Magnetic Monopoles in Non-Abelian Gauge Theory

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Contents

Notation and conventions:

- We take the $(-,+,+,+)$ signature for the Minkowski metric $\eta_{\mu\nu}$, such that

$$ds^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + g_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad \mathrm{d}^4x\sqrt{\eta} = \mathrm{d}t\mathrm{d}^3x\sqrt{g}, \tag{1}$$

  where g_{ij} is the flat space metric on $\mathbb{R}^3$. We will often employ spherical coordinates (r,θ,φ) for which

$$g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}r^2 + \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2, \quad \mathrm{d}^3x\sqrt{g} = \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi r^2\sin\theta. \tag{2}$$

- The generators of a Lie group G are taken to be anti-hermitian matrices in the Lie algebra $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{su}(2)$, we take the conventions

$$\mathfrak{su}(2) = \mathrm{Span}(\tau_1,\tau_2,\tau_3), \quad \tau_a := \frac{\sigma_a}{2i}, \tag{3}$$

where σ_a are the Pauli matrices, with relations

$$[\tau_a, \tau_b] = \epsilon_{abc}\tau_c, \quad \{\tau_a, \tau_b\} = -\frac{\delta_{ab}}{2}. \tag{4}$$

- For gauge theory, we define the covariant derivative and field strength tensor as

$$D_\mu = \partial_\mu + A_\mu = \partial_\mu - eW_\mu, \quad F_{\mu\nu} = [D_\mu, D_\nu] = -eG_{\mu\nu}. \tag{5}$$

  Here, we absorbed the gauge coupling into the definition of A, F, such that $\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} = -\frac{1}{4e^2}F^a_{\mu\nu}F^{a\mu\nu}$.

Setting: We will compute monopoles and dyons as minima of the Hamiltonian of an effective action. After an appropriate gauge fixing procedure, we assume that this effective action takes the form of the classical gauge theory action with renormalized parameters, along with a potential that imposes spontaneous symmetry breaking.

References: For the beginning of section 2, I rely mostly on discussions and derivations in (Goddard und Olive, 1978, §4.4). Section 2.3 comes from (Shnir, 2005, §5.2). Finally, in section 3, I take from (Goddard und Olive, 1978, §5), (Preskill, 1984, §4) and finally (Coleman, 1983, §4.2).The original sources for the main results of section 2 are also cited within the notes.

1 Quick review of non-abelian gauge theory

1.1 Yang-Mills and Higgs fields

Consider a compact lie group G with Lie algebra $\mathfrak{g} = \mathrm{Span}(\tau_a)_{a=1}^{\mathrm{rank}(\mathfrak{g})}$. The Yang-Mills connection is a map

$$\mathbb{R}^{1,3} \mapsto \wedge^1(\mathbb{R}^{1,3}) \otimes \mathfrak{g}, \quad x \mapsto A = A^a_\mu(x)\mathrm{d}x^\mu\tau_a \tag{6}$$

Given a field $x \mapsto \phi(x) \in \mathbb{V}_\rho$ in a unitary representation $\tau_a \mapsto \rho(\tau_a)$ of $\mathfrak{g}$, where $\rho(\tau_a)^\dagger = -\rho(\tau_a)$, its covariant derivative is

$$D_\mu\phi := \partial_\mu\phi + A^a_\mu\,\rho(\tau_a)\cdot\phi, \qquad D_\mu\,\rho(g) = \rho(g)D_\mu, \forall g \in G. \tag{7}$$

In particular, this implies that both $(D_\mu\phi)^\dagger D_\nu\phi$ and $\phi^\dagger\phi$ are gauge invariant. The field strength is then defined by

$$[D_\mu, D_\nu] =: \rho(F_{\mu\nu}) \Longrightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{8}$$

In particular, the Jacobi identity for triple commutators of D_μ is equivalent to the Bianchi identity for $F_{\mu\nu}$:

$$\sum_{\text{cyclic}(1,2,3)} [D_{\mu_1}, [D_{\mu_2}, D_{\mu_3}]] = 0 \iff D_\nu (\star F)^{\mu\nu} = 0, \quad (\star F)^{\mu\nu} := \frac{\sqrt{\eta}}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}. \tag{9}$$

Real Higgs field: As a slight simplification, we will assume that the Higgs field is real, i.e. $\phi \in \mathbb{V}_\rho \cong \mathbb{R}^n$. This means that the $\mathfrak{g}$-invariant scalar product on $\mathbb{V}_\rho$ is simply $\phi^\dagger \phi = \frac{1}{2}\phi^\mathsf{T} \phi$.

1.2 The (effective) action and its Hamiltonian

The Lagrangian density is given by

$$\mathrm{d}^4 \sqrt{\eta}\, \mathcal{L} = \frac{1}{2e^2} F^a \wedge \star F^a - \frac{1}{2} D\phi^\mathsf{T} \wedge \star D\phi + \frac{\vartheta}{4\pi} F^a \wedge F^a - \mathrm{d}^4 x \sqrt{\eta}\, U(\phi). \tag{10}$$

By varying $\mathcal{L}$ and integrating by parts we the fundamental equation

$$\mathrm{d}^4 \sqrt{\eta}\, \delta\mathcal{L} = \mathrm{d}\Theta + \mathrm{EoM}, \tag{11}$$

where the equations of motion are

$$\mathrm{EoM} = \mathrm{d}^4 x \sqrt{\eta}\, \delta A^a_\mu \left(\frac{1}{e^2} D_\nu F^{a\mu\nu} - \epsilon^{abc} \phi^b D^\mu \phi^c \right) \tag{12}$$

$$+ \, \mathrm{d}^4 x \sqrt{\eta}\, \delta\phi^\mathsf{T} \left(D^\mu D_\mu \phi + \frac{\mathrm{d}U}{\mathrm{d}\phi} \right), \tag{13}$$

and the exact term takes the form

$$\Theta(\delta A, \delta\phi) = \delta A^a \wedge \left(\frac{1}{e^2} \star F^a + \frac{\theta}{2\pi} F^a \right) - \delta\phi^\mathsf{T} \star D\phi. \tag{14}$$

Hamiltonian: We would now like to compute the Hamiltonian on a timeslice $\Sigma_t := \{x \in \mathbb{R}^4 \,|\, x^0 = t\}$ using Noether's method: inserting $\delta = \partial_0$ in (11), we obtain

$$H = \int_{\Sigma_t} \left(\Theta(\partial_0 A, \partial_0 \phi) - \mathrm{d}^3 x \sqrt{g} \mathcal{L} \right) \tag{15}$$

The goal of the rest of the talk is to find monopole and dyon configurations that minimize this Hamiltonian. To this end, we make a few simplifying assumptions:

- first, we set all fields to be *static*, $\partial_0 A = \partial_0 \phi = 0$, such that the Hamiltonian is the same as the mass and the opposite of the Lagrangian.
- Next, we impose that $\rho(A_0)\phi = 0$ such that $D_0\phi = 0$.

With these assumptions, we can expand the Yang-Mills two-form into magnetic and electric parts as

$$\begin{aligned} G^a &= E_i^a \mathrm{d}x^0 \wedge \mathrm{d}x^i + \epsilon_{ijk} B_i^a \mathrm{d}x^j \wedge \mathrm{d}x^k, \\ \star G^a &= B_i^a \mathrm{d}x^0 \wedge \mathrm{d}x^i - \epsilon_{ijk} E_i^a \mathrm{d}x^j \wedge \mathrm{d}x^k, \end{aligned}$$

and we obtain a simple form

$$H = \int_{\Sigma_t} \mathrm{d}^3x \sqrt{g} \left(\frac{1}{2} E_i^a E_i^a + \frac{1}{2} B_i^a B_i^a + \frac{1}{2} D_i \phi^\mathsf{T} D_i \phi + V(\phi) \right). \tag{16}$$

Electric charge and Witten effect: The presence of a theta term requires us to distinguish between the flux of the electric field and the electric charge. Indeed, inserting $\delta_\epsilon A_\mu^a = D_\mu \epsilon^a$ and $\delta_\epsilon \phi = \rho(\epsilon)\phi$ in (11) produces the conserved charge

$$\mathcal{Q}_\epsilon = -e \int_{\Sigma_t} \mathrm{d} \left(\frac{1}{e^2} \epsilon^a \star F^a + \frac{\vartheta}{2\pi} \epsilon^a F^a \right) = \oint_{\partial \Sigma_t} \left(\epsilon^a \star G^a + \frac{\vartheta}{2\pi e^2} \epsilon^a G^a \right), \tag{17}$$

where $\partial\Sigma_t \cong S^2$ is defined to be the two-sphere at spatial infinity $r \to \infty$. Taking constant Lie algebra elements $\partial_\mu \epsilon = 0$, we obtain the electric charges

$$\mathcal{Q}_\epsilon = \epsilon^a \int_{S^2} \mathrm{d}\theta \mathrm{d}\phi \, \sin\theta \lim_{r\to\infty} r^2 \left(E_r^a + \frac{\vartheta}{2\pi e^2} B_r^a \right). \tag{18}$$

Clearly, if either $\vartheta = 0$ or B_r^a does not contain monopoles, then the electric charge remains proportional to the electric flux. Otherwise, it is in general a linear combination of electric and magnetic fluxes: this is the Witten effect Witten (1979). Despite this subtlety, we will continue to denote electric flux by $q \propto E$ and magnetic flux by $g \propto B$ throughout the talk.

2 Monopoles & dyons in the Georgi-Glashow model

As an explicit example, we will consider the Georgi-Glashow model, with gauge algebra $\mathfrak{g} = \mathfrak{su}(2)$, adjoint matter $\phi = \phi^a \tau_a \in \mathfrak{su}(2)$ and a potential

$$U(\phi) = \frac{1}{2}\mu^2 \phi^a \phi^a + \frac{\lambda}{8}\left((\phi^a \phi^a)^2 + v^4 \right) = \frac{\lambda}{8}(\phi^a \phi^a - v^2)^2, \tag{19}$$

where μ, v have units of mass and λ is dimensionless. To complete the square, we needed a negative mass-squared term, $\mu^2 = -\frac{1}{2}\lambda v^2$, which can arise in the quantum effective action of a theory with $(\phi^a\phi^a)^2$ interactions. Thus, the minimal of this potential are reached at

$$U(\phi) = 0 \iff \phi^a\phi^a = v^2 \iff \frac{\phi}{v} \in S^2 \cong \mathrm{SO}(3)/\mathrm{SO}(2). \tag{20}$$

In this case, the Higgs vacuum spontaneously breaks the SO(3) or SU(2) symmetry down to SO(2) = U(1). The remaining term in the Hamiltonian (16) is $\frac{1}{2}D_i\phi^a D_i\phi^a \geq 0$, which reaches its minimum for

$$D_i\phi = \partial_i\phi + [A_i, \phi] = 0. \tag{21}$$

For $\phi \in \mathfrak{su}(2)$, we can solve this equation for A_i (see Goddard und Olive (1978), Eq. (4.30)),

$$A_i \mathrm{d}x^i = -\frac{1}{v^2}\epsilon_{abc}\phi^a \mathrm{d}\phi^b \tau_c + \frac{\phi}{v}a, \tag{22}$$

where $a = a_i \mathrm{d}x^i$ is an arbitrary abelian connection. The field strength is then given by

$$F_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j = -\frac{1}{v^2}\epsilon_{abc}\mathrm{d}\phi^a \wedge \mathrm{d}\phi^b \tau_c + \frac{\phi}{v}\mathrm{d}a + \frac{\mathrm{d}\phi \wedge a}{v}. \tag{23}$$

In particular, we can take the restriction of F to the $\mathfrak{so}(2)$ subalgebra preserved after the SSB by contracting it with ϕ:

$$\frac{\phi^a F^a}{v} = -\frac{1}{v^3}\epsilon_{abc}\phi^a \mathrm{d}\phi^b \wedge \mathrm{d}\phi^c + \mathrm{d}a. \tag{24}$$

The last term proportional to a vanishes because $2\phi^a \mathrm{d}\phi^a = \mathrm{d}(\phi^a\phi^a) = 0$. We can then compute the magnetic flux of the Yang-Mills field along the ϕ and through the S^2 at spatial infinity:

$$\frac{1}{v}\oint_{\partial\Sigma_t} \phi^a G^a = \frac{1}{v^3 e}\int_{S^2} \epsilon_{abc}\, \mathrm{d}\phi^a \wedge \mathrm{d}\phi^b\, \phi^c. \tag{25}$$

This integral is known is called *topological charge* for the map $S^2 \mapsto S^2$, $\frac{x^i}{r} \mapsto \frac{\phi^a}{v}$. This means that it is invariant under small deformations of ϕ on the S^2,

$$\phi \to \phi + \delta\phi, \quad \phi^a\delta\phi^a = 0. \tag{26}$$

Indeed we find

$$\delta\left(\epsilon_{abc}\mathrm{d}\phi^a \wedge \mathrm{d}\phi^b\, \phi^c\right) = \delta\phi^c\, \epsilon_{abc}\mathrm{d}\phi^a \wedge \mathrm{d}\phi^b + \mathrm{d}\Big(\epsilon_{abc}\delta\phi^a \mathrm{d}\phi^b \phi^c\Big). \tag{27}$$

The second term is exact (a total derivative), and therefore vanishes by Stokes' theorem, whereas the second term vanishes because $\epsilon_{abc}\mathrm{d}\phi^a \wedge \mathrm{d}\phi^b \propto \phi^a$. Thus, the magnetic flux depends only on the topology of the $S^2 \mapsto S^2$ map, which is known to be classified by the second fundamental group $\pi_2(S^2) \cong \mathbb{Z}$. More concretely, in spherical coordinates

$$(x^i) = r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \tag{28}$$

topologically distinct maps are classified by the winding number of φ along the equator, and are of the form

$$(\phi_n^a(\theta,\varphi)) = v(\sin\theta\cos n\varphi, \sin\theta\sin n\varphi, \cos\theta), \tag{29}$$

which is single-valued w.r.t. $\varphi \to \varphi+2\pi$ as long as $n \in \mathbb{Z}$. For these solutions, we find

$$\frac{1}{v^3 e}\int_{S^2} \epsilon_{abc}\,\mathrm{d}\phi_n^a \wedge \mathrm{d}\phi_n^b\,\phi_n^c = \frac{n}{e}\int \sin\theta\mathrm{d}\theta \wedge \mathrm{d}\varphi = \frac{4\pi n}{e}. \tag{30}$$

We thus obtain Dirac's quantization condition for the magnetic charge of a U(1) gauge field from the topology of the Higgs field!

Relation to the Dirac monopole: In the form (22), the connection looks very different from the U(1) Dirac monopole. To retrieve the a Dirac monopole form, we can gauge transform A w.r.t. a SU(2) element that maps $\phi_n(\theta,\varphi)$ to the north pole:

$$\phi_n(\theta,\varphi) = R_n\, v\tau_3\, R_n^{-1}, \quad R_n(\theta,\varphi) := e^{n\varphi\tau_3}e^{\theta\tau_2}e^{-n\varphi\tau_3}. \tag{31}$$

Applying this to Eq. (22) gives

$$R_n^{-1}\,W\,R_n - \frac{1}{e}R_n^{-1}\mathrm{d}R_n = \frac{1}{e}\left\{n\frac{1-\cos\theta}{2}\mathrm{d}\varphi + a\right\}\tau_3. \tag{32}$$

Thus, in the gauge where $\phi_n = v\tau_3$, the gauge field W_μ^3 is a Dirac monopole with winding number n. Note that, similar to the $\mathfrak{g} = \mathfrak{u}(1)$ case, the gauge transformation R_n has a singularity at the the south pole $x^3 = -r$:

$$R_n = \sqrt{\frac{r}{2}}\left\{\sqrt{r+x^3} + 2in\frac{x^1\tau_2 - x^2\tau_1}{\sqrt{r+x^3}}\right\} \tag{33}$$

Of course we can always avoid such a singular gauge-transformation via the Wu-Yang procedure: we gauge transform by $R_n(\theta,\varphi)$ for $x^3 \geq 0$, and by $R_n(\pi-\theta,\pi+\varphi)$ for $x^3 \leq 0$, such that the two fields at the equator $x^3 = 0$ are related by the gauge transformation $R_n(\pi/2,\pi+\varphi)^{-1}R_n(\pi/2,\varphi) = e^{n\varphi\tau_3}$.

2.1 The 't Hooft-Polyakov monopole

Consider the monopole solution (29) with unit winding number,

$$\phi \overset{r\to\infty}{\sim} v\frac{x^a}{r}\tau_a = v\tau_r, \quad A \overset{r\to\infty}{\sim} -[\tau_r, \mathrm{d}\tau_r], \tag{34}$$

where $r\tau_r := x^a\tau_a$. If we extend this solution into the bulk, the energy diverges at small distances because of the monopole magnetic field:

$$B_i^a = \frac{1}{2}\epsilon_{ijk}G_{jk} = \frac{x_i\tau_r}{er^3} \Longrightarrow \mathrm{d}^3x\sqrt{g}B_i^aB_i^a = \frac{\mathrm{d}^2\Omega}{e^2}\,\mathrm{d}\left(-\frac{1}{r}\right). \tag{35}$$

To obtain a simple, finite energy monopole solution, note that the asymptotic forms of ϕ and G are invariant under the symmetry $\mathrm{SO}(3)_{\mathrm{diag}} \subset \mathrm{Poincare} \times \mathrm{SU}(2)$ consisting of simultaneous rotation of the spatial S^2 around the origin and the gauge group generators. We can preserve this symmetry using 't Hooft's ansatz ('t Hooft (1974)):

$$\phi = vh(\xi)\tau_r, \quad W_i\,\mathrm{d}x^i = \frac{1-K(\xi)}{e}[\tau_r, \mathrm{d}\tau_r], \tag{36}$$

where $\xi := evr$ is a dimensionless variable, and $K \to 0$, $h \to 1$ as $r \to \infty$. Inserting this ansatz into (16), we can find monopole solutions by minimizing the Hamiltonian w.r.t. the two functions h, K, and imposing boundary conditions at $r \to 0$ that ensure finite energy. This gives a coupled system of two non-linear second order differential equations. While we do not know how to solve this system analytically, numerical methods have been used to approximate these solutions.

2.2 The Bogomol'nyi bound and a monopole that saturates it

Consider the Hamiltonian (16) in the case of a pure monopole, i.e. $E_i^a = 0$. Then the first two terms are a sum of squares $\sum_{i,a=1}^3 (B_i^a)^2 + (D_i\phi^a)^2$, which we can complete as

$$H = \int_{\Sigma_t} \mathrm{d}^3x \left\{ \sum_{i,a} (B_i^a - D_i\phi^a)^2 + U(\phi) \right\} + \int_{\Sigma_t} \mathrm{d}^3x \sum_{i,a} B_i^a D_i\phi^a. \tag{37}$$

Using the Einstein convention again for index summation, we can easily integrate the second term by parts using

$$B_i^a D_i\phi^a = D_i(\phi^a B_i^a) - \phi^a D_i B_i^a = \partial_i(\phi^a B_i^a). \tag{38}$$

From the first to the second equality, we used the Bianchi identity $D_i\tilde{G}^{0i} = D_iB_i = 0$ and the fact that $\phi^a B_i^a$ is gauge invariant. We thereby obtain

$$H = \int_{\Sigma_t} \left\{(B_i^a - D_i\phi^a)^2 + U(\phi)\right\} + \oint_{\Sigma_t} \phi^a G^a. \tag{39}$$

Since the first two terms are positive semi-definite, we obtain a *bound* for the mass of the monopole — the *Bogomol'nyi bound* (Bogomolny (1976)):

$$H \geq vg = n\frac{ev}{\alpha} = n\frac{m_W}{\alpha}. \tag{40}$$

Here, $\alpha = e^2/(4\pi)$ is the fine structure constant, and $m_W = ev$ is the mass of the $W^1 \pm iW^2$ gauge fields after spontaneous symmetry breaking. Since $\alpha \ll 1$ in the infrared, we conclude that monopoles are much heavier than the massive gauge bosons in this model.

The *Bogomol'nyi-Prasad-Sommerfeld states* (see Prasad (1975); Bogomolny (1976)) are any solutions of the field equations that saturate this bound, i.e.

$$H = vg \iff D_i\phi^a = B_i^a, \quad \text{and} \quad U(\phi) = 0. \tag{41}$$

Note that this is consistent with the local equations for the Higgs vacuum, since $B_i^a \propto r^{-4}x_i x^a = \mathrm{O}(r^{-2})$ vanishes at spatial infinity. The BPS equation (41) is much easier to solve than the full-fledged energy minimization problem, and even admits analytic solutions. Taking 't Hooft's ansatz (36), we find

$$B = v\frac{1-K^2}{\xi^2}\mathrm{d}\xi\,\tau_r - vK'\mathrm{d}\tau_r, \quad D\phi = vh'\mathrm{d}\xi\,\tau_r + vhK\mathrm{d}\tau_r. \tag{42}$$

and the BPS equation is equivalent to a first order system

$$\frac{\mathrm{d}K}{\mathrm{d}\xi} = -hK, \quad \frac{\mathrm{d}h}{\mathrm{d}\xi} = \frac{1-K^2}{\xi^2}, \tag{43}$$

the solution of which is

$$K = \frac{\xi}{\sinh\xi}, \quad h = \coth\xi - \frac{1}{\xi}. \tag{44}$$

2.3 The Julia-Zee dyon and generalized Bogomol'nyi bound

Recall that if we impose stationarity $\partial_t A = \partial_t \phi = 0$ and a Hamiltonian of the form (16), then we can only introduce an electric field by turning on the timelike component of the gauge field, $W_0 = W_0^a\tau_a$ that commutes with the scalar field, $D_0\phi = -e[A_0, \phi] = 0$. In this case, the electric field allows for a neat generalization of the Bogmol'nyi bound to dyons, by inserting $1 = \cos^2\alpha + \sin^2\alpha$ next to $D_i\phi$ and completing two squares:

$$\begin{aligned} H = &\int_{\Sigma_t} \mathrm{d}^3x\sqrt{g}\left\{(B_i^a - \cos\alpha D_i\phi^a)^2 + (E_i^a - \sin\alpha D_i\phi^a)^2 + U(\phi)\right\} \\ &+ \int_{\Sigma_t} \mathrm{d}^3x\sqrt{g}\left(\cos\alpha B_i^a D_i\phi^a + \sin\alpha E_i^a D_i\phi^a\right) \end{aligned}$$

Integrating by parts, we obtain $E_i^a D_i\phi^a = \partial_i(E_i\phi^a) - \phi^a D_i F_i^a$, and $D_i E_i = -D_i G^{0i} \propto D_0\phi = 0$, and the generalized Bogomol'nyi bound is

$$H \geq v(\cos\alpha g + \sin\alpha q), \tag{45}$$

where q is the flux of the electric field through the S^2 at spatial infinity (and not the electric charge when a Theta term is present). Furthermore, the dyonic BPS states that saturate this bound must satisfy

$$H = v(\cos\alpha g + \sin\alpha q) \iff B_i^a = \cos\alpha D_i\phi^a,\ E_i^a = \sin\alpha D_i^a. \tag{46}$$

We can obtain a very simple solution to these BPS equations by noting that the electric field takes the form

$$E_i = \partial_i W_0 - e[W_i, W_0] \equiv D_i W_0. \tag{47}$$

The last identification follows from the fact that W_0^a transforms like the tensor ϕ^a (no inhomogeneous part) under time-independent gauge transformations, so the action of D_i is well defined. Thus, the electric BPS equation is simply $D_i(W_0 - \sin\alpha\phi) = 0$. In particular, if we impose $W_0 = \sin\alpha\,\phi$, then the dyon BPS equation reduces to the monopole BPS equation for $\cos\alpha\,\phi$ and there exists the simple solution

$$W_0 = v\tan\alpha\, h(\xi)\tau_r, \quad W_i \mathrm{d}x^i = \frac{1-K(\xi)}{e}[\tau_r, \mathrm{d}\tau_r], \quad \phi = v\frac{h(\xi)}{\cos\alpha}\tau_r, \tag{48}$$

Reinserting this ansatz into the computation of the electric flux constrains the angle α. Using

$$\frac{\cos\alpha}{v}\phi^a E_i^a = \frac{\cos\alpha\sin\alpha}{v}\phi^a D_i\phi^a = \frac{v\tan\alpha}{2}\partial_i(h^2) = v\tan\alpha\, h\partial_i h, \tag{49}$$

we obtain that

$$q = \frac{\cos\alpha}{v}\oint_{\partial\Sigma_t} \mathrm{d}^2x\sqrt{g}n^i\phi^a E_i = g\tan\alpha \iff H = g\cos\alpha + q\sin\alpha = \sqrt{g^2+q^2}, \tag{50}$$

This BPS state is a particular case of the *Julia-Zee* dyon (c.f. Julia und Zee (1975)), which is a natural generalization of 't Hooft's ansatz (36) to $W_0 = \frac{vq}{g}h_0(\xi)\tau_r$, where $h_0(\xi)$ satisfies the same asymptotics as $h(\xi)$.

Optimal Bogomol'nyi bound for dyons: Going back to the general case $H/v \geq g\cos\alpha + q\sin\alpha$, we will now show that the constraint $\tan\alpha = q/g$ for the BPS dyon defines the optimal bound of the dyon mass. To this end, we rewrite the bound in the complex (q, g) plane as:

$$g\cos\alpha + q\sin\alpha = \operatorname{Re} e^{-i\alpha}(g + iq). \tag{51}$$

The angle α thus parameterizes the orbit of $q + ig$ in the complex plane and the bound of H, being the real part, is maximized when it is equal to the norm $|g + iq| = \sqrt{g^2 + q^2}$. We have thus obtained the *general Bogomol'nyi bound* for $\mathfrak{su}(2)$ gauge theory:

$$H \geq v\sqrt{g^2 + q^2}. \tag{52}$$

3 Topological classification of monopoles

After a detailed analysis of the Georgi-Glashow model $\mathfrak{g} = \mathfrak{su}(2)$, $\rho = \mathrm{adj}$, we now return to the general setup of §1. The *moduli space of vacua* is defined as

$$\mathcal{M} := \{\phi \in \mathbb{V}_\rho \,|\, U(\phi) = 0\}\,. \tag{53}$$

In the case of spontaneous symmetry breaking to a subgroup $H \subset G$ any solution to $U(\phi) = 0$ must be invariant under H: $\rho(h)\phi = \phi$, $\forall h \in H$. If we assume that the action of G on $\mathcal{M}$ is *transitive*, then we can map a single solution $U(\phi_0) = 0$ to all other solutions with a group transformation, and $\mathcal{M}$ is a *symmetric space*:

$$\mathcal{M} = \{\rho(g)\phi_0 \,|\, g \in G\} \cong G/H. \tag{54}$$

Note that we can find explicit choices of $\mathfrak{g}, \rho, U$ for which the transitivity assumption fails, see e.g. (Goddard und Olive, 1978, p. 1402).

In addition to $U(\phi) = 0$ a minimal energy monopole solution (16) must satisfy

$$D_\mu\phi = \partial_\mu\phi + \rho(A_\mu)\phi \overset{r\to\infty}{\approx} 0 \tag{55}$$

The most general solution to this equation is a map $S^2 \mapsto G/H$, and its topology is classified by

$$\pi_2(G/H) \cong \pi_1(H). \tag{56}$$

To label solutions ϕ by elements of $\pi_1(H)$, we use the transitivity assumption to write the scalar field as

$$\phi|_{x^3\geq 0} \overset{r\to\infty}{\approx} \rho(g_+(\theta,\varphi))\phi_0, \quad \phi|_{x^3\leq 0} \overset{r\to\infty}{\approx} \rho(g_-(\theta,\varphi))\phi_0, \tag{57}$$

where $g_\pm$ is only singular at $x^3 = \mp r$. Applying the gauge transformation $g_\pm(\theta, \varphi)^{-1}$ to the connection in the regions $x^3 \lessgtr 0$, we get

$$A_\pm = g_\pm^{-1} A\, g_\pm + g_\pm^{-1} \mathrm{d} g_\pm = \pm \frac{1 \mp \cos\theta}{2} \mathrm{d}\varphi\, N, \tag{58}$$

where $N = \frac{1}{4\pi} \int_{S^2} F$ is proportional to the magnetic charge of the gauge field. Furthermore,

$$0 = \rho(g_\pm)^{-1} D\phi = \rho(A_\pm)\phi_0 = 0 \iff N \in \mathfrak{h}, \tag{59}$$

where $\mathfrak{h}$ is the Lie algebra of H. Then in the overlap region $x^3 = 0$ we get

$$A_- = hA_+h^{-1} + h\mathrm{d}h^{-1}, \quad h(\varphi) = e^{\varphi N} \in H, \tag{60}$$

and the Dirac quantization condition amounts to imposing that

$$h : S^1 \mapsto H \in \pi_1(H) \iff e^{2\pi N} = 1. \tag{61}$$

Examples:

- In §2, we had $G = \mathrm{SO}(3)$ or $\mathrm{G} = \mathrm{SU}(2)$, and $H = \mathrm{U}(1)$, with $\pi_1(H) = \mathbb{Z}$.
- If G is of rank r, then we can break the symmetry down to $\mathrm{U}(1)^n$, $n \leq r$, with $\pi_1(\mathrm{U}(1)^n) = \mathbb{Z}^n$.
- For higher rank gauge groups, we can consider symmetry breaking to $H = \mathrm{SU}(2)$ or $H = \mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$. In the former case, there are no monopoles because $\pi_1(\mathrm{SU}(2)) = \{1\}$, while in the latter case, there is one non-trivial monopole configuration coming from the second element of $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$.
- More generally, if $\pi_1(\tilde{H}) = \{1\}$ and $H := \tilde{H}/Z(\tilde{H})$, then $\pi_1(H) = Z(\tilde{H})$. This applies to e.g. $\tilde{H} = \mathrm{SU}(N)$, $Z(\mathrm{SU}(N)) = \mathbb{Z}_N$.

For more examples, see e.g. (Preskill, 1984, §5), or (Coleman, 1983, p. 57).

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