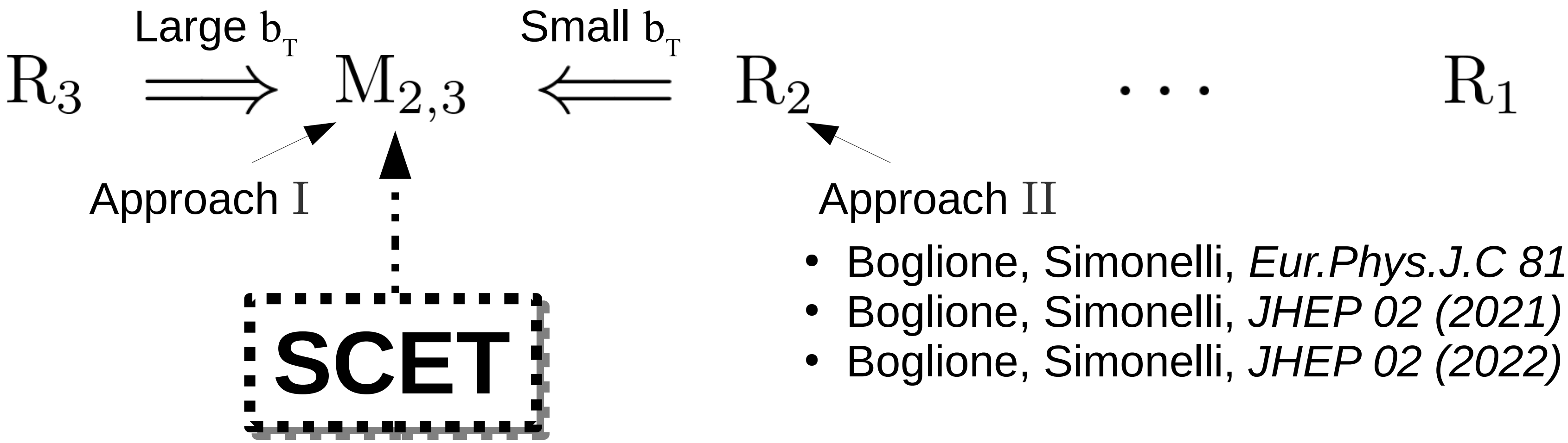
Having a well-defined factorization theorem in a matching region is a unusual and remarkable fact!

- Very helpful for phenomenological description of experimental data
- Extremely useful for constraining the non-perturbative behavior of generalized FJFs

Constraining the long-distance behavior of TMF FFs would require a factorization theorem M_{12}, but this has not been investigated, yet.

Kinematic structure of SIA$^{\text{thr}}$

ii) Approach I coincides with the result obtained in **SCET**



$$\begin{tikzcd} R_3 \arrow[r, Rightarrow, "\text{Large } b_T"] & M_{2,3} & R_2 \arrow[l, Rightarrow, "\text{Small } b_T"'] & \cdots & R_1 \end{tikzcd}$$

- Boglione, Simonelli, *Eur.Phys.J.C 81 (2021)*
- Boglione, Simonelli, *JHEP 02 (2021) 076*
- Boglione, Simonelli, *JHEP 02 (2022) 013*

$$d\sigma^{(I)}_{M_{2,3}} \sim H\ J(\tau)\ S_{\text{thr}}\ \mathcal{G}^{\text{asy}}_{h/j}(\tau, z, P_T) = H\ J(\tau)\ S_{\text{thr}}\ \mathcal{C}_R(\tau, P_T)\ \boxed{\frac{\mathcal{G}^{\text{asy}}_{h/j}(\tau, z, P_T)}{\mathcal{C}_R(\tau, P_T)} = D_{h/j}(z, P_T)}$$

...and this coincides with Eq.(2.21) of Makris, Ringer, Waalewijn, *JHEP 02 (2021) 070*

Kinematic structure of SIA$^{\text{thr}}$

iii) The two approaches coincide in the small b_T limit, i.e. they are totally **equivalent at perturbative level**

$$\begin{tikzcd}
R_3 \arrow[r, Rightarrow, "\text{Large } b_T"] & M_{2,3} & R_2 \arrow[l, Rightarrow, "\text{Small } b_T"'] & \cdots & R_1
\end{tikzcd}$$

Approach I (SCET)

- Makris, Ringer, Waalewijn, *JHEP 02 (2021)*

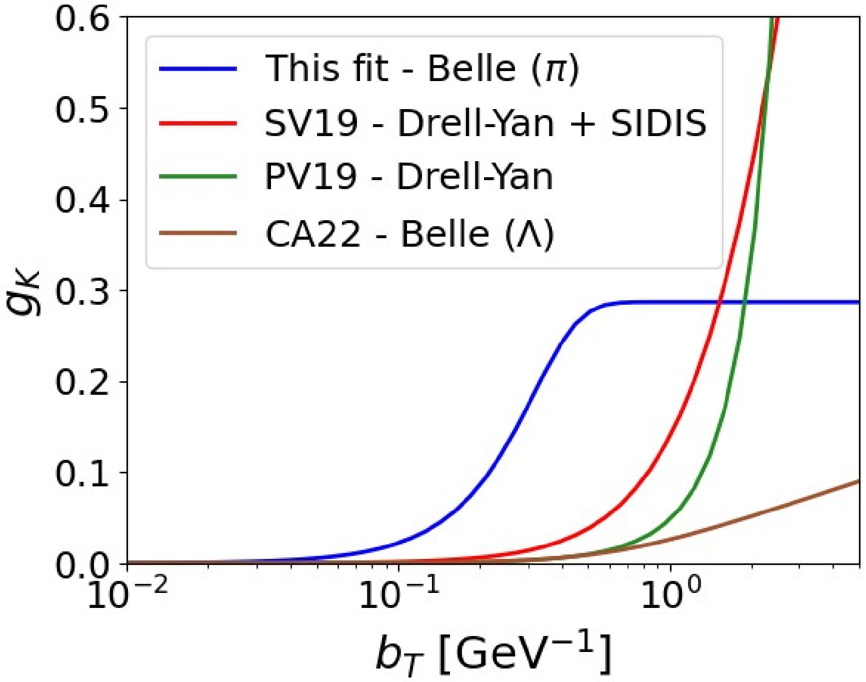
Approach II (CSS - based)

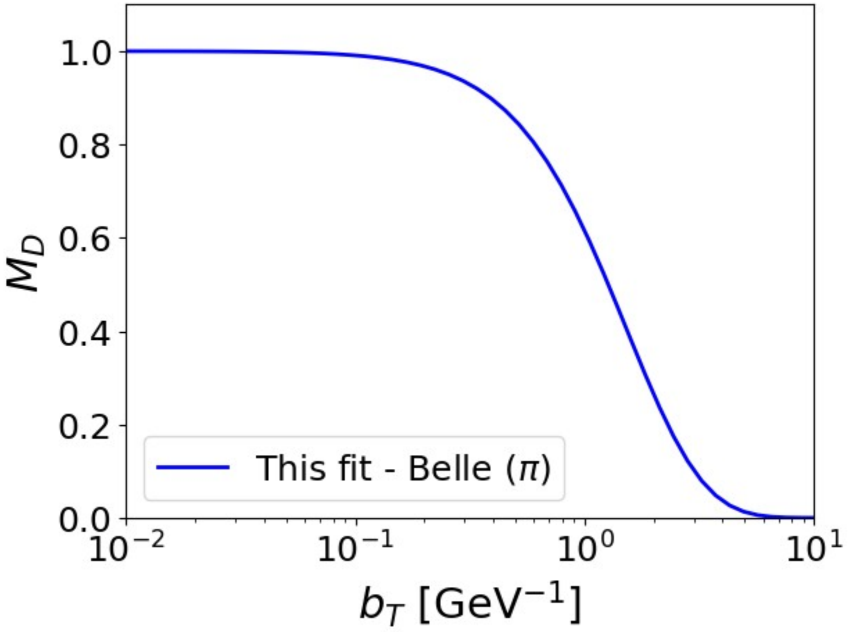
- Boglione, Simonelli, *Eur.Phys.J.C 81 (2021)*
- *JHEP 02 (2021) 076*
- *JHEP 02 (2022) 013*

SCET: $d\sigma^{(I)}_{M_{2,3}} \sim H\ J(\tau)\ S_{\text{thr}}\ \mathcal{C}_R(\tau, P_T)\, D_{h/j}(z, P_T)$

CSS: $d\sigma^{(II)}_{R_2} \sim H\ J(\tau)\ \dfrac{\mathcal{S}(\tau)}{\mathcal{Y}_L(\tau)}\ D_{h/j}(z, P_T)$

The differences are only in the large distance behavior (NON-PERTURBATIVE)


This fit - Belle (π)
SV19 - Drell-Yan + SIDIS
PV19 - Drell-Yan
CA22 - Belle (Λ)
g_K
0.6
0.5
0.4
0.3
0.2
0.1
0.0
10^{-2}
10^{-1}
10^{0}
b_T [GeV^{-1}]


M_D
1.0
0.8
0.6
0.4
0.2
0.0
This fit - Belle (π)
10^{-2}
10^{-1}
10^{0}
10^{1}
b_T [GeV^{-1}]