Seiberg-Witten theory

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Contents

1 Premises

The discussion that follows stems from the work of Seiberg and Witten exposed in the two seminal papers [1] and [2]. Here I will follow mostly the description given in the lecture notes by Bertolini [3] and those by Bilal ([4] and most of all [5]). An additional read can be found at [6]. There are also two video lectures by Witten himself on the topic [7, 8].

I will try and introduce as little knowledge of Supersymmetry as possible in the following. However, some general and some more specific concepts are obligatory and I will introduce them when needed. For starters, let me immediately give away the punchline of the whole discussion:

> *the presence of monopoles and dyons in the spetrum allows the determination of the exact non-perturbative low-energy effective Lagrangian of the $\mathcal{N} = 2$ supersymmetric Yang-Mills theory with gauge group $SU(2)$.*

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This theory is the supersymmetric generalization of the usual Yang-Mills one, and it contains, on top of the gauge bosons, two Weyl fermions and a complex scalar.

2 Supersymmetry basics

As it is known, supersymmetry is a symmetry that relates fermionic and bosonic degrees of freedom. It can be generated by postulating that, in addition to the Poincaré generators of spacetime translations P_μ and rotations $M_{\mu\nu}$, and possibly internal symmetry generators, there exist a set of additional anticommuting fermionic generators Q^I_α, $\bar{Q}^I_{\dot{\alpha}}$ transforming in the $\left(\frac{1}{2},0\right)$ and the $\left(0,\frac{1}{2}\right)$ representation of the Lorentz group, respectively[1]. Here α and $\dot{\alpha}$ are Dirac indices and $I,\ J = 1,\ldots,\mathcal{N}$ indicate how many of these new generators we add. From here on, we will restrict to $\mathcal{N}=2$, which is the case we are interested in. Then, one must add to the Poincaré algebra the following (anti-)commutation relations[2]:

$$\left[P_\mu, Q^I_\alpha\right] = \left[P_\mu, \bar{Q}^I_{\dot{\alpha}}\right] = 0 \tag{1}$$

$$\left[M_{\mu\nu}, Q^I_\alpha\right] = i\left(\sigma_{\mu\nu}\right)_\alpha{}^\beta Q^I_\beta \tag{2}$$

$$\left[M_{\mu\nu}, \bar{Q}^{I\dot{\alpha}}\right] = i\left(\bar{\sigma}_{\mu\nu}\right)^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{I\dot{\beta}} \tag{3}$$

$$\left\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\right\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{IJ} \tag{4}$$

$$\left\{Q^1_\alpha, Q^2_\beta\right\} = 2\sqrt{2}\varepsilon_{\alpha\beta} Z \tag{5}$$

$$\left\{\bar{Q}^1_{\dot{\alpha}}, \bar{Q}^2_{\dot{\beta}}\right\} = 2\sqrt{2}\varepsilon_{\dot{\alpha}\dot{\beta}} Z^* . \tag{6}$$

The ones we care about the most, here, are the last three. In particular, taking a state with mass m in its rest frame, $P_\mu = (m,0,0,0)$, Eq. (4) becomes

$$\left\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\right\} = 2m\,\delta_{\alpha\dot{\beta}}\delta^{IJ} \ . \tag{7}$$

Then one can define

$$a_\alpha = \frac{1}{\sqrt{2}}\left(Q^1_\alpha + \varepsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right) \tag{8}$$

$$b_\alpha = \frac{1}{\sqrt{2}}\left(Q^1_\alpha - \varepsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right) \ , \tag{9}$$

[1]This is the only way to elude the Coleman-Mandula theorem [9] stating that, under very generic assumption, (locality, causality, positivity of energy, finiteness of number of particles, etc...), the only possible symmetries of the S-matrix are, besides C, P, and T , the Poincaré group in direct product with internal symmetries. See also [10].

[2]here $\sigma^\mu = \left(\sigma^0, \sigma^i\right)$, $\bar{\sigma}^\mu = \left(\sigma^0, -\sigma^i\right)$, where $\sigma^0_{\alpha\dot{\beta}} = \delta_{\alpha\dot{\beta}}$ and σ^i are the Pauli matrices, and $\left(\sigma^{\mu\nu}\right)_\alpha{}^\beta = \frac{1}{4}\left(\sigma^\mu_{\alpha\dot{\gamma}}\left(\sigma^\nu\right)^{\dot{\gamma}\beta} - (\mu\leftrightarrow\nu)\right)$, $\left(\bar{\sigma}^{\mu\nu}\right)^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{4}\left(\left(\bar{\sigma}^\mu\right)^{\dot{\alpha}\gamma}\sigma^\nu_{\gamma\dot{\beta}} - (\mu\leftrightarrow\nu)\right)$ are $(-i$ times) the infinitesimal generators of Lorentz rotations on the left- and right-handed Weyl spinor representations, respectively.

satisfying the oscillator algebra

$$\left\{a_\alpha, a_\beta^\dagger\right\} = 2(m + \sqrt{2}|Z|)\delta_{\alpha\beta} \tag{10}$$

$$\left\{b_\alpha, b_\beta^\dagger\right\} = 2(m - \sqrt{2}|Z|)\delta_{\alpha\beta} \ , \tag{11}$$

with the other combinations vanishing. Eq. 11 implies[3]

$$m \geq \sqrt{2}|Z| \ . \tag{13}$$

The states we are interested in are those for which the bound is saturated, i.e. $m = \sqrt{2}|Z|$. These are called BPS states[4]. In this case, the b generators are trivially realized, and we are only left with a_α and $a_\alpha^\dagger$. The action of $a^\dagger$ is to raise the spin by 1/2 while a lowers it by the same quantity. Then, we can define a so called Clifford vacuum $|m, s\rangle$ as the state with mass m and spin s that is annihilated by a_α, $\alpha = 1, 2$. Because of the anticommutation relation between two $a^\dagger$ vanishes, we can act with either of them only once. Then, if we start with a spin 0 Clifford vacuum, we can produce the following multiplet:

$$\left(0, \frac{1}{2}, \frac{1}{2}, 1\right) \ . \tag{14}$$

Actually, CPT requires that we add by hand the multiplet obtained starting with a spin -1 state, so that the actual result is

$$\left(0, \frac{1}{2}, \frac{1}{2}, 1\right) \underset{CPT}{\oplus} \left(-1, -\frac{1}{2}, -\frac{1}{2}, 0\right) \ . \tag{15}$$

In the massless case, the creation-annihilation operators are defined in a slightly different way, but following similar steps one can show that half of them is trivially realized and, starting from a Clifford vacuum with helicity 0, we obtain the same multiplet as in Eq. (15). Obviously, there are a lot of other possibilities other than Eq. (15). For example, another multiplet we will encounter is the so called hypermultiplet, obtained by starting with a spin $-\frac{1}{2}$ state, and corresponding to

$$\left(-\frac{1}{2}, 0, 0, +\frac{1}{2}\right) \underset{CPT}{\oplus} \left(-\frac{1}{2}, 0, 0, +\frac{1}{2}\right) \ . \tag{16}$$

As we can see, the hypermultiplet contains two Weyl fermions and two complex scalars.

[3] This can be seen as follows: taking the expectation value of the 11 or 22 component of (11) (the only non vanishing ones) on a state $|\psi\rangle$, we obtain

$$\langle\psi|b_\alpha b_\alpha^\dagger + b_\alpha^\dagger b_\alpha|\psi\rangle = \langle\psi|b_\alpha b_\alpha^\dagger|\psi\rangle + \langle\psi|b_\alpha^\dagger b_\alpha|\psi\rangle = \|b_\alpha^\dagger\,|\psi\rangle\|^2 + \|b_\alpha\,|\psi\rangle\|^2 \geq 0 \ . \tag{12}$$

[4] named after Bogomol'nyi–Prasad–Sommerfield, also see Jeremy's talk.

3 $\mathcal{N} = 2$ $SU(2)$ SuperYM Lagrangian

The generalization of the $SU(2)$ Yang-Mills Lagrangian to $\mathcal{N} = 2$ SUSY takes the following form:

$$\mathcal{L} = \frac{1}{g^2}\,\mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{32\pi^2}g^2 F_{\mu\nu}\tilde{F}^{\mu\nu} + \overline{D_\mu\phi}D^\mu\phi - \frac{1}{2}\left[\phi,\bar{\phi}\right]^2 + \right.$$
$$\left. -i\lambda\sigma^\mu D_\mu\bar{\lambda} - i\psi\sigma^\mu D_\mu\bar{\psi} + i\sqrt{2}\left[\bar{\phi},\psi\right]\lambda + i\sqrt{2}\left[\phi,\bar{\lambda}\right]\bar{\psi}\right] \ . \tag{17}$$

The main features of this Lagrangian can be summarized as follows:

- As advertised, it contains a gauge field $A_\mu = A^a_\mu T^a$, where T^a are the $SU(2)$ generators, a complex scalar ϕ^a, and two Weyl fermions λ^a and ψ^a. As supersymmetry commutes with internal symmetries, and all of the fields belong to the same SUSY multiplet, they have to belong to the same representation of the gauge group, too. That is why the scalar field as well as the two fermions belong to the adjoint representation, justifying the trace that must be taken for all terms of the Lagrangian.

- Moreover, as SUSY exchanges the different field within themselves, in order to respect it the Lagrangian must be expressed in terms of only two couplings, the gauge coupling g and the anomalous coefficient θ. In the SUSY literature, these two are usually packed in a single complex coefficient τ, defined as:

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \ . \tag{18}$$

- In addition, this theory enjoys a $U(1)$ global symmetry, the R-symmetry, that is the abelian part of the $U(2)$ symmetry that rotates the two SUSY generators $Q^{1,2}$. The charge assignments can be deduced from the Yukawa potentials, and can be normalized so that $R(\lambda,\psi) = 1$ and $R(\phi) = 2$. Because it couples to Weyl fermions, this symmetry is anomalous, and it is broken to $\mathbb{Z}_8$.

- Lastly, the β-function of the gauge coupling turns out to be negative, and the theory enjoys asymptotic freedom.

Now, as showed in the lecture by Jeremy, an $SU(2)$ gauge theory with a charged scalar ϕ with scalar potential

$$V = \frac{\lambda}{4}\left(\mathrm{Tr}\,\phi^2 - a^2\right)^2 \ , \tag{19}$$

undergoes a Higgs mechanism that breaks $SU(2) \to U(1)$ and admits solitons solutions which carry monopole and/or dyonic charge under the residual $U(1)$. This is the Georgi-Galshow model. The $\mathcal{N} = 2$ $SU(2)$ SYM in

Eq. (17) has the same boson content as such a model, and its scalar potential $V \propto \left[\phi, \bar{\phi}\right]^2$ is minimized whenever ϕ is in the Cartan sublagebra of $SU(2)$, i.e. $\langle\phi\rangle = a\sigma_3$ for any (complex) value of a. This vev breaks, too, $SU(2) \to U(1)$, and our scope is the study of the effective theory describing the low energy modes after the breaking. In particular, we expect the spectrum to contain monopole and dyon solutions. Using Noether's theorem, one can actually compute the electric and magnetic charges under the unbroken $U(1)$, which turn out to be:

$$q = -\frac{1}{ag}\int \mathrm{d}x^3 \partial_i \left(F_a^{0i}\phi^a\right) = g n_e\,, \quad p = -\frac{1}{ag}\int \mathrm{d}x^3 \partial_i \left(\tilde{F}^{a0i}\bar{\phi}^a\right) = \frac{4\pi}{g} n_m\ , \tag{20}$$

where the second equalities for both equations are just the consequence of Dirac's quantization condition[5]:

$$qp = 4\pi n\ . \tag{21}$$

The fundamental observation is that such charges can be related to the Z charge. Indeed, as the Q_α operators are the generators of supersymmetry, they can, too, be computed as the spatial integral of the time component of the conserved current of SUSY transformation, called supercurrent. Explicitly, one can then show that:

$$\left\{Q^1_\alpha, Q^2_\beta\right\} = \frac{2\sqrt{2}}{g^2}\varepsilon_{\alpha\beta}\int \mathrm{d}x^3 \partial_i \left[\left(F^{a0i} - i\tilde{F}^{a0i}\right)\bar{\phi}_a\right] \equiv 2\sqrt{2}\varepsilon_{\alpha\beta} Z \tag{22}$$

$$\left\{\bar{Q}^1_{\dot{\alpha}}, \bar{Q}^2_{\dot{\beta}}\right\} = \frac{2\sqrt{2}}{g^2}\varepsilon_{\dot{\alpha}\dot{\beta}}\int \mathrm{d}x^3 \partial^i \left[\left(-F_{a0i} + i\tilde{F}_{a0i}\right)\bar{\phi}_a\right] \equiv 2\sqrt{2}\varepsilon_{\dot{\alpha}\dot{\beta}} Z^* \tag{23}$$

and finally

$$\mathrm{Re}\, Z = a n_e \quad \mathrm{Im}\, Z = a\tau n_m\ . \tag{24}$$

Now, if $Z \neq 0$, the bound in Eq. (13) implies the presence of massive states in the spectrum. Semiclassically, one can show that the soliton solutions in the spectrum do satisfy the bound, meaning that they have a mass

$$m = \sqrt{2}|a(n_e + \tau n_m)|\ . \tag{25}$$

Then, they need to satisfy it also in the full quantum theory. Indeed, in the region where the semiclassic expansion is valid, these state are annihilated by the b ladder operators. Being an algebraic condition, this property cannot be spoiled by quantum correction, otherwise spurious additional degrees of

[5]the presence of a factor of 4π instead of the usual 2π is justified by the fact that the unit charge here is actually $g/2$, which is the charge a field in the fundamental representation of $SU(2)$ would have under the unbroken $U(1)$

freedom would appear in the spectrum. However, this does not mean that the quantities within Eq. (25) do not individually undergo renormalization. Including quantum corrections, indeed, we can generalize Eq. (25) to

$$m = \sqrt{2}|an_e + a_D n_m| \ , \tag{26}$$

which we can interpret as the definition of the quantities a and a_D. In the classical limit, the former is just the vev of the field ϕ, while the latter is reduced to $a_D = \tau a$.

4 Low energy theory

After the symmetry is broken we will have to deal with an effective theory. However, the theory is still $\mathcal{N} = 2$ supersymmetric, a feature that already imposes strong constraints on what the Lagrangian can look like. Let us look at which are the fields that can appear. As the theory is Higgsed, the only surviving gauge boson is the one associated with the T^3 generator, resulting in an abelian gauge theory. In a similar way, and as can be straightforwardly deduced by the shape of the Lagrangian in Eq. (17), the non-zero $\langle\phi\rangle$ gives a mass to the components along $T^{1,2}$ of the fermions λ and ψ, while the same components of the scalar ϕ are eaten by the gauge bosons. In the end, we are left with the fields $\{A^3_\mu\, , \phi^3\, , \psi^3\, , \lambda^3\}$. $\mathcal{N} = 2$ SUSY requires that all of the couplings can still be expressed in terms of a single complex number $\tau(a)$, so that the Lagrangian is:

$$\mathcal{L}_{\text{eff}} = \text{Im}\left[\tau(a)\left(\frac{1}{2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \partial_\mu\bar{\phi}\partial^\mu\phi\right) + \text{fermionic contributions}\right] \ , \tag{27}$$

where[6] $\mathcal{F}_{\mu\nu} = F_{\mu\nu} + \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ and $\tau(a) = \frac{\theta}{2\pi} + \frac{4\pi i}{g(a)^2}$. The fermionic pieces play no role here and we will ignore them from now on. Knowing $\tau(a)$ for all values of a would then amount to solve the theory completely, and this is indeed what we are after. Actually, as we are dealing with an effective field theory, one can show that the Lagrangian in (27) is but the expansion up to second order in the fields of the whole expression. Still, $\mathcal{N} = 2$ SUSY fixes all the couplings so that they can be derived in terms of a single function $\mathcal{F}(a)$, called *prepotential*, which is crucially holomorphic. In particular,

$$\tau(a) \equiv \frac{\partial^2\mathcal{F}(a)}{\partial\phi\partial\phi} \ . \tag{28}$$

It is more convenient to parametrize the space of vacua, which is usually referred to as the *moduli space*, through the coordinate:

$$u \equiv \frac{1}{2}\left\langle \text{tr}\,\phi^2\right\rangle \ . \tag{29}$$

[6]this is just a fancy way to rewrite $\mathcal{L}_{EM} \propto -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}$ as can be checked by expanding explicitly

In the classical region, $u \sim a^2$, while in the full quantum moduli space this relation will be modified

$$u = a^2 + \text{quantum correction} \ . \tag{30}$$

This coordinate is invariant under the Z_2 symmetry $\phi \to -\phi$ which represents the action of the Weyl group of $SU(2)$ and which is left unbroken. Now, we are armed with all the tools to study the structure of the moduli manifold. First, one can show that $\mathcal{F}(a)$ cannot be a holomorphic function for all values of a. Indeed, if it were, it would mean that $\mathrm{Im}\,\tau(a) = \mathrm{Im}\,\partial^2\mathcal{F}(a)\big/\partial\phi\partial\phi$ is a harmonic function. As such, it could not be positive everywhere, unless it were a constant (which we will show is not the case). So, there would be regions of the moduli space where $\mathrm{Im}\,\tau(a)$ would be negative, making the effective gauge coupling g^2 negative, too. This would imply the propagation of negative norm states, which is unacceptable. The only way out lies in the possibility that $\mathcal{F}(a)$ is defined only locally, i.e. there exist different local description in different patches in the values of a. For example, we will have some description in the classical region where $u \to \infty$, while a different one is needed for small u. Because of asymptotic freedom, these regions correspond to larger values of g, eventually infinite on some singularities. The study of these singularities will guide us towards the structure of the whole moduli space. To understand how different local descriptions can emerge, we now have to introduce the notion of electric-magnetic duality in this effective theory. Defining $\phi_D \equiv \partial\mathcal{F}/\partial\phi$ we can rewrite Eq. (27) as

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\,\mathrm{Im}\left[\tau(a)\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right] + \frac{1}{2}\partial_\mu \begin{pmatrix}\phi_D\\ \phi\end{pmatrix}^\dagger J\partial^\mu \begin{pmatrix}\phi_D\\ \phi\end{pmatrix} \quad \text{where} \quad J = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix} \ . \tag{31}$$

The scalar kinetic term is invariant under transformations that leave the matrix J invariant, i.e.

$$\begin{pmatrix}\phi_D\\ \phi\end{pmatrix} \to M \begin{pmatrix}\phi_D\\ \phi\end{pmatrix} \quad \text{where} \quad M^\dagger J M = J \ . \tag{32}$$

This is just $Sp(2,\mathbb{R})$, the continuous version of the duality group defined above, and it is generated by

$$S = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \quad \text{and} \quad T_b = \begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} \ . \tag{33}$$

Now we need to see how this group acts on the Maxwell term. First, let us introduce a Lagrangian multiplier $A_{D\mu}$. We can now write the Maxwell term as

$$\mathcal{S} = \int \mathrm{d}x^4\, \mathrm{Im}\left[\frac{1}{2}\tau(a)\left(F_{\mu\nu} + i\tilde{F}_{\mu\nu}\right)^2\right] + \int \mathrm{d}x^4\, A_{D\mu}\partial_\nu F^{\mu\nu} \ , \tag{34}$$

where the path integral is now done over A_D and F. Indeed, integrating over A_D imposes $\partial_\nu F^{\mu\nu}$, which requires $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and brings us again to the usual Maxwell term with path integral over A_μ. However, we can first integrate over F instead. By integration by parts, the second term in Eq. (34) becomes $F_{D\mu\nu}F^{\mu\nu}$, where $F_{D\mu\nu} \equiv \partial_\mu A_{D\nu} - \partial_\nu A_{D\mu}$ is the field strength corresponding to $A_{D\mu}$. Adding and subtracting $\mathrm{Im}\left[\frac{1}{2\tau(a)}\left(F_{D\mu\nu} + i\tilde{F}_{D\mu\nu}\right)^2\right]$ we can complete the square

$$\mathcal{S} = \int \mathrm{d}x^4\, \mathrm{Im}\left[\frac{1}{2}\tau(a)\left\{\left(F_{\mu\nu} + i\tilde{F}_{\mu\nu}\right) + \frac{1}{\tau(a)}\left(F_{D\mu\nu} + i\tilde{F}_{D\mu\nu}\right)^2\right\}^2 + \right.$$
$$\left. - \frac{1}{2\tau(a)}\left(F_{D\mu\nu} + i\tilde{F}_{D\mu\nu}\right)^2\right] \; . \tag{35}$$

Finally we can perform the path integral over F to get

$$\mathcal{S} = \int \mathrm{d}x^4\, \mathrm{Im}\left[-\frac{1}{2\tau(a)}\left(F_{D\mu\nu} + i\tilde{F}_{D\mu\nu}\right)^2\right] \; . \tag{36}$$

So the action of S, which amounts to the exchange $A_\mu \leftrightarrow A_{D\mu}$, acts on the coupling $\tau(a)$ as $\tau(a) \to \tau_D = -\frac{1}{\tau(a)}$. This suggests which are the patches that we should use to describe the theory: when u is large, we can exploit asymptotic freedom and safely take the classical approximation, where the theory is described by the fields A_μ and ϕ and the vev a is a good coordinate, while for small values of u we can get a perturbative description in terms of the fields $A_{D\mu}$ and ϕ_D and the vev a_D. T_b does not act on the fields but on the coupling only, by shifting the θ angle by $2\pi b$. In order for it to be a symmetry, we need $b \in \mathbb{Z}$, so the group reduces to $SL(2,\mathbb{Z})$, and we define for later $T \equiv T_1$. Let us make a final remark. Imagine we add a hypermultiplet with electric charge n_e to this theory. Recall that a hypermultiplet contains two complex scalar fields, which we dub $h_{1,2}$. Now, $\mathcal{N} = 2$ SUSY would dictate that its coupling to the complex scalar in the vector multiplet be, too, proportional to n_e, i.e. its coupling to the vector gauge boson A^3_μ:

$$\mathcal{L}_{\text{eff}} \supset \sqrt{2} n_e h_1 \phi h_2 \; . \tag{37}$$

Then, after the symmetry breaking, this induces a mass term for the hypermultiplet $m = \sqrt{2} a n_e$, which means this state satisfies the BPS bound at the classical level. With a similar argument as before, as the vacuum respects $\mathcal{N} = 2$ SUSY, this means that the bound is satisfied at the quantum level, too. If we had a magnetic monopole with magnetic charge n_m, instead, and we applyied a S duality transformation, we would get, in the theory now described by the dual field ϕ_D, a term

$$\mathcal{L}_{\text{eff}} \supset \sqrt{2} n_m h_1 \phi_D h_2 \; . \tag{38}$$

with mass $m = \sqrt{2} a_D n_m$ with, most importantly $a_D = \partial \mathcal{F}/\partial a$.

Thus, on a vector (a_D, a), a generic duality $A \in Sp(2, \mathbb{Z})$ acts as $(a_D, a) \to A \cdot (a_D, a)$. Since masses are physical observables, Eq. (26) has to be left invariant by the action of A, meaning that the vector of charges (n_m, n_e) must transform as $(n_m, n_e) \cdot A^{-1}$. Consistently, then, an S transformation acts by

$$S : (n_m, n_e) \to (-n_e, n_m) \ , \tag{39}$$

and a T transformation as

$$T : (n_m, n_e) \to (n_m, n_e - n_m) \ . \tag{40}$$

Notice that the latter is consistent with the Witten effect[7].

5 Singularities and monodromies

We are now in the position to fully understand the singularity structure of the moduli space. In particular, we want to obtain the functions $a(u)$ and $a_D(u)$, and invert one of them to plug into the other and obtain $\mathcal{F}(a)$ and more importantly $\tau(a) = \partial^2 \mathcal{F}(a)/\partial a \partial a$. Let us start by looking at the semiclassical region $u \to \infty$, where we can safely use the classical relation $u = a^2$ and the one loop expression for[8] $\mathcal{F}$:

$$\mathcal{F}_{\text{1-loop}} = \frac{i}{2\pi} a^2 \log \frac{a^2}{\Lambda^2} \ , \tag{41}$$

where Λ is the strong coupling scale. A poweful non-renormalization theorem ensures that $\mathcal{N} = 2$ SUSY is one-loop exact, so that the perturbative contributions are all captured by Eq. (41). Then, from $a_D = \partial \mathcal{F}/\partial a$, we can get

$$a_D = \frac{i}{\pi} a \left(\log \frac{a^2}{\Lambda^2} + 1 \right) \ . \tag{42}$$

We can now take a counterclockwise contour in the u-plane, $u \to e^{2\pi i} u$ with very large $|u|$. As in this region $u = a^2$, we see that this corresponds to $a \to -a$. For a_D, instead

$$a_D \to \frac{i}{\pi}(-a) \left(\log \frac{e^{2\pi i} a^2}{\Lambda^2} + 1 \right) = -a_D + 2a \ . \tag{43}$$

So this transformation acts on the vector (a_D, a) as

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to M_\infty \begin{pmatrix} a_D \\ a \end{pmatrix} \quad \text{with} \quad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \ . \tag{44}$$

[7] see Quentin's first lecture

[8] this result has been obtained by Seiberg in [11]

Notice that $M_\infty \in Sp(2,\mathbb{Z})$ as expected, and in particular $M_\infty = -T_{-2}$. A matrix M of this kind is called *monodromy* around the loop. This shows that there is a singularity at $u = \infty$, which must be matched by other isolated singularities somewhere else on the plane, with corresponding matrices M_i so that

$$M_\infty = M_1 M_2 \dots M_k \ , \tag{45}$$

since the contour at infinity can be deformed to encircle all of the other singularities.

What do these matrices look like? Singularities in the moduli space correspond to states becoming massless. For example, in the classical theory, we have a singularity in $u = 0$ that corresponds to the gauge bosons $A_\mu^{2,3}$ becoming massless. This singularity signals that the effective theory is no longer valid. In this example, this is because to have a correct description at $u = 0$ we need a theory where these particles must be included as degrees of freedom. Quantum effects, however, may move the singularities away from the origin. Nonetheless, we have a way to get a hang on what they look like. We already showed that, given a singularity with magnetic and electric charges (n_m, n_e), a monodromy matrix M acts on it as

$$(n_m, n_e) \to (n_m, n_e) M^{-1} \ . \tag{46}$$

Now, the monodromy matrix around a singularity has to be characterized by the singularity itself. More specifically, it must leave the vector (n_m, n_e) of the singularity invariant, i.e. (n_m, n_e) has to be one of its eigenvectors with unit eigenvalue. Requiring $M \in Sp(2,\mathbb{Z})$ together with the latter condition fixes

$$M(n_m, n_e) = \begin{pmatrix} 1 + 2n_m n_e & 2n_e^2 \\ -2n_m^2 & 1 - 2n_m n_e \end{pmatrix} \ . \tag{47}$$

We then have to solve Eq. (45) with matrices in the shape of (47). To this end, the last step consists in counting how many singularities the moduli space under consideration has.

As mentioned, the global R-symmetry is broken by the fermion charges down to Z_8, transforming ϕ as[9] $\phi \to e^{\pi n/2}\phi$. Thus, on a generic point in the moduli space where $u \sim \mathrm{tr}\big[\phi^2\big] \neq 0$, this symmetry is broken down to Z_4. However, the theory has to be invariant under the full Z_8, meaning that the physics has to be the same sending $u \to -u$. This implies that, except for the fixed points 0 and ∞, the singularities must come in pairs. So the least one can do is put a singularity at $u = 0$, which is also what is suggested by the classical theory. However, this would mean that, in shrinking the loop from around infinity to around 0, we encounter no other singularity, and

[9] Remember that ϕ has $R-$charge=2

$M_0 = M_\infty$. However, M_∞ does not change a^2 (see Eq. (44)), so a^2 is not affected by any monodromy and it is a good coordinate on the whole moduli space. However, this would mean that we need just one patch to describe the moduli space, in contrast with what was proven earlier.

The next simplest case is the one where we have three singularities, ∞ and $\pm u_0$. Notice that in the limit $\Lambda \to 0$, a always lies in the classical region (i.e. $a \gg \Lambda$ always) and one should recover the classical theory with only two singularities at $u = 0$ and $u = \infty$. Thus, $u_0 \propto \Lambda^2$, and we can safely rescale so that $u_0 = \Lambda^2$.

What are the states that become massless here? A non-zero value for u introduces a characteristic scale in the theory. If it were the gauge bosons to become massless, we would recover the full $SU(2)$ gauge group. However, since $\mathcal{N} = 2$ SUSY is conserved on the moduli space, the Lagrangian of the theory there would still be the one in Eq. (17), which has no characteristic scale.

The only other states in the theory, at least classically, are monopoles and dyons. For two singularities, the condition in Eq.(45) becomes

$$M_\infty = M_{\Lambda^2} M_{-\Lambda^2} \ . \tag{48}$$

Requiring $M_{\pm\Lambda^2}$ to be of the form in Eq. (47) one finds the solutions:

$$M_{\Lambda^2} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \ , \quad M_{-\Lambda^2} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \ , \tag{49}$$

corresponding to two states with charges

$$(n_m, n_e) = \pm(1, 0) \ , \quad (n'_m, n'_e) = \pm(1, -1) \tag{50}$$

So we finally see the nature of the states becoming massless at these singularities: a monopole with charge $\pm(1, 0)$ and a dyon with charge $\pm(1, -1)$. Incidentally, one can prove that solving Eq. (45) with matrices of the form of Eq. (47) with $k > 2$ is not possible.

Now, an holomorphic function is defined univocally by its singularities. Thus, by looking for expressions with singularities in $\pm u_0$ and at ∞, Seiberg and Witten found the two functions:

$$a(u) = \frac{\sqrt{2}}{\pi} \int_{-\Lambda^2}^{\Lambda^2} \mathrm{d}x \, \frac{\sqrt{x-u}}{\sqrt{x^2 - \Lambda^4}} \tag{51}$$

$$a_D(u) = \frac{\sqrt{2}}{\pi} \int_{-\Lambda^2}^{u} \mathrm{d}x \, \frac{\sqrt{x-u}}{\sqrt{x^2 - \Lambda^4}} \tag{52}$$

Inverting Eq. (51) to get $u(a)$ and plugging it into Eq. (52) we obtain the dependence of $a_D(a)$, whence we can get the expression for $\tau(a) = \frac{\mathrm{d}a_D(a)}{\mathrm{d}a}$! It is worth checking what these expression look like close to the singularities.

- For $u \to \infty$ we have $a \sim \sqrt{u}$ and $a_D \sim \frac{i}{\pi}\sqrt{u}\log\frac{u}{\Lambda^2} \sim \frac{i}{\pi}a\log\frac{a^2}{\Lambda^2}$, reproducing the semiclassical result. One can check that there is no value of (n_m, n_e) that gives a vanishing mass, in agreement with the fact that no additional particle becomes massless for $u \to \infty$.

- For $u \to \Lambda^2$ we have $a \sim \frac{i}{\pi}a_D\log\frac{a_D^2}{\Lambda^2}$ and $a_D \sim (u - \Lambda^2)$. Close to Λ^2, a is singular while a_D vanishes. This is the correct behavior for a magnetic monopole of charge n_m becoming massless at $u = \Lambda^2$

- For $u \to -\Lambda^2$ we have $a - a_D \sim (u + \Lambda^2)$ and $a \sim \frac{i}{\pi}(a_D - a)\log\frac{a_D - a}{\Lambda}$. This means that at $u = \Lambda^2$ we have a singularity when $a_D = a$, meaning that a particle with $n_m = -n_e$ becomes massless here, again in agreement with the dyon we found before.

Finally, one can check that for no values of u the vev a vanishes, showing that the point $a = 0$ is not part of the moduli space anymore, and nowhere in the moduli space extra massless gauge bosons arise.

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