Scattering in quantum electromagnetodynamics

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1 Motivation

- Quantum gravity

No global symmetries conjecture of quantum gravity implies that the magnetic 1-form symmetry of electrodynamics must be broken in the UV: $dF \neq 0$. If the conjecture is true, there exist magnetic monopoles in nature, although they can be very heavy.

- Spectacular effects

It is known that the interactions of electrically charged fermions with a scalar magnetic monopole are not well-defined in the framework of quantum mechanics. In particular, the Hamiltonian of the problem fails to be Hermitian at the position of the monopole. For 't Hooft-Polyakov monopoles, this circumstance is known to be associated with the Rubakov-Callan process which is decay of the proton at a strong interaction rate in the presence of magnetic monopole.

- Yang-Mills dynamics

In the Abelian 't Hooft gauges, the non-Abelian gauge symmetry of the Yang-Mills theory is reduced to the number of Abelian subgroups. The gauge fixing procedure introduces singularities which turn out to be magnetically charged under the surviving $U(1)$ gauge groups. Understanding the dynamics of these magnetic monopoles can shed some light on the dynamics of the Yang-Mills theory and QCD in particular.

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2 Basic features of scattering

2.1 Non-perturbativity

Dirac quantization condition [1] tells us that for any particle with magnetic charge g, the following identity holds:

$$eg = 2\pi n\,, \quad n \in \mathbb{Z}\,. \tag{1}$$

Since the unit of electric charge is small $e \simeq 0.3$, the interaction between a given electric and a given magnetic charge as well as the interaction between any two given magnetic charges are strong. This means that the perturbation theory is of no help while calculating the scattering amplitudes involving magnetically charged particles. The scattering theory has to be built differently. For an illustration, let us go half a century back in time and consider the work by D. Zwanziger [2], where he finds a local Lagrangian for quantum electromagnetodynamics (QEMD)[1]:

$$\begin{aligned}\mathcal{L} \;=\; \frac{1}{2n^2}\,\big\{[n\cdot(\partial\wedge B)]\cdot[n\cdot(\partial\wedge A)^d] \;-\; [n\cdot(\partial\wedge A)]\cdot[n\cdot(\partial\wedge B)^d] \;-\\ [n\cdot(\partial\wedge A)]^2 \;-\; [n\cdot(\partial\wedge B)]^2\big\} \;-\; j_e\cdot A \;-\; j_m\cdot B \;+\; \mathcal{L}_G\,,\end{aligned} \tag{2}$$

where j_e and j_m are electric and magnetic currents, respectively, A and B are vector potentials of the electromagnetic field, n is a fixed four-vector and $\mathcal{L}_G$ is the gauge-fixing part:

$$\mathcal{L}_G \;=\; \frac{1}{2n^2}\left\{[\partial\,(n\cdot A)]^2 + [\partial\,(n\cdot B)]^2\right\}\,. \tag{3}$$

Due to the presence of the fixed vector n in its formulation, this Lagrangian is not Lorentz-invariant! As it was shown both in path integral formalism [3, 4] and in a toy model with perturbative magnetic charge [5], one has to take into account all quantum corrections to recover Lorentz-invariance of the theory. Although formal Feynman rules can be formulated, the tree-level (or any finite order) amplitudes do not make sense in this case, since they are not even Lorentz-invariant. Lagrangian approach is not useful for studying the scattering of electric and magnetic charges.

An option which is left and which has been successfully pursued in the literature is to assume that the magnetic charge is static, i.e. that electric particles scatter on the classical electromagnetic field created by the heavy magnetic monopole. In this case, it is convenient to make use of the axial symmetry of the problem and expand the scattering amplitude as a series of partial waves, each of which corresponds to the fixed value of total angular momentum j.

2.2 Non-locality

Again, let us start with the Dirac quantization condition (1). It tells us that the presence of only one magnetic monopole anywhere in the Universe forces all electric charges to be quantized. One can already suspect that some kind of non-locality is at play here. This property also arises in the existing field-theoretical constructions of QEMD. While Zwanziger theory Lagrangian (2) is local, the connection between the two four-potentials A and B describing the electromagnetic field in this formulation is non-local. In particular, their commutators are proportional to $(n\cdot\partial)^{-1}$ operator:

$$[A^\mu(t,\vec{x}), B^\nu(t,\vec{y})] = i\epsilon^{\mu\nu}{}_{\rho 0}\, n^\rho\,(n\cdot\partial)^{-1}(\vec{x}-\vec{y})\;, \tag{4}$$

$$[A^\mu(t,\vec{x}), A^\nu(t,\vec{y})] = [B^\mu(t,\vec{x})\,, B^\nu(t,\vec{y})] = -i\,({g_0}^\mu n^\nu + {g_0}^\nu n^\mu)\,(n\cdot\partial)^{-1}(\vec{x}-\vec{y})\;. \tag{5}$$

Another formulation of QEMD due to J. Schwinger [6] reveals its non-local properties already at the level of the theory Hamiltonian.

So why is QEMD essentially non-local? To answer this question, let us investigate a system of particles with charges (e_i, g_i) which move in the asymptotically distant past or future [7]. Such particles can be approximated as moving along the straight lines $\lim_{t\to\pm\infty} x_i^\mu(t) = u_i^\mu t$ with the constant four-velocity u_i^μ. The electromagnetic field which is spatially far from particle trajectories can be calculated classically. Solving the Maxwell equations

$$\partial_\mu F^{\mu\nu} = j_e^{\,\nu}\,, \tag{6}$$

$$\partial_\mu F^{d\,\mu\nu} = j_m^{\,\nu}\,, \tag{7}$$

[1]The notations we use are: $(a\wedge b)^{\mu\nu} = a^\mu b^\nu - a^\nu b^\mu$, $(a\cdot G)^\nu = a_\mu G^{\mu\nu}$, $G^d_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}/2$.

one obtains

$$\lim_{t\to\pm\infty} F(x) = F^{\text{free}}(x) + \int d^4y\, G(x-y)\left(\partial\wedge j_e - (\partial\wedge j_m)^d\right) , \tag{8}$$

where $G(x-y)$ is the Green function (retarded or advanced) for the wave equation, $F^{\text{free}}(x)$ is a solution of the free Maxwell equations. Due to the simplicity of the motion of the particles in the asymptotic region, the integral in the right-hand side of Eq. (8) is nothing more than the well-known Liénard-Wiechert solution for uniformly moving electric charge plus the dual solution for the magnetic charge:

$$\lim_{t\to\pm\infty} F(x) = F^{\text{free}}(x) + \frac{1}{4\pi}\sum_i \frac{e_i(x\wedge u_i) - g_i(x\wedge u_i)^d}{\left[(x\cdot u_i)^2 - x^2\right]^{3/2}} . \tag{9}$$

The source contribution to F vanishes as t^{-2}, so that the corresponding energy-momentum tensor $T_{\mu\nu}$ scales as t^{-4}. Since the volume scales as t^3, total energy and momentum associated to the charges vanish for $t\to\pm\infty$, however there is a finite contribution to the angular momentum tensor:

$$M^{\mu\nu} = \lim_{t\to\pm\infty}\int d^3x\, \left(x^\mu T^{\nu 0} - x^\nu T^{\mu 0}\right) , \tag{10}$$

where $T = (F\cdot F + F^d\cdot F^d)/2$. Let us keep only the source contribution to F since we are interested in the self-fields of the charges. After a straightforward calculation, one arrives at the following expression for the asymptotic angular momentum tensor:

$$M^{\mu\nu} = \sum_{i>j} \pm\frac{e_i g_j - e_j g_i}{4\pi}\, \frac{\epsilon^{\mu\nu}{}_{\kappa\lambda} u_i^\kappa u_j^\lambda}{\left[(u_i\cdot u_j)^2 - 1\right]^{3/2}} , \tag{11}$$

where the sum is over all possible pairs of particles. The term for each pair comes with the plus sign if the pair is incoming (in-state) and with the minus sign if the pair is outgoing (out-state). In the non-relativistic limit, summing over all pairs of particles (i,j) with the relative velocities $\vec{v}_{ij}$ one obtains:

$$M^{0k} = 0 , \quad \vec{J} = \sum_{i>j} \pm\frac{e_i g_j - e_j g_i}{4\pi}\, \frac{\vec{v}_{ij}}{|\vec{v}_{ij}|} , \tag{12}$$

where $J_k = \epsilon_{klm} M^{lm}/2$ is angular momentum. The angular momentum does not depend on the values of the relative speeds and agrees with the expression derived in the first workshop seminar for the static system of electric and magnetic charges.

The presence of extra angular momentum in the in- and out-states associated to the pairs of particles implies that these pairs are quantum mechanically entangled: the quantum state of the pair cannot be reduced to the product of one-particle states. This non-locality is connected to the failure of the QFT Lagrangian methods in providing the scattering matrix: note that the usage of quantum fields for calculating scattering processes is motivated by the cluster decomposition principle [8], which is basically the principle of locality. This means that for the calculation of the electric-magnetic S-matrix we have to resort to on-shell methods.

2.3 No crossing symmetry

The absence of the usual crossing symmetry for the scattering in QEMD follows immediately from the non-locality we have just discussed. Since each given pair of particles (i,j) in the in- or out-state is entangled as long as $e_i g_j - e_j g_i \neq 0$, one cannot CPT-transform one particle from the pair without affecting the other. Moreover, all the incoming pairs have different sign in front of their additional angular momenta compared to all the outgoing pairs. This means that no single particle from the in-state can be transferred to the out-state without modifying the amplitude unless all particles in the in- and out-states are either purely electric or purely magnetic.

3 Pairwise helicity

3.1 Two-particle irreducible representations of the Poincaré group

The asymptotic properties of in- and out-states can be inferred from the analysis of the irreducible projective unitary representations of the Poincaré group [9]. Normally, in- and out-states of the S-matrix approach the product of states transforming under one-particle irreducible representations of the Poincaré group. In fact, this is the basis of the normal "electric" QFT, where the particles are defined by these representations. For example, any state of a single massive particle in the reference frame where it is at rest generically has extra degrees of freedom associated to the rotations in the $SU(2)$ double covering of the $SO(3)$ rotation group. All the massive particles are then parameterized not only by their mass, but also by their spin s which fixes the $(2s+1)$-dimensional representation of the Poincaré group. Similarly, it can be derived that the massless particles are parameterized by an extra degree of freedom called helicity which is an integral multiple of $1/2$.

Let us apply Wigner's method to the particle states in QEMD. Allowed extra degrees of freedom will follow from the little group transformations, i.e. the Poincaré group actions which do not affect particle momenta. Wigner classified all the little groups for one-particle states. Let us now follow Zwanziger [7] and consider multi-particle states, too. An arbitrary two-particle state can be transformed into COM frame (or simply the frame where momenta are collinear) via a Lorentz transformation:

$$|p_1, p_2\rangle \quad \longrightarrow \quad |k_1, k_2\rangle \,, \tag{13}$$

so that $\vec{k}_1 \upuparrows \vec{k}_2$. It is now easy to see that the little group which preserves both momenta is the $U(1)$ subgroup of the Poincaré group: both momenta are invariant under rotations around the axis directed along $\vec{k}_1$. This means that one can associate an extra helicity with each electric-magnetic pair which is called pairwise helicity q_{ij}. Pairwise helicity parameterizes a finite-dimensional representation of the little group and thus is an integral multiple of $1/2$. Under a generic Lorentz transformation Λ, any two-particle state must transform as follows:

$$U(\Lambda)|p_1, p_2; \sigma_1, \sigma_2; q_{12}\rangle \;=\; e^{iq_{12}\phi_{12}}\, D_{\sigma_1'\sigma_1}(W_1)\, D_{\sigma_2'\sigma_2}(W_2)\, |\Lambda p_1, \Lambda p_2; \sigma_1', \sigma_2'; q_{12}\rangle \,, \tag{14}$$

where σ_1, σ_2 are little group parameters (spins, their projections or helicities) of the one-particle states $|p_1\rangle$ and $|p_2\rangle$, respectively; W_1 and W_2 are the corresponding little groups and $D(W_i)$ are their representations. Extension of the transformation law (14) to generic multi-particle states is straightforward. Indeed, there is no subgroup of the Poincaré group that would leave more than 2 momenta invariant, which means that a generic n-particle state transforms as a product of n one-particle states and $n(n-1)/2$ two-particle states.

3.2 Pairwise helicity in QEMD

Let us now connect the pairwise helicity q_{12} to the observables in QEMD. Consider a pair of massive spinless particles with charges (e_1, g_1) and (e_2, g_2) in the in- or out-state. In the frame where the first particle is at rest $p_1 = (m, 0, 0, 0)$ and the second is moving along the z-axis $p_2 = (\sqrt{m_2^2 + \vec{p}^2}, 0, 0, p)$, the expression for non-vanishing component of the angular momentum tensor derived in sec. 2.2 is

$$M_{12} = \pm \frac{e_1 g_2 - e_2 g_1}{4\pi} \,, \tag{15}$$

where plus sign is for in-state and minus sign for the out-state. Thus, under the rotation around the z-axis the two-particle state transforms as follows:

$$|p_1, p_2\rangle \quad \longrightarrow \quad e^{iM_{12}\phi_{12}}\, |p_1, p_2\rangle \,. \tag{16}$$

Comparing this transformation with the transformation given by eq. (14), we obtain the value for the pairwise helicity in terms of electric and magnetic charges:

$$q_{12} = \pm \frac{e_1 g_2 - e_2 g_1}{4\pi} \,. \tag{17}$$

Note that we have derived the Dirac quantization condition once again, since the representation theory restricts pairwise helicity q_{12} to the integral multiples of $1/2$:

$$q_{12} = \frac{n}{2} \quad \Rightarrow \quad e_1 g_2 - e_2 g_1 = 2\pi n \,, \quad n \in \mathbb{Z} \,. \tag{18}$$

4 Electric-magnetic S-matrix

4.1 Transformation properties of the S-matrix

Now that we determined the Lorentz transformations of in- and out-states of electric-magnetic scattering, we can infer the corresponding transformations of the S-matrix:

$$S(p'_1,\dots,p'_m|p_1,\dots,p_n) \equiv \langle p'_1,\dots,p'_m;-|p_1,\dots,p_n;+\rangle =$$
$$\langle p'_1,\dots,p'_m;-|U(\Lambda)^\dagger\, U(\Lambda)|p_1,\dots,p_n;+\rangle =$$
$$e^{i(\Sigma_+ + \Sigma_-)}\prod_{i=1}^{m}\mathcal{D}(W_i)^\dagger \prod_{i'=1}^{n}\mathcal{D}(W_{i'})\; S(\Lambda p'_1,\dots,\Lambda p'_m|\Lambda p_1,\dots,\Lambda p_n)\,, \tag{19}$$

where

$$\Sigma_+ \equiv \sum_{i>j}^{n}\mu_{ij}\phi_{ij}\,, \qquad \Sigma_- \equiv \sum_{i'>j'}^{m}\mu_{i'j'}\phi_{i'j'}\,, \tag{20}$$

(i,j) and (i',j') are indices for incoming and outgoing particles, respectively, and $\mu_{ab} = (e_a g_b - e_b g_a)/4\pi$. Thus, S-matrix transforms as follows:

$$S(\Lambda p'_1,\dots,\Lambda p'_m|\Lambda p_1,\dots,\Lambda p_n) = e^{-i(\Sigma_+ + \Sigma_-)}\prod_{i=1}^{m}\mathcal{D}(W_i)\prod_{i'=1}^{n}\mathcal{D}(W_{i'})^\dagger\, S(p'_1,\dots,p'_m|p_1,\dots,p_n)\,. \tag{21}$$

4.2 Helicity amplitude method

The transformation law (21) restricts the explicit form of the S-matrix in each particular process. To see what these restrictions are, it is convenient to work with spinor-helicity variables, since they transform under the little group in a simple way. S-matrix depends on momenta p_i which transform in the $(1/2,1/2)$ representation of the Lorentz group (note the isomorphism $so(1,3)\simeq su(2)\oplus su(2)$). This means that every momentum can be represented as a bispinor:

$$p^{\alpha\dot\alpha} = \sigma_\mu^{\alpha\dot\alpha}p^\mu = \begin{pmatrix} p^0-p^3 & -p_1+ip_2 \\ -p_1-ip_2 & p_0+p_3\end{pmatrix}\,, \tag{22}$$

where $\sigma_\mu = (1,\vec\sigma)$. Note that $\det p^{\alpha\dot\alpha} = m^2$. For massless particles, $\det p^{\alpha\dot\alpha} = 0$, which ensures that one can decompose the momentum matrix into the product of two spinors:

$$p^{\alpha\dot\alpha} = \lambda^\alpha\tilde\lambda^{\dot\alpha} \equiv p\rangle[p\,. \tag{23}$$

For massive particles, the momentum can be represented as rank two matrix, which means that it can be decomposed into the sum of two rank one matrices as follows:

$$p^{\alpha\dot\alpha} = \lambda_I^\alpha\tilde\lambda^{\dot\alpha\,I} \equiv p_I\rangle[p^I\,, \quad I=1,2\,. \tag{24}$$

In the spinor-helicity variables, we will denote the momentum of the ith particle p_i simply by i. Then our building blocks for the S-matrix are $i\rangle$, $j_I\rangle$, $[i$ and $[j^I$, where i runs over all the massless particles and j runs over all the massive ones. Under the action of the corresponding little groups, these spinors transform in a simple way:

$$i\rangle \;\to\; e^{i\phi/2}|i\rangle\,, \quad [i \;\to\; e^{-i\phi/2}[i\,, \quad j_I\rangle \;\to\; W_K^I|j_I\rangle\,, \quad [j^I \;\to\; (W^\dagger)_I^K[j^I\,, \tag{25}$$

where $W\in SU(2)$. For simplification, let us work in the out-out framework, in which all of the particles in the process are formally considered outgoing. This is possible if we always keep in mind which particles are really incoming and do not pair particles one of which is in the in-state and another is in the out-state. Helicities and spins of the particles in the scattering process put a constraint on how many helicity spinors of each type we are supposed to have in the S-matrix. For example, if the particle 1 is massless and has helicity h_1, we will have n spinors $1\rangle$ and m spinors $[1$, such that $m-n=2h_1$. The massive particle of spin s enters the S-matrix as a symmetric rank $2s$ tensor which forms an irreducible spin-s representation of the $SU(2)$ group. For this reason, the S-matrix is always symmetrized over I indices. Keeping this in mind, we will omit them and write massive spinors in bold, like $\mathbf{i}\rangle$.

Let us first consider an example which involves only electrically charged particles. In this case, we already have all the helicity spinors necessary to construct the S-matrix. Our example will be a massive particle decaying into the two massless particles. Suppose that the massive particle has spin 1/2. The S-matrix can then be given by the following expressions:

$$S \propto \langle \mathbf{1}2\rangle[23]^n\langle 23\rangle^m \quad \text{or} \quad [\mathbf{1}2][23]^n\langle 23\rangle^m \quad \text{or} \quad 2 \leftrightarrow 3\,, \tag{26}$$

from which we infer that precisely one of the massless particles must be a fermion. In general, it is clear that using this method one can find no-go theorems (or selection rules) for various processes. In particular, using the same setup of one massive and two massless particles, one can prove that a massive spin one particle cannot decay to a pair of photons and that a massive spin three particle cannot decay to a pair of gravitons [10].

4.3 Electric-magnetic spinor helicity variables

Let us now adapt this method to the electric-magnetic scattering [11]. In this case, we have to account for the extra little group transformations associated to the pairwise helicities. Since these transformations are $U(1)$ rotations, similar to the massless particle case, we have to introduce new spinor helicity variables associated with null momenta. The required null momenta must be linear combinations of the particle momenta in the pair, so that they have the same Lorentz transformation properties. It is especially easy to build the reference null momenta in the COM reference frame:

$$\left(k_{ij}^{\pm}\right)^{\mu} = p_c\,(1,0,0,\pm 1)\,, \tag{27}$$

where p_c is the COM momentum of the (i,j) pair. Boosted to any other frame, the null momenta are given by the following covariant expressions in terms of the particle momenta:

$$p_{ij}^{+} = \frac{1}{E_i^c + E_j^c}\left[\left(E_j^c + p_c\right)p_i - \left(E_i^c - p_c\right)p_j\right]\,, \tag{28}$$

$$p_{ij}^{-} = \frac{1}{E_i^c + E_j^c}\left[\left(E_i^c + p_c\right)p_j - \left(E_j^c - p_c\right)p_i\right]\,. \tag{29}$$

Now that we defined the pairwise null momenta, it is straightforward to define the corresponding spinor helicity variables, in full analogy to the case of the massless particles discussed above:

$$\left(p_{ij}^{\pm}\right)^{\alpha\dot{\alpha}} = \sigma_{\mu}^{\alpha\dot{\alpha}}\left(p_{ij}^{\pm}\right)^{\mu} = p_{ij}^{\pm}\rangle[p_{ij}^{\pm}\,. \tag{30}$$

Their little group transformations are given by the following $U(1)$ rotations:

$$p_{ij}^{\pm}\rangle \;\to\; e^{\pm i\phi/2}|p_{ij}^{\pm}\rangle\,, \quad [p_{ij}^{\pm} \;\to\; e^{\mp i\phi/2}[p_{ij}^{\pm}\,. \tag{31}$$

Note that $p_{ij}^{+}\rangle$ and $p_{ij}^{-}\rangle$ have opposite pairwise helicities $\pm 1/2$. Explicit contractions with the other spinors in the massless limit are:

$$\left[p_{ij}^{+}i\right] = \langle i p_{ij}^{+}\rangle = [\eta_i p_{ij}^{-}] = \langle p_{ij}^{-}\eta_i\rangle = 0\,, \tag{32}$$

$$\left[p_{ij}^{-}i\right] = \langle i p_{ij}^{-}\rangle = \sqrt{2p_c}[\eta_i p_{ij}^{+}] = \sqrt{2p_c}\langle p_{ij}^{+}\eta_i\rangle = 2p_c\,, \tag{33}$$

where η_i are Parity-conjugate massless spinors which appear in the massless limit of the massive spinors $\mathbf{i}$ [10]. Note that in constructing the S-matrix we require that the helicity weights under each individual particle as well as the pairwise helicity weights are matched for both the initial and the final states, since only the diagonal Lorentz transformation for which each particle and each pair of particles are transformed simultaneously is physical.

4.4 Selection rules

Let us construct the S-matrix of a massive vector decaying to two different massless fermions with the pairwise helicity $\mu_{23} = -1$. For the massive vector, we need two spinors $\mathbf{1}\rangle$. For the massless fermions, it is enough to take spinors $2\rangle$ and $3\rangle$. Now, there are four spinor indices from the normal spinors which need to be contracted with the pairwise spinors. In total, we should have 4 pairwise spinors, three of which should have negative helicity and one positive, since we require that $\mu_{23} = -1$. The scattering matrix for positive helicity fermions is then:

$$S\left(\mathbf{1}^{s=1}|2^{-1/2}, 3^{-1/2}\right)_{\mu_{23}=-1} \sim \langle 2p_{23}^{-}\rangle\langle p_{23}^{+}3\rangle\langle \mathbf{1}p_{23}^{-}\rangle^2 \,, \tag{34}$$

up to a little group invariant. We also see that the decay to the different helicity fermions $h_2 = -h_3 = 1/2$ is forbidden in this case, since $[p_{23}^{-}3] = 0$.

Now let us consider the same example where the pairwise helicity of the massless fermions is $\mu_{23} = -2$. In this case, the situation is the opposite: the S-matrix for the same helicity fermions vanishes, while for the different helicity fermions it does not vanish:

$$S\left(\mathbf{1}^{s=1}|2^{-1/2}, 3^{+1/2}\right)_{\mu_{23}=-2} \sim \langle 2p_{23}^{-}\rangle[p_{23}^{+}3]\langle \mathbf{1}p_{23}^{-}\rangle^2 \,, \tag{35}$$

up to a little group invariant. Same helicity fermions are forbidden in this case since $\langle p_{23}^{-}3\rangle = [p_{23}^{+}2] = 0$.

Finally, let us deal with a more general case of a massive particle decaying into two other massive particles. The S-matrix is a contraction of the massive part:

$$\left(\langle\mathbf{1}|^{2s_1}\right)^{\{\alpha_1\ldots\alpha_{2s_1}\}}\ \left(\langle\mathbf{2}|^{2s_2}\right)^{\{\beta_1\ldots\beta_{2s_2}\}}\ \left(\langle\mathbf{3}|^{2s_3}\right)^{\{\gamma_1\ldots\gamma_{2s_3}\}} \tag{36}$$

with a massless part involving the pairwise spinors:

$$S^q_{\{\alpha_1\ldots\alpha_{2s_1}\}\{\beta_1\ldots\beta_{2s_2}\}\{\gamma_1\ldots\gamma_{2s_3}\}} = \sum_{i=1}^{C} a_i\left(|p_{23}^{-}\rangle^{\hat{s}-\mu_{23}}\ |p_{23}^{+}\rangle^{\hat{s}+\mu_{23}}\right)_{\{\alpha_1\ldots\alpha_{2s_1}\}\{\beta_1\ldots\beta_{2s_2}\}\{\gamma_1\ldots\gamma_{2s_3}\}} \,, \tag{37}$$

where $\hat{s} = s_1 + s_2 + s_3$ and C counts all the ways to group the spinors into α, β, γ indices. Since we cannot have negative powers of pairwise spinors, a selection rule follows:

$$|q| \leq \hat{s} \,, \tag{38}$$

which restricts the charges and spins of individual particles.

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