Lecture V:
EM Duality in String Theory

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Contents

1 Basics of String Theory

1.1 Particle Spectrum

We begin with an extremely abridged review of the particle spectra present in the low energy effective actions of critical superstring theory.

There are 5 distinct 10D superstring theories: Type IIA, Type IIB, $E_8\times E_8$ Heterotic, SO(32) Heterotic, and Type I. In this lecture, we will be concerned with the second of these theories, but we will start quite general.

The superstring is a 1D object that maps out a 2D surface - the **worldsheet** - as it evolves in time. A classical string configuration is determined by its spacetime coordinates $X^M(t,\sigma)$. Here t and σ are coordinates on the worldsheet. One can think of the X^M coordinates as bosonic fields living on the 2D worldsheet. For the superstring, we supplement these fields with worldsheet spinor fields $\Psi^M(t,\sigma) = (\psi_L^M(t,\sigma), \psi_R^M(t,\sigma))$. String dynamics are then defined by introducing an action for the $\{X^M, \psi_L^M, \psi_R^M\}$. We will not go into such detail here, but merely state that the relevant equations of motion are

$$\partial_+\partial_- X^M(t,\sigma) = 0 \tag{1}$$

$$\partial_-\psi_L(t,\sigma) = 0 \tag{2}$$

$$\partial_+\psi_R(t,\sigma) = 0 \tag{3}$$

where we have introduced left- & right-moving coordinates $\sigma^\pm = t \pm \sigma$ & $\partial_\pm = \partial_t \pm \partial_\sigma$. These equations of motion explain the subscript convention for the spinors - $\psi_{L/R}$ is a left/right moving worldsheet spinor. We can also decompose $X^M(t,\sigma) = X_L(\sigma^+) + X_R(\sigma^-)$.

To define the Type II theories, we introduce equal amounts of left- & right-moving fields with $M = 0,..,9$. The heterotic theories have different numbers of left- and right-moving fields.

The equations of motion must be supplemented with boundary conditions. We can consider two types of strings - closed strings and open strings. Closed strings have the obvious boundary condition

$$X^M(t, \sigma + \pi) = X^M(t, \sigma) \tag{4}$$

For open strings, one has two choices

- Neumann:

  $$\partial_\sigma X^M|_{\sigma=0,\pi} = 0 \tag{5}$$

  This condition implies no momentum is flowing out of the string endpoints.

- Dirichlet:

  $$X^N|_{\sigma=0,\pi} = X_0^N \tag{6}$$

  $$\tag{7}$$

  Where we pick some p such that $N = 1, .., 9 - p$. This implies that the string endpoints are fixed to move on some p-dimensional surface. This clearly breaks the Poincare invariance. The surfaces are the Dp-branes of string theory.

Closed worldsheet fermions have two choices of boundary conditions:

- Neveu-Schwarz (NS) - or anti-periodic:

  $$\psi^M_{L/R}(t, \sigma + \pi) = -\psi^M_{L/R}(t, \sigma) \tag{8}$$

- Ramond (R) - or periodic:

  $$\psi^M_{L/R}(t, \sigma + \pi) = \psi^M_{L/R}(t, \sigma) \tag{9}$$

Importantly, *boundary conditions for left/right moving sectors can be chosen independently*. There are corresponding NS & R boundary conditions for open string worldsheet fermions, but we will not need them.

Using these boundary conditions, one can expand the coordinates $\{X^M, \psi^M_L, \psi^M_R\}$ in terms of oscillators. After quantizing the theory, the creation operators for these modes define quantum string states. The massless states from this procedure then define the particle content of the effective quantum field theory.

There is one additional wrinkle - to obtain a well-defined theory, a projection operation must be enacted on the string states (the **GSO projection**). We will not go into detail on this procedure except to say that there are two inequivalent projections, leading to the two Type II theories.

We consider the bosonic massless spectrum of the Type II's. This is consists of the R-R and NS-NS sectors:

The **NS-NS sector** is identical between the Type II's - it consists of

- The graviton $g_{\mu\nu}$
- The 2-form B_2
- The dilaton ϕ

The vacuum expectation value of the dilaton ϕ sets the coupling constant of the string theories: $g_s = \langle e^{\phi} \rangle$.

The **R-R sector** of the Type II's consists of rank-(p+1) antisymmetric tensor fields (**(p+1)-forms**) C_{p+1}. These are gauge potentials that generalize the 1-form potential $A_1 = A_M dx^M$ of electromagnetism. In terms of coordinate components,

$$C_{(p+1)} = \frac{1}{(p+1)!} C_{M_1 \cdots M_{p+1}} dx^{M_1} \wedge \cdots \wedge dx^{M_{p+1}} \tag{10}$$

Similar to the EM field strength $F = dA_1$, the (p+1)-forms have (p+2)-form field strengths $F_{p+2} = dC_{p+1}$. Since the exterior derivative operator satisfies $d^2 = 0$, the field strengths are invariant under gauge transformations $C_{p+1} \to C_{p+1} + d\Lambda_p$.

The precise p-form content differs between the two Type II's - IIA has only odd-form potentials and IIB has only even-form potentials.:

- IIA: C_1 & C_3
- IIB: C_0, C_2, & C_4

The above bosonic field content is supplemented with fermions to furnish two distinct $\mathcal{N} = 2$ 10D supergravity theories. But there is a small puzzle - none of states defined above are charged under the p-form potentials, so what are they sourced by?

1.2 Non-perturbative Spectrum

To answer the lingering question from the previous section, let us first recall some basics of electromagnetism. The EM 1-form potential A_1 is sourced by charged particles, so to the Maxwell action we add an interaction term

$$S \supset e \int A_1 = e \int_W A_\mu \frac{dx^\mu}{dt} dt \tag{11}$$

where t is the proper time along the charged particle's worldline. Note that we integrate over a 1D surface - the worldline. The generalization of this to our (p+1)-form potentials is

$$S \supset Q_p \int_{W_{p+1}} C_{p+1} = \frac{1}{(p+1)!} \int C_{M_1 \cdots M_{p+1}} \frac{\partial x^{M_1}}{\partial \sigma^0} \cdots \frac{\partial x^{M_{p+1}}}{\partial \sigma^p} d^{p+1}\sigma \tag{12}$$

The integration is over a (p+1)-dimensional "worldvolume", implying that the source of the C_{p+1} is an object with p spatial dimensions - *these are the **Dp-branes** of string theory*!

From the R-R spectrum in the previous section, we deduce that the Type II's have distinct Dp-brane states:

- IIA: D0-& D2-branes
- IIB: D(-1)-, D1-, & D3-branes

For $p > 1$, the Dp-branes are extended membranes. The D1-brane is a type of string that we will refer to as the **D-string**. Similarly, henceforth we will call the fundamental string of IIB the **F-string**. The D0-brane is a 0-dimensional object - a particle. The D(-1)-brane appears a bit odd - it has total spacetime dimension 1+(-1) = 0. This is an object that is localized in space as well as time - it is an instanton, the **D-instanton**.

The Dp-branes are non-perturbative (half-)BPS states in the Type II theories. Their tensions are given by

$$T_p = \frac{1}{g_s (2\pi)^p (\alpha')^{(p+1)/2}} \tag{13}$$

One can compare this to the tension of the F-string is

$$T_F = \frac{1}{2\pi\alpha'} \tag{14}$$

1.3 Electric & Magnetic Charges

The Dp-branes introduced in the previous section are *electrically* charged under the C_{p+1} potentials. The question we want to answer now is - are there objects that are *magnetically* charged under the C_{p+1}?

Once again, we turn to electromagnetism to motivate the answer. Gauss' Laws in the presence of monopoles read

$$\begin{aligned} g &= \int_{S^2} \vec{B} \cdot d\vec{S} \\ e &= \int_{S^2} \vec{E} \cdot d\vec{S} \end{aligned} \tag{15}$$

These can be written in a more covariant language using the differential form version of the equation of motion:

$$dF = \star J_m \tag{16}$$

$$d \star F = \star J_e \tag{17}$$

Where $J_{e(m)}$ is the electric (magnetic) current 1-form with components $J_\mu = (\rho, \vec{j})$. The **Hodge star operator** $\star$ is an operator that maps p-forms to (D-p)-forms in D-dimensions. In 4D, it simply takes a 2-form into a 2-form. Integrating the above (and using the generalized Stokes theorem) gives

$$\begin{aligned} g &= \int_{S^2} F \\ e &= \int_{S^2} \star F \end{aligned} \tag{18}$$

In 10D, the Hodge star maps the (p+2)-form field strengths F_{p+2} to (8-p)-forms $F_{8-p} = \star F_{p+2}$. Integrating over a (8-p)-dimensional sphere that encloses the Dp-brane in the transverse (9-p) space, we have

$$Q_p = \int_{S^{8-p}} F_{8-p} = \int_{S^{8-p}} \star F_{p+2} \tag{19}$$

Generalizing the other electromagnetism equation, we determine the magnetic charge enclosed in a (p+2)-dimensional sphere as

$$\tilde{Q}_{6-p} = \int_{S^{p+2}} F_{p+2} \tag{20}$$

In 10D, a (p+2)-dimensional sphere can surround a (6-p)-dimensional brane, hence the notation for the magnetic charge. Thus we deduce that *the magnetic dual of a Dp-brane is a D(6-p)-brane*! Here we are have motivated these branes from the standpoint of electric-magnetic duality, but they do exist within the Type II theory. We can now make the follow table of potentials and charged states for IIB:

C_{p+1}	Electric	Magnetic
C_0	D(-1)-brane	D7-brane
C_2	D-string	D5-brane
C_4	D3-brane	D3-brane
B_2	F-string	NS5-brane

Note that the 5-form field strength F_5 is self-dual under the Hodge star: $F_5 = \star F_5$, so the D3-brane carries a self-dual charge and is electrically and magnetically charged under C_4. We have also added an entry for B_2. The F-string is electrically charged under B_2 and the magnetic dual is a different object called an **NS5-brane**. NS5-branes are not D-branes and in fact their tension scales as $T_{NS5} \sim g_s^{-2}$. These are actually solitons.

We end this section by noting that the charge quantization argument of Dirac can be generalized to the Dp-branes to give

$$Q_p \tilde{Q}_{6-p} \in 2\pi\mathbb{Z} \tag{21}$$

2 Self-Duality of Type IIB

2.1 IIB Supergravity & SL(2,$\mathbb{R}$)

Before describing the non-perturbative self-duality of IIB, we first note that the low-enery supergravity theory of IIB (dubbed IIB supergravity) has a global SL(2,$\mathbb{R}$) symmetry. In terms of the spectra outlined above, the IIB supergravity action[1] is

$$
\begin{aligned}
S_{IIB} &= S_{NS} + S_R + S_{CS} && (22)\\
S_{NS} &= \frac{1}{2\kappa_{10}} \int d^{10}x\sqrt{-G}e^{-2\phi}\Big(R + 4\partial_M\phi\partial^M\phi - \frac{1}{2}|H_3|^2\Big) && (23)\\
S_R &= \frac{1}{2\kappa_{10}} \int d^{10}x\sqrt{-G}\Big(-\frac{1}{2}|F_1|^2 - \frac{1}{2}|\tilde{F}_3|^2 - \frac{1}{4}|\tilde{F}_5|^2\Big) && (24)\\
S_{CS} &= -\frac{1}{2}\frac{1}{2\kappa_{10}}\int_{10D} C_4 \wedge H_3 \wedge F_3 && (25)
\end{aligned}
$$

We have labeled the sectors "NS"=Neveu-Schwarz, "R" = Ramond, and "CS" = Chern-Simons. Where $2\kappa_{10} = (2\pi)^7(\alpha')^4$ and

$$
\begin{aligned}
F_{p+1} &= dC_p && (26)\\
\tilde{F}_3 &= F_3 - C_0 \wedge H_3 && (27)\\
\tilde{F}_5 &= F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3 && (28)
\end{aligned}
$$

This is the action in the **String frame**. Performing a conformal rescaling, we get the action in the **Einstein frame**:

$$
\begin{aligned}
S_{IIB} = \frac{1}{2\kappa_{10}} \int d^{10}x\sqrt{-G_E}\Big(R_E - \frac{\partial_M\tau^*\partial^M\tau}{2(\mathrm{Im}(\tau)^2} - \frac{1}{2}\mathcal{M}_{ij}G_3^i \cdot G_3^j - \frac{1}{4}|\tilde{F}_5|^2\Big)\\
- \frac{\epsilon_{ij}}{8\kappa_{10}^2}\int_{10D} C_4 \wedge G_3^i \wedge G_3^j && (29)
\end{aligned}
$$

We have also repackaged the fields:

$$
\begin{aligned}
\tau &= a + ie^{-\phi} && (30)\\
\mathcal{M}_{ij} &= e^{\phi}\begin{pmatrix} |\tau|^2 & -C_0 \\ -C_0 & 1 \end{pmatrix} && (31)\\
A_2 &= \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} && (32)\\
G_3 &= \begin{pmatrix} H_3 \\ F_3 \end{pmatrix} = dA_2 && (33)
\end{aligned}
$$

[1]There is no known fully covariant action for IIB supergravity. This is due to the self-dual field strength $\tilde{F}_5$. However, one can vary the action above and then impose the self-duality constraint $\tilde{F}_5 = \star\tilde{F}_5$

This results in an action that is manifestly invariant under

$$
\begin{aligned}
\tau &\to \frac{a\tau + b}{c\tau + d} \\
\mathcal{M} &\to (\Lambda^{-1})^T \mathcal{M} \Lambda^{-1} \\
A_2^i &\to \Lambda^i_j A_2^j \\
G_3^i &\to \Lambda^i_j G_3^j \\
\tilde{F}_5 &\to \tilde{F}_5 \\
G_E &\to G_E
\end{aligned}
\tag{34}
$$

When

$$
\Lambda = \begin{pmatrix} d & a \\ b & c \end{pmatrix} \tag{35}
$$

$$
ad - bc = 1 \tag{36}
$$

thus $\Lambda \in$ SL(2,$\mathbb{R}$) and the transformations above describe the action of SL(2,$\mathbb{R}$) on the IIB supergravity fields.

We can consider a particular element of the transformation group

$$
\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{37}
$$

which sends

$$
\tau \to \tau' = -\frac{1}{\tau} \tag{38}
$$

If we set $C_0 = 0$ and using $\langle e^{\phi} \rangle = g_s$, we find

$$
g_s' = \frac{1}{g_s} \tag{39}
$$

This is a nonperturbative mapping - a duality - between strong and weak coupling. This is an example of an **S-duality**. Broadly speaking, S-duality relates a theory with coupling g_s to another theory with coupling $1/g_s$. Here we see that IIB is S-dual to itself - this is a “self-duality”.

2.2 IIB & SL(2,$\mathbb{Z}$)

Before expounding the physics of the IIB self-duality, we first refine the duality since the full continuous SL(2,$\mathbb{R}$) duality symmetry above is valid only in the classical theory. The actual duality symmetry of IIB is the SL(2,$\mathbb{Z}$) subgroup. This can be argued from several angles, including charge quantization [1] - the F-string has one unit of B_2 charge. Under a duality transformation, the F-string is transformed into a string with d units of B_2 charge. Charge quantization demands that this be an integer - the maximal subgroup of SL(2,$\mathbb{R}$) with integer d is the set of matrices

$$\begin{pmatrix} a & \alpha b \\ c/\alpha & d \end{pmatrix} \tag{40}$$

for *integers* a, b, c, d and some real constant α that can be absorbed into a redefinition of C_2. Thus we arrive at SL(2,$\mathbb{Z}$). From this point on, we will refer to the $\mathbb{Z}_2$ subgroup of SL(2,$\mathbb{Z}$) generating $g_s \leftrightarrow g_s^{-1}$ as the S-duality group while the full duality group will be simply called SL(2,$\mathbb{Z}$)[2].

As hinted at in the explanation above, the SL(2,$\mathbb{Z}$) duality symmetry of IIB does not just act on the fields, but on the states of the theory as well. In general, we do not know how the duality symmetry acts on a generic state [1]. However, we do know how it acts on BPS states, whose tensions are protected by supersymmetry and therefore are known functions for all values of g_s. We can use this information as a check on the veracity of the SL(2,$\mathbb{Z}$) duality.

Let us examine the two strings already present in the theory - the F-string and the D-string. In the Einstein frame, these have tensions

$$\begin{aligned} T_F &= \frac{\sqrt{g_s}}{2\pi\alpha'} \\ T_D &= \frac{1}{2\pi\sqrt{g_s}\alpha'} \end{aligned} \tag{41}$$

so that under S-duality $T_F \leftrightarrow T_D$. However, this is not yet satisfactory. At weak coupling, the F-string is light and defines the low energy particle

[2] The story is actually more subtle - including the fermionic sector involves extending the duality group to Mp(2,$\mathbb{Z}$), the metaplectic group [2]. Many thanks to Timo for pointing this out.

spectrum. What happens to this particle spectrum at strong coupling? It turns out that there is an exact match in the spectrum of massless states between F-strings and D-strings. We take this as evidence for the veracity of SL(2,$\mathbb{Z}$) duality - exchanging strong coupling with weak coupling means exchanging a light F-string with a light D-string, but the massless spectrum remains unchanged.

We also comment that S-duality exchanges D5-branes with the NS5-branes - in Einstein frame

$$\begin{aligned} T_{NS5} &= \frac{1}{(2\pi)^5\sqrt{g_s}(\alpha')^3} \\ T_{D5} &= \frac{\sqrt{g_s}}{(2\pi)^5(\alpha')^3} \end{aligned} \tag{42}$$

So indeed $T_F \leftrightarrow T_D$.

In the following sections, we explore the applications and implications of this duality.

3 Applications of SL(2,$\mathbb{Z}$) Duality

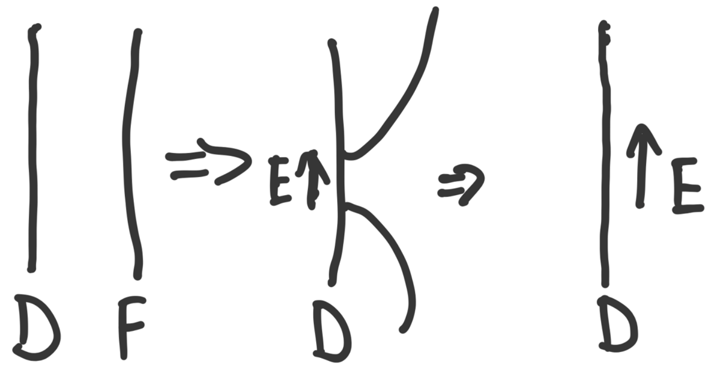
3.1 BPS states of IIB

A full SL(2,$\mathbb{Z}$) duality implies the existence of strings that are electrically charged under both B_2 & C_2 - the so-called (p, q) strings. These have tension

$$\tau_{p,q} = \frac{1}{2\pi\alpha'\sqrt{\tau_2}}|p + \tau q| \tag{43}$$

which is invariant under SL(2,$\mathbb{Z}$).

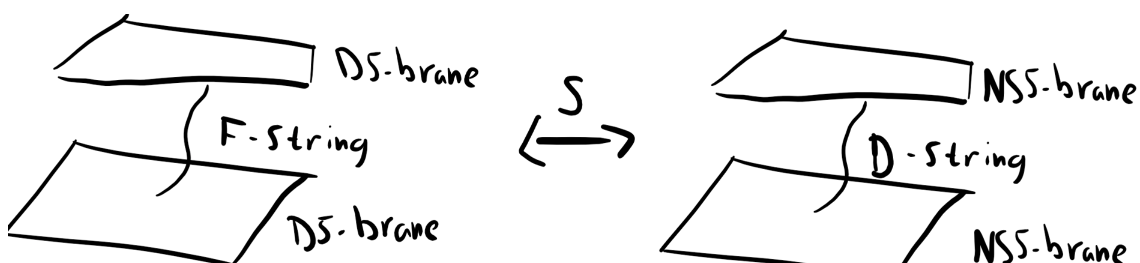
How do we interpret these states? An obvious interpretation is that these are simply states with p F-strings and q D-strings. But this is too naive - such a configuration is not supersymmetric and is unstable. This can be argued from BPS bounds. However, such a state can lower its energy. If we have a single D-string with a single F-string, then the F-string can “dissolve” in the D-string, which results in the D-string with flux.



We reproduce the relevant diagram from [3] above. This is an example of a **D-brane bound state**.

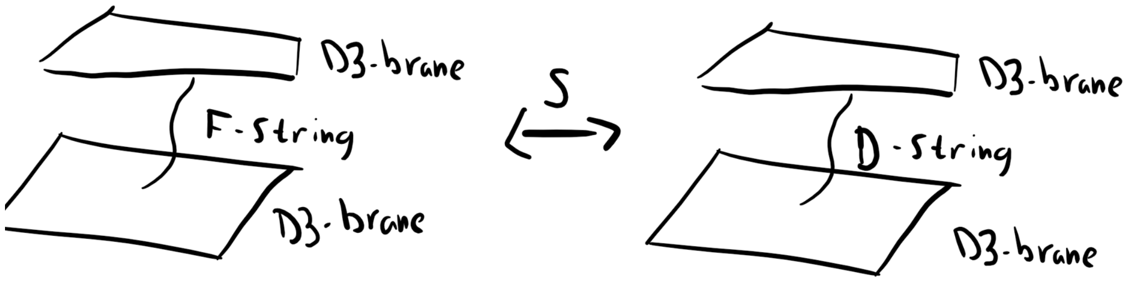
The story above generalizes to D7-branes, in which case we have (p, q) 7-branes. These play an important role in describing IIB in terms of F-theory, as discussed below.

We can also get information on allowed Dp-brane configurations from SL(2,$\mathbb{Z}$) [4, 5]. For example, by definition, the F-string can end on a pair of D5-branes:



Assuming the duality, D-strings are allowed to end on NS5-branes!

This also holds for D3-branes. The C_4 potential is invariant under SL(2,$\mathbb{Z}$) and so the D3-brane is self-dual. We are allowed to have a configuration:



So now we find D-strings can end on D3-branes!

The above are interesting and nontrivial results - F-strings can end on D-branes, but a priori it is not obvious that D-branes can end on other D-branes. However, SL(2,$\mathbb{Z}$) duality demands that this be so.

As a final comment, let us return to the D3-branes. The theory on a stack of N D3-branes is a 3+1 dimensional $\mathcal{N} = 4$ U(N) SYM theory with gauge coupling $g_{D3}^2 = 2\pi g$. Since the D3-brane is self-dual, this implies that the SL(2,$\mathbb{Z}$) is a duality of the gauge theory. This is nothing but Montonen-Olive duality of the low-energy theory! More on this below. Extending this idea, an F-string looks like a point source to the worldvolume theory of the D3-brane - this is just an electric charge for the gauge field. The S-dual picture is a D-string ending on the brane, which acts as a magnetic source.

3.2 Geometrization of SL(2,$\mathbb{Z}$): M-Theory & F-Theory

The discussion here is based on [6, 7, 8]. A simple observation is possible from the preceding sections - the duality group of IIB, SL(2,$\mathbb{Z}$), is the same[3] as the group of large diffeomorphisms[4] of a torus (or **elliptic curve**). In particular, the τ field defined above transforms exactly like the complex structure of a torus under a modular transformation.

This could simply be a coincidence, but it turns out that this is a hint for understanding IIB from M-theory and F-theory.

We first turn to 11D M-Theory. Let us compactify two of the M-theory dimensions ($\mathbb{R}^{1,10} \to \mathbb{R}^{1,8} \times \mathbf{T}^2$) and let R_1 and R_2 denote the radii of the $\mathbf{T}^2$

[3]Note that the modular group of a torus is actually PSL(2,$\mathbb{Z}$) = SL(2,$\mathbb{Z}$)/$\{\pm 1\}$). I leave the resolution of this slight mismatch as an exercise for the determined reader.

[4]Also called the modular group or mapping class group (MCG).

amd θ their relative angle. The area A and complex structure τ of the $\mathbf{T}^2$ are then

$$A = R_1 R_2 \sin\theta \tag{44}$$

$$\tau = \frac{R_2}{R_1} e^{i\theta} \tag{45}$$

If one takes the limit of vanishing area $A \to 0$ while keeping τ fixed, then the 9D M-theory grows an extra dimension. This 10D theory is the 10D IIB theory with complex coupling $\tau = C_0 + ie^{-\phi}$. This is quite incredible - the ***duality* symmetry of IIB is understood as a *geometric* symmetry in M-theory**. Namely, the modular symmetry of the M-theory torus.

F-theory takes this interpretation even further. One way to view F-theory is as IIB with D7-branes. As described above, D7-branes are magnetically charged under the 0-form potential C_0 and their presence implies a spacetime dependent profile for τ. The idea of F-theory is to interpret this varying τ geometrically as a torus and its varying profile as defining an elliptic fibration[5] over the compactification manifold. Thus we arrive at the usual statements of F-theory as a 11+1D theory where two of the spatial dimensions are a $\mathbf{T}^2$. The strength of this geometric picture is that it allows one to go beyond perturbative IIB theory.

These pictures all tie together neatly - F-theory on $\mathbf{T}^2$ is the $A \to 0$ limit of M-theory on $\mathbf{T}^2$, which is 10D IIB with coupling τ. This then extends to lower dimensional compactifications. However, one should take care - there is no supergravity theory with signature (-1,11), and IIB is not the Kaluza-Klein reduced theory of F-theory on $\mathbf{T}^2$.

For more details and far better discussion, see [6, 7].

3.3 Holography & Conformal Field Theory

This section is based on the recent paper [9].

[5]Loosely speaking, one can think of a fibration as stacking one manifold (the fiber) over another (the base).

The broad[6] statement of holography is the duality

$$\text{(D+1)-dimensional gravity in AdS} \leftrightarrow \text{ D-dimensional CFT} \tag{46}$$

The quintessential example is the duality between IIB on $\mathrm{AdS}_5 \times S^5$ and 4D $\mathcal{N} = 4$ Super Yang-Mills (SYM) [17, 18, 19]. The $\mathcal{N} = 4$ SYM theory is has a complex coupling constant

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} \tag{47}$$

that enjoys an SL(2,$\mathbb{Z}$) duality symmetry

$$\gamma \cdot \tau \to \frac{a\tau + b}{c\tau + d} \tag{48}$$

This is not a surprise - the bulk dual is IIB, so this boundary symmetry is the holographic dual of IIB's SL(2,$\mathbb{Z}$) symmetry. We also argued this above from the self-duality of D3-branes under SL(2,$\mathbb{Z}$) duality. This duality implies that inequivalent values τ are points in the SL(2,$\mathbb{Z}$) fundamental domain $\mathcal{F} = \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$, where $\mathbb{H}$ is the upper half plane.

If one has a boundary SYM observable $\mathcal{O}(\lambda)$ that depends on the 't Hooft coupling $\lambda = g^2 N$, then the statement of holography is that there is a dual observable $\mathcal{O}_{SUGRA}$ in the supergravity theory such that

$$\mathcal{O}(\lambda \to \infty) = \mathcal{O}_{SUGRA} \tag{49}$$

The basic idea of [9] is quite simple - since a SL(2,$\mathbb{Z}$) duality symmetry is present, one should harness this symmetry from the outset to simplify . On the CFT side, the authors use SL(2,$\mathbb{Z}$) spectral theory to decompose SYM observables and derive several interesting results. On the holographic side, they are able to harness an ensemble of SYM theories to find a new interpretation of IIB supergravity.

3.3.1 CFT Operators

Observables in the SYM CFT are functions of the complex coupling constant τ. In particular, "non-perturbatively well-defined" observables $\mathcal{O}(\tau)$

[6] Perhaps not the broadest definition - other possibilities include attempts to extend this to dS spacetimes [10, 11] as well as more unusual dualities [12, 13, 14, 15, 16]

are modular functions:

$$\mathcal{O}(\gamma \cdot \tau) = \mathcal{O}(\tau) \tag{50}$$

and should be square-integrable with respect to the Petersson inner product

$$(f, g) = \int_{\mathcal{F}} \frac{dxdy}{y^2} f(\tau)(g(\tau))^* \tag{51}$$
$$\tau = x + iy$$

SL(2,$\mathbb{Z}$) spectral theory then gives an expansion of $\mathcal{O}(\tau)$ in terms of its average over τ and its overlaps with the real-analytic Eisenstein series $E_s(\tau)$ and the Maass cusp forms $\phi_n(\tau)$:

$$\mathcal{O}(\tau) = \bar{\mathcal{O}} + \frac{1}{4\pi i} \int_{\mathrm{Re}s=\frac{1}{2}} (\mathcal{O}, E_s) E_s(\tau) ds + \sum_{n=1}^{\infty} (\mathcal{O}, \phi_n) \phi_n(\tau) \tag{52}$$

The authors of [9] use this spectral decomposition to prove several conjectured expressions for SYM observables and draw several interesting conclusions. For example, the Fourier expansion of the above decomposition implies that the k-instanton sectors are completely determined by the k=0,1 sectors. In a sense, the instantons of SYM are redundant.

3.3.2 Ensembles

Formally, one can consider an ensemble of SYM boundary theories, where each member of the ensemble corresponds to a single point in the fundamental domain of τ. It is then possible to define averaged quantities with respect to this ensemble - for example, for an observable $\mathcal{O}(\tau)$, the ensemble average is $\langle \mathcal{O} \rangle$. An incredible result of [9] is the statement that

$$\mathcal{O}_{SUGRA} = \langle \mathcal{O} \rangle \tag{53}$$

Thus one can think of IIB supergravity from two separate perspectives - as the low energy limit of IIB string theory *and* as an ensemble average over the duality group SL(2,$\mathbb{Z}$). This echoes of results found in holography of lower dimensions, where ensemble averages of boundary theories result in extremely simple bulk theories.

3.4 Consistency of the Duality

Unfortunately this section remains unwritten due to time constraints, but I encourage the reader to look at the interesting discussion in [20].

4 Further Reading

Much of the basic discussion above is adapted from the numerous fantastic texts on string theory: [21, 3, 8, 1, 22, 23]. For F-theory [24, 25, 26], I recommend the excellent reviews by our very own Timo Weigand [7, 6]. For holography, some of my favorite reviews are: [27, 28, 29, 30, 31]. For lower dimensional holography, I recommend [32].

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