Continuum Limit of Heavy Quarks Moments and their Perturbative Expansion

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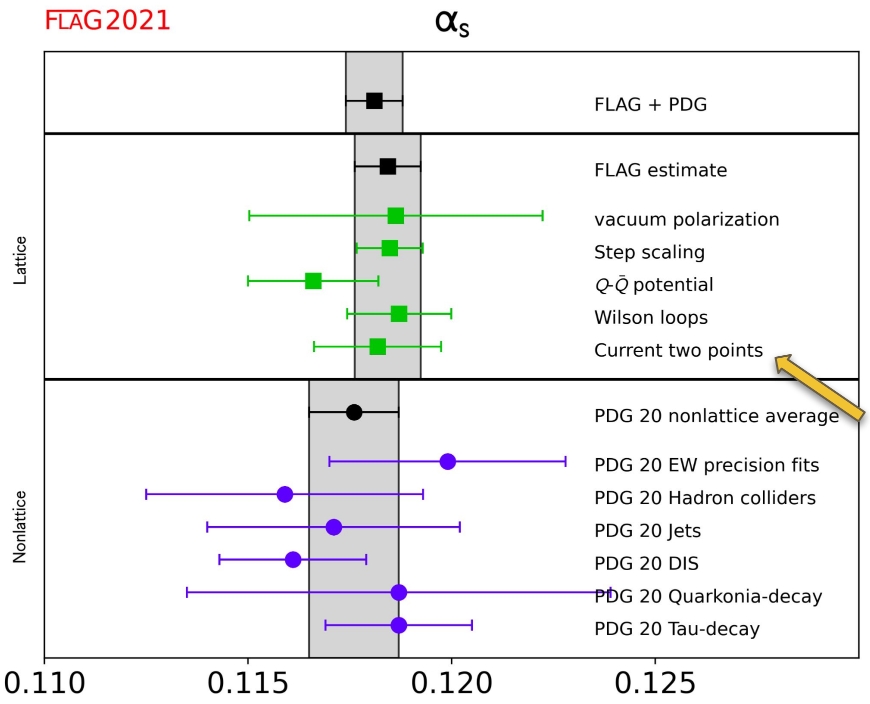
Thanks to N. Husung, T. Korzec,
S. Schaefer, B. Strassberger.

The Strong Coupling

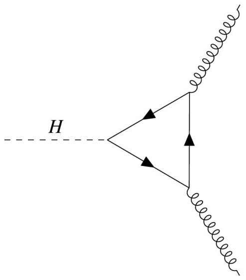
- Strong Coupling α in the Lagrangian, fundamental parameter of SM. It is an *input* of the theory.



$$\mathcal{L}_{SM} \supset \mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \sum_{f=1}^{N_f} \overline{\psi}_f \left[i\partial\!\!\!/ - g A\!\!\!/ - m_f \right] \psi_f$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

- Error in determination of α (or g) directly propagates in predictions.
- Parametric uncertainty on α is among important uncertainties for example for H → gg and H → bb, for total and partial hadronic Z widths, as well as implications for EW vacuum stability and top quark physics.
- Study **systematics** of **"moments method"**
- Notice Lattice results dominate world average: $\alpha_s(M_Z)$ = 0.1179 (10)
- Experimental input is needed on the lattice, but this has very different systematic effects w.r.t. the one of experiments measuring α_s



Moments in Momentum Space

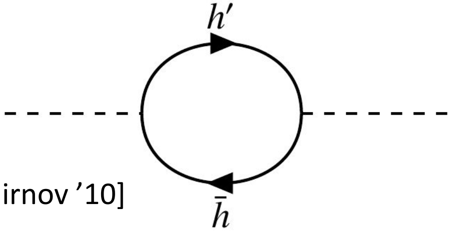
- Moments method, pioneered by Bochkarev, de Forcrand [*hep-lat/9505025*] and HPQCD in 2008 [*hep-lat/0805.2999*].
- The observables are derivatives of the vacuum polarization with heavy quarks (*h, h'*) at CoM energy $q^2 = 0$.
- *m* ↔ scale of observable, *m* is some generic mass, can be some PT scheme or an RGI-mass (more later).

$$\Pi(q^2, m) = i \int \mathrm{d}^4x\, e^{iq\cdot x} \langle 0| \mathscr{T} \left\{ J^\dagger(x,m) J(0,m) \right\} |0\rangle$$

$$\mathcal{M}_n(m) = \frac{1}{n!} \left(\frac{\partial}{\partial q^2} \right)^n \Pi(q^2, m) \Big|_{q^2=0}$$

Object known to high (4-loop) orders in PT

[Maier, Maierhöfer, Marquard, Smirnov '10]



$$[\mathcal{M}_n] = \mathrm{En.}^{4-n} \quad \Rightarrow \quad \mathcal{M}_n(m) = \overline{m}_{\overline{\mathrm{MS}}}(\mu)^{4-n} \sum_{i\geq 0} c_n^{(i)}(\mu/\overline{m}_{\overline{\mathrm{MS}}}(\mu))\, \alpha^i_{\overline{\mathrm{MS}}}(\mu)$$

$$c_n^{(i)}(\mu/\overline{m}_{\overline{\mathrm{MS}}}(\mu)) = c_n^{(i,0)} + \sum_{k=1}^{i-1} c_n^{(i,k)} \log^k(\mu/\overline{m}_{\overline{\mathrm{MS}}}(\mu))$$

If $J_\mu(x) = V_\mu(x)$ (vector operator) the moment can both be calculated perturbatively and inferred from the experimental R-ratio *R(s)*, which is tied to physical value of quark mass: α determined at that scale.

$$\mathcal{M}_n(m) = \int \frac{\mathrm{d}s}{s^{n+1}} R(s,m)\,, \quad R(s,m) = \frac{\sigma_{e^+e^-\to h\bar{h}}(s)}{\sigma_{e^+e^-\to\mu^+\mu^-}(s)}$$

Moments in Position Space

- Project to zero spatial momentum, then derivatives give $\int t^n\, G(t)$, with $G(t)$ time-slice correlator
- Here we Wick rotate to Euclidean, t and q_0 are Euclidean

once $q_\mu = q^\mu = (q_0, \vec{0})$

$$\Pi(q^2, m) = i \int \mathrm{d}^4x\, e^{iq\cdot x} \langle 0|\, \mathscr{T}\left\{J^\dagger(x,m) J(0,m)\right\} |0\rangle \quad \rightarrow \quad \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{itq_0} \int \mathrm{d}^3\vec{x} \left\langle J^\dagger(x,m) J(0,m) \right\rangle = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{itq_0} G(t,m)$$

$$\mathcal{M}_n(m) = \left(\frac{\partial}{\partial i q_0}\right)^n \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{itq_0} G(t,m) \bigg|_{q_0=0} = \int_{-\infty}^{\infty} \mathrm{d}t\, t^n \mathrm{e}^{itq_0} G(t,m) \bigg|_{q_0=0} = \int_{-\infty}^{\infty} \mathrm{d}t\, t^n G(t,m)$$

O.P.E.: $$J_A(x) J_B(0) \underset{x\to 0}{\sim} \sum_l O_l C^{(l)}_{A,B}(x)$$

$$O_1 = \mathbb{1}\,,\ C^{(1)}_{A,B} \sim \frac{1}{|x|^6} \text{ up to logs} \implies G(t) \underset{t\to 0}{\sim} \frac{1}{|t|^3} \implies \text{for } n > 3 \quad \exists \lim_{t\to 0} \left\{ G(t)\, t^n \right\} \Rightarrow \boxed{n = 4,\ 6,\ 8,\ 10, \ldots}$$

Lattice Transcription

★ The **lattice transcription** of the moments is:

$$\mathcal{M}_n(aM_{RGI}, \sqrt{8t_0}M_{RGI}) = \lim_{T\to\infty, L\to\infty} a \sum_{t=-T/2r}^{t=T/2r} t^n \left(\frac{a}{L}\right)^3 a^3 \sum_{\vec{x},\vec{y}}^{L-a} \langle J(t,\vec{x},\mu_{tm}) J^\dagger(0,\vec{y},\mu_{tm}) \quad , \quad r<1$$

Doublet of mass-degenerate twisted mass Wilson fermions, at full twist

★ $J_{PS}(x) = i\mu_{tm}\overline{\psi}_h(x)\gamma_5\psi_{h'}(x)$ is renormalization independent in certain regularizations (no *Z*-factors!). At full twist, PCAC relation maps into exact vector current WI yielding:

$$Z_P Z_\mu = 1$$

★ Full twist also ensures automatic *O(a)*-improvement.

$$S_F = a^4 \sum_x \overline{\psi}(x) \left\{ \sum_{\nu=0}^{3} \left(\gamma_\nu \frac{\nabla_\nu + \nabla_\nu^*}{2} - \frac{a}{2}\nabla_\nu^*\nabla_\nu + c_{sw}\, a \sum_{\mu=0}^{3} \frac{i}{4}\sigma_{\mu\nu}\hat{F}_{\mu\nu} \right) \mathbb{1}_f + m\mathbb{1}_f + i\mu_{tm}\gamma_5\tau_3 \right\} \psi(x)$$

Wilson term

twisted mass term

$$\begin{cases} \chi &= \exp\left(i\omega\frac{\tau^3}{2}\gamma_5\right)\psi \\ \overline{\chi} &= \overline{\psi}\exp\left(i\omega\frac{\tau^3}{2}\gamma_5\right) \end{cases}$$

Full twist at: $\omega = \pi/2$

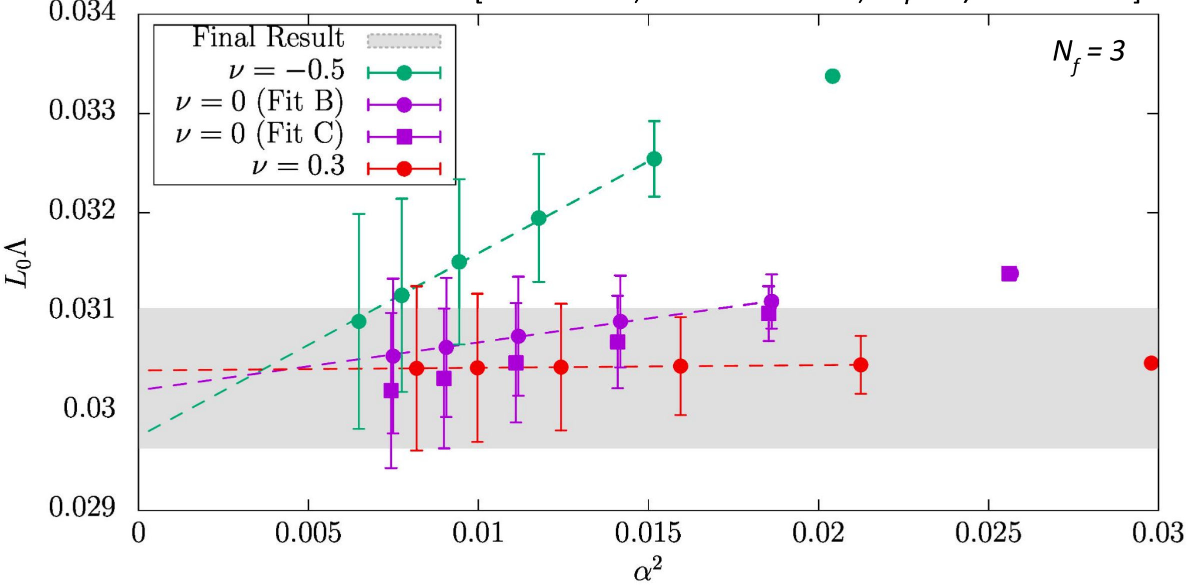
★ On the lattice: vary the mass, i.e. vary the scale of α → study variation of truncated part.

In What Domain is PT Accurate?

$$\mathcal{M}_n(m,\mu) \overset{\alpha\to 0}{\sim} \sum_{i=0}^{L} c_n^{(i)}(\mu/\overline{m}_{\overline{\text{MS}}})\,\alpha^i_{\overline{\text{MS}}}(\mu) + O(\alpha^{L+1}(\mu))$$

- ★ One cannot always simply assume a flat enough behavior!
- ★ Lambda parameter computed by ALPHA collaboration in a modified Schrödinger Functional scheme: even at "small" values for α (e.g. α≈0.13), truncated terms may be large.
- ★ Extrapolation to high energy needed.

[ALPHA Coll., dalla Brida et al., *hep-lat/1803.10230*]

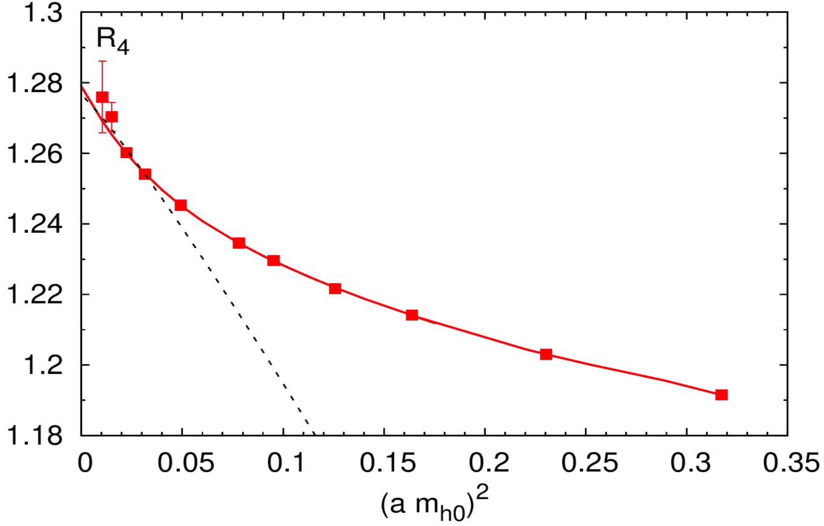


Disclaimer: Study the Systematics

Different discretization: (highly improved) staggered quarks

- N_f=2+1
- M=M_c

Petreczky, Weber, *arXiv:hep-lat/1901.06424*



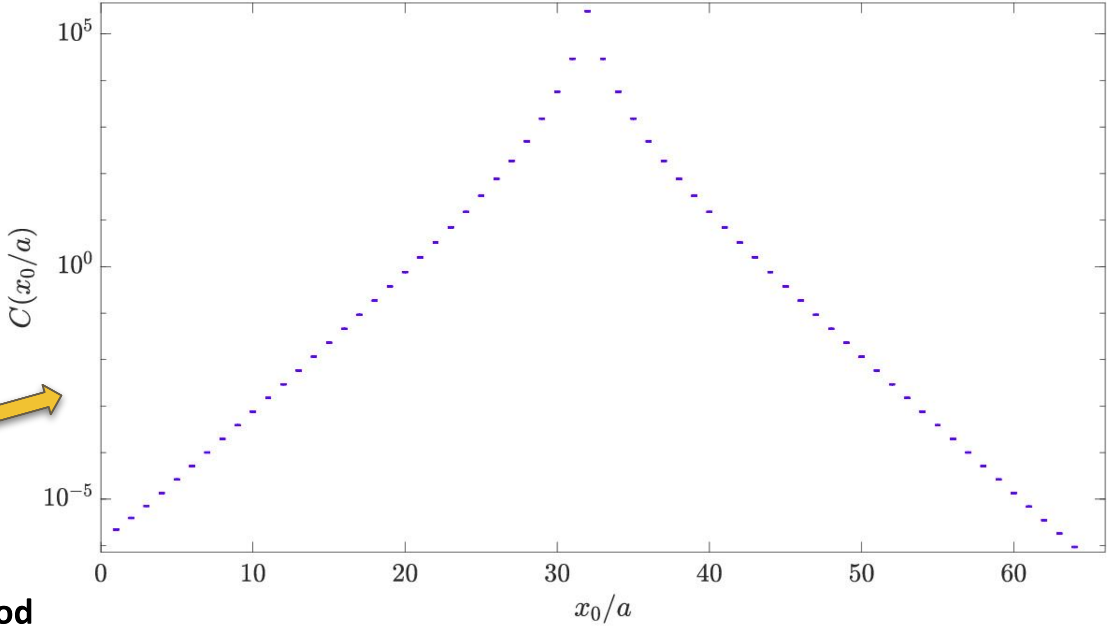
- ❑ Very <u>difficult extrapolation</u>, no range with just ~ a^2 behavior
- ❑ We are not trying to get a competitive α result for the FLAG
- ❑ Rather, we want to be able to really study the two big issues with this method: <u>truncation errors</u> and <u>lattice artefacts</u>

Do this in the **quenched model**, where it is more feasible to reach small lattice spacings and hopefully get reliable continuum extrapolations.

Lattice Setup

- ➢ **Plaquette** gauge action
- ➢ P.b.c. in space, **open b.c. in time** to avoid **frozen topological charge** at small *a*
- ➢ Full twist doublet, with **non-perturbative c_{sw}** to reduce cutoff effects
- ➢ Stochastic evaluation of trace and sum over space with **U(1) noise sources**
- ➢ **Source** placed at **1 fm from boundary**, checked absence of boundary effects
- ➢ Full twist, set **κ to its critical value** [1]
- ➢ Autocorrelation analysis done with **Γ-method**
- ➢ Scale set through **gradient flow** t_0 [2]

Example correlator: no asymmetry around source (no <u>boundary effects</u>) can be seen within precision; β = 6.7859, $M/M_c \simeq$ 1.6



[1] Lüscher, Sint, Sommer, Weisz, Wolff. [*arXiv:hep-lat/9609035*]
[2] Lüscher, *arXiv:hep-lat/1006.4518*

Measurements

Run Name	β	$l^3 \times t$	N_{cnfg}	t_0/a^2	$a[fm]$	$\tau_{\text{int}}(t_0)[\text{cfg}]$
q_beta616	6.1628	$32^3 \times 96$	128	5.1604(98)	0.071	0.78
q_beta628	6.2885	$36^3 \times 108$	137	7.578(22)	0.059	1.37
q_beta649	6.4956	$48^3 \times 144$	109	13.571(50)	0.044	1.55
sft4	6.7859	$64^3 \times 192$	200	29.390(98)	0.030	1.00
sft5	7.1146	$96^3 \times 320$	80	67.74(23)	0.020	0.55
sft6	7.3600	$128^3 \times 320$	98	124.21(91)	0.015	1.03
sft7	7.700	$192^3 \times 480$	31	286.3(4.7)	0.010	–

Gauge run details, $l = L/a$, $t = T/a$.

[Ensembles sft from:

Husung, Krah, Sommer

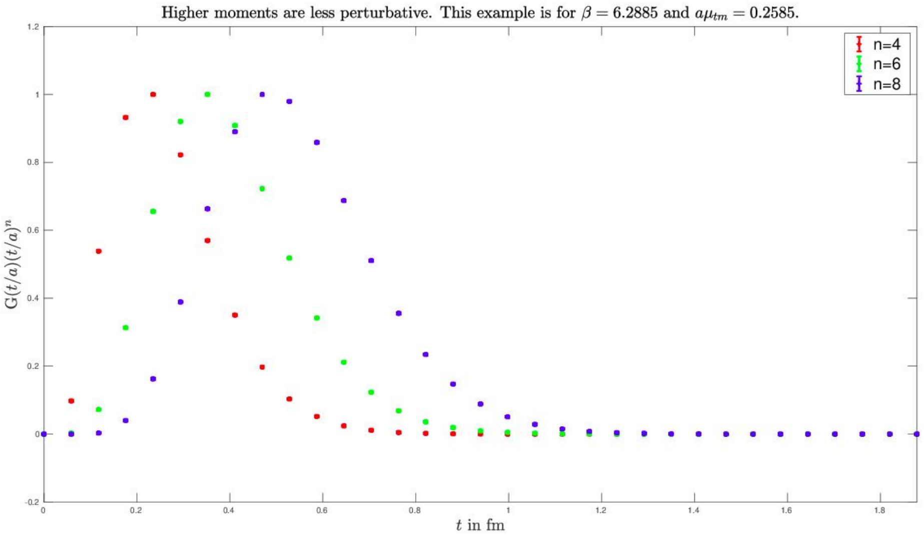
arXiv:hep-lat/1711.01860]

Physical **Volume of L≃2 fm**, time direction about **T≃6 fm**

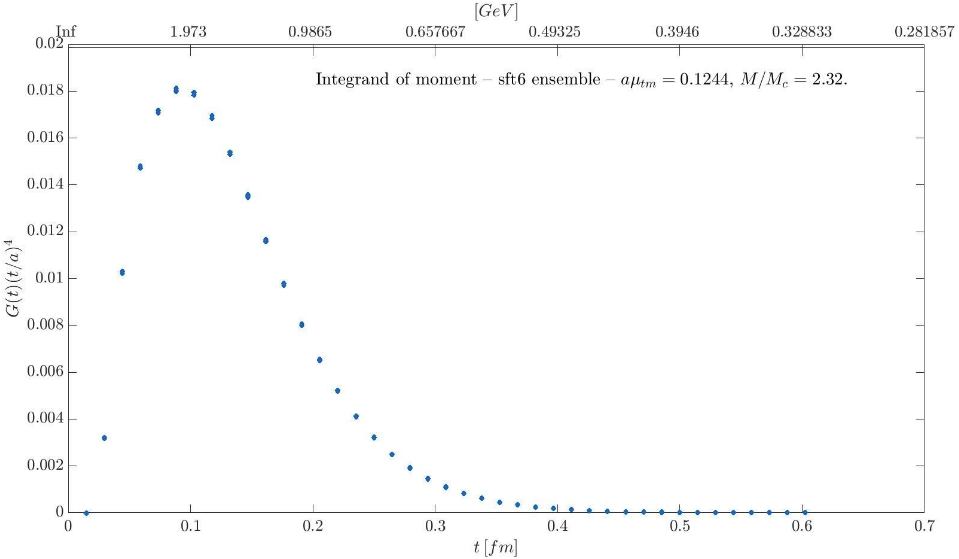
We measure for a range of masses:

$$M_{RGI}/M_{RGI,\,\text{charm}} \simeq 3.48,\ 2.32,\ 1.55,\ 1.16,\ 0.77.$$
$$M_{RGI} \simeq 5.75\,, 3.83\,, 2.56\,, 1.92\,, 1.28\,\text{GeV}$$

Quick Glance at Observables



- Lhs: integrands normalized so their height is 1
- Increasing *n* makes moments less perturbative
- Energy scale of moments is somewhat worrying



Qualitative, rough approximation: for large *t*

$$G(t) \sim e^{-2m_h t},$$

We see peak position increases with n and decreases with mass:

$$\partial\left\{t^n e^{-2m_h t}\right\} = 0 \implies t^{n-1} e^{-2m_h t}\{n - 2m_h t\} = 0 \implies t_{peak} \simeq \frac{n}{2m_h}$$

Constant Mass Trajectory

- Line of constant "physics": at every *a* we tune the bare mass in order to keep some renormalized mass fixed.
- We keep the renormalization group invariant mass fixed (scheme independent!):

$$M_{RGI} = \lim_{\mu\to\infty} \overline{m}_X(\mu) \left[2b_0\, \overline{g}_X^2(\mu)\right]^{-d_0/(2b_0)}$$

RGI-mass (parameters) are like running to infinite energy.

$$z := \sqrt{8t_0} M_{RGI} = \frac{\sqrt{8t_0}}{a} a M_{RGI} = \frac{\sqrt{8t_0}}{a} \underbrace{\frac{M_{RGI}}{\overline{m}_{SF}(\mu)\, Z_P^{SF}(a\mu,\beta)}}_{\text{Literature, from [1]}} a\mu_{tm} + O((a\mu_{tm})^2)$$

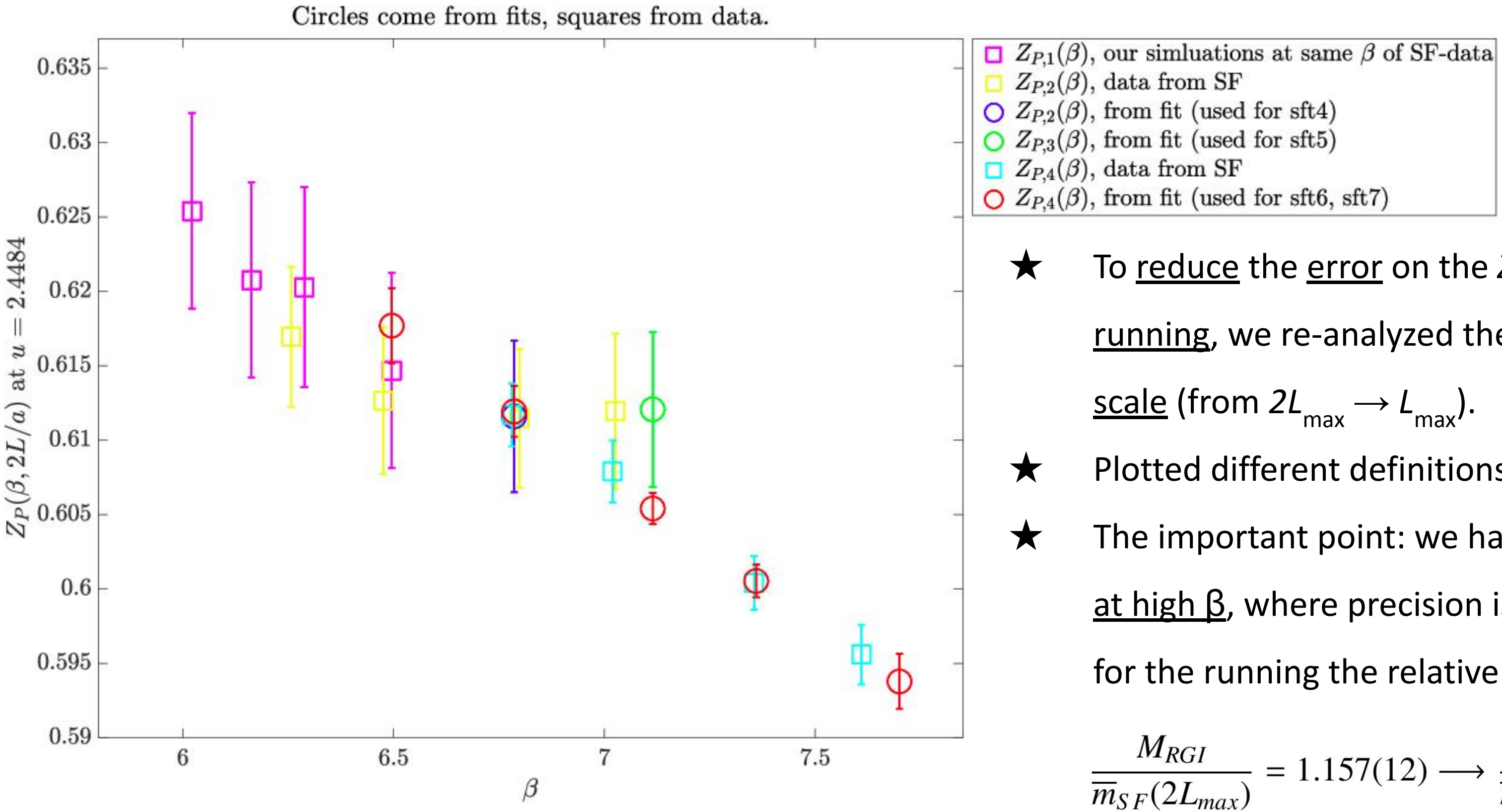
Choose (z) — Measure ($\sqrt{8t_0}/a$) — Literature, from [1] — Tune ($a\mu_{tm}$)

Some numbers: $$M_{RGI} = (\hbar c)\frac{z}{\sqrt{8t_0}} \simeq 5.75\,, 3.83\,, 2.56\,, 1.92\,, 1.28\,\text{GeV}$$

$$M_c^{RGI}\big|_{N_f=0} = 1.654(45)\,\text{GeV}$$ [Rolf and Sint, *hep-ph/0110139*]

[1] Capitani, Lüscher, Sommer, Wittig, [*hep-lat/9810063]*

Reanalysis of Quenched SF Data

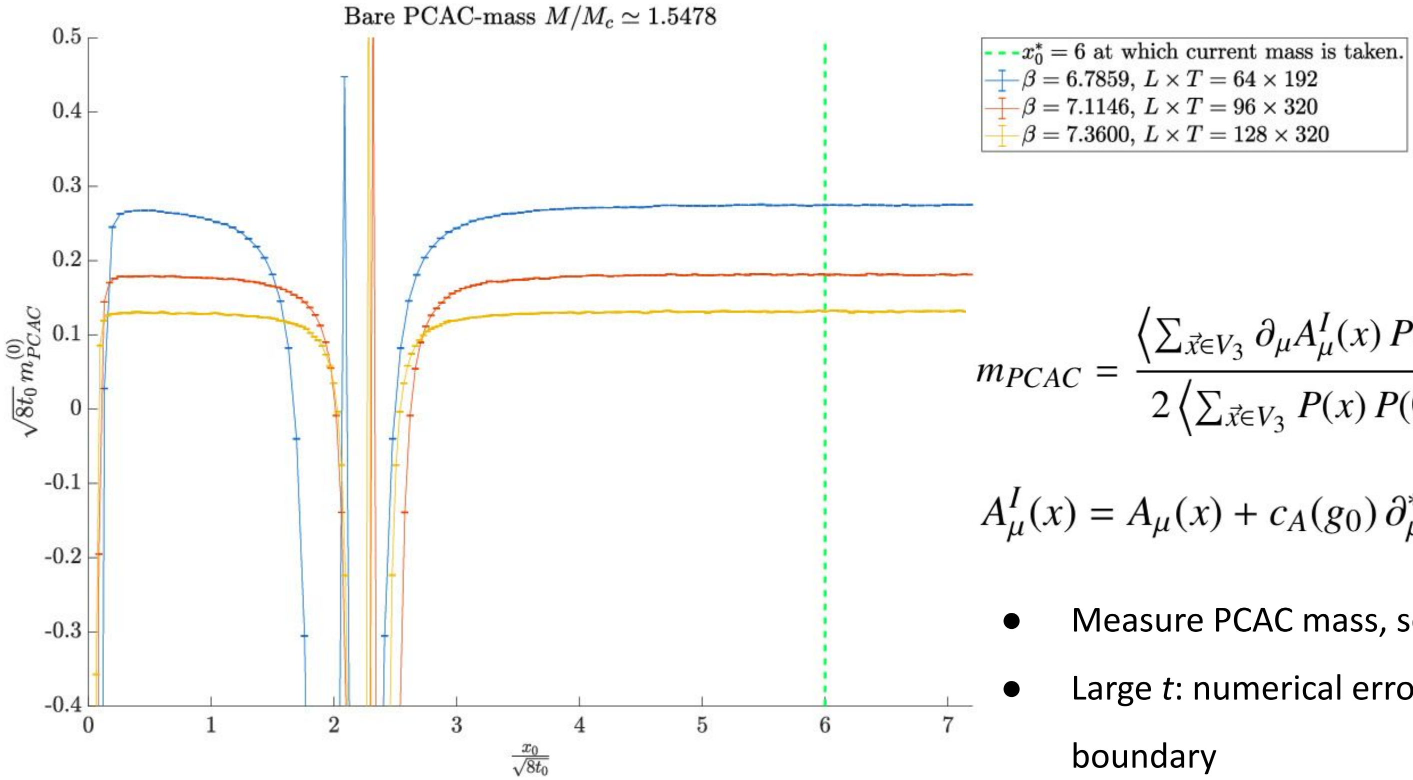


- ★ To <u>reduce</u> the <u>error</u> on the Z_P-<u>factor</u> and the <u>running</u>, we re-analyzed the data of [1] at a <u>lower scale</u> (from $2L_{max} \longrightarrow L_{max}$).
- ★ Plotted different definitions of Z_P, labeled $Z_{P,i}$.
- ★ The important point: we have <u>rather small errors at high β</u>, where precision is most important. Also for the running the relative error decreases:

$$\frac{M_{RGI}}{\overline{m}_{SF}(2L_{max})} = 1.157(12) \longrightarrow \frac{M_{RGI}}{\overline{m}_{SF}(L_{max})} = 1.379(11)$$

[1] Capitani, Lüscher, Sommer, Wittig, [*hep-lat/9810063*]

PCAC Data I

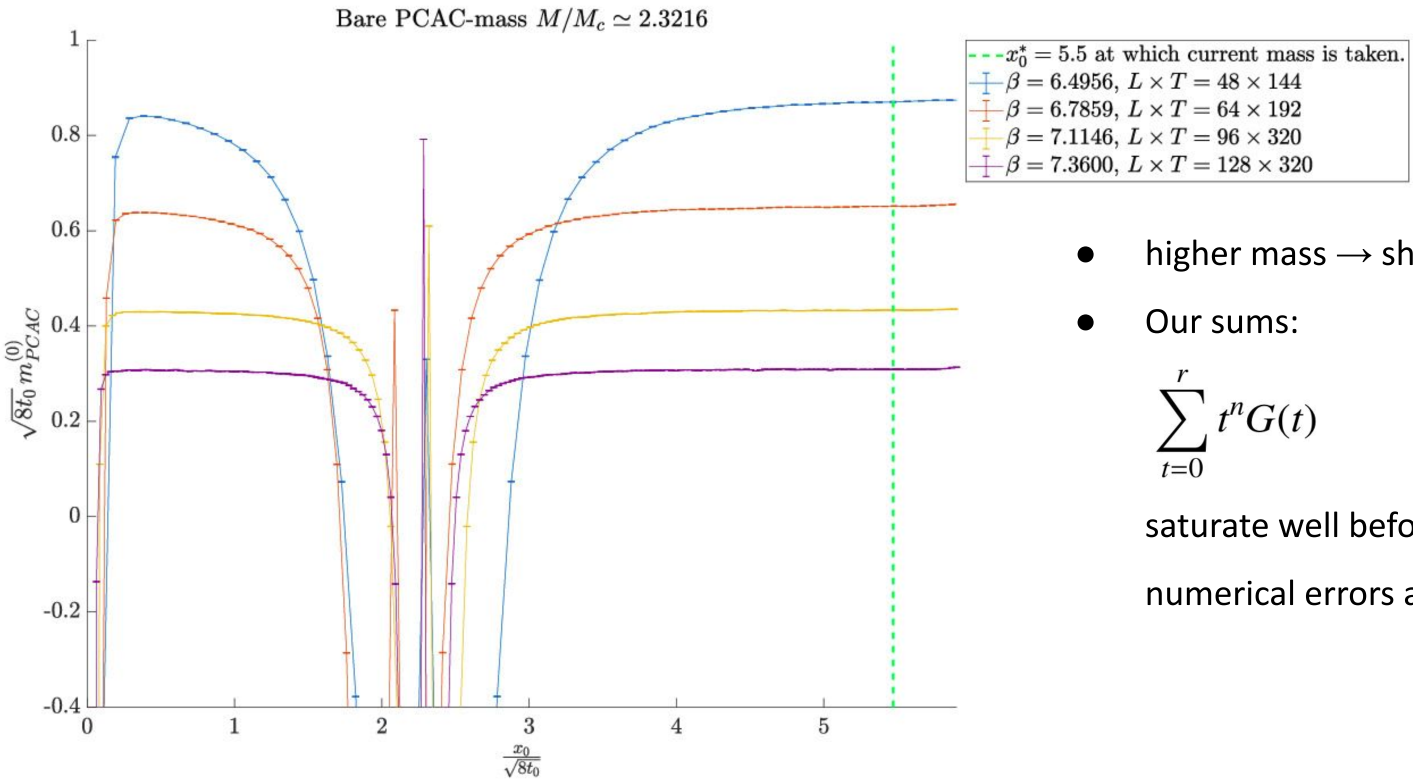


$$m_{PCAC} = \frac{\left\langle \sum_{\vec{x}\in V_3} \partial_\mu A_\mu^I(x)\, P(0) \right\rangle}{2\left\langle \sum_{\vec{x}\in V_3} P(x)\, P(0) \right\rangle} = \frac{\left\langle \sum_{\vec{x}\in V_3} \partial_0 A_0^I(x)\, P(0) \right\rangle}{2\left\langle \sum_{\vec{x}\in V_3} P(x)\, P(0) \right\rangle},$$

$$A_\mu^I(x) = A_\mu(x) + c_A(g_0)\, \partial_\mu^* P(x)$$

- Measure PCAC mass, select one value in plateau.
- Large t: numerical errors grow + states from $t = T$ boundary

PCAC Data II

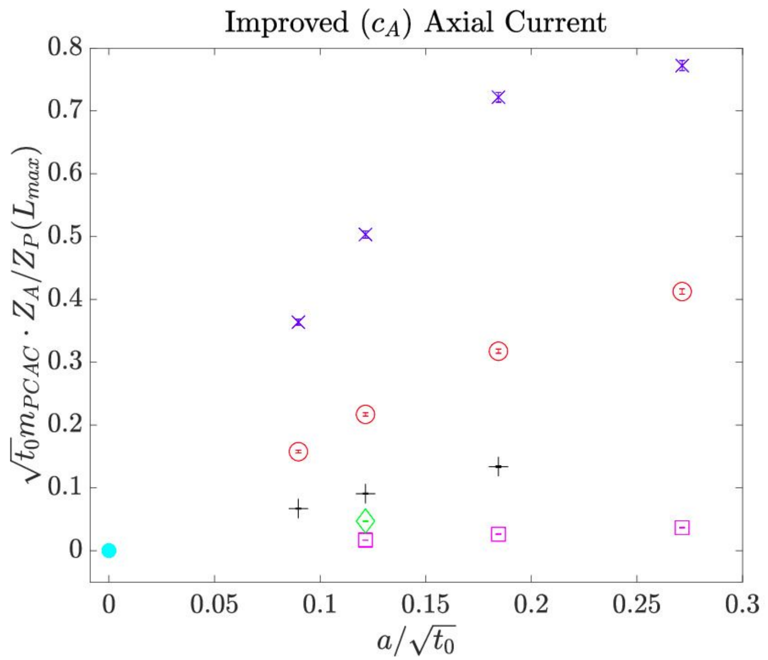


- higher mass → shorter plateau
- Our sums:

$$\sum_{t=0}^{r} t^n G(t)$$

saturate well before the point where numerical errors appear.

Monitoring Full Twist

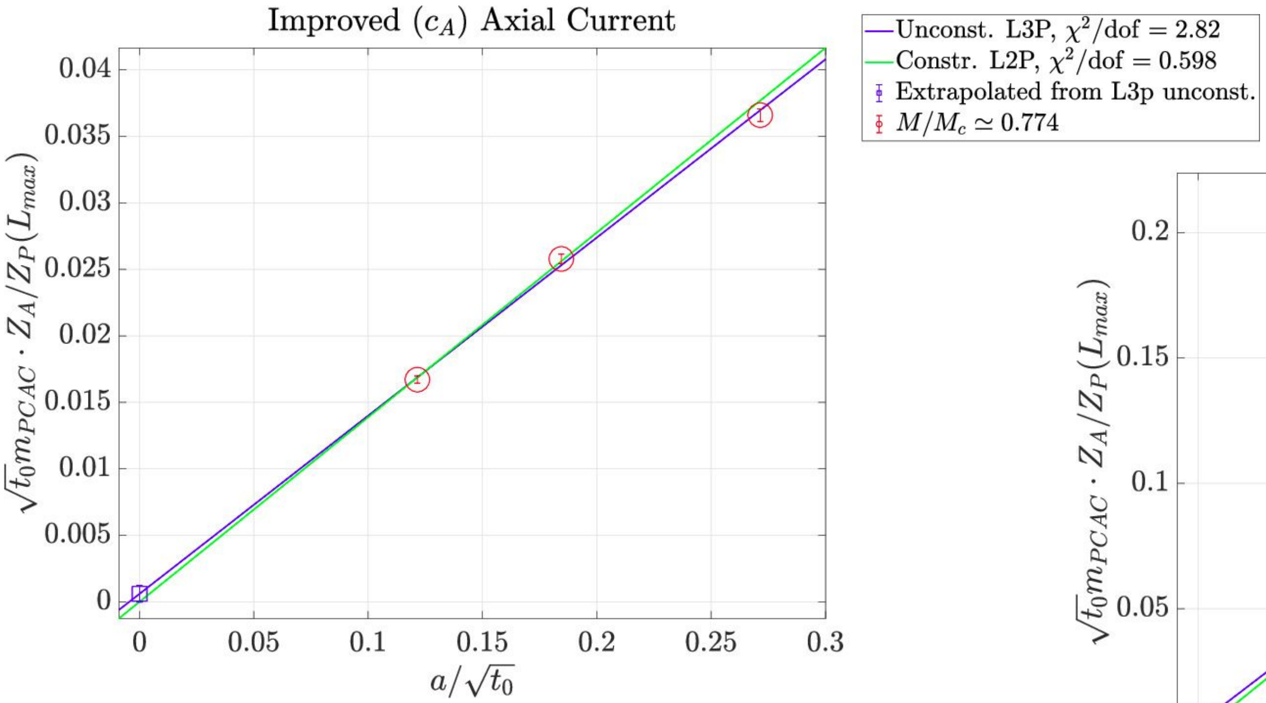


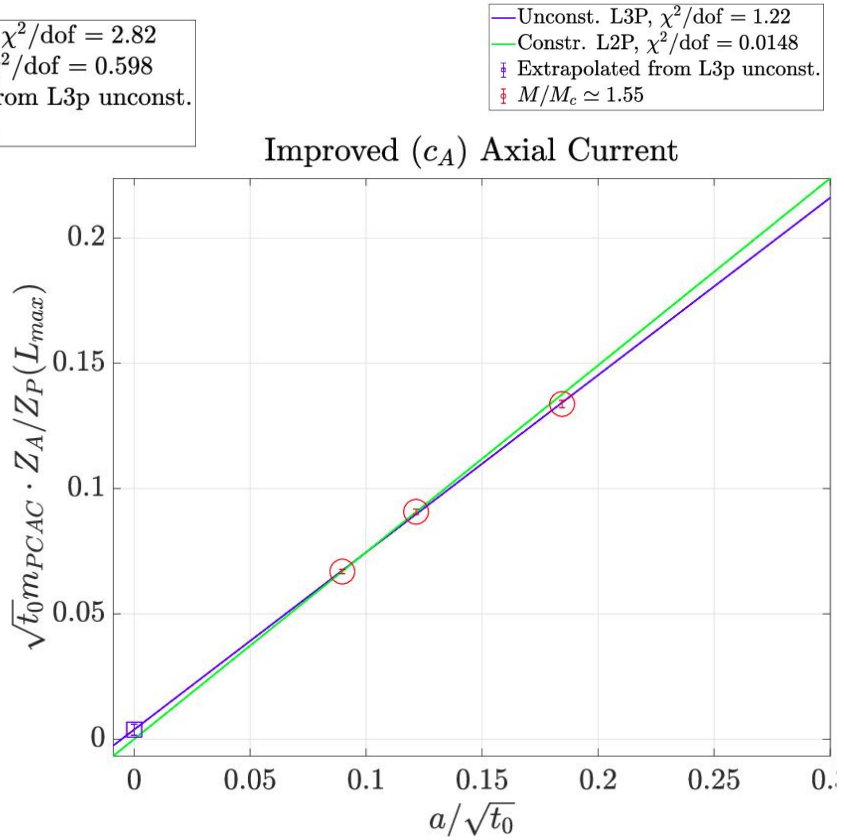
Note:
⟨ *J(x) J(0)* ⟩is *O(a)* improved, but the PCAC-mass is an *O(a)* effect at full twist

- ❖ Renormalized PCAC mass vs *a*
- ❖ All errors included: statistic, Z_A, Z_P
- ❖ also systematic error on κ, since there it was determined at fixed *L/a* = 16. Possible NP effects $O\left((a/L)^3 f(L/r)\right)$, but they are under control:
- ❖ *L/a*=8 ⟶ *L/a*=16 gives 2.0e-05 effect, propagate into m_{PCAC} ⟶ 5.5e-04 effect

$$m_{PCAC} = \frac{\left\langle \sum_{\vec{x}\in V_3} \partial_\mu A^I_\mu(x)\, P(0) \right\rangle}{2\left\langle \sum_{\vec{x}\in V_3} P(x)\, P(0) \right\rangle} = \frac{\left\langle \sum_{\vec{x}\in V_3} \partial_0 A^I_0(x)\, P(0) \right\rangle}{2\left\langle \sum_{\vec{x}\in V_3} P(x)\, P(0) \right\rangle}, \quad A^I_\mu(x) = A_\mu(x) + c_A(g_0)\, \partial^*_\mu P(x)$$

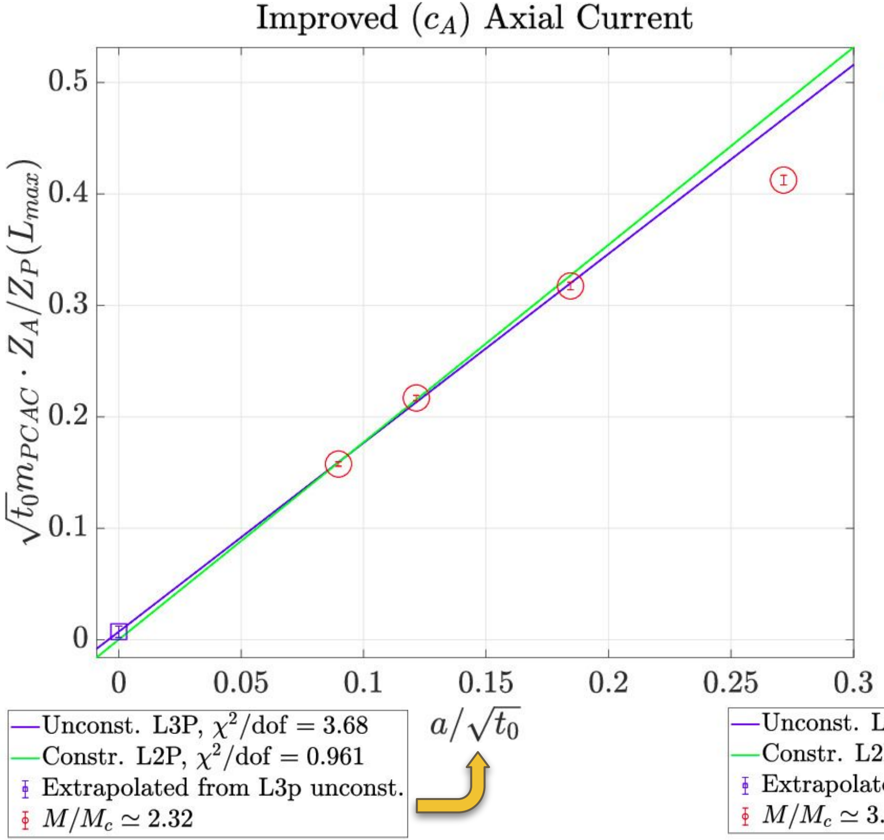
PCAC vs a - I



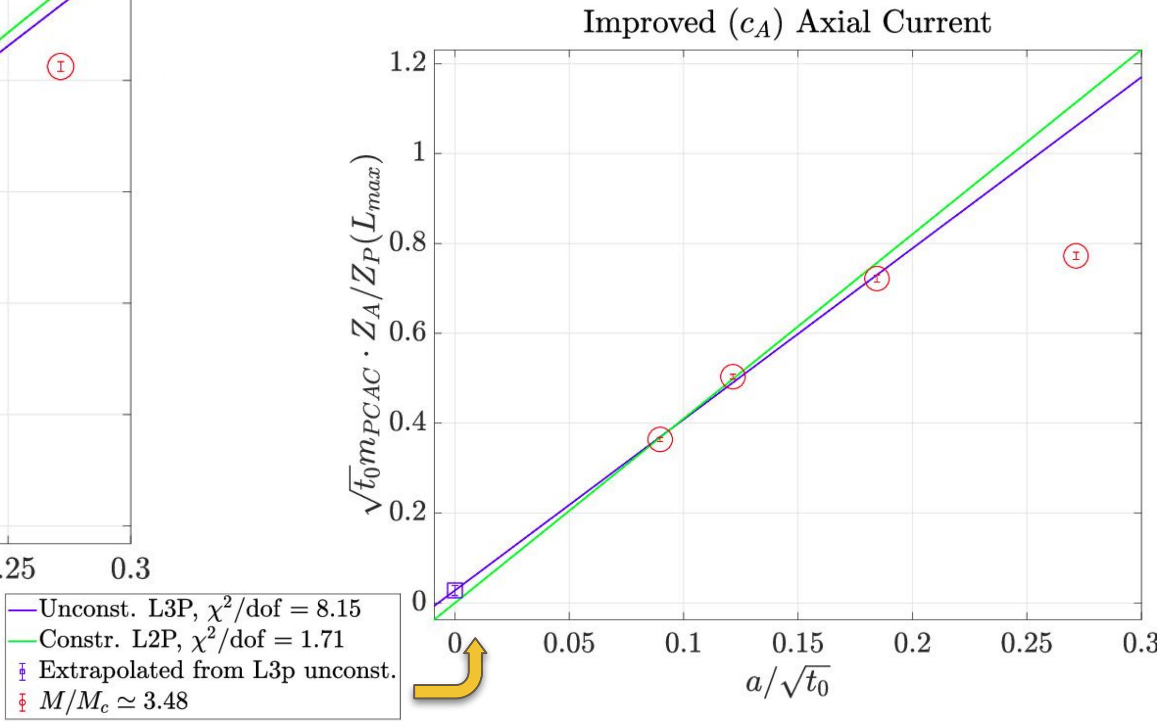


- Fits constrained through 0 show <u>linear behavior</u>

PCAC vs a - II



- Higher masses: We see violations to the linear behavior, but this is expected at large *a*

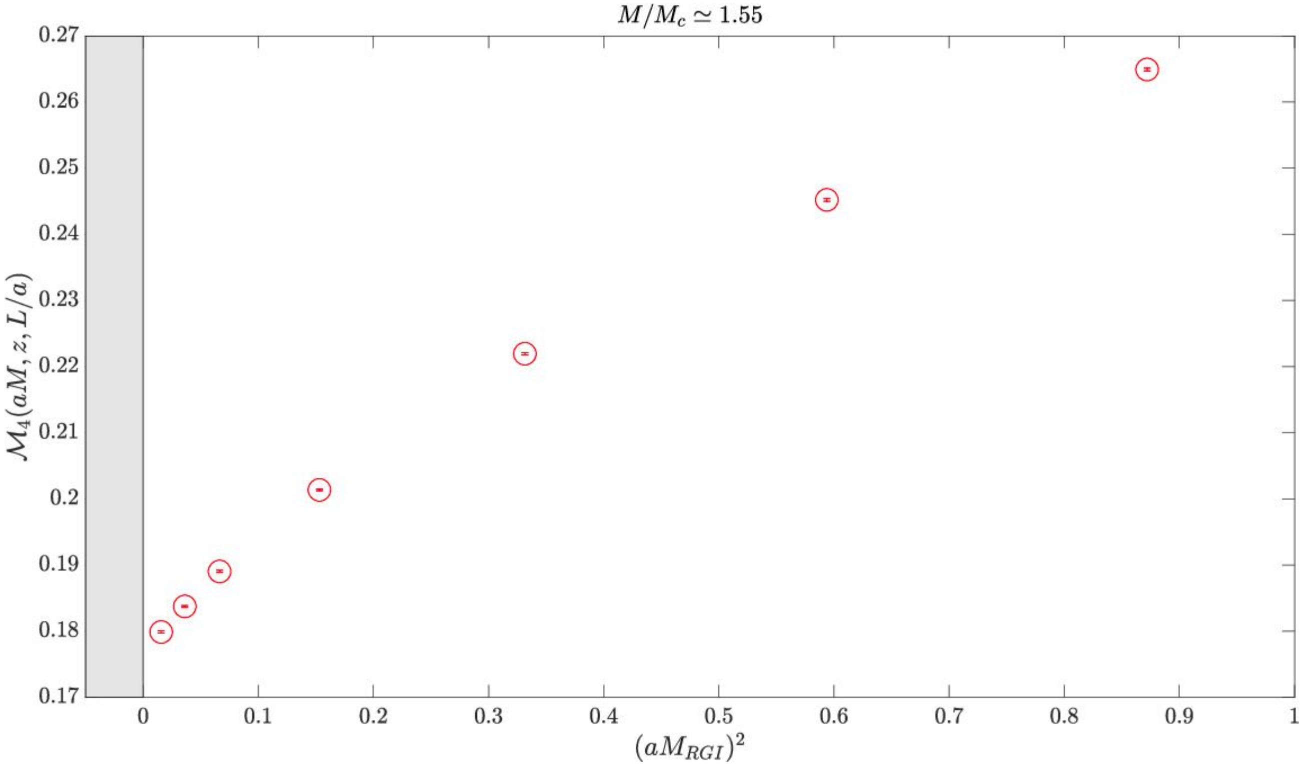


Continuum Limits for *n = 4* ?

Even with:

- **full twist**
- non-perturbative $\boldsymbol{c_{sw}}$
- **quenched** quarks

Cutoff effects with *a=0.01* fm are huge, *15%* for 4 leftmost points! One modification needed (which all collaborations using moments have done): **normalize by tree-level**



Tackling Cutoff Effects - I

★ We compute the **finite volume, finite *a* tree-level** (TL) **analytically** and divide the moments by it:

$$R_n(\sqrt{8t_0}M_{RGI}, aM_{RGI}) = \begin{cases} \dfrac{\mathcal{M}_n(\sqrt{8t_0}M_{RGI}, aM_{RGI})}{\mathcal{M}_n^{TL}(a\mu_{tm})}, & n = 4 \\ \left(\dfrac{\mathcal{M}_n(\sqrt{8t_0}M_{RGI}, aM_{RGI})}{\mathcal{M}_n^{TL}(a\mu_{tm})}\right)^{\frac{1}{n-4}}, & n > 4 \end{cases}$$

Leading cutoff effects suppressed by a power of the coupling

$$\mathcal{M}_n(a\mu_{tm}) = \mathcal{M}_n^{(0)}(a\mu_{tm}) + \alpha\mathcal{M}_n^{(1)}(a\mu_{tm}) + \ldots \rightarrow R_n(a\mu_{tm}) = 1 + \alpha R_n^{(1)}(a\mu_{tm}) + \ldots$$

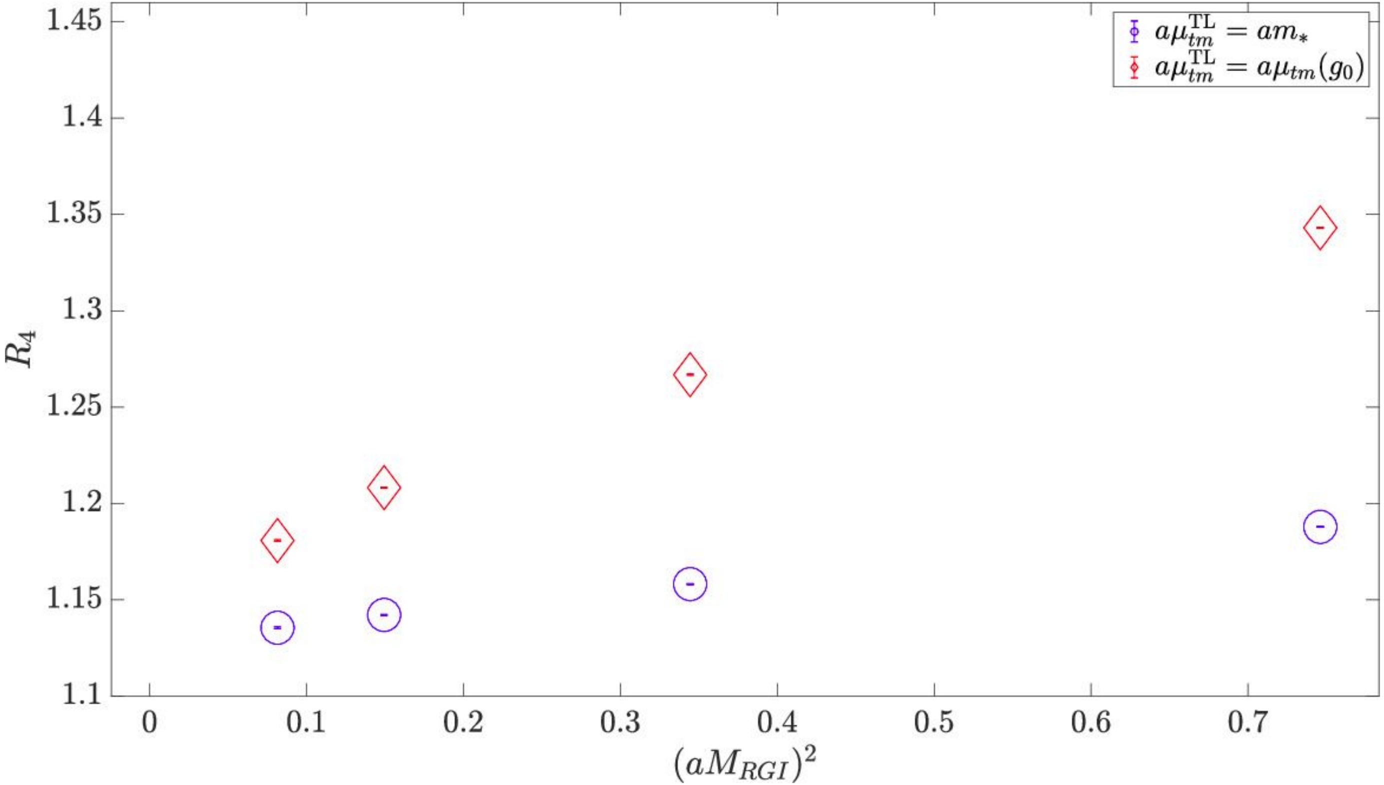
$O(a^2\mu_{tm}^2) \rightarrow O(\alpha a^2\mu_{tm}^2)$ up to logs (see Husung, Marquard, Sommer [*hep-lat/1912.08498*])

★ Caveat: for *n=4*, potentially *log(a)* corrections arise at TL (more later).

★ For *n > 4* take ratios of moments to get rid of strong mass dependence and mitigate some error sources:

$$\lim_{a\to 0} \frac{R_n(\sqrt{8t_0}M_{RGI}, aM_{RGI})}{R_{n+2}(\sqrt{8t_0}M_{RGI}, aM_{RGI})} = \sum_{i\geq 0}^{L} c_n^{(i)}(\mu/\overline{m}_{\overline{\rm MS}}(\mu))\, \alpha_{\overline{\rm MS}}^{i}(\mu) + O(\alpha^{L+1})$$

Tackling Cutoff Effects - II

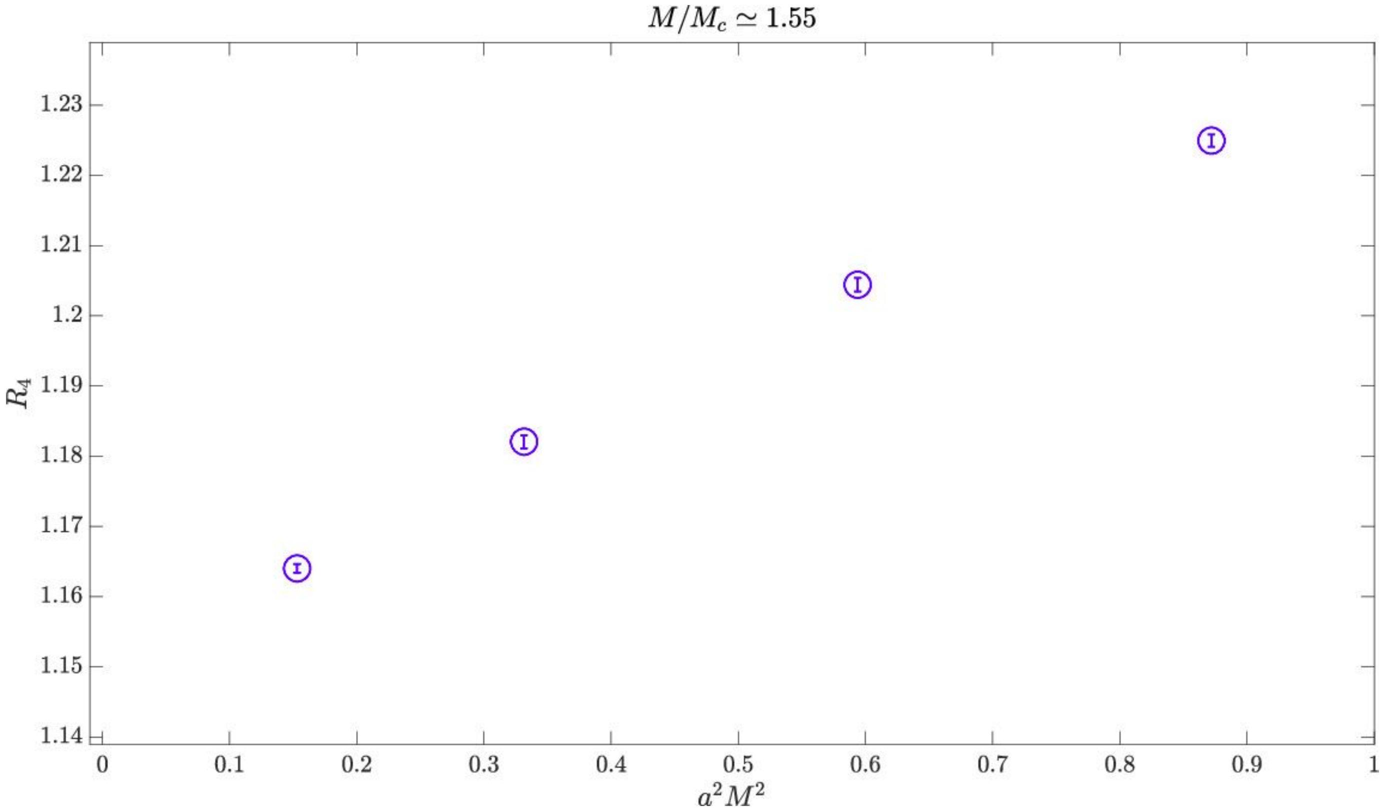


The TL normalization is performed with

$$m_* = \overline{m}_{\overline{\rm MS}}(m_*)$$

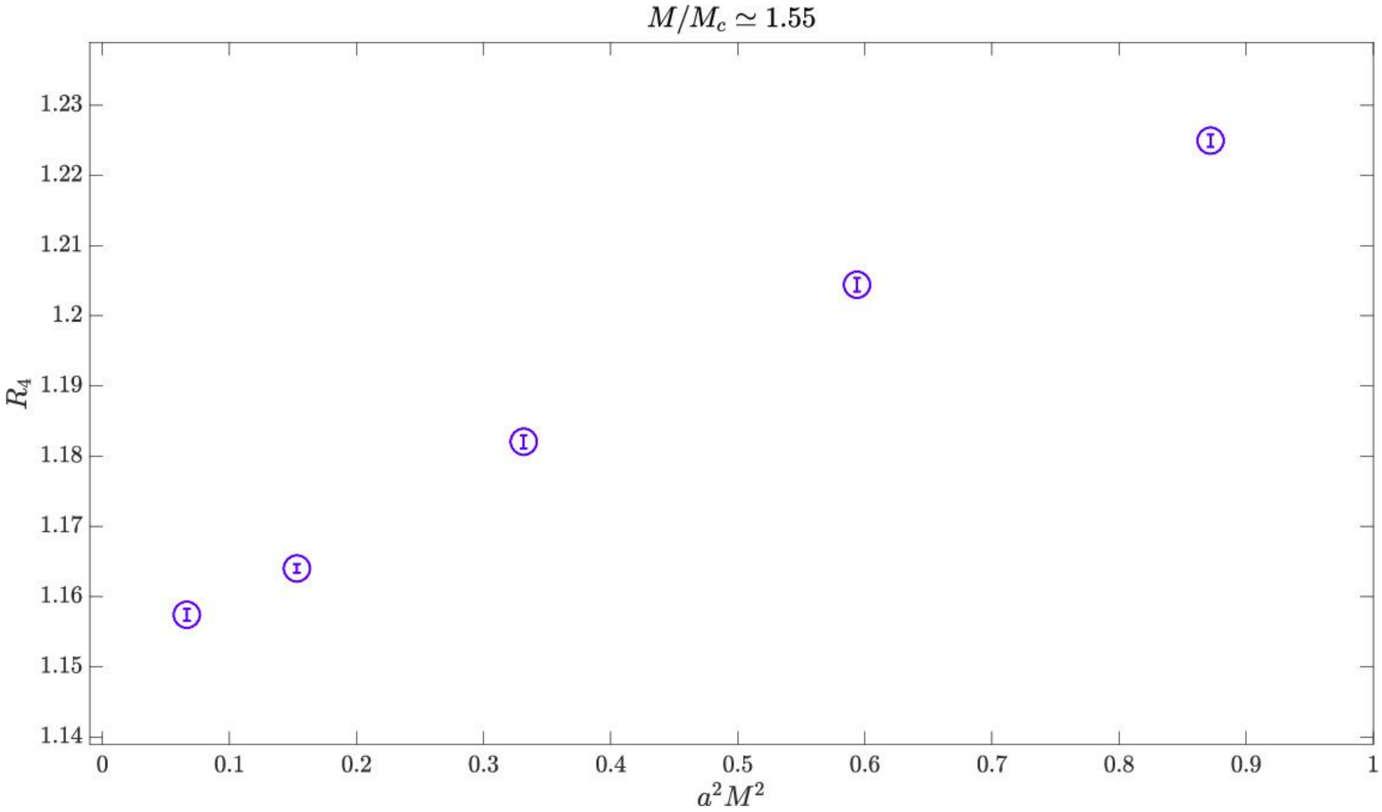
not with the bare parameter value of the non-perturbative simulation

Continuum Extrapolation of R_4: $M/M_c \sim 1.6$



- Smallest lattice spacing: ***a = 0.030*** **fm**
- Now cutoff effect for 4 leftmost points: about *5%*

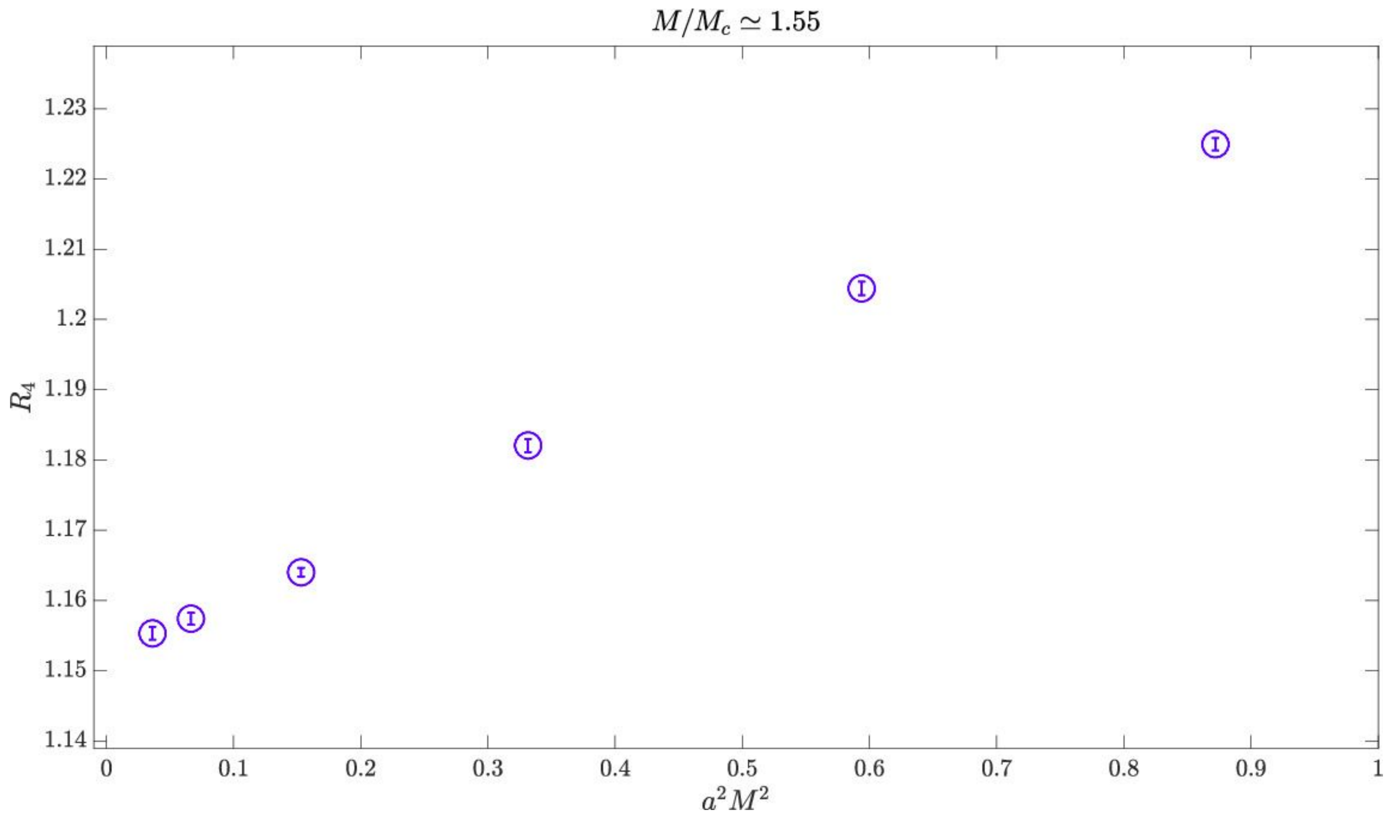
Continuum Extrapolation of R_4: $M/M_c \sim 1.6$



- Smallest lattice spacing:

  ***a* = *0.020* fm**

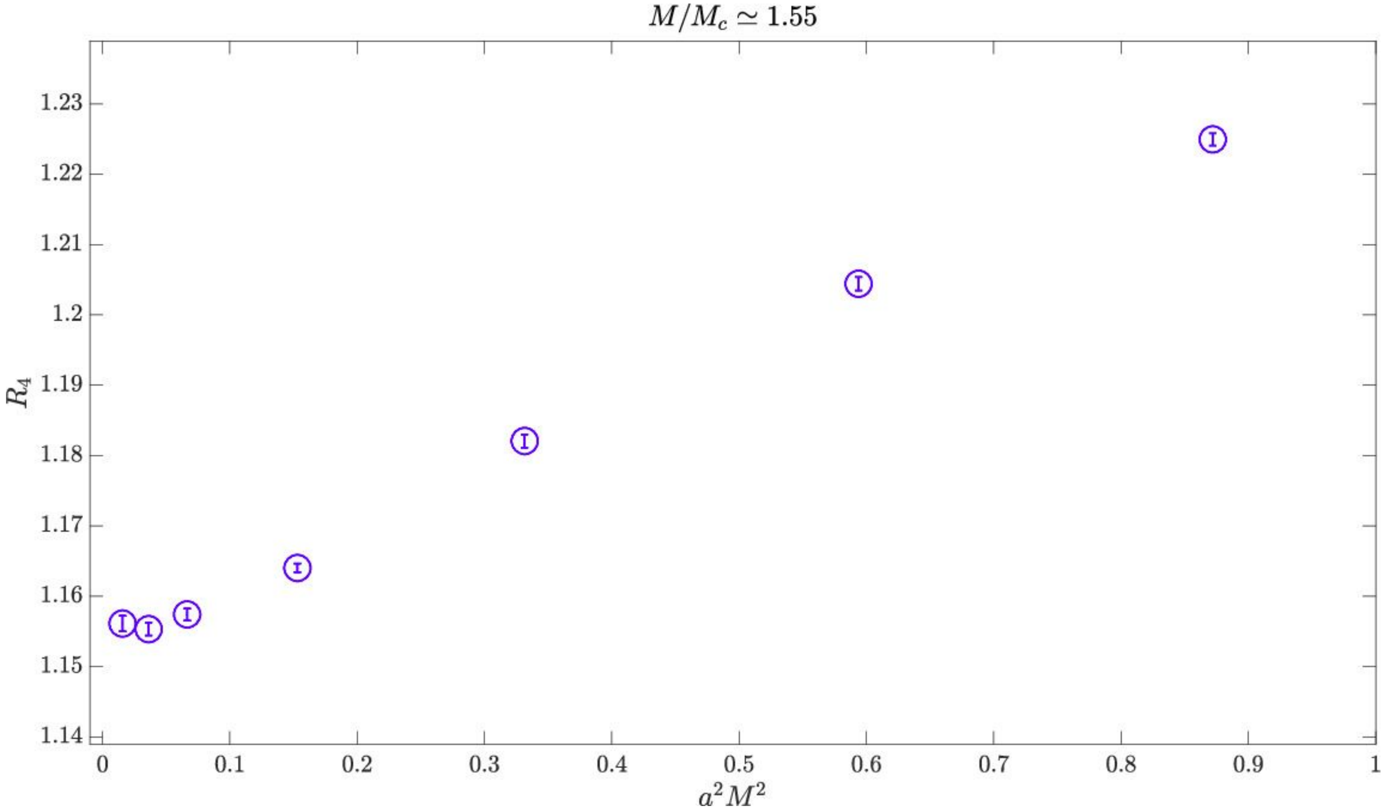
Continuum Extrapolation of R_4: $M/M_c \sim 1.6$



- Smallest lattice spacing:

  ***a* = *0.015* fm**

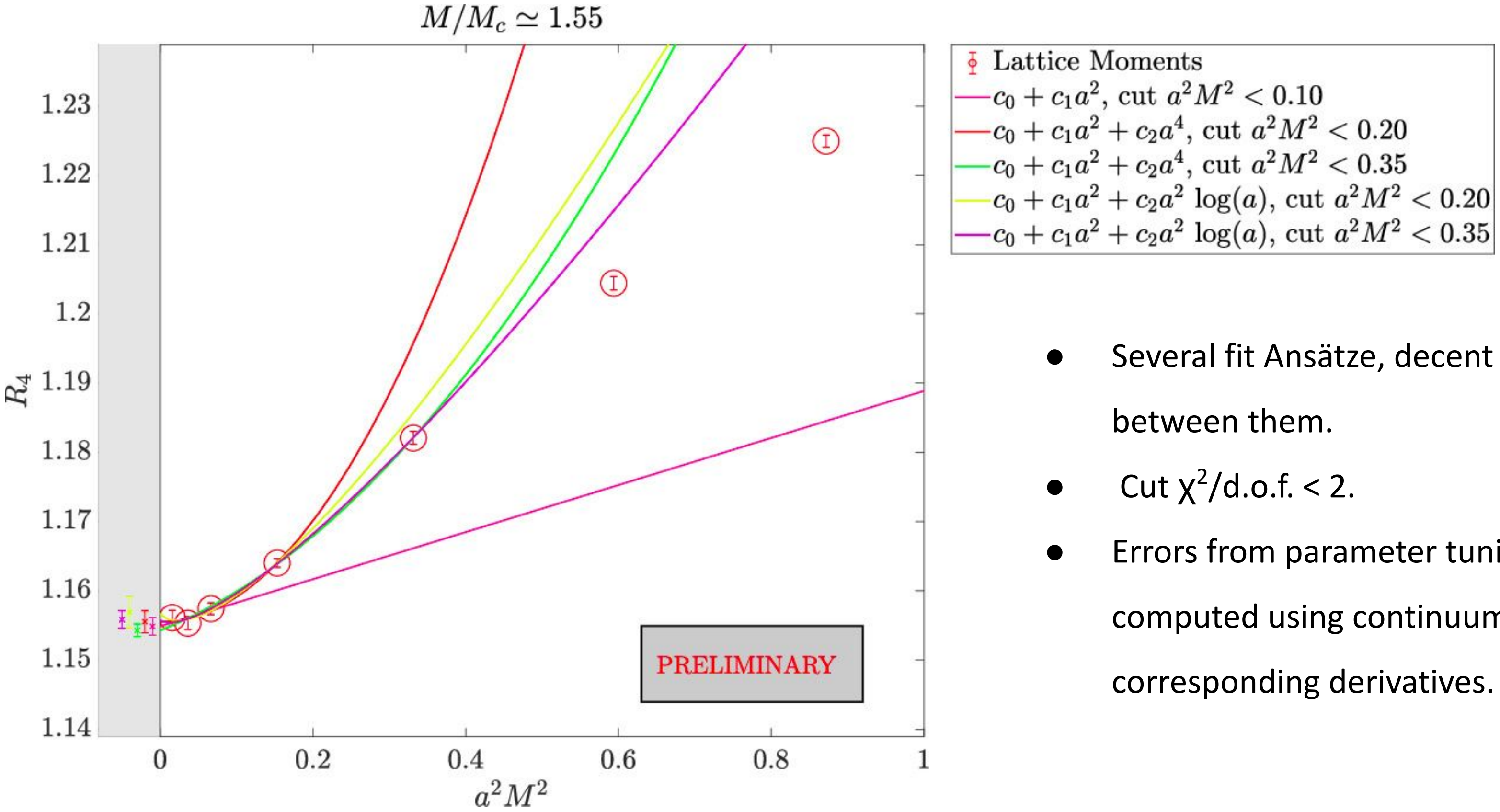
Continuum Extrapolation of R_4: $M/M_c \sim 1.6$



- Smallest lattice spacing:

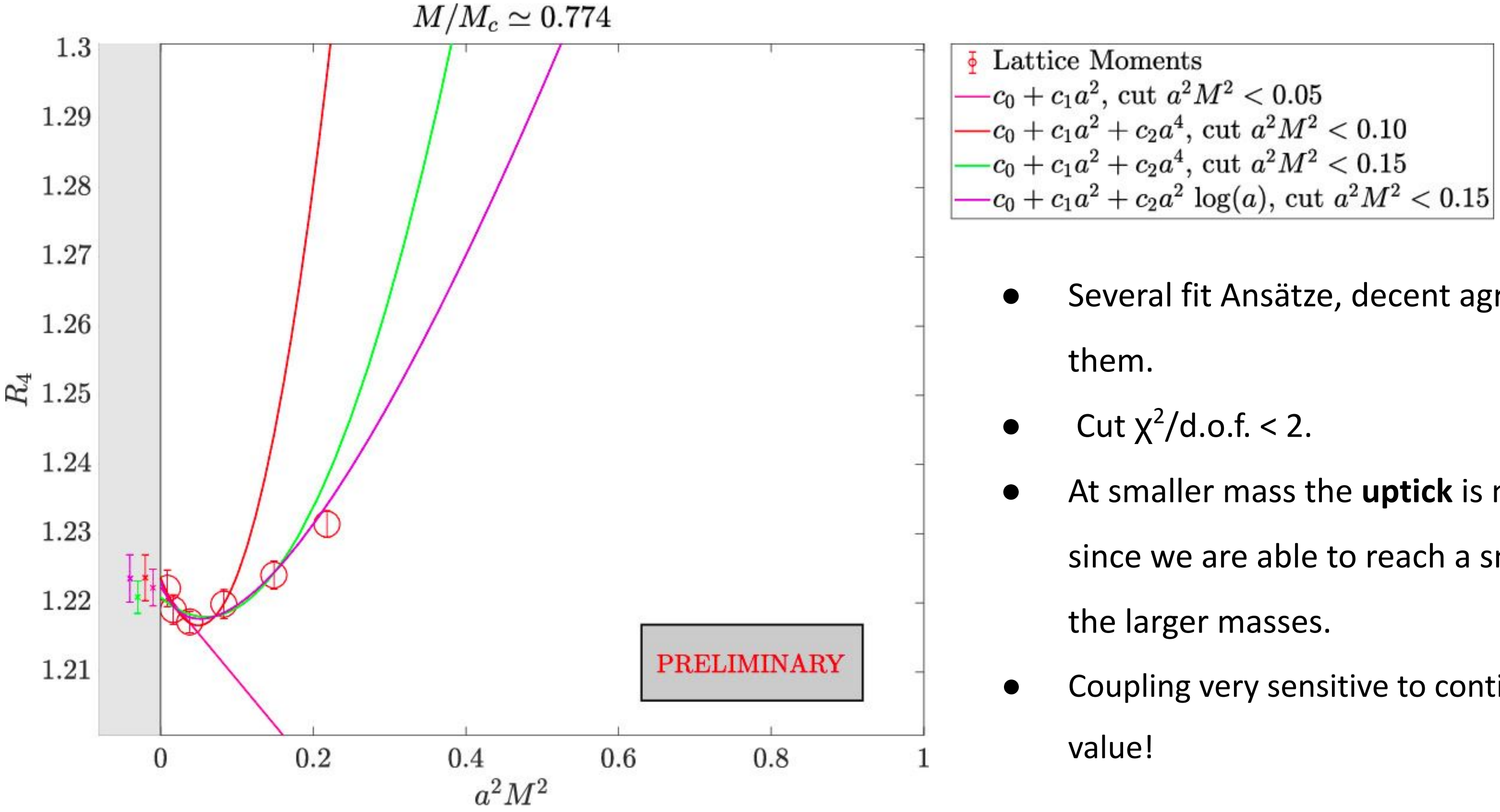
  ***a* = *0.010* fm**

Continuum Extrapolation of R_4: $M/M_c \sim 1.6$



- Several fit Ansätze, decent agreement between them.
- Cut χ^2/d.o.f. < 2.
- Errors from parameter tuning (Z_P, t_0/a^2, κ) computed using continuum PT for corresponding derivatives.

Continuum Extrapolation of R_4: $M/M_c \sim 0.8$



- Several fit Ansätze, decent agreement between them.
- Cut χ^2/d.o.f. < 2.
- At smaller mass the **uptick** is much more visible, since we are able to reach a smaller *aM* w.r.t. the larger masses.
- Coupling very sensitive to continuum extrapolated value!

Intermezzo: Why log Fits?

- We find the presence of $(aM)^2 log(aM)$ terms at tree-level (see also [Cè et al. (2021)] for g-2).
- Preliminary analysis: these **logs seem to be present** in the true $a \rightarrow 0$ asymptotics, although weaker.
- This is a work in progress and will be discussed at the Lattice Conference 2022.
- Presently, $\boldsymbol{R_4}$ **cannot be extrapolated** to the continuum with reasonable precision.
- Note: this is the reason JLQCD [Nakayama et al. 2016] **did not use** $\boldsymbol{R_4}$, but only higher moments.

❖ Naive Symanzik expansion, a^2 terms go like:

$$a^2 \int \mathrm{d}t\, t^{n-5}$$

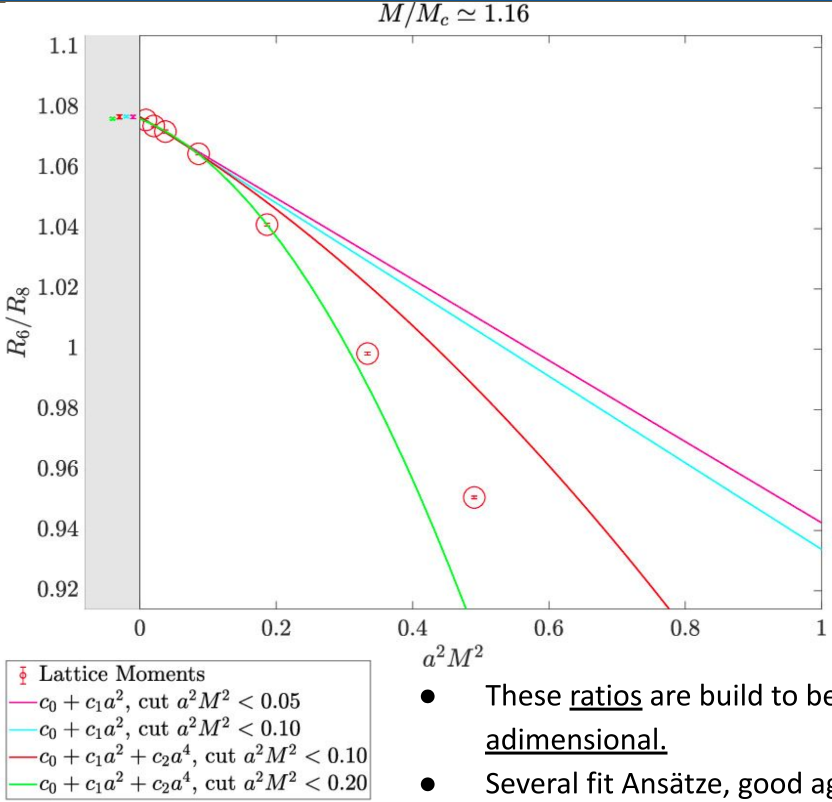
Not integrable for $n = 4$
Integrable for $n = 6, 8, \ldots$

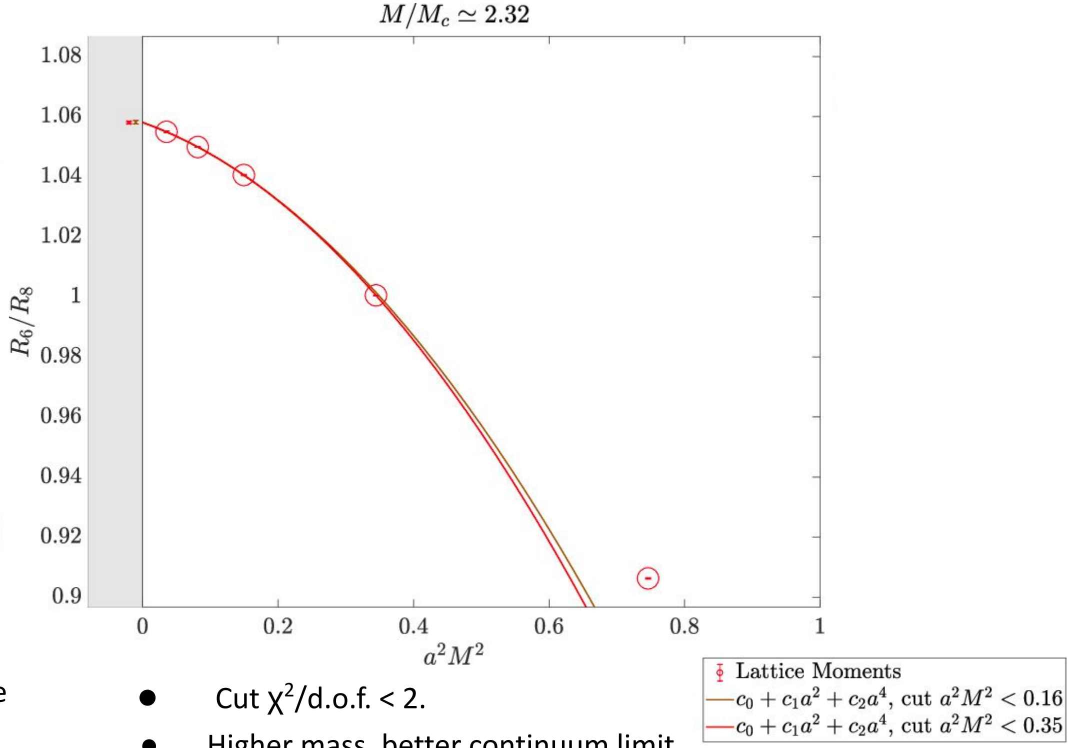
❖ Or, more formally, but still naively:

- ➢ $I(0, t) \longrightarrow$ integral for $n=4$ from 0 to t (i.e. the moment if $t \longrightarrow \infty$)
- ➢ $\tau = t/a$
- ➢ ΔI the **difference** between the continuum **integral** and the lattice **sum**, we find:

$$\frac{\Delta I(0,t)}{I_{\text{cont}}(0,t)} \sim \tau^{-2} \sum_i k_i \int_{\tau_0}^{\tau} 1/s\,\mathrm{d}s + u_i(\tau_0)$$
$$= \tau^{-2} \sum_i k_i \left[\log(\tau) + u_i(1)\right]$$

Continuum Extrapolations: R_6/R_8

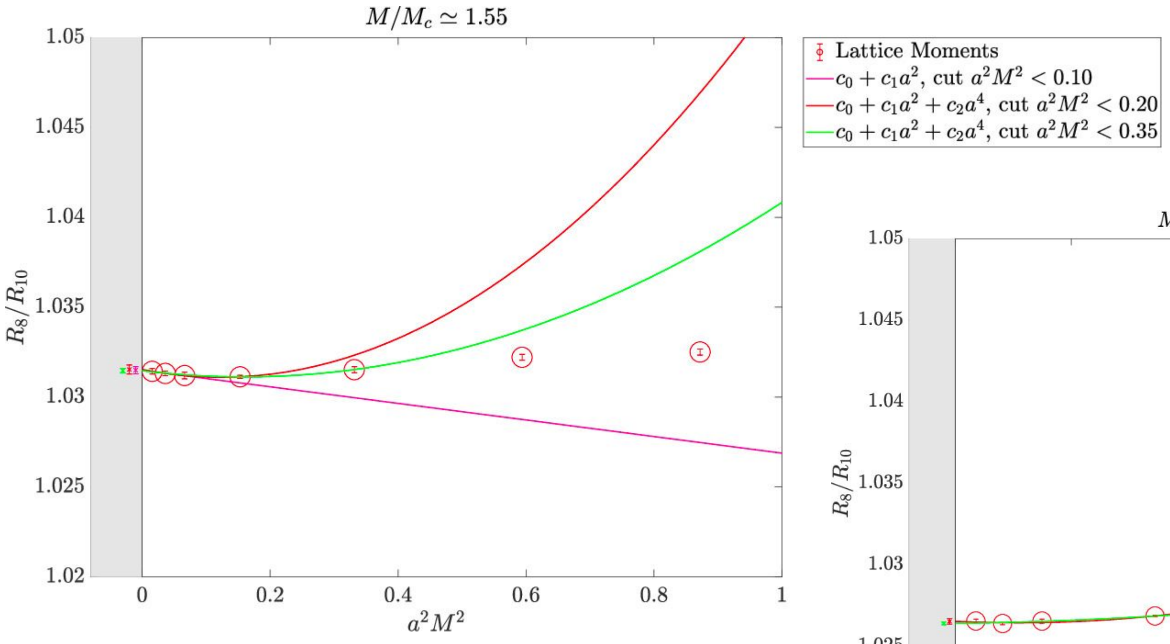


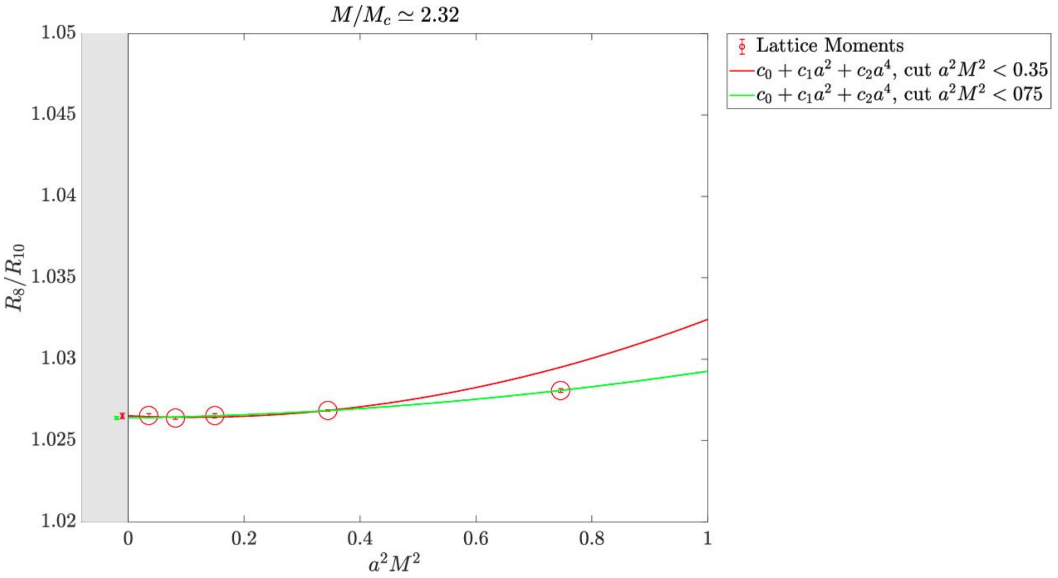


- These ratios are build to be adimensional.
- Several fit Ansätze, good agreement between them.

- Cut χ^2/d.o.f. < 2.
- Higher mass, better continuum limit.
- But less perturbative!

Continuum Extrapolations: R_8/R_{10}





- Highest moments, best continuum limit.
- Least perturbative

Extracting α

- Perturbative expansion of moments with known coefficients:

$$\lim_{a\to 0} R_4(\sqrt{8t_0}M_{RGI}, aM_{RGI}) \overset{\alpha\to 0}{\sim} \sum_{i=0}^{L} r_4^{(i)}\left(\mu/\overline{m}_{\overline{\rm MS}}(\mu)\right) \alpha^i_{\overline{\rm MS}}(\mu) + \mathcal{O}\left(\alpha^{L+1}(\mu)\right)$$

- For each RGI-mass, fix scale parameter s from $\mu_s = s\overline{m}_{\overline{\rm MS}}(\mu_s)$ and invert to obtain α at this scale. Then repeat for different s.

$$\lim_{a\to 0} R_4(\sqrt{8t_0}M_{RGI}, aM_{RGI}) \overset{\alpha\to 0}{\sim} \sum_{i=0}^{3} r_4^{(i)}(s)\, \alpha^i_{\overline{\rm MS}}(\mu_s) + \mathcal{O}(\alpha^4(\mu_s))$$

- Now we have two handles to turn to see how the truncated term varies.

$$r_n^{(i)}\left(\mu/\overline{m}_{\overline{\rm MS}}(\mu)\right) = r_n^{(i,0)} + \sum_{k=1}^{i-1} r_n^{(i,k)} \log^k\left(\mu/\overline{m}_{\overline{\rm MS}}(\mu)\right) \longrightarrow r_n^{(i)}(s) = r_n^{(i,0)} + \sum_{k=1}^{i-1} r_n^{(i,k)} \log^k(s)$$

$$\mathcal{R}(\alpha) \overset{\alpha\to 0}{\sim} \sum_{n=0}^{\infty} c_n(s)\,\alpha(\mu)^n \iff \lim_{\alpha\to 0}\left|\mathcal{R}(\alpha) - \sum_{n=0}^{N} c_n(s)\,\alpha(\mu)^n\right| = \mathcal{O}(\alpha^{N+1}), \quad \forall N < \infty$$

Asymptotic series!
$s \ll 1$ or $s \gg 1$ means coefficents may spoil quality of series

Scale Variations and Truncation Error

$$\mathcal{M}_n(m,\mu) \overset{\alpha\to 0}{\sim} \sum_{i=0}^{L} c_n^{(i)}(\mu/\overline{m}_{\overline{\text{MS}}})\, \alpha^i_{\overline{\text{MS}}}(\mu) + \mathcal{O}(\alpha^{L+1}(\mu))$$

- Spurious μ dependence
- Hard to estimate, systematic, scale dependent truncation error

- An <u>observable does not depend on μ</u>, but in its asymptotic perturbative expansion there will, in general, be a spurious due to the truncation.
- Variation of *s* as an estimate of the truncation?

$$r_n^{(i)}(s) = r_n^{(i,0)} + \sum_{k=1}^{i-1} r_n^{(i,k)} \log^k(s)$$

- … but no knowldege about $r_n^{(i,0)}$
- Estimate of truncation error is tricky. Using $r_n^{(4,0)} \simeq c\, r_n^{(3,0)}$, $c = 2 \div 5$, gives smaller error estimate w.r.t. to variation of $s \in [0.5, 2]$ done at fixed mass. Is this reliable?
- FAC (Fastest Apparent Converegence) scale: fix *s* so that first scale dependent coefficient is zero, i.e.:

$$R_4(\sqrt{8t_0}M_{RGI},0) \overset{\alpha\to 0}{\sim} 1 + r_4^{(1)}\,\alpha_{\overline{\text{MS}}}(\mu_{s_{opt}}) + r_4^{(3)}(s_{opt})\,\alpha^3_{\overline{\text{MS}}}(\mu_{s_{opt}}) + \mathcal{O}(\alpha^4) \iff r_4^{(2)}(s_{opt}) = 0$$

The Λ Parameter

$$\frac{\mathrm{d}\bar{g}}{\mathrm{d}\ln\mu} = \beta(\bar{g}(\mu)) \to \mathrm{d}\ln\mu = \int^{\bar{g}} \mathrm{d}x \frac{1}{\beta(x)} + C \Longrightarrow \Lambda_{RGI} = \mu(b_0\bar{g}^2)^{-b_1/(2b_0^2)} \exp\left\{-\frac{1}{2b_0\bar{g}^2} - \int_0^{\bar{g}} \mathrm{d}x\left[\frac{1}{\beta(x)} + \frac{1}{b_0x^3} - \frac{b_1}{b_0^2x}\right]\right\}$$

- Contains info of <u>coupling</u> and <u>running.</u>
- Is an <u>integration constant</u> of RGEs.

$$\left(\alpha(\mu),\, m(\mu)\right) \longleftrightarrow \left(\Lambda_{RGI},\, M_{RGI}\right)$$

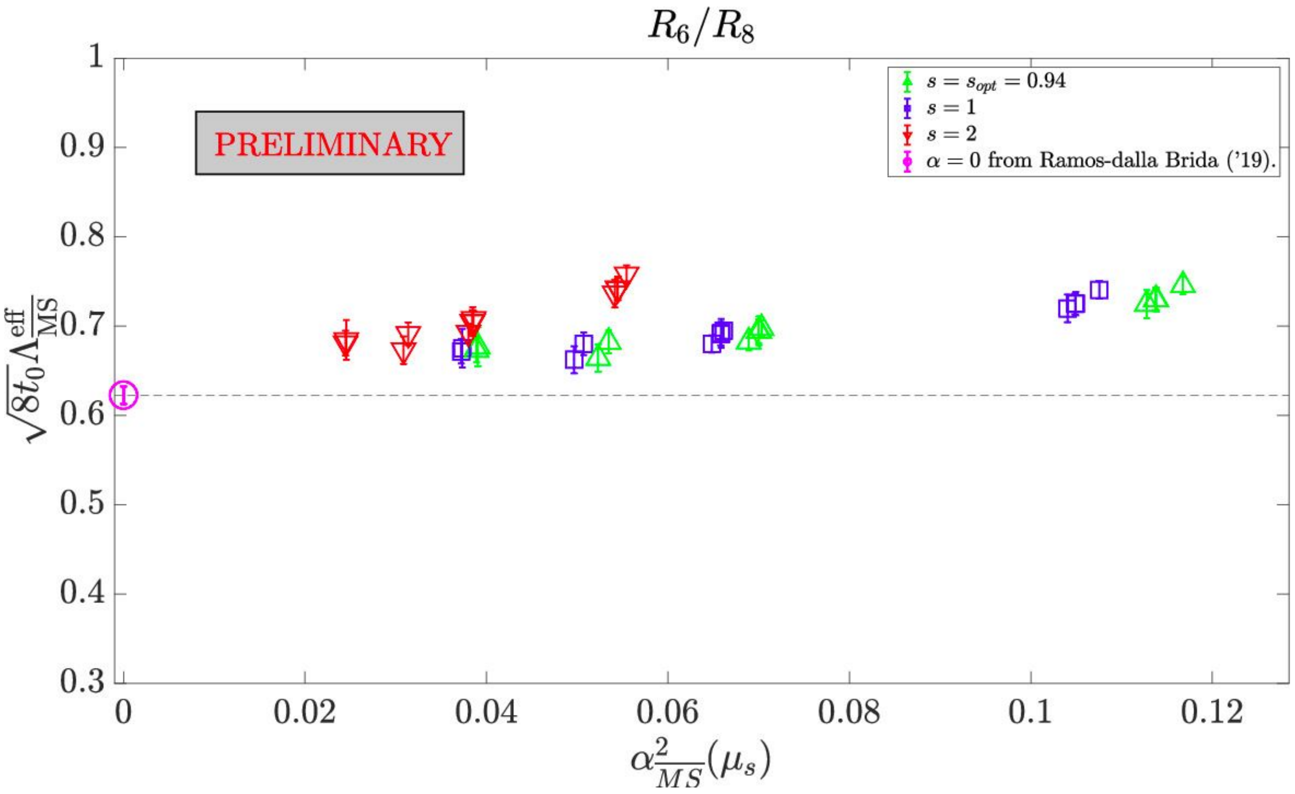
- Run to infinite energy via 5L β-function and 4L τ (mass anomalous dimension):

$$\frac{\sqrt{8t_0}\Lambda_{RGI}^{\overline{\mathrm{MS}}}}{\sqrt{8t_0}M_{RGI}} = s\frac{(b_0\bar{g}_{\overline{\mathrm{MS}}}(\mu_s)^2)^{-b_1/(2b_0^2)}}{(2b_0\bar{g}_{\overline{\mathrm{MS}}}(\mu_s)^2)^{-d_0/(2b_0)}} \exp\left\{-\frac{1}{2b_0\bar{g}(\mu_s)^2} - \int_0^{\bar{g}_{\overline{\mathrm{MS}}}(\mu_s)} \mathrm{d}x\left[\frac{1-\tau(x)_{\overline{\mathrm{MS}}}}{\beta_{\overline{\mathrm{MS}}}(x)} + \frac{1}{b_0x^3} - \frac{b_1}{b_0^2x} + \frac{d_0}{b_0x}\right]\right\}$$

- Given the coupling at μ_s and our choice of mass z, we can compute Λ. Given the above and $O(\alpha^3)$ knowledge of moments one has:

$$\Lambda_{RGI}^{\mathrm{eff}} = \Lambda_{RGI} + O\left(\alpha_{\overline{\mathrm{MS}}}^2(\mu_s)\right)$$

Λ Plot from R_6/R_8

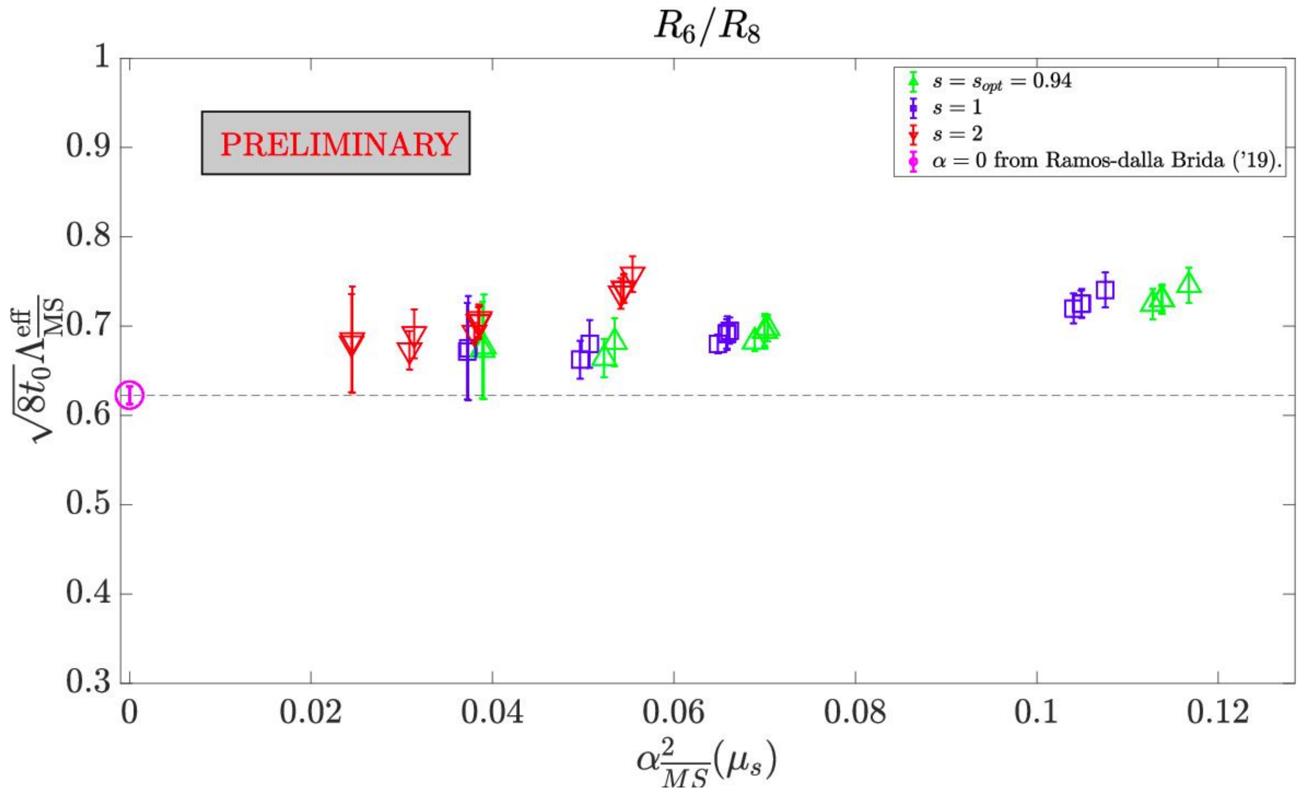


$$\Lambda^{\text{eff}}_{RGI} = \Lambda_{RGI} + O\left(\alpha^2_{\overline{\text{MS}}}(\mu_s)\right)$$

$$\mu_s = s\overline{m}_{\overline{\text{MS}}}(\mu_s)$$

- Largest mass dropped here, really not possible to take a continuum limit.
- Still, better than R_4, also **no *log(a)*** term here.

Λ Plot from R_6/R_8 : Extra Error

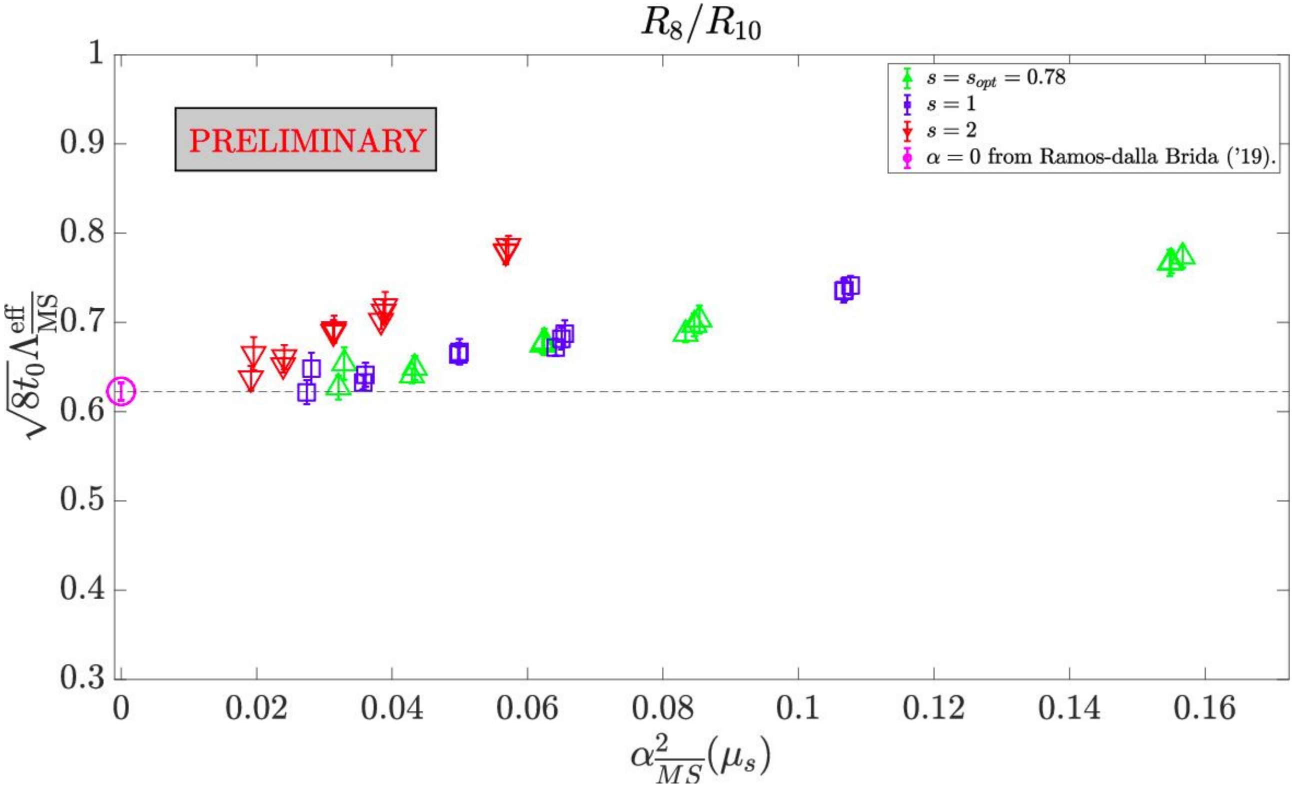


$$\Lambda^{\rm eff}_{RGI} = \Lambda_{RGI} + O\left(\alpha^2_{\overline{\rm MS}}(\mu_s)\right)$$

$$\mu_s = s\overline{m}_{\overline{\rm MS}}(\mu_s)$$

- Still, better than R_4, also **no *log(a)*** term here.
- **Extra systematic error: ½ the distance between leftmost and extrapolated point.**

Λ Plot from R_8/R_{10}

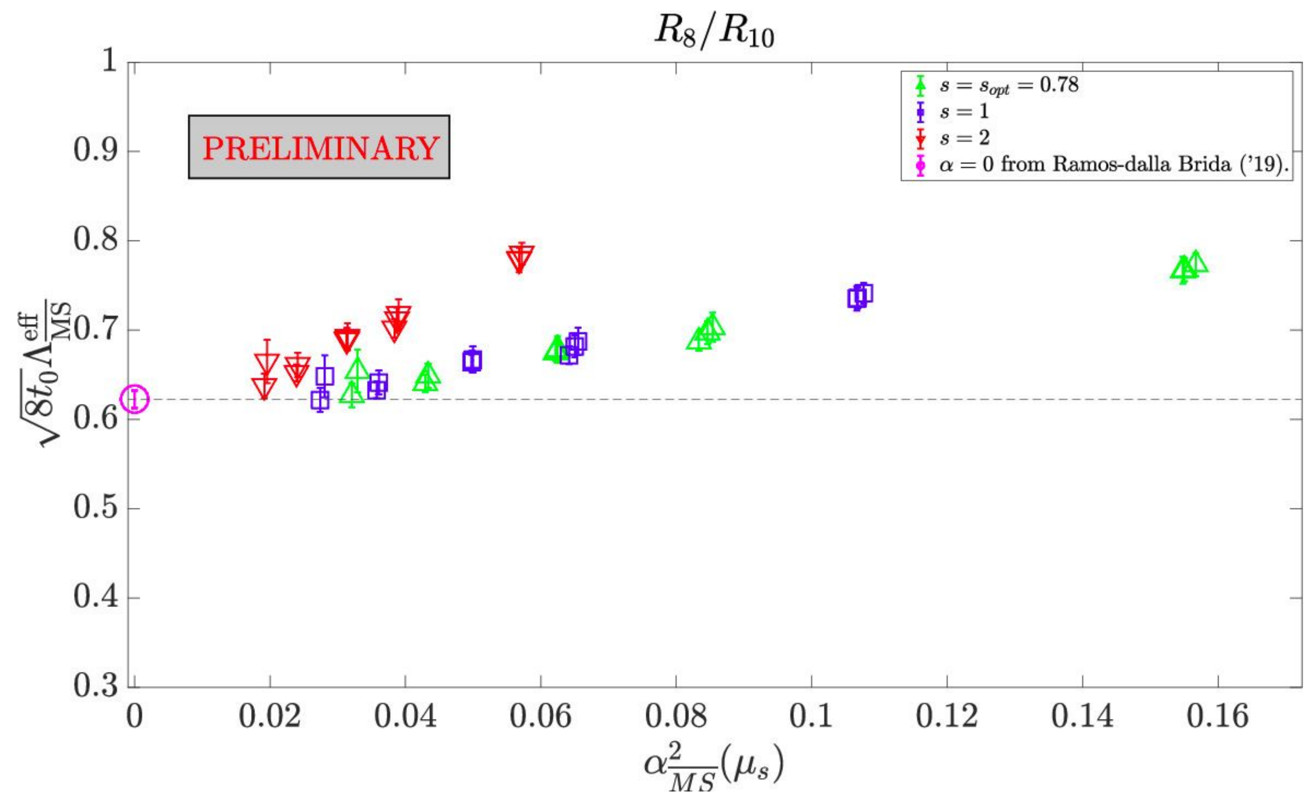


$$\Lambda^{\text{eff}}_{RGI} = \Lambda_{RGI} + O\left(\alpha^2_{\overline{\text{MS}}}(\mu_s)\right)$$

$$\mu_s = s\overline{m}_{\overline{\text{MS}}}(\mu_s)$$

- Continuum limit more reliable. Also **no *log(a)*** here.
- Is **least perturbative** (maybe even FV effects, but this affects rightmost points, at lightest mass).

Λ Plot from R_8/R_{10} : Extra Error



$$\Lambda^{\text{eff}}_{RGI} = \Lambda_{RGI} + O\left(\alpha^2_{\overline{\text{MS}}}(\mu_s)\right)$$

$$\mu_s = s\bar{m}_{\overline{\text{MS}}}(\mu_s)$$

- **Extra systematic error: ½ the distance between leftmost and extrapolated point.**
- Our results are compatible with a linear extrapolation constrained to go through the Ramos-dalla Brida result.

Summary and Outlook

- Even with non-perturbative c_{sw} and fully twisted, quenched Wilson fermions taking the **continuum limit** is very **challenging**.
- We are looking into **understanding log-enhanced cutoff effects of R_4** due to the short distance region of the correlation function.
- Performing **global fits** of all *M, a* together might help.
- Extracting the Λ-parameter from R_4 with controlled errors is very demanding. It might be beyond our capabilities in the pure gauge theory.
- Extracting the Λ-parameter from ratios of R_n , with ***n=6,8,10***, looks much better.
- But the higher moments are **less perturbative.**
- All in all, **window problem** looks **tough** for **moments method**.

Thank You!

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Questions Please!

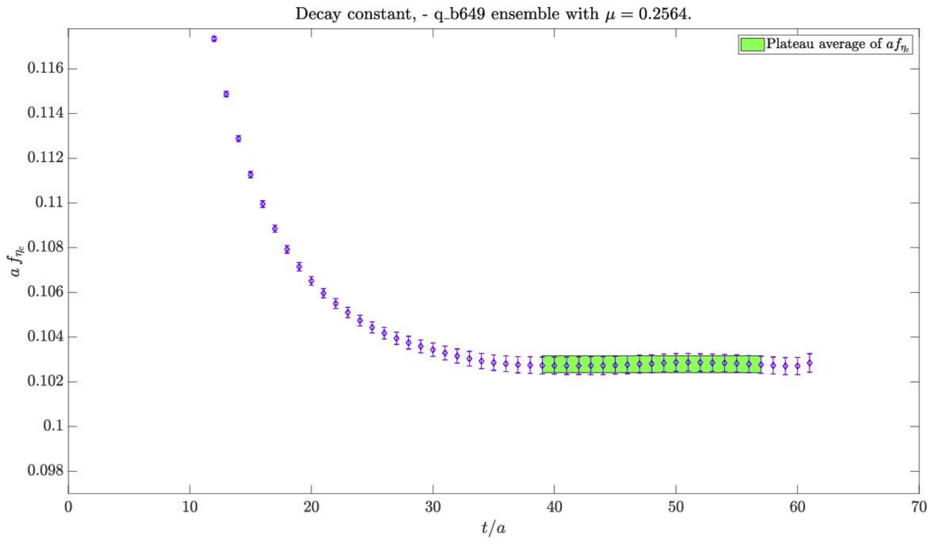
BUP 0.1: Checks - I

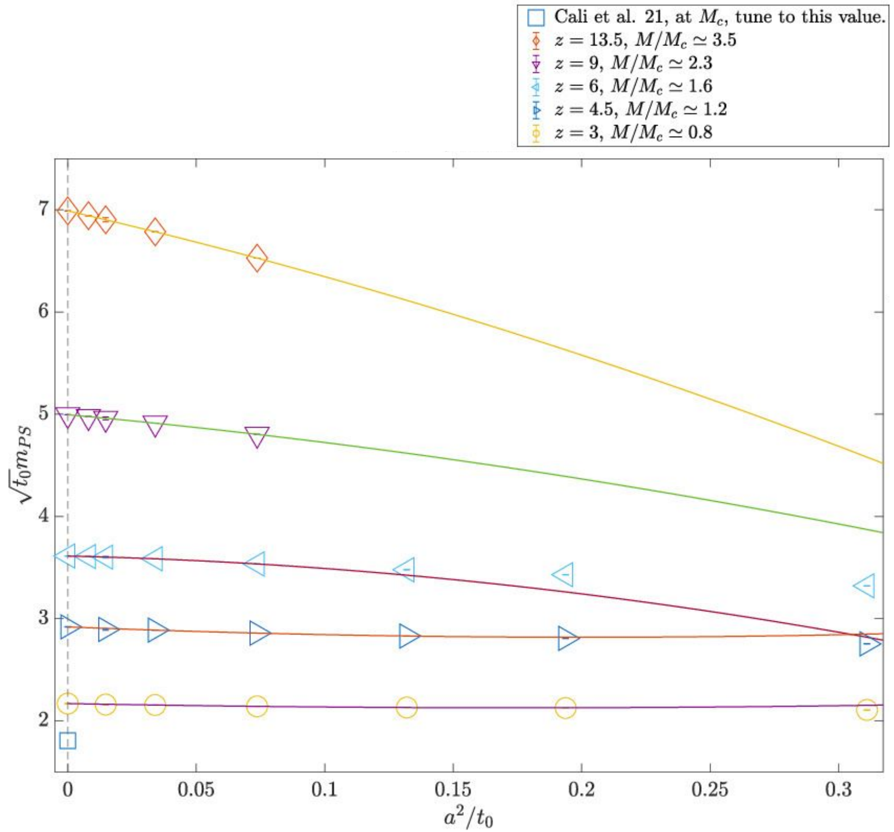
We cross checked several things:

A. variation of results when varying cutoff of sum

B. double check of perturbative inversion and running

C. measuring of decay constant and pseudoscalar mass:





BUP 0.2: Checks - II

No hints of issues, no mistunings, all consistent.

D. Finite Volume Effects? We computed analytically the continuum TL (where FV effects are expected to be larger):

$$\frac{\Delta G_n^L}{G_n(\infty)} \overset{mL\to\infty}{\sim} \frac{\pi}{2}\Gamma(3/2)\Gamma\left(\frac{n-2}{2}\right)\left(\frac{2}{mL}\right)^{\frac{3-n}{2}} e^{-mL}\left(1+O\left(\frac{1}{mL}\right)\right).$$

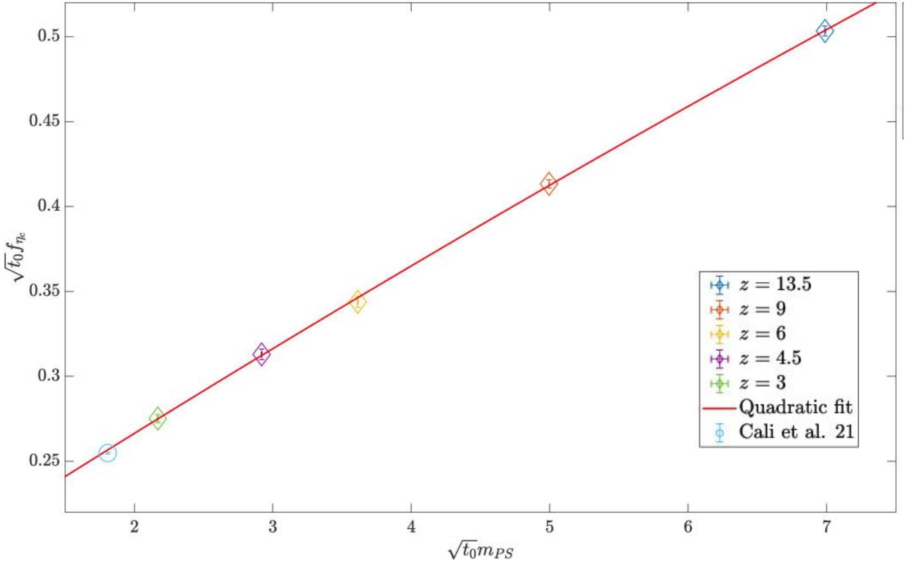


Table 4: Relative TL-FV effects, normalized by $L = \infty$ value, as function of $y = Lm_*$. For $M/M_c > 1.1$, we have $y > 14$.

y	5	6	7	8	9	10	11	12	13	14	15
$n = 4$	0.015	0.0060	0.0024	0.00093	0.00036	1.4e-04	5.5e-05	2.1e-05	8.0e-06	3.1e-06	1.2e-06
$n = 6$	0.037	0.018	0.0083	0.0037	0.0016	7.1e-04	3.0e-04	1.3e-04	5.2e-05	2.1e-05	8.7e-06
$n = 8$	0.19	0.11	0.058	0.030	0.015	0.0071	0.0033	1.5e-03	6.8e-04	3.0e-04	1.3e-04
$n = 10$	1.39	0.97	0.61	0.36	0.20	0.11	0.054	0.027	0.013	0.0063	3.0e-03