Multivariable evolution in initial state parton shower

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Motivation

Our main objective with **DEDUCTOR** is to advance the theory of parton shower algorithms.
Our strategy has two main cornerstones:

Parton shower is **perturbative QCD**

- Based on Feynman graphs
- Evolution of the **QCD density matrix** (evolution at amplitude level)
- **Quantum colour** and spin,…
- Can be defined order-by-order systematically
- …

Parton shower is **quantum statistical physics**

- The shower evolution and the shower cross sections are **solution of a renromalization group equation**
- **Defining/generalizing/abusing** the framework as much as possible
- Defining and understanding **different shower schemes**
- **More concrete goal:** Fitting the angular ordered shower into this framework
- Summation of large logarithms
- Threshold logarithms
- …

N^kLO calculations

$$\sigma[O_J] = \overbrace{\big(1\big|\mathcal{O}_J\,\mathcal{F}_{\rm bare}\mathcal{D}(\mu_{\rm R}(\vec\mu),\vec\mu)}^{\text{Singularities cancel each other here}}\ \underbrace{\mathcal{D}^{-1}(\mu_{\rm R}(\vec\mu),\vec\mu)\big|\rho(\mu_{\rm R}(\vec\mu))\big)}_{=|\rho_{\rm H}(\vec\mu))}$$

$$+\,\mathcal{O}(\alpha_s^{k+1}L^{2k+2})$$

$$+\,\mathcal{O}(\Lambda^2_{QCD}/\mu_J^2)$$

Subtractions

Hard part, finite in d=4 dimension

- **We alway relate the renormalization scale to the shower scales**

$$\mu_{\rm R} = \mu_{\rm R}(\vec\mu)$$

- **Thus we can simplify the notation as**

$$\mathcal{D}(\vec\mu) = \mathcal{D}(\mu_{\rm R}(\vec\mu),\vec\mu)$$

Usually $\mathcal{D}^{-1}(\vec\mu)$ is constructed by hand and $\mathcal{D}(\vec\mu)$ is its inverse.

$$\mathcal{D}^{-1}(\vec\mu)|\rho(\mu_{\rm R})) = \overbrace{|\rho^{(0)}(\mu_{\rm R}))}^{\text{Born term}} + \frac{\alpha_s(\mu_{\rm R})}{2\pi}\overbrace{\Big[|\rho^{(1)}(\mu_{\rm R})) - \mathcal{D}^{(1)}(\vec\mu)|\rho^{(0)}(\mu_{\rm R}))\Big]}^{\text{NLO contributions}}$$

$$+\left[\frac{\alpha_s(\mu_{\rm R})}{2\pi}\right]^2\underbrace{\Big\{|\rho^{(2)}(\mu_{\rm R})) - \mathcal{D}^{(1)}(\vec\mu)|\rho^{(1)}(\mu_{\rm R})) - \big[\mathcal{D}^{(2)}(\vec\mu) - \mathcal{D}^{(1)}(\vec\mu)\mathcal{D}^{(1)}(\vec\mu)\big]|\rho^{(0)}(\mu_{\rm R}))\Big\}}_{\text{NNLO contributions}}$$

$$+\,\mathcal{O}(\alpha_s^3)$$

Shower Cross Section

The fixed order cross section is fine as long as we can calculate at "**all order level**". But life is not that easy…
Defining the normalised singular operator as

$$\mathcal{X}(\vec{\mu}) = \mathcal{F}_{\mathrm{bare}}\mathcal{D}(\vec{\mu})\mathcal{F}^{-1}(\mu_{\mathrm{R}}(\vec{\mu}))\mathcal{V}^{-1}(\vec{\mu})$$

$$\mathcal{V}(\vec{\mu}) = \underbrace{\left[\mathcal{F}_{\mathrm{bare}}\mathcal{D}(\vec{\mu})\right]_{\mathbb{P}}}\mathcal{F}^{-1}(\mu_{\mathrm{R}}(\vec{\mu}))$$

- **Finite operator**
- **Doesn't change** the number of partons and their flavours
- Operates only in the color space

- **truncated** at NLO, NNLO level
- prefers **large scale,** $\mu_i^2 \approx Q^2$

$$\sigma[O_J] = \big(1\big|\underbrace{\mathcal{O}_J\,\mathcal{X}(\vec{\mu})}\overbrace{\mathcal{V}(\vec{\mu})\mathcal{F}(\mu_{\mathrm{R}})\big|\rho_{\mathrm{H}}(\vec{\mu})\big)}$$

- prefers **small scale**, $\mu_i^2 \ll \mu_J^2$
- that is **in conflict** with the hard part

- Choose a **hard scale**, $\vec{\mu}_{\mathrm{H}} \sim \sqrt{Q^2}$
- Choose a **cutoff scale**, $\vec{\mu}_{\mathrm{f}} \sim 1\mathrm{GeV} \ll \mu_J$
- Insert a unit operator before the measurement operator as,

$$1 = \mathcal{X}(\vec{\mu}_{\mathrm{f}})\mathcal{X}^{-1}(\vec{\mu}_{\mathrm{f}})$$

$$\sigma[O_J] = \underbrace{\big(1\big|\mathcal{O}_J\,\mathcal{X}(\vec{\mu}_{\mathrm{f}})}_{=(1|\mathcal{O}_J}\overbrace{\mathcal{X}^{-1}(\vec{\mu}_{\mathrm{f}})\mathcal{X}(\vec{\mu}_{\mathrm{H}})}^{\mathcal{U}(\vec{\mu}_{\mathrm{f}},\vec{\mu}_{\mathrm{H}})}\mathcal{V}(\vec{\mu}_{\mathrm{H}})\mathcal{F}(\mu_{\mathrm{R}})\big|\rho_{\mathrm{H}}(\vec{\mu}_{\mathrm{H}})\big)$$

No resolvable radiation come from $\mathcal{D}(\vec{\mu}_{\mathrm{f}})$ operator, thus these operators commute,

$$\mathcal{O}_J\mathcal{X}(\vec{\mu}_{\mathrm{f}}) \approx \mathcal{X}(\vec{\mu}_{\mathrm{f}})\mathcal{O}_J \quad .$$

$$\mathcal{U}(\vec{\mu}_{\mathrm{f}},\vec{\mu}_{\mathrm{H}}) = \mathbb{T}\exp\left\{\int_C d\vec{\mu}\cdot\vec{\mathcal{S}}(\mu)\right\}$$

$$\vec{\mathcal{S}}(\vec{\mu}) = \lim_{\epsilon\to 0}\mathcal{X}^{-1}(\vec{\mu})\frac{d\mathcal{X}(\vec{\mu})}{d\vec{\mu}}$$

LC+ decomposition (approx.)

Despite of the name it is **not an approximation of the colour space**, it is an **approximation of the shower evolution operator**.

LC+ part
- Diagonal operator in the color space
- **Exact** in the **collinear** limit
- Some soft interferences are included but not all
- **Easy to exponentiate**

Glauber/Coulomb gluon part
- Imaginary part of the 1-loop soft singularities
- Highly non-trivial in color space
- Can be treated **perturbatively** or **fully exponentiated**

$$\vec{\mathcal{S}}^{[1]}(\vec{\mu}) = \overbrace{\vec{\mathcal{S}}^{[1]}_{\mathrm{LC}+}(\vec{\mu})} + \underbrace{\vec{\mathcal{S}}^{[1]}_{\mathrm{soft}}(\vec{\mu})} + \mathrm{i}\pi\,\overbrace{\vec{\mathcal{S}}^{[1]}_{\mathrm{i}\pi}(\vec{\mu})}$$

Wide angle soft part
- Only **wide angle soft** singularities
- Only single log contribution
- Leads to only $1/N_c^2$ suppressed terms
- Can be treated **perturbatively**

This decomposition **preserves unitary**,

$$(1|\mathcal{S}^{[1]}(\vec{\mu}) = (1|\mathcal{S}^{[1]}_{\mathrm{LC}+}(\vec{\mu}) = (1|\mathcal{S}^{[1]}_{\mathrm{soft}}(\vec{\mu}) = (1|\mathcal{S}^{[1]}_{\mathrm{i}\pi}(\vec{\mu}) = 0$$

and it allows us to treat the wide angle soft part perturbatively in a very efficient and flexible way.

- No approximation of the colour group, it is the **full SU(3)** algebra
- Can handle **any colour interferences**

$$\{c\}_m \neq \{c'\}_m$$

- At the end of the shower we calculate the **full SU(3)** colour overlap without approximation,

$$\langle \{c'\}_m | \{c\}_m \rangle$$

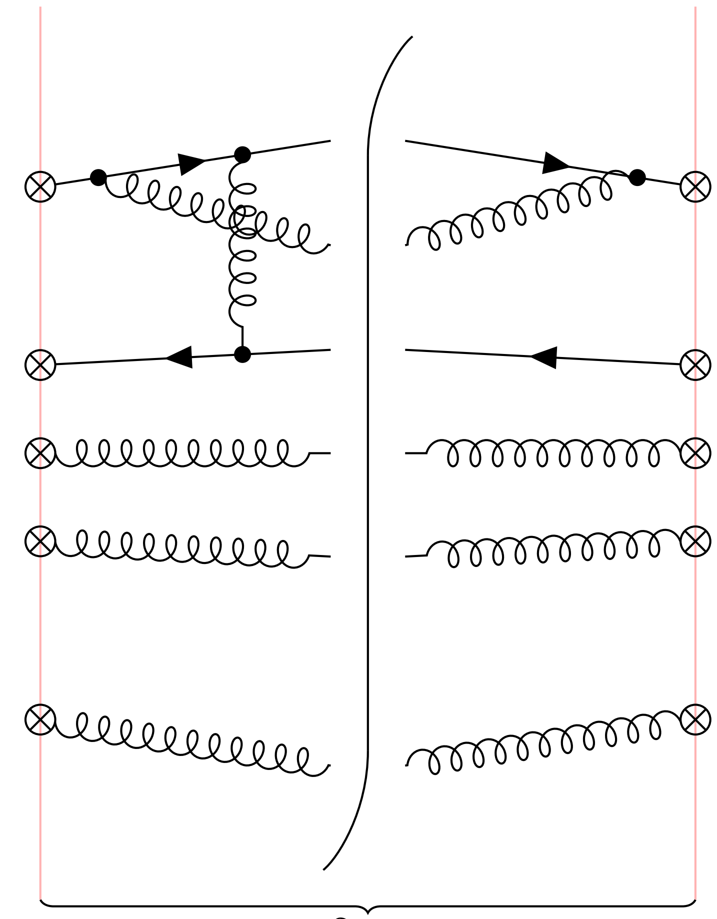
We have a **very fast algorithm** to do this, and can deal with hundreds of partons.

- **No** need of **tweaking** the $C_A/2,\ C_F$ colour factors.

Infrared sensitive operator

We can consider a more constructive approach to build the full infrared sensitive operator. This operator basically represents the QCD density operator of a $m \to X$ *(anything)* process.

$$\mathcal{D}(\mu_{\rm R},\vec{\mu}) = 1 + \sum_{n=1}^{k} \left[\frac{\alpha_{\rm s}(\mu_{\rm R}^2)}{2\pi}\right]^n \sum_{\substack{n_{\rm R}=0 \\ n_{\rm R}+n_{\rm V}=n}}^{n} \sum_{n_{\rm V}=0}^{n} \mathcal{D}^{(n_{\rm R},n_{\rm V})}(\mu_{\rm R},\vec{\mu})$$



$$\left[\frac{\alpha_{\rm s}(\mu^2)}{2\pi}\right]^2 \mathcal{D}^{(1,1)}(\mu^2)$$

The structure is rather straightforward:

$$\begin{aligned}
&\big(\{\hat{p},\hat{f},\hat{s}',\hat{c}',\hat{s},\hat{c}\}_{m+n_{\rm R}}\big|\mathcal{D}^{(n_{\rm R},n_{\rm V})}(\mu_{\rm R},\vec{\mu})\big|\{p,f,s',c',s,c\}_m\big) \\
&= \sum_{G\in{\rm Graphs}} \int d^d\{\ell\}_{n_{\rm V}}\ {}_D\big\langle\{\hat{s},\hat{c}\}_{m+n_{\rm R}}\big|\boldsymbol{V}_L(G;\{\hat{p},\hat{f}\}_{m+n_{\rm R}},\{\ell\}_{n_{\rm V}},\mu_{\rm R})\big|\{s,c\}_m\big\rangle \\
&\qquad \times \big\langle\{s,c\}_m\big|\boldsymbol{V}_R^\dagger(G;\{\hat{p},\hat{f}\}_{m+n_{\rm R}},\{\ell\}_{n_{\rm V}},\mu_{\rm R})\big|\{\hat{s},\hat{c}\}_{m+n_{\rm R}}\big\rangle_D \\
&\quad \times \sum_{I\in{\rm Regions(G)}} \big(\{\hat{p},\hat{f}\}_{m+n_{\rm R}}\big|\mathcal{P}_G(I)\big|\{p,f\}_m\big)\, \underbrace{\Theta_G(I;\{\hat{p},\hat{f}\}_{m+n_{\rm R}},\{\ell\}_{n_{\rm V}};\vec{\mu})}
\end{aligned}$$

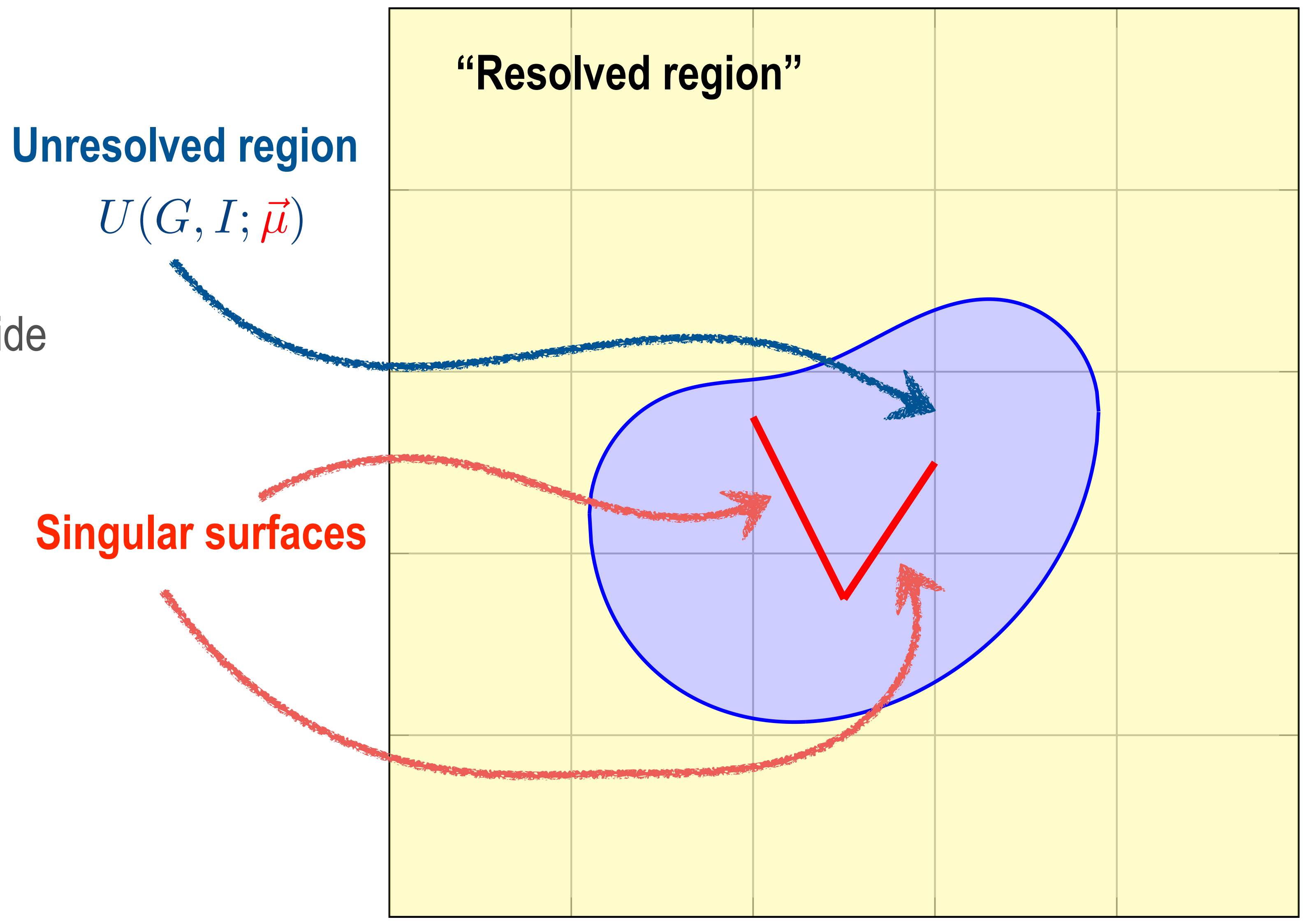
This function is the main focus of this talk.

Infrared sensitive operator

In general the $\Theta_G(I;\{\hat{p},\hat{f}\}_{m+n_{\mathrm{R}}},\{\ell\}_{n_{\mathrm{V}}};\vec{\mu})$ functions defines the unresolved region.

The singular surfaces may not extend outside of the unresolved region.

There can be **no naked singularity**!



$$\Theta_G(I;\{\hat{p},\hat{f}\}_{m+n_{\mathrm{R}}},\{\ell\}_{n_{\mathrm{V}}};\vec{\mu}) = \begin{cases} 1 & \text{if } \left(\{\hat{p},\hat{f}\}_{m+n_{\mathrm{R}}},\{\ell\}_{n_{\mathrm{V}}}\right) \in U(G,I;\vec{\mu}) \\ 0 & \text{otherwise} \end{cases}$$

Infrared sensitive operator

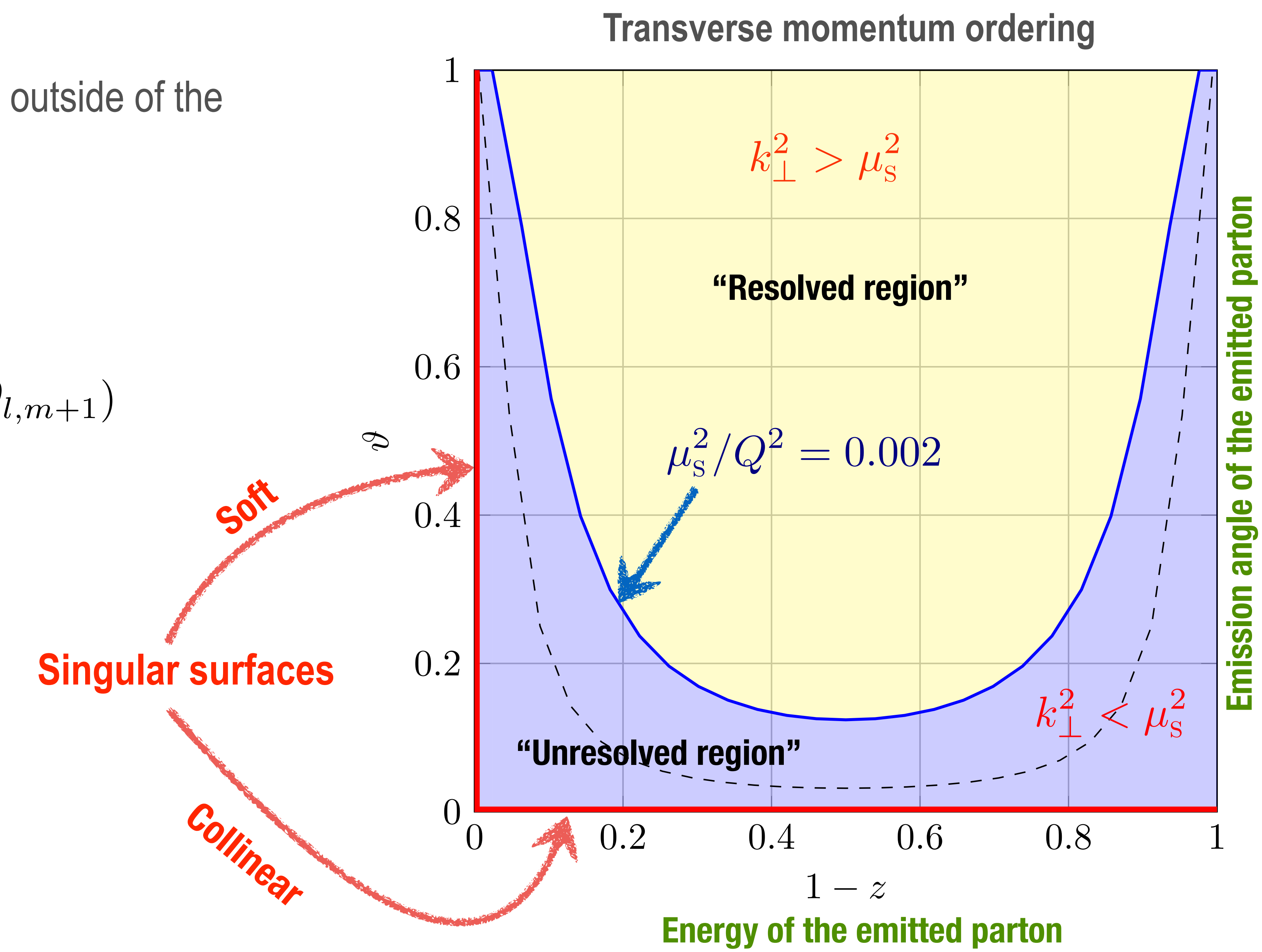
We have to introduce an **ultraviolet cutoff to capture only the IR part** of the amplitudes. At first order level in the real graphs it is just a cut on an infrared sensitive variable of the splitting:

$$\Theta_G(I; \{\hat{p}, \hat{f}\}_{m+n_{\mathrm{R}}}, \{\ell\}_{n_{\mathrm{V}}}; \mu_{\mathrm{S}}^2) \sim \theta(k_\perp^2 < \mu_{\mathrm{S}}^2)$$

The singular surfaces may not extend outside of the unresolved region.

There can be **no naked singularity**!

$$\vartheta = \frac{1}{2}(1 - \cos\theta_{l,m+1})$$



Unresolved regions of LC+and soft operators

We define the unresolved regions differently in the LC+ and "soft" terms:

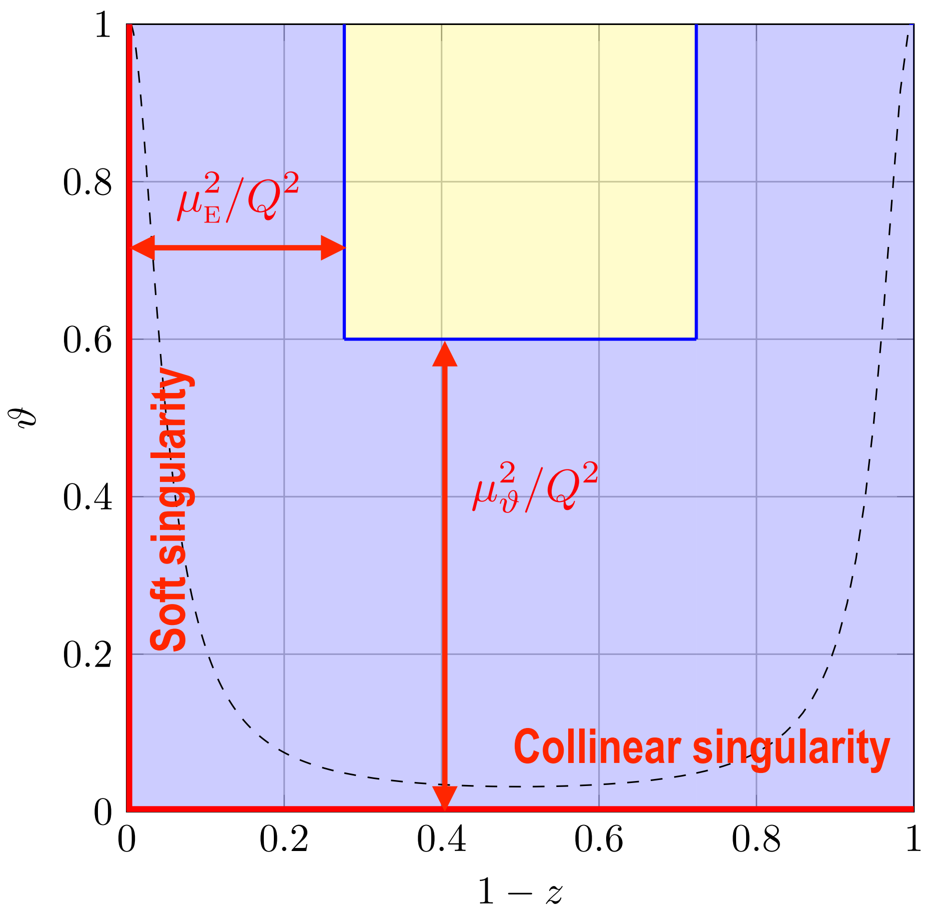
$$\mathcal{D}^{[1]}_{\mathrm{LC+}}(\vec{\mu}) \propto \left[1 - \theta(\vartheta Q^2 > \mu_\vartheta^2)\theta((1-z)Q^2 > \mu_{\mathrm{E}}^2)\right]$$

Depends on both scales since we have soft and collinear singularities.

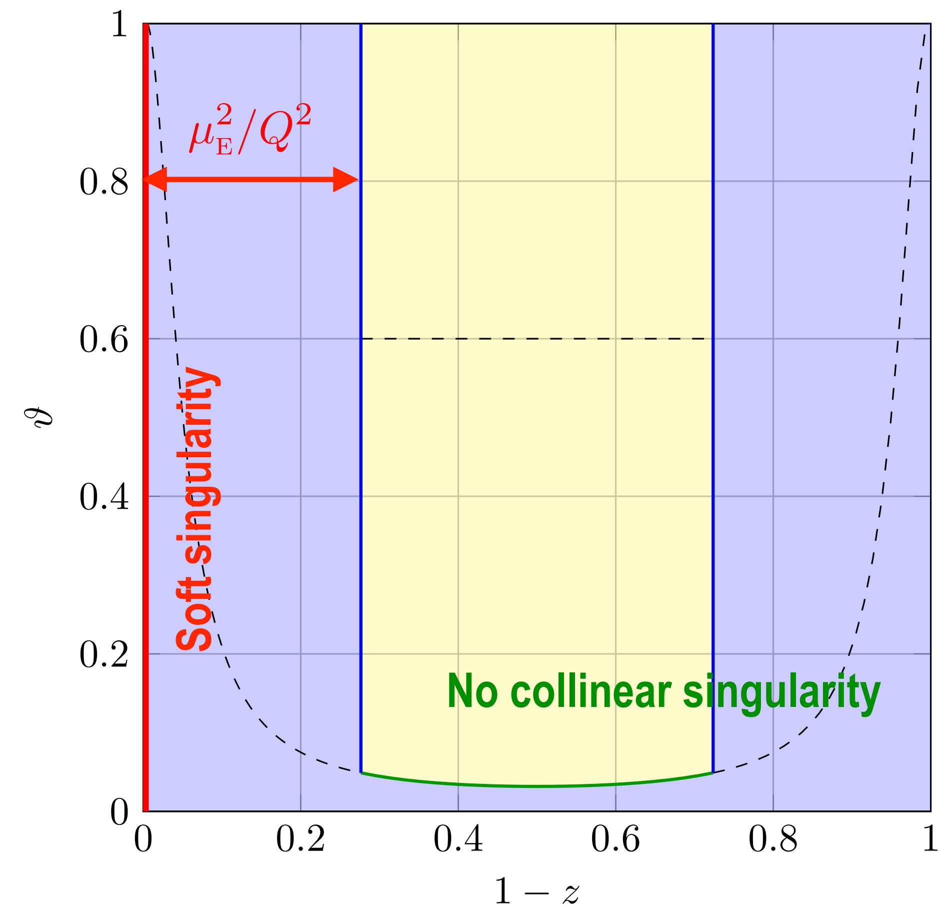
$$\mathcal{D}^{[1]}_{\mathrm{soft}}(\vec{\mu}) \propto \left[1 - \theta((1-z)Q^2 > \mu_{\mathrm{E}}^2)\right]$$

Depends only on the energy scale, since we don't have collinear singularities.

Unresolved region in $\mathcal{D}^{[1]}_{\mathrm{LC+}}(\vec{\mu})$



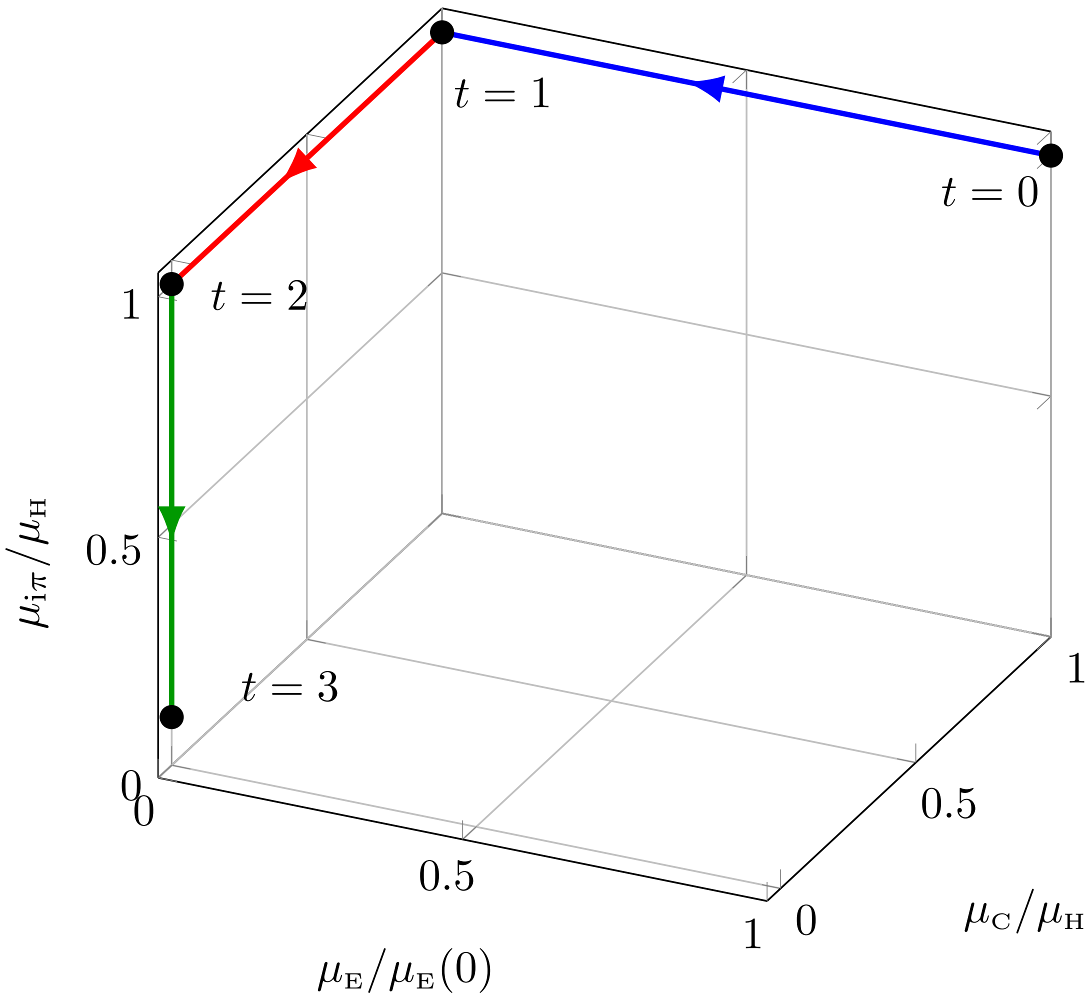
Unresolved region in $\mathcal{D}^{[1]}_{\mathrm{soft}}(\vec{\mu})$



Parton shower evolution

Now, the shower operator is a exponentiated contour integral of the splitting kernels between the hard and soft scales.

- As long as we work **at all order** it is **independent of the** chosen **contour**.
- At finite order the error is always higher order. Thus at a given order of accuracy the shower is still independent of the contour.
- Proof is in the paper.
- Now, we have three scales, $\vec{\mu} = (\mu_{\mathrm{C}}, \mu_{\mathrm{E}}, \mu_{\mathrm{i}\pi})$:
  - μ_{C} is sensitive for the collinear splitting ($\vartheta, k_\perp, \Lambda, \ldots$)
  - μ_{E} is sensitive in the soft region ($1 - z, E, \ldots$)
  - $\mu_{\mathrm{i}\pi}$ is the scale only for the $\mathrm{i}\pi$ terms
- Choose your contour



$$\mathcal{U}(t_{\mathrm{f}}, t_{\mathrm{H}}) = \mathbb{T}\exp\left\{-\int_{t_{\mathrm{H}}}^{t_{\mathrm{f}}} dt\left[\frac{d\mu_{\mathrm{C}}(t)}{dt}\mathcal{S}_{\mathrm{C}}(\vec{\mu}) + \frac{d\mu_{\mathrm{E}}(t)}{dt}\mathcal{S}_{\mathrm{E}}(\vec{\mu}) + \mathrm{i}\pi\frac{d\mu_{\mathrm{i}\pi}(t)}{dt}\mathcal{S}_{\mathrm{i}\pi}(\vec{\mu})\right]\right\}$$

Parton shower evolution

With this **three segment path** the shower cross section factorizes as

This is basically the **usual LC+ shower** evolution
- Full collinear physics is considered.
- Some soft radiation is considered.
- Simple in colour, and relative easy to implement

$$\sigma[O_J] = \underbrace{\big(1\big|\mathbb{T}\exp\Big\{-\mathrm{i}\pi\int_2^3 dt\,\frac{d\mu_{\mathrm{i}\pi}(t)}{dt}\mathcal{S}_{\mathrm{i}\pi}(\vec{\mu})\Big\}}_{=(1|}\mathcal{O}_J\,\overbrace{\mathbb{T}\exp\Big\{-\int_1^2 dt\,\frac{d\mu_{\mathrm{C}}(t)}{dt}\mathcal{S}_{\mathrm{C}}(\vec{\mu})\Big\}}\underbrace{\mathbb{T}\exp\Big\{-\int_0^1 dt\,\frac{d\mu_{\mathrm{E}}(t)}{dt}\mathcal{S}_{\mathrm{E}}(\vec{\mu})\Big\}}\mathcal{V}(\vec{\mu}_{\mathrm{H}})\mathcal{F}(\mu_{\mathrm{R}})\big|\rho_{\mathrm{H}}(\vec{\mu}_{\mathrm{H}})\big)$$

Glauber/Coulomb gluon effect
- It completely **drops out,** since $\big(1\big|\mathcal{S}_{\mathrm{i}\pi}(\vec{\mu}) = 0$.
- This means it is a **genuine higher order effect**, and from the first order shower it **can be "transformed out"**.
- Maybe this is the reason why we didn't find big effect when it was implemented interleaved with the other kernels.

This is the **soft shower** operator
- No collinear physics
- **Only wide angle soft** radiations
- Complicated in colour, but it always acts on the hard state.
- It can be implemented **perturbative** in the general case.
- For **Drell-Yan** process this is a **unit operator**.

There is no surprise here, the structure is very similar to that we had in e+e- case. But here we have to pay attention to the **PDF operator** $\mathcal{F}(\mu_{\mathrm{R}})$ and the **inclusive splitting operator** $\mathcal{V}(\vec{\mu})$.

PDF renormalization

The bare PDF need to be renormalised:

Defines the **factorization scheme**
- Finite operator in d=4 dimension
- Unit operator in MSbar scheme

$$\mathcal{F}_{\rm bare} = \big[\mathcal{F}(\mu_{\rm R}) \circ \overbrace{\mathcal{K}(\mu_{\rm R})} \circ \underbrace{\mathcal{Z}_F(\mu_{\rm R})}\big] = \mathcal{F}(\mu_{\rm R}) + \frac{\alpha_{\rm s}(\mu_{\rm R})}{2\pi}\big[\mathcal{F}(\mu_{\rm R}) \circ \big(\mathcal{K}^{(1)}(\mu_{\rm R}) + \mathcal{Z}_F^{(1)}(\mu_{\rm R})\big)\big] + \cdots$$

Usual $\overline{\rm MS}$ poles

When this operator acts on a basis state it might have more familiar expression:

$$\begin{aligned}\mathcal{F}_{\rm bare}\big|\{p,f\}_m\big) = \big|\{p,f\}_m\big)\Big[& f_a(\eta_a,\mu_{\rm R}) f_b(\eta_b,\mu_{\rm R}^2) \\ & + \frac{\alpha_{\rm s}(\mu_{\rm R}^2)}{2\pi}\sum_{a'}\int_0^1 \frac{dz}{z}\left(K_{a,a'}(z,\mu_{\rm R}^2) + \frac{1}{\epsilon_{\overline{\rm MS}}} P_{a,a'}(z)\right) f_{a'}(\eta_a/z,\mu_{\rm R}^2) f_b(\eta_b,\mu_{\rm R}^2) \\ & + f_a(\eta_a,\mu_{\rm R})\frac{\alpha_{\rm s}(\mu_{\rm R}^2)}{2\pi}\sum_{b'}\int_0^1 \frac{dz}{z}\left(K_{b,b'}(z,\mu_{\rm R}^2) + \frac{1}{\epsilon_{\overline{\rm MS}}} P_{b,b'}(z)\right) f_{b'}(\eta_b/z,\mu_{\rm R}^2)\Big] + \cdots\end{aligned}$$

Inclusive splitting operator

After cancelling all the singularities and choosing the renormalization scale as $\mu_{\rm R}^2 = \mu_{\rm C}^2$, we have found

$$\mathcal{V}(\vec{\mu}) = \underbrace{\left[\mathcal{F}_{\rm bare}\mathcal{D}(\vec{\mu})\right]_{\mathbb{P}}}\mathcal{F}^{-1}(\mu_{\rm R}(\vec{\mu}))$$

- **Finite operator**
- **Doesn't change** the number of partons and their flavours

It **vanishes** in the $\vec{\mu}, m_\perp \to 0$ limit.
- Complicated in color
- It can be exponentiated

$$\boldsymbol{V}^{\rm a}_{aa'}(z;\{p\}_m) = \overbrace{\boldsymbol{P}^{\rm a,NS}_{aa'}(z;\{p\}_m)} + K_{aa'}(z,\mu_{\rm C}^2) + \underbrace{\left[\hat{P}^{(\epsilon)}_{aa'}(z)\right]_{\rm MSR} + \left[\hat{P}_{aa'}(z)\,\log\left(\max\left\{w_{\rm C}(z)\,,\frac{m_\perp^2}{\mu_{\rm C}^2}\right\}\right)\right]_{\rm MSR}}$$

It **remains finite** in the $\vec{\mu}, m_\perp \to 0$ limit.
We **must change** the PDF factorization scheme.

$$K_{aa'}(z,\mu_{\rm R}^2) = -\left[\hat{P}^{(\epsilon)}_{aa'}(z) + \hat{P}_{aa'}(z)\,\log\left(\max\left\{w_{\rm C}(z)\,,\frac{m_\perp^2}{\mu_{\rm R}^2}\right\}\right)\right]_{\rm MSR}$$

The factorization scheme **depends on the ordering** of the shower

$$w_{\rm C}(z) = \begin{cases} 1 & \text{for } k_\perp \text{ ordering, } {\rm C} = \perp \\ (1-z)r_{\rm a} & \text{for } \Lambda \text{ ordering, } {\rm C} = \Lambda \\ (1-z)^2/z & \text{for angular ordering, } {\rm C} = \vartheta \end{cases}$$

Shower dependent PDF

This leads to the following DGLAP equation:

$$\mu_{\rm R}^2 \frac{df_a(\eta,\mu_{\rm R}^2)}{d\mu_{\rm R}^2} = \frac{\alpha_{\rm s}(\mu_{\rm R}^2)}{2\pi}\sum_{a'}\int_0^1 \frac{dz}{z}\left[\hat{P}_{a,a'}(z)\theta(w_{\rm C}(z)\mu_{\rm R}^2 > m_\perp)\right]_{\rm MSR} f_{a'}(\eta/z,\mu_{\rm R}^2)$$

One should do a full PDF fit with shower oriented PDF schemes. It is rather unlikely that it will happen in my lifetime, thus it might be a better approach to **relate the shower oriented PDF to the MSbar** one.

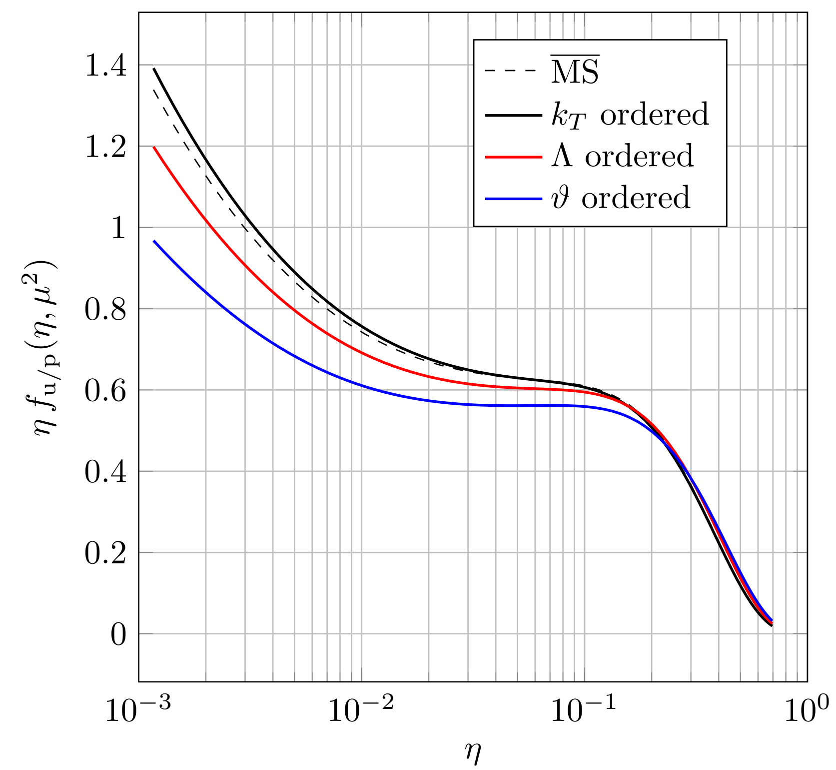
- Transverse momentum ordering ***almost*** corresponds to MSbar PDF. Only the $P_{a,b}^{(\epsilon)}(z)$ needs to be included, but it is rather negligible.
- Parameterising the ordering schemes by a continuous parameter λ, we can relate the corresponding PDF by solving :

$$\frac{df_a(\eta,\mu_{\rm R}^2,\lambda)}{d\lambda} = \frac{\alpha_{\rm s}(\mu_{\rm R}^2)}{2\pi}\sum_{a'}\int_0^1 \frac{dz}{z}\left[\log(1-z)\hat{P}_{a,a'}(z)\theta((1-z)^\lambda\mu_{\rm R}^2 > m_\perp)\right]_{\rm MSR} f_{a'}(\eta/z,\mu_{\rm R}^2,\lambda)$$

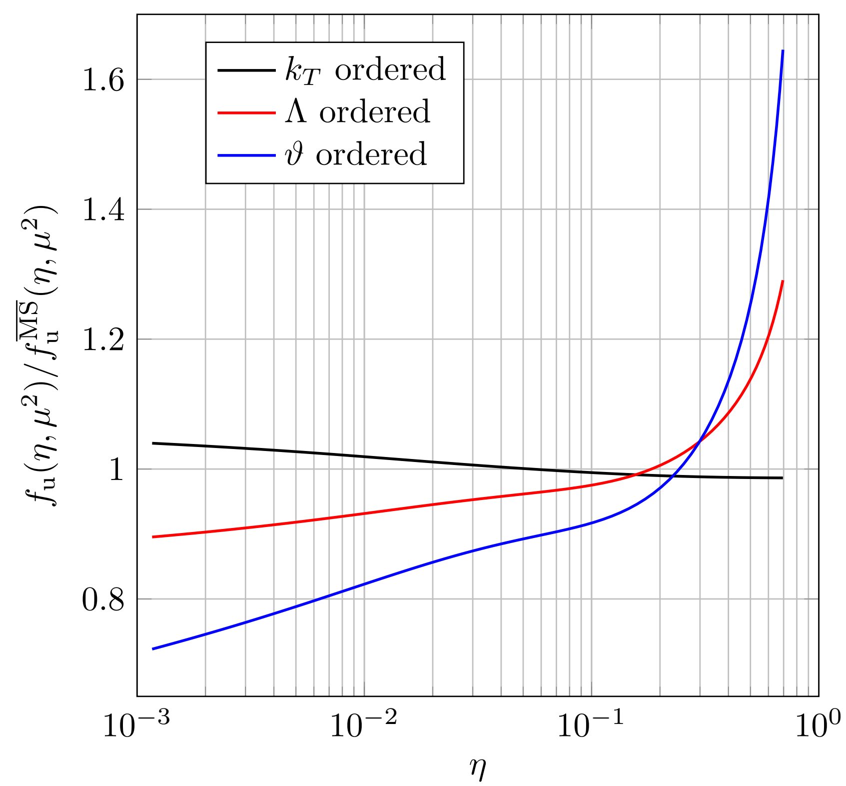
- $\lambda = 0$ is almost the MSbar PDF and that is our **boundary condition**.
- $\lambda = 1$ gives the PDF for Λ-ordered shower.
- $\lambda = 2$ gives the PDF for angular ordered shower.

Shower oriented PDF

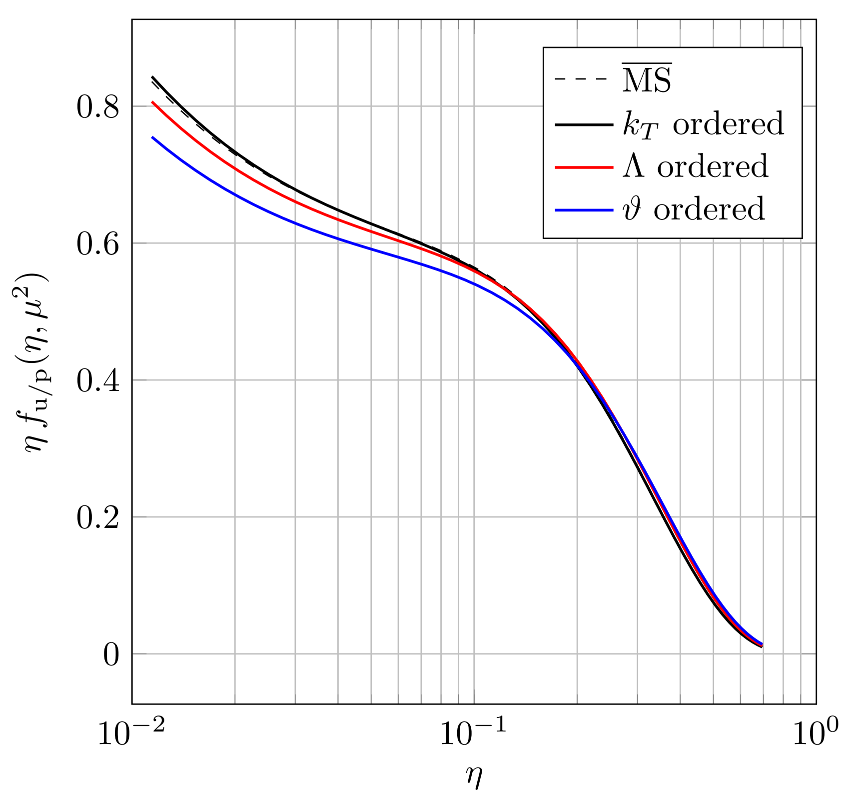
up-quark distribution, $\mu \approx 50$ GeV



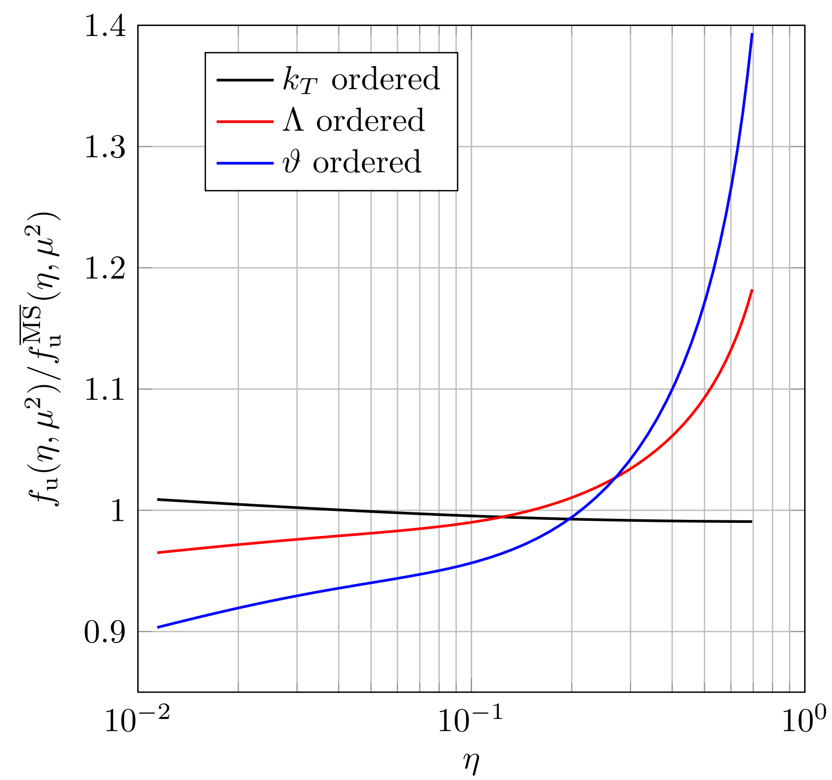
$f_{\mathrm{u/p}}/f^{\overline{\mathrm{MS}}}_{\mathrm{u/p}}$, $\mu \approx 50$ GeV



up-quark distribution, $\mu \approx 1$ TeV



$f_{\mathrm{u/p}}/f^{\overline{\mathrm{MS}}}_{\mathrm{u/p}}$, $\mu \approx 1$ TeV



Threshold logarithms

The inclusive spitting operator can be exponentiated and it sums up the **threshold logarithms**.

$$\mathcal{V}(\vec{\mu}_{\rm H}) = \underbrace{\mathcal{V}(\vec{\mu}_{\rm f})}_{\approx 1}\mathcal{U}_{\mathcal{V}}(\vec{\mu}_{\rm f},\vec{\mu}_{\rm H}) \approx \mathcal{U}_{\mathcal{V}}(\vec{\mu}_{\rm f},\vec{\mu}_{\rm H})$$

since

$$\vec{\mu}_{\rm H} \sim Q$$
$$\vec{\mu}_{\rm f} \sim 1\mathrm{GeV}$$

Here the evolution operator is

$$\mathcal{U}_{\mathcal{V}}(\vec{\mu}_{\rm f},\vec{\mu}_{\rm H}) = \mathbb{T}\exp\left\{\int_C d\vec{\mu}\cdot\vec{\mathcal{S}}_{\mathcal{V}}(\vec{\mu})\right\} \qquad \vec{\mathcal{S}}_{\mathcal{V}}(\vec{\mu}) = \mathcal{V}^{-1}(\vec{\mu})\frac{d\mathcal{V}(\vec{\mu})}{d\vec{\mu}}$$

With the shower oriented PDF we sum up some of the threshold logarithms with the PDF evolution, thus the full threshold effects is given as

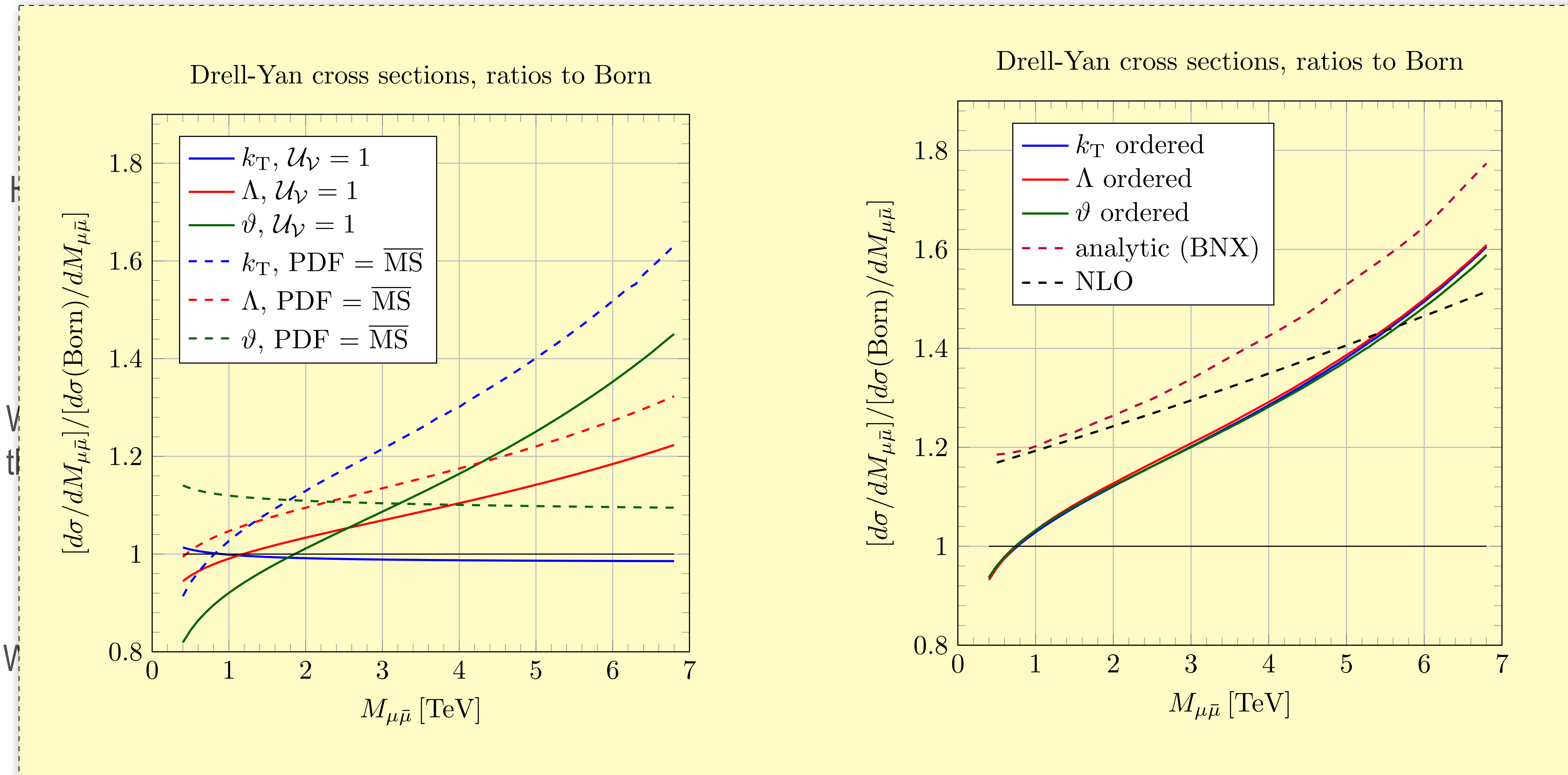
$$\mathcal{U}_{\mathcal{V}}(\vec{\mu}_{\rm f},\vec{\mu}_{\rm H})\mathcal{F}(\mu_{\rm R}(\vec{\mu}_{\rm H}))\mathcal{F}^{-1}_{\overline{\rm MS}}(\mu_{\rm R}(\vec{\mu}_{\rm H}))$$

We can test this by calculating the rate of the total cross section for Drell-Yan process,

$$\frac{\big(1\big|\mathcal{U}_{\mathcal{V}}(\vec{\mu}_{\rm f},\vec{\mu}_{\rm H})\mathcal{F}(\mu_{\rm R}(\vec{\mu}_{\rm H}))\big|\rho_{\rm H}\big)}{\big(1\big|\mathcal{F}_{\overline{\rm MS}}(\mu_{\rm R}(\vec{\mu}_{\rm H}))\big|\rho_{\rm H}\big)}$$

Threshold logarithms

The inclusive spitting operator can be exponentiated and it sums up the **threshold logarithms**.



$$\overline{\big(1\big|\mathcal{F}_{\overline{\text{MS}}}(\mu_R(\vec{\mu}_H))\big|\rho_H\big)}$$

Conclusion, Outlook

- We generalised the concept of the ordering in parton shower algorithms by using multiple variables to define the resolved and unresolved phase space regions.
  - This could be very useful in the NLO and higher order shower, where the structure of the singular surfaces is more complicate.
  - In this framework an angular ordered dipole shower makes perfect sense.
  - With **three scales** we can significantly simplify the colour evolution.
  - From the first order shower the $\mathrm{i}\pi$ **can be eliminated** completely
- With initial state parton we have to take care about the PDF. ***MSbar PDF is not good for everything***.
  - The PDF factorization scheme **depends on the ordering**.
  - This is important since we have to **match the DGLAP** evolution to the evolution of the PDF in the parton shower
- From theory point of view every ordering is good as long as it obeys the "**no naked singularity** principle".
  - As long as we work at "all order" level every ordering is accurate.
  - This is not true anymore when the shower is only LO or NLO.
  - We have lots of freedom in a LO shower framework, but we can **achieve good accuracy** for a certain class of observables by choosing the ordering, mappings and partitioning **wisely.**
  - Even the $\mathrm{i}\pi$ effect can be consider by picking the ordering carefully and make sure that the $\mathrm{i}\pi$ operator is interleaved with the standard shower generators "correctly". Of course it is observable dependent.

Inclusive splitting operator

The inclusive splitting operator is a finite operator only in the colour space:

$$\mathcal{V}(\vec{\mu})\big|\{p,f,c,c'\}_m\big) = \left[1 + \frac{\alpha_s(\mu_R^2)}{2\pi}\sum_{a'}\int_0^1 \frac{dz}{z}\frac{f_{a'}(\eta_a/z,\mu_R^2)}{f_a(\eta_a,\mu_R^2)} \boldsymbol{V}^{\mathrm{a}}_{a,a'}(z,\{p\}_m) + (a \leftrightarrow b)\right]\big|\{p,f,c,c'\}_m\big) + \cdots$$

Contributions of the **real emissions**

- Integrated over the unresolved region
- Singular operator

$$\boldsymbol{V}^{\mathrm{a}}_{a,a'}(z,\{p\}_m) = \lim_{\epsilon\to 0}\left[K_{a,a'}(z,\mu_R) + \frac{1}{\epsilon_{\overline{\mathrm{MS}}}}P_{a,a'}(z) + \overbrace{\hat{\boldsymbol{P}}_{a,a'}(z,\{p\}_m,\epsilon)} + \underbrace{\delta_{a,a'}\delta(1-z)\boldsymbol{\Gamma}_a(\{p\}_m,\epsilon)}\right]$$

Contributions of the **virtual graphs**

- Singular operator

The real and virtual poles (soft and collinear) cancel each others and the finite part of the virtual operator is fixed by the **momentum sum rule,**

$$\sum_a \int_0^1 dz\, z\, \hat{\boldsymbol{P}}_{a,a'}(z,\{p\}_m,\epsilon) + \boldsymbol{\Gamma}_{a'}(\{p\}_m,\epsilon) = 0$$

Or with fancy notation we have

$$\boldsymbol{P}_{a,a'}(z,\{p\}_m,\epsilon) = \left[\hat{\boldsymbol{P}}_{a,a'}(z,\{p\}_m,\epsilon)\right]_{\mathrm{MSR}} = \hat{\boldsymbol{P}}_{a,a'}(z,\{p\}_m,\epsilon) - \delta_{a,a'}\delta(1-z)\sum_c\int_0^1 d\bar{z}\,\bar{z}\,\hat{\boldsymbol{P}}_{c,a'}(\bar{z},\{p\}_m,\epsilon)$$

Inclusive splitting operator

The inclusive splitting operator is a finite operator only in the colour space:

$\mathcal{V}(\vec{\mu})\Big|$ ... $\cdots$

$\boldsymbol{V}^{\mathrm{a}}_{a,a}$ ... $,\epsilon)\Big]$

...hs

Even fancier notation:

$$\left[\boldsymbol{A}_{aa'}(z;\{p\}_m)\right]_{\mathrm{MSR}} = \boldsymbol{A}_{aa'}(z;\{p\}_m)\ ,\ \ a \neq a'$$

and

$$\left[\boldsymbol{A}_{aa}(z;\{p\}_m)\right]_{\mathrm{MSR}} = \frac{1}{z}\left[z\,\boldsymbol{A}_{aa}(z;\{p\}_m)\right]_+ - \delta(1-z)\sum_{c\neq a}\int_0^1 d\bar{z}\ \bar{z}\,\boldsymbol{A}_{ca}(\bar{z};\{p\}_m)\ .$$

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