Lattice QCD calculation of the hadronic light-by-light contribution to the magnetic moment of the muon

Harvey Meyer
J. Gutenberg University Mainz

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Cluster of Excellence

THE LOW-ENERGY FRONTIER OF THE STANDARD MODEL

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$(g-2)_\mu$: an early test of quantum electrodynamics

- in classical electromagnetism, the angular momentum $\boldsymbol{L}$ of a charged particle is associated with a magnetic moment $\boldsymbol{\mu} \propto \boldsymbol{L}$.
- the electron and its heavier cousin the muon carry an intrinsic angular momentum, $\boldsymbol{s}$ = spin, $s_z = \pm\hbar/2$.
- for the magnetic moment associated with the spin, one writes

$$\boldsymbol{\mu} = g \cdot \frac{e}{2m} \cdot \boldsymbol{s}, \qquad\qquad (e = \text{charge}, \quad m = \text{mass})$$

- $g = 2$ in Dirac's theory (1928)
- $a_\mu \equiv (g-2)_\mu/2 = F_2(0) = \frac{\alpha}{2\pi} + O(\alpha^2) \simeq 0.00116$

  (Schwinger 1948; $a_\mu = a_e$ to this order).
- corrections to $(g-2)_{\text{lepton}}$ from new heavy particle $\propto (m^2_{\text{lepton}}/M^2_{\text{heavy}})$.

$(g-2)_\mu$: a history of testing the Standard Model

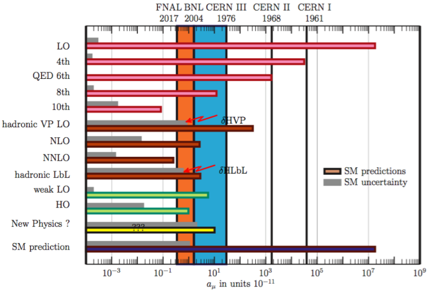


Fig. from Jegerlehner 1705.00263

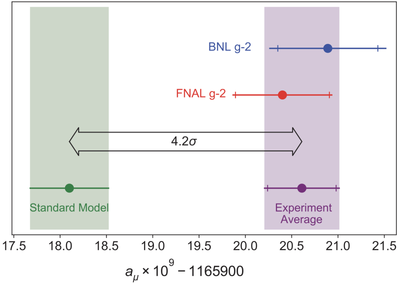
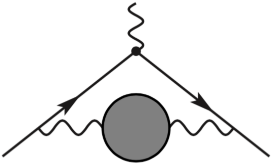


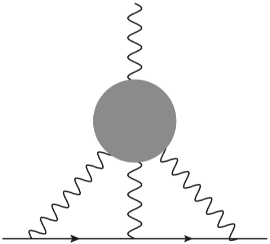
Fig. from Muon g-2 collab, PRL 126, 141801 (2021)

- After 2020 Theory White Paper and announcement by Fermilab Muon $(g-2)$ experiment (7 April 2021): $a_\mu^{\rm exp} - a_\mu^{\rm SM} = (251 \pm 59)\cdot 10^{-11}$
- 4.2σ, with practically equal contributions to the error from theory and experiment.

Source of dominant uncertainties in SM prediction for $(g-2)_\mu$



Hadronic vacuum polarisation



Hadronic light-by-light scattering

HVP: O(α^2), about $7000 \cdot 10^{-11}$

$\Rightarrow$ target accuracy: $\lesssim 0.5\%$

HLbL: O(α^3), about $100 \cdot 10^{-11}$

$\Rightarrow$ target accuracy: $\lesssim 15\%$.

Recall: $a_\mu^{\rm exp} - a_\mu^{\rm SM} = (251 \pm 59) \cdot 10^{-11}$.

Approaches to $a_\mu^{\rm HLbL}$

1. **Model calculations:** (the only approach until 2014)
   - based on pole- and loop-contributions of hadron resonances

2. **Dispersive representation:** the Bern approach has been worked out furthest.
   - identify and compute contributions of most important intermediate states
   - determine/constrain the required input (transition form factors, $\gamma^*\gamma^* \to \pi\pi$ amplitudes, ...) dispersively

3. **Experimental program:** provide input for model & dispersive approach, e.g. $(\pi^0, \eta, \eta') \to \gamma\gamma^*$ at virtualities $Q^2 \lesssim 3\,\text{GeV}^2$; active program at BES-III.

4. **Lattice calculations:**
   - RBC-UKQCD T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, Ch. Lehner, ...
   - Mainz N. Asmussen, E.-H. Chao, A. Gérardin, J. Green, J. Hudspith, HM, A. Nyffeler, ...

Wisdom gained from model calculations Prades, de Rafael, Vainshtein 0901.0306

- heavy (charm) quark loop makes a small contribution

$$a_\mu^{\mathrm{HLbL}} = (\frac{\alpha}{\pi})^3\, N_c \mathcal{Q}_c^4 c_4 \frac{m_\mu^2}{m_c^2} + \ldots, \qquad c_4 \approx 0.62.$$

- Light-quarks: (A) charged pion loop is negative, proportional to m_π^{-2}:

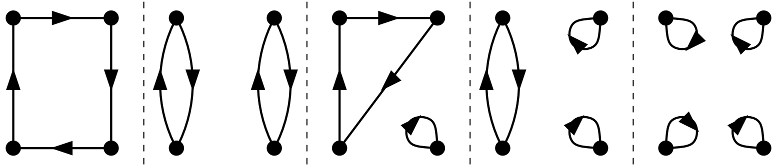
$$a_\mu^{\mathrm{HLbL}} = (\frac{\alpha}{\pi})^3\, c_2 \frac{m_\mu^2}{m_\pi^2} + \ldots, \qquad c_2 \approx -0.065.$$

(B) The neutral-pion exchange is positive, $\log^2(m_\pi^{-1})$ divergent:
Knecht, Nyffeler, Perrottet, de Rafael PRL88 (2002) 071802

$$a_\mu^{\mathrm{HLbL}} = (\frac{\alpha}{\pi})^3\, N_c \frac{m_\mu^2}{48\pi^2 (F_\pi^2/N_c)} \Big[\log^2 \frac{m_\rho}{m_\pi} + \mathrm{O}\Big(\log \frac{m_\rho}{m_\pi}\Big) + \mathrm{O}(1)\Big].$$

- For real-world quark masses: using form factors for the mesons is essential, and resonances up to 1.5 GeV can still be relevant ⇒ medium-energy QCD.

Wick-contraction topologies in HLbL amplitude $\langle 0|T\{j_x^\mu j_y^\nu j_z^\lambda j_0^\sigma\}|0\rangle$



First two classes of diagrams turn out to be dominant, with a cancellation between them.

Example: $\Pi = \langle (j_u - j_d)(j_u - j_d)(j_u - j_d)(j_u - j_d)\rangle$ does not contain the π^0 pole (π^0 only couples to one isovector, one isoscalar current).

Write out the Wick contractions: $\Pi = 2 \cdot \Pi^{(4)} + 4 \cdot \Pi^{(2,2)}$

In kinematic regime where π^0 dominates: $|\Pi| \ll \Pi^{(4)} \Rightarrow \Pi^{(2,2)} \approx -\frac{1}{2}\Pi^{(4)}$.

Including charge factors: $\Big[(Q_u^2 + Q_d^2)^2 \Pi^{(2,2)}\Big] = -\frac{25}{34}\Big[(Q_u^4 + Q_d^4)\Pi^{(4)}\Big]$.

Large-N_c argument by J. Bijnens, 1608.01454; see also 1712.00421.

Direct lattice calculation of HLbL in $(g-2)_\mu$

At first, this was thought of as a QED+QCD calculation
[pioneered in Hayakawa et al., hep-lat/0509016].

Today's viewpoint: the calculation is considered a QCD four-point Green's function, to be integrated over with a weighting kernel which contains all the QED parts.

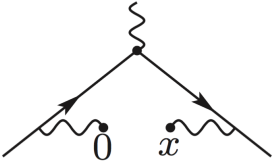
RBC-UKQCD: calculation of $a_\mu^{\rm HLbL}$ using coordinate-space method in muon rest-frame; photon+muon propagators:

- either on the $L \times L \times L$ torus (QED$_L$) (1510.07100–present)
- or in infinite volume (QED$_\infty$) (1705.01067–present).

Mainz:

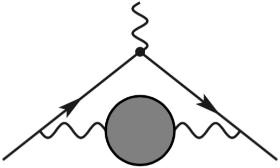
- manifestly covariant QED$_\infty$ coordinate-space approach, averaging over muon momentum using the Gegenbauer polynomial technique (1510.08384–present).

Analogy: hadronic vacuum polarization in x-space HM 1706.01139



QED kernel $H_{\mu\nu}(x)$

$\Rightarrow$



$a_\mu^{\rm hvp}$

$$a_\mu^{\rm hvp} = \int d^4x \; H_{\mu\nu}(x) \left\langle j_\mu(x) j_\nu(0) \right\rangle_{\rm QCD},$$

$$j_\mu = \frac{2}{3}\bar u \gamma_\mu u - \frac{1}{3}\bar d \gamma_\mu d - \frac{1}{3}\bar s \gamma_\mu s + \dots; \qquad H_{\mu\nu}(x) = -\delta_{\mu\nu}\mathcal{H}_1(|x|) + \frac{x_\mu x_\nu}{x^2}\,\mathcal{H}_2(|x|)$$

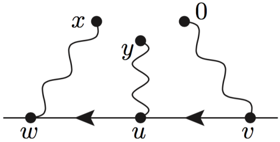
Kernel known in terms of Meijer's functions: $\mathcal{H}_i(|x|) = \frac{8\alpha^2}{3m_\mu^2} f_i(m_\mu |x|)$ with

$$f_2(z) = \frac{G^{2,2}_{2,4}\left(z^2 \middle| \begin{matrix} \frac{7}{2}, 4 \\ 4, 5, 1, 1 \end{matrix}\right) - G^{2,2}_{2,4}\left(z^2 \middle| \begin{matrix} \frac{7}{2}, 4 \\ 4, 5, 0, 2 \end{matrix}\right)}{8\sqrt{\pi} z^4},$$

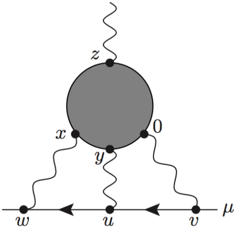
$$f_1(z) = f_2(z) - \frac{3}{16\sqrt{\pi}} \cdot \left[G^{2,3}_{3,5}\left(z^2 \middle| \begin{matrix} 1, \frac{3}{2}, 2 \\ 2, 3, -2, 0, 0 \end{matrix}\right) - G^{2,3}_{3,5}\left(z^2 \middle| \begin{matrix} 1, \frac{3}{2}, 2 \\ 2, 3, -1, -1, 0 \end{matrix}\right) \right].$$

Coordinate-space approach to $a_\mu^{\rm HLbL}$, Mainz version

QED kernel $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$



$\Rightarrow$



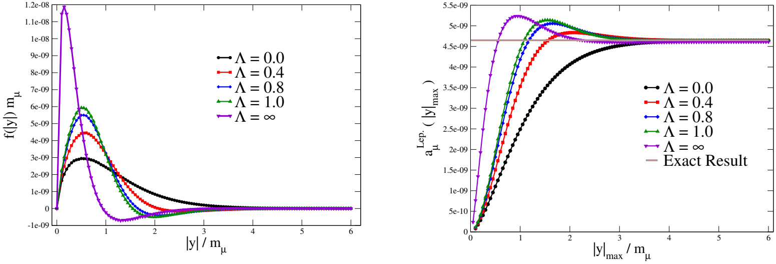
$$a_\mu^{\rm HLbL} = \frac{me^6}{3} \underbrace{\int d^4y}_{=2\pi^2|y|^3 d|y|} \Big[\int d^4x \, \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\rm QED} \, \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{=\rm QCD\ blob} \Big].$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z \; z_\rho \, \Big\langle \, j_\mu(x)\, j_\nu(y)\, j_\sigma(z)\, j_\lambda(0) \Big\rangle.$$

- $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ computed in the continuum & infinite-volume
- no power-law finite-volume effects & only a 1d integral to sample the integrand in $|y|$.

[Asmussen, Gérardin, Green, HM, Nyffeler 1510.08384, 1609.08454]

Tests of the framework and adjustments to the kernel



Integrands (Lepton loop, method 2) Correspondings integrals

- The QED kernel $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ is parametrized by six 'weight' functions of the variables $(x^2, x\cdot y, y^2)$.
- $$\bar{\mathcal{L}}^{(\Lambda)}_{[\rho,\sigma];\mu\nu\lambda}(x,y) = \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) - \partial^{(x)}_\mu (x_\alpha e^{-\Lambda m_\mu^2 x^2/2}) \bar{\mathcal{L}}_{[\rho,\sigma];\alpha\nu\lambda}(0,y) - \partial^{(y)}_\nu (y_\alpha e^{-\Lambda m_\mu^2 y^2/2}) \bar{\mathcal{L}}_{[\rho,\sigma];\mu\alpha\lambda}(x,0),$$
- Using this kernel, we have reproduced (at the 1% level) known results for a range of masses for:
  1. the lepton loop (spinor QED, shown in the two plots);
  2. the charged pion loop (scalar QED);
  3. the π^0 exchange with a VMD-parametrized transition form factor.

Rearrangement of integrals: 'method 2'

For the fully-connected calculation we use the following master equation for the integrand:

$$f^{(\mathrm{Conn.})}(|y|) = -\sum_{j\in u,d,s} \hat{Z}_{\mathrm{V}}^4 Q_j^4 \frac{m_\mu e^6}{3} 2\pi^2 |y|^3 \times$$

$$\int_x \left(\mathcal{L}'_{[\rho,\sigma]\mu\nu\lambda}(x,y) \int_z z_\rho \widetilde{\Pi}^{(1),j}_{\mu\nu\sigma\lambda}(x,y,z) + \bar{\mathcal{L}}^{(\Lambda)}_{[\rho,\sigma];\lambda\nu\mu}(x,x-y) x_\rho \int_z \widetilde{\Pi}^{(1),j}_{\mu\nu\sigma\lambda}(x,y,z) \right),$$

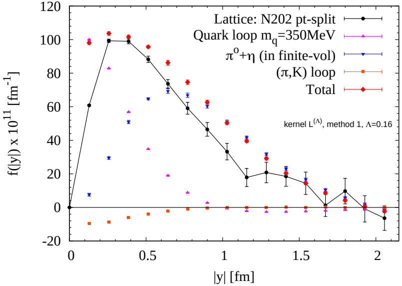
with hadronic contribution

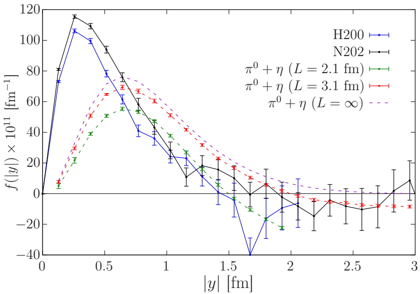
$$\widetilde{\Pi}^{(1),j}_{\mu\nu\sigma\lambda}(x,y,z) = -2\mathrm{Re}\left\langle \mathrm{Tr}\left[S^j(0,x)\gamma_\mu S^j(x,y)\gamma_\nu S^j(y,z)\gamma_\sigma S^j(z,0)\gamma_\lambda \right] \right\rangle_U .$$

- $S^j(x,y)$ is the flavour j-quark propagator from source y to sink x;
- Q_j is the charge factor ($Q_u = \frac{2}{3}$, $Q_d = -\frac{1}{3}$, $Q_s = -\frac{1}{3}$);
- $\langle\cdot\rangle_U$ denotes the ensemble average.

$$\mathcal{L}'_{[\rho,\sigma];\mu\nu\lambda}(x,y) = \bar{\mathcal{L}}^{(\Lambda)}_{[\rho,\sigma];\mu\nu\lambda}(x,y) + \bar{\mathcal{L}}^{(\Lambda)}_{[\rho,\sigma];\nu\mu\lambda}(y,x) - \bar{\mathcal{L}}^{(\Lambda)}_{[\rho,\sigma];\lambda\nu\mu}(x,x-y).$$

Integrand at $m_\pi = m_K \simeq 415$ MeV



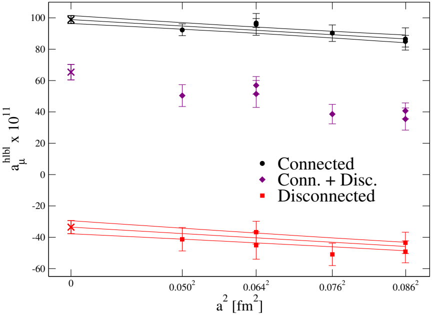


- Partial success in understanding the integrand in terms of familiar hadronic contributions.
- Reasonable understanding of magnitude of finite-size effects. ($L_{\rm H200} = 2.1$ fm, $L_{\rm N202} = 3.1$ fm)

2006.16224 Chao et al. (EPJC)

$a_\mu^{\rm HLbL}$ at $m_\pi = m_K \simeq 415$ MeV

[Chao, Gérardin, Green, Hudspith, HM 2006.16224 (EPJC)]



$$a_\mu^{\rm hlbl,SU(3)_f} = (65.4 \pm 4.9 \pm 6.6) \times 10^{-11}.$$

Guesstimating the result at physical quark masses: correct for π^0 exchange

$$a_\mu^{\rm hlbl,SU(3)_f} - a_\mu^{{\rm hlbl},\pi^0,{\rm SU(3)_f}} + a_\mu^{{\rm hlbl},\pi^0,{\rm phys}} = (104.1 \pm 9.1) \times 10^{-11}.$$

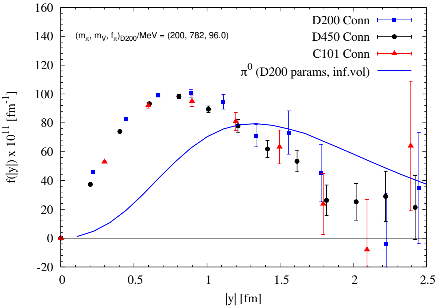
Estimate based on lattice QCD calculation of $\pi^0 \to \gamma^*\gamma^*$ transition form factor [Gérardin, HM, Nyffeler 1903.09471 (PRD)].

$N_\mathrm{f} = 2 + 1$ CLS ensembles used towards physical quark masses

	(4)	(22)	(31)	(211)	(1111)	β	$(a \text{ GeV})^2$	$(\frac{m_\pi}{\text{GeV}})^2$	$(\frac{m_K}{\text{GeV}})^2$	$m_\pi L$	$\hat{Z}_\mathrm{V}$
A653	l, s	l, s	0	0	0	3.34	**0.2532**	**0.171**	**0.171**	**5.31**	**0.70351**
A654	l, s	l, s	l				**0.2532**	**0.107**	**0.204**	**4.03**	**0.69789**
U103	l, s	l, s	0	0	0		0.1915	0.172	0.172	4.35	0.71562
H101	l, s	l, s	0	0	0		0.1915	0.173	0.173	5.82	0.71562
U102	l	l	l			3.40	0.1915	0.127	**0.194**	3.74	0.71226
H105	l, s	l, s	l, s				0.1915	0.0782	0.213	3.92	0.70908
C101	l, s	l, s	l, s	l	l, s		0.1915	0.0488	0.237	4.64	0.70717
B450	l, s	l, s	0	0	0	3.46	0.1497	0.173	0.173	5.15	0.72647
D450	l	l	l				0.1497	**0.0465**	0.226	**5.38**	**0.71921**
H200	l, s	l, s	0	0	0		0.1061	0.175	0.175	4.36	0.74028
N202	l, s	l, s	0	0	0		0.1061	0.168	0.168	6.41	0.74028
N203			l	l		3.55	0.1061	0.120	0.194	5.40	0.73792
N200	l	l	l				0.1061	0.0798	0.214	4.42	0.73614
D200	l	l	l				0.1061	0.0397	0.230	4.15	0.73429
N300	l, s	l, s	0	0	0	3.70	0.06372	0.178	0.178	5.11	0.75909

En-Hung Chao, Renwick Hudspith, Antoine Gérardin, Jeremy Green, HM, Konstantin Ottnad
2104.02632 (EPJC)

Integrand of connected contribution at $m_\pi \approx 200$ MeV

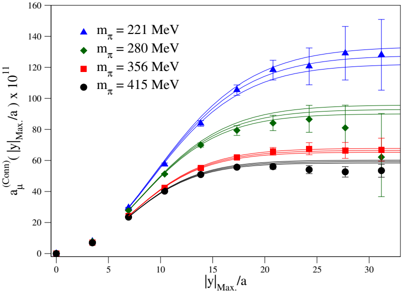


- using four local vector currents
- based on 'Method 2'.

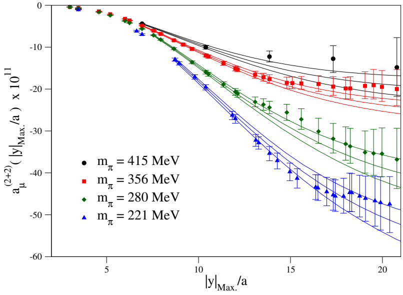
2104.02632

Truncated integral for $a_\mu^{\rm HLbL}$

Connected



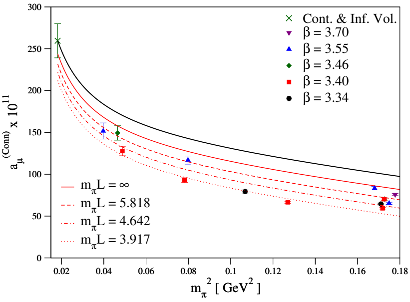
(2+2) Disconnected



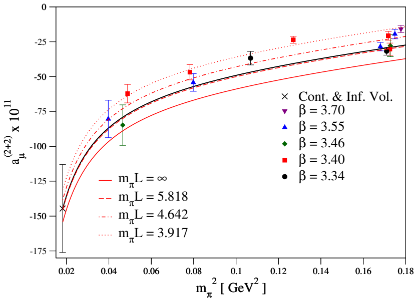
- Extend reach of the signal by two-param. fit $f(y) = A|y|^3 \exp(-M|y|)$;
- provides an excellent description of the π^0 exchange contribution in infinite volume.
- We see a clear increase of the magnitude of both connected and disconnected contributions.

Chiral, continuum, volume extrapolation

Connected contribution

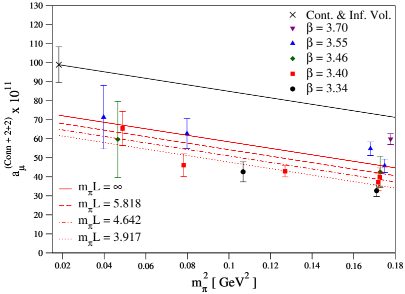


disconnected contribution

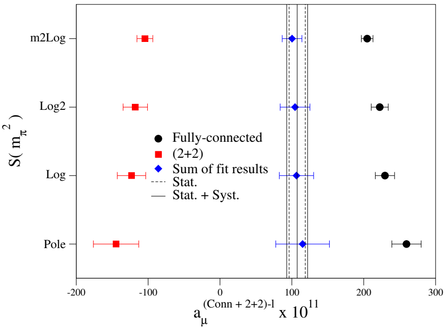


Total light-quark contribution:

- vol. dependence: $\propto \exp(-m_\pi L/2)$
- pion-mass dependence fairly mild (!)



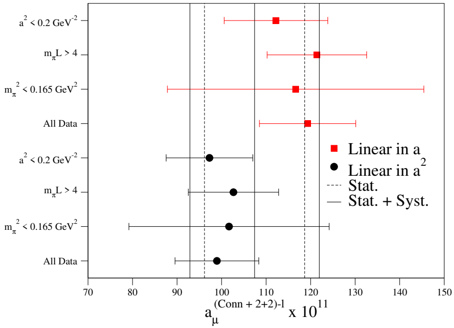
Separate extrapolation of conn. & disconn.



$$\text{Ansatz}: \quad Ae^{-m_\pi L/2} + Ba^2 + CS(m_\pi^2) + D + Em_\pi^2$$

- chirally singular behaviour cancels in sum of connected and disconnected.

Extrapolation to the sum of conn. & disconn.



$$\text{Ansatz}: \quad Ae^{-m_\pi L/2} + Ba^2 + D + Em_\pi^2$$

- results very stable with respects to cuts in a, m_π or $m_\pi L$.
- largest systematic comes from choice of continuum limit ansatz.
- final result: central value from fitting these results with a constant; systematic error set to $\sqrt{(1/N)\sum_{i=1}^{N}(y_i - \bar{y})^2}$ as a measure of the spread of the results.

Overview table

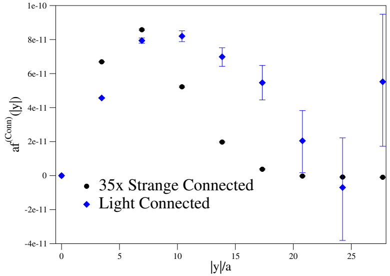
Contribution	Value$\times 10^{11}$
Light-quark fully-connected and $(2+2)$	107.4(11.3)(9.2)(6.0)
Strange-quark fully-connected and $(2+2)$	$-0.6(2.0)$
$(3+1)$	0.0(0.6)
$(2+1+1)$	0.0(0.3)
$(1+1+1+1)$	0.0(0.1)
Total	106.8(15.9)

- error dominated by the statistical error and the continuum limit.
- all subleading contributions have been tightly constrained and shown to be negligible.

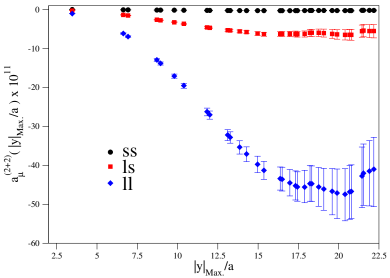
[Chao et al, 2104.02632]

Strange contribution

Ensemble C101 ($48^3 \times 96$, $a = 0.086$ fm, $m_\pi = 220$ MeV)

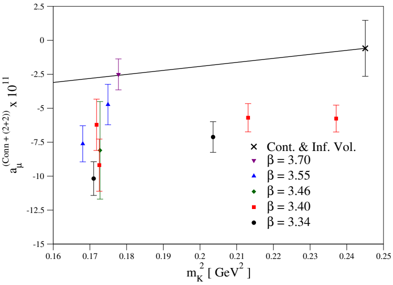


NB. Strange integrand has a factor 17 suppression due to charge factor.



(2,2) disconnected contributions.

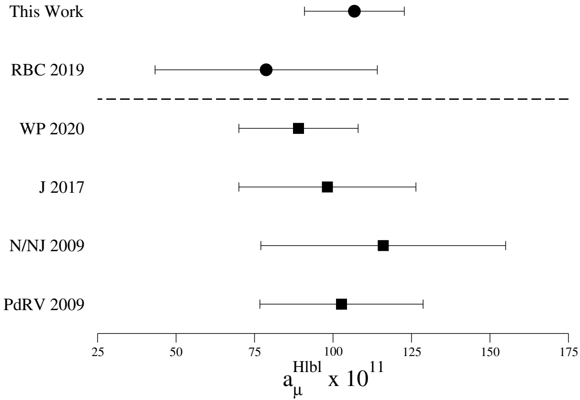
Extrapolation of strange contributions



Sum of connected-strange + (2,2) topology with ss and sl quark-line content.

Final strange contribution is very small as a result of cancellations.

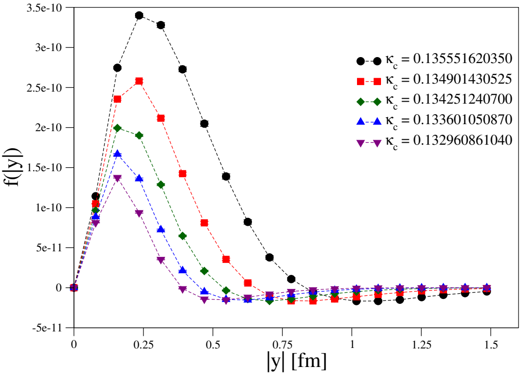
Compilation of a_μ^{HLbL} determinations



Good consistency of different determinations (not including charm here).

Fig from Chao et al, 2104.02632 (EPJC).

The charm contribution at the SU(3)$_\mathrm{f}$ point

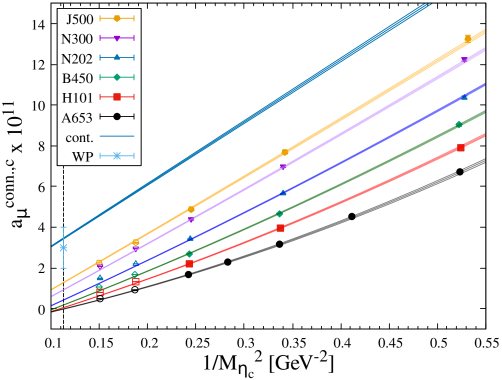


Integrand for the connected charm contribution (J500, $a = 0.039\,\mathrm{fm}$)

- direct calculation at physical charm mass difficult due to lattice artefacts
- $\rightsquigarrow$ perform a combined extrapolation in $1/m_c^2$ and the lattice spacing.

Chao, Hudspith, Gérardin, Green, HM arXiv:2204.08844

Extrapolation in charm mass and lattice spacing



This particular fit:

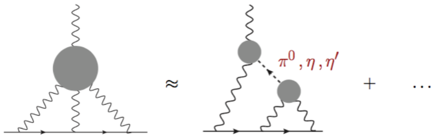
$$a_\mu(a, m_{\eta_c}) = Aa + \frac{B + Ca^2}{m_{\eta_c}^2} + Da^2 + E\frac{a^2}{m_{\eta_c}^4}$$

Final result (average of several fits): $a_\mu(\text{charm}) = (2.8 \pm 0.5) \times 10^{-11}$.

Conclusion on $a_\mu^{\rm HLbL}$

- Results from the Bern dispersive framework and from two independent lattice QCD calculations are in good agreement and have comparable uncertainties.
- It is now practically excluded that $a_\mu^{\rm HLbL}$ can by itself explain the tension between the SM prediction and the experimental value of a_μ.
- Epilogue: $a_\mu^{\rm HLbL}$ is a tale of many cancellations, both between the exchange of different mesons and also between Wick-contraction topologies in lattice QCD.

Models for a_μ^{HLbL}



Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
π, K loops +subl. N_C	–	–	–	0 ± 10	–	–	–
quark loops	21 ± 3	9.7 ± 11.1	–	–	–	2.3 (c-quark)	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = AN '09, JN = Jegerlehner, AN '09

Table from A. Nyffeler, PhiPsi 2017 conference

One further estimate: NB. much smaller axial-vector contribution

$$a_\mu^{\mathrm{HLbL}} = (103 \pm 29) \times 10^{-11} \qquad \text{Jegerlehner 1809.07413}$$