Feynman integrals, Calabi-Yau Motives and Integrable Systems

DESY – Theory Seminar

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universitätbonn

Based on work with

Kilian Bönisch, Claude Duhr, Fabian Fischbach, Florian Loebbert, Christoph Nega, Franzika Porkert, Reza Safari, Lorenzo Tancredi

[1]=arXiv:1912.06201v2, [2]=arXiv:2008.10574v1,
[3]=arXiv:2108.05310, published in JHEP
[4]=arXiv:2208.xxxx and [5]=arXiv:2208.xxxx, in progress

Introduction perturbative QFT

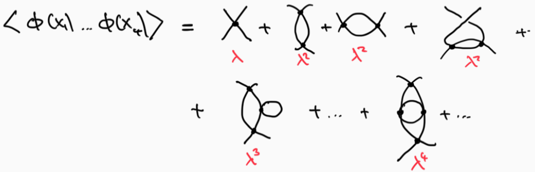
$$Z[J] = \int \mathcal{D}\phi \exp\left[\frac{i}{\hbar}\int \mathrm{d}^D x(\mathcal{L} + J\phi)\right] \ .$$

E.g. with $\mathcal{L} = \int \mathrm{d}^D x \left[\frac{1}{2}(\partial_u \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4\right]$.

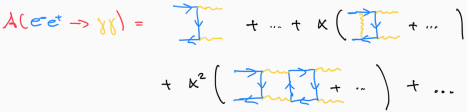
All physical correlators are of the form

$$\langle\phi(x_1)..\phi(x_n)\rangle = Z[J]^{-1}\left(\frac{\delta}{\delta J(x_1)}\right)..\left(\frac{\delta}{\delta J(x_n)}\right) Z[J]\bigg|_{J=0}$$

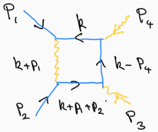
In interacting theories $\lambda \neq 0$ this is expanded asymptotically in Feynman graphs



Realistic theories: Probability for e^- e^+ to annihilate to two photons $P(e^- e^+ \to \gamma\gamma) \sim |\mathcal{A}(e^- e^+ \to \gamma\gamma)|^2$, $\alpha \sim \frac{1}{137}$



Scalar part e.g. for e.g. the box integral I: Propagators $\frac{1}{q^2 - m^2 + i \cdot 0}$



$\sum_{i=1}^{4} p_i = 0$ momentum conservation

$$\sim \int \frac{d^D k}{(k^2-m^2)\,(k+p_1)^2\,((k+p_1+p_2)^2-m^2)((k-p_4)^2-m^2)}$$

$D = D_0 - 2\epsilon$, $I = \sum_{k=-n}^{\infty} I_k \epsilon^n$ with I_k functions of masses and Lorentz invariant products of the external momenta that we need to know!

Emerging relation Feyman Integrals and Periods

Feynman integrals $\Leftrightarrow$ Periods of algebraic varities

Planar Feynman graph	Max. Cut Integrals	Period - Geometry
1-loop	rational functions	Pts in Fano 1-fold

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The full Feynman integral has boundaries: Periods integrals are replaced by chain integrals; hom. diff. eqs. are replaced by inhom. diff. eqs.

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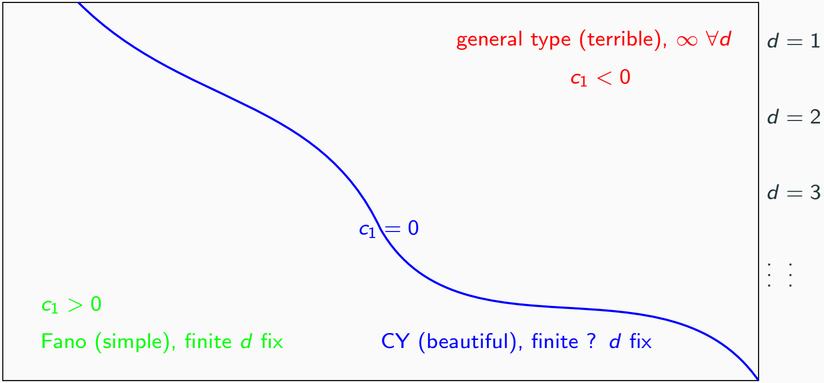
Kodaira map of algebraic varieties



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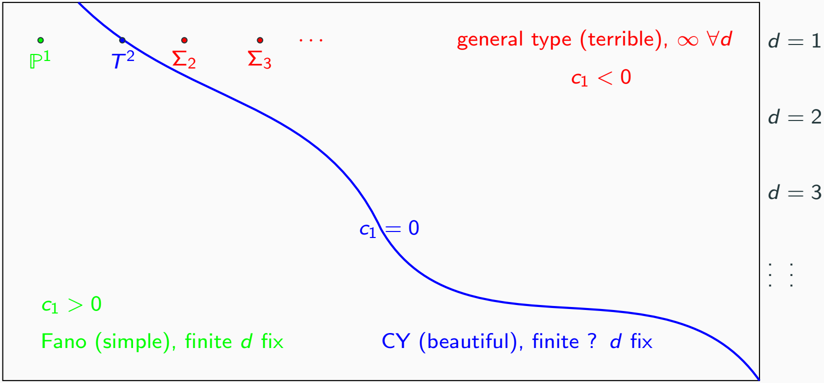
$$\begin{array}{ccccc} l=0 & l=1 & l=2 & l=3 & \cdots \\ g=0 & g=1 & g=2 & g=3 & \cdots \end{array}$$



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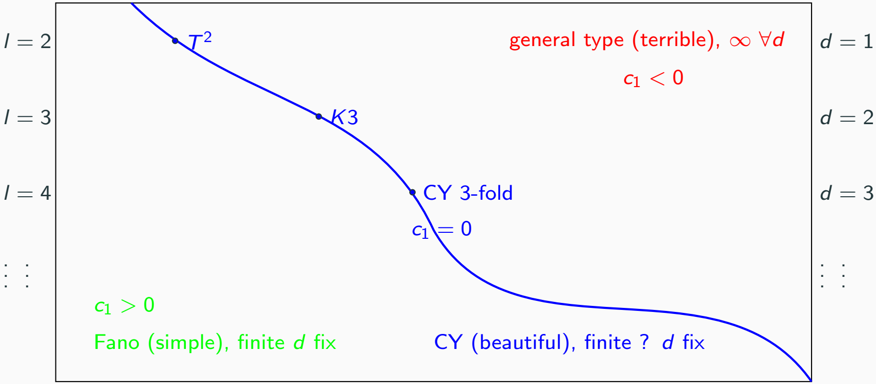

Detailed dictionary

	$l=(n+1)$**-loop banana integrals in** $D=2$ **dimensions**	**Calabi-Yau (CY) geometry**
1	Maximal cut integrals in $D=2$ dimensions	$(n,0)$-form periods of CY manifolds or CY motives
2	Dimensionless ratios $z_i = {m_i}^2/p^2$	Unobstructed compl. moduli of M_n, or equi'ly Kähler moduli of the mirror W_n
3	Integration-by-parts (IBP) reduction	Griffiths reduction method
4	Integrand-basis for maximal cuts of of master integrals in $D=2$	Middle (hyper) cohomology $H^n(M_n)$ M_n
5	Complete set of differential operators annihilating a given maximal cut in $D=2$ dimensions	Homogeneous Picard-Fuchs differential ideal (PFI) / Gauss-Manin (GM) connection

Relative Calabi-Yau periods via Symanzik representation

In the Feynman representation the contribution of an l-loop graph yields an integral with a rational integrand defined by the graph polynomials $\mathcal{U}(\underline{x})$ and $\mathcal{F}(\underline{x}, \underline{p}, \underline{m})$, $\underline{p}$ independent momenta, $\underline{m}$ masses

$$I_{\sigma_{n-1}}(\underline{p}, \underline{m}) = \int_{\sigma_{n-1}} \prod_i x_i^{\nu_i - 1} \frac{\mathcal{U}^{\omega - \frac{D}{2}}}{\mathcal{F}^{\omega}} \mu_{n-1}$$

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$\sigma_{n-1} = \{[x_1 : \ldots : x_n] \in \mathbb{P}^{n-1} | x_i \in \mathbb{R}_{\geq 0} \, \forall \, 1 \leq i \leq n\}$ an open domain.

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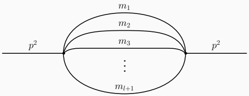

This graph leads in $t = \frac{p^2}{\mu^2}$, $\xi_i = \frac{m_i}{\mu}$ $(z_i = \frac{m_i^2}{p^2})$ to period integrals

$$I_{\sigma_l} = \int_{\sigma_l} \frac{\mu_l}{\mathcal{F}(t, \xi_i; x)} = \int_{\sigma_l} \frac{\mu_l}{\left(t - \left(\sum_{i=1}^{l+1} \xi_i^2 x_i\right)\left(\sum_{i=1}^{l+1} x_i^{-1}\right)\right) \prod_{i=1}^{l+1} x_i}$$

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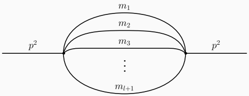

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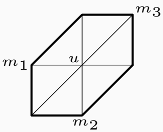
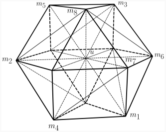
The Newton polytopes of $\mathcal{F}$ is reflexive, hence $\mathcal{F} = 0$ defines a Calabi-Yau manifold. For example for $l = 2, 3$ they look like





Maximals cut integrals

By closing the chain σ_l to a T^l cycle one gets a maximal cut integral in $D_0 = 2$

$$I_{T^{l-1}}(\underline{z};0) = \int_{T^l} \frac{\mu_l}{\mathcal{F}(1,\underline{z})} = \int_{T^{l-1}} \oint_{S^1} \frac{\mu_l}{\mathcal{F}(1,\underline{z})} = 2\pi i \int_{\Gamma_T = T^{l-1}} \Omega_{l-1}(\underline{z}) \ .$$

Here cycle T^l is defined as

$$T^l := \{[x_1 : \ldots : x_{l+1}] \in \mathbb{P}^l \,|\, |x_i| = 1 \text{ for all } 1 \leq i \leq l+1\} \,.$$

The last identification relies on the Griffiths residue form for the holomorphic n-form Ω for complete intersections

$$\Omega(\underline{z}) = \frac{1}{(2\pi i)^r} \oint_{S_1^1} \cdots \oint_{S_r^1} \frac{\wedge_{i=1}^m \mu_{n_i}}{P_1 \cdots P_r} \ ,$$

where S_k^1 encircles the constraints $P_k = 0$ in the ambient space. The crucial point is that the integral over the S^1 cycle of T^l leads to a closed period integral of Ω_{l-1} over T^{l-1} on a CY family M_{l-1}

$$M_{l-1}^{\mathrm{HS}} = \{\underline{x} \in \mathbb{P}^l | \mathcal{F}(1,\underline{z};\underline{x}) = 0\} \ .$$

Maximals cut integrals

Performing all l residua integrals one gets with $|k| = \sum_{i=1}^{l+1} k_i$

$$I_{T^{l-1}}(\underline{z};0) = (2\pi i)^l \sum_{n=0}^{\infty} \sum_{|k|=n} \begin{pmatrix} n \\ k_1 \ldots k_{l+1} \end{pmatrix}^2 \prod_{i=1}^{l+1} z_i^{k_i} \ .$$

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- The Feynman integrals is a chain integral. It fulfils an associated inhomogeous extension of the latter.
- The hypersurface M_{l-1}^{HS} defines a singular family of Calabi-Yau motives with $l+1$ complex parameters. To get a workable smooth model one could deform $F(1, \underline{z}; \underline{x})$ (toric resolution). However, one needs l^2 (complex) moduli to achieve that. This leads to a highly redundant model that is very hard to solve. We provide a better CY motive latter.

A better motive for the Banana integral

Consider the complete intersection of two polynomials of degree $(1,\ldots,1)$ in the cartesian product of $(\mathbb{P}^1)'s$

$$\mathbb{P}_{l+1} := \otimes_{i=1}^{l+1}\mathbb{P}^1_{(i)}\,.$$

Such a complete intersection manifold in a product of manifolds is denoted in short as

$$M^{\mathrm{CI}}_{l-1} = \left(\begin{array}{c\|cc} \mathbb{P}^1_{(1)} & 1 & 1 \\ \vdots & \vdots & \vdots \\ \mathbb{P}^1_{(l+1)} & 1 & 1 \end{array}\right) \subset \left(\begin{array}{c\|c} \mathbb{P}^1_{(1)} & 1 \\ \vdots & \vdots \\ \mathbb{P}^1_{(l+1)} & 1 \end{array}\right) =: F_l \subset \mathbb{P}_{l+1}\,.$$

A better motive for the Banana integral

$$P_1 \ = a_0\, w_0^{(1)} + \sum_{m=1}^{l+1} a_{2m-1}\, w_m^{(1)} = a_0 \prod_{k=1}^{l+1} x_1^{(k)} + \sum_{m=1}^{l+1} a_{2m-1}\ x_2^{(m)} \prod_{k\neq m}^{l+1} x_1^{(k)}$$

$$P_2 \ = \tilde{a}_0\, w_0^{(2)} + \sum_{m=1}^{l+1} a_{2m}\, w_m^{(2)} = \tilde{a}_0 \prod_{k=1}^{l+1} x_2^{(k)} + \sum_{m=1}^{l+1} a_{2m}\ x_1^{(m)} \prod_{k\neq m}^{l+1} x_2^{(k)} \ .$$

On these parameters the $(\mathbb{C}^*)^{l+1}$-scaling symmetries given in [**?**]

$$\begin{aligned}
\ell^{(1)} = &\quad (-1,-1;1,1,0,0,\cdots,0,0,0,0)\\
\ell^{(2)} = &\quad (-1,-1;0,0,1,1,\cdots,0,0,0,0)\\
\vdots\\
\ell^{(l)} = &\quad (-1,-1;0,0,0,0,\cdots,1,1,0,0)\\
\ell^{(l+1)} = &\quad (-1,-1;0,0,0,0,\cdots,0,0,1,1)
\end{aligned}$$

act and yield the $(l+1)$ second order GKZ operators in the Batyrev large radius coordinates $z_k = \prod_{i=1}^{2(l+2)} a_i^{\ell_i^{(k)}}/(a_0\tilde{a}_0)$, $k = 1,\ldots,l+1$.

A better motive for the Banana integral

To compare with hypersurface representation tset

$$a_0 = h, \quad \tilde{a}_0 = 1$$
$$a_{2k-1} = z_k, \quad a_{2k} = h, \qquad k = 1, \ldots, l+1 \tag{1}$$

and construct a birational map from the complete intersection geometry to the hypersurface geometry. Solving for $P_1 = 0$ one gets $h = -\sum_{k=1}^{l+1} \frac{m_k^2}{p^2} W_k$, while P_2 becomes $P_2 = 1 + h \sum_{k=1}^{l+1} 1/W_k$. Here we passed to toric $\mathbb{C}^*$-coordinates $W_k = x_1^{(k)}/x_2^{(k)}$ for $k = 1, \ldots, l+1$ and arrive at

$$P_2 = p^2 - \left(\sum_{i=1}^{l+1} m_i^2 \, W_i\right) \left(\sum_{i=1}^{l+1} \frac{1}{W_i}\right) = \mathcal{F} \, .$$

Periods on Calabi-Yau n-folds

Periods are integrals

$$\Pi_{ij}(\underline{z}) = \int_{\lambda_i} \Lambda^j(\underline{z})$$

that define a pairing between between homology and cohomology (n odd) well defined by the theorem of Stokes:

$$Pi : H_n(M_n, \mathbb{Z}) \times H^n(M_n, \mathbb{C}) \to \mathbb{C}\,.$$

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$\{\alpha_I, \beta^J\} = \underline{\Lambda}$, $\int_M \alpha_I \wedge \beta^J = \delta^J_I$
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$\underline{\lambda}$ is topol. and so is $\underline{\Lambda}$ via $\int_{A^I} \alpha_J \;=\; \int_{B_J} \beta^I \;=\; \delta^I_J$. A basis moving with the comp. str. in $\underline{\Lambda}$ are the meromorphic forms $\Omega(z), \partial_z\Omega(z), \ldots$.

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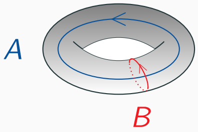
Calabi-Yau 1-fold $\quad p_3 = wy^2 - 4x^3 - g_2(z)xw^2 - g_3(z)w^3 = 0 \subset \mathbb{P}^2$



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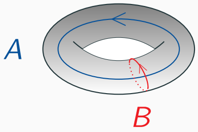

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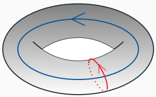
B

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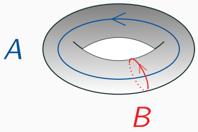

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Well studied in part because they solve Keplers problem

Periods on Calabi-Yau n-folds

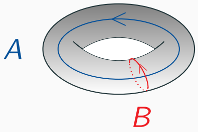
Periods are integrals

$$\Pi_{ij}(\underline{z}) = \int_{\lambda_i} \Lambda^j(\underline{z})$$

that define a pairing between between homology and cohomology (n odd) well defined by the theorem of Stokes:

$$Pi : H_n(M_n, \mathbb{Z}) \times H^n(M_n, \mathbb{C}) \to \mathbb{C}\,.$$

Calabi-Yau 1-fold $\quad p_3 = wy^2 - 4x^3 - g_2(z)xw^2 - g_3(z)w^3 = 0 \subset \mathbb{P}^2$



$\Omega(z) = \oint \frac{2dx\wedge dy}{p_3} = \frac{dx}{y}$, $\partial_z\Omega(z) \sim \frac{xdx}{y}$

$E_1(z) = \oint_A \Omega$, $E_2(z) = \oint_B \Omega$ Elliptic integrals.

Well studied in part because they solve Keplers problem

Fullfill linear diff eq. of 2cd order. Picard(1891)-Fuchs(1881) eq.

Periods on 3-folds

Consider the mirror quintic W

$$\hat{p}_5 = \sum_{i=0}^{4} x_k^5 - 5z^{-\frac{1}{5}} \prod_{k=0}^{4} z_i = 0 \subset \hat{\mathbb{P}}^4$$

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Hodge diamond of elliptic curve

Hodge diamond of W

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array} \longrightarrow \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

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The period vector $\Pi(z) = \left(\int_{A^0} \Omega, \int_{A^1} \Omega(z), \int_{B^0} \Omega(z), \int_{B^1} \Omega(z)\right)^T$ fullfils a 4th order Picard-Fuchs diff. eq. ($\theta = zd/dz$)

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$$\boxed{[\theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4)]\Pi(z) = 0}$$

Periods on 3-folds

Local $\rightarrow$ global: How to find the periods over cycles in $H_3(W, \mathbb{Z})$?
Find the basis in which mondromies $\Pi \mapsto M_* \Pi$ around the singular points $*$ are in $\mathrm{Sp}(4, \mathbb{Z})$

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$$\mathcal{P} \left\{ \begin{array}{cccc} 0 & 5^{-5} & \infty & * \\ 0 & 0 & \frac{1}{5} & \\ 0 & 1 & \frac{2}{5} & z \\ 0 & 2 & \frac{3}{5} & \\ 0 & 1 & \frac{4}{5} & \end{array} \right\}$$

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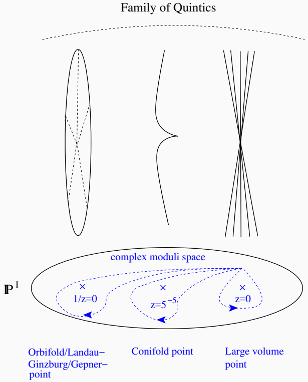

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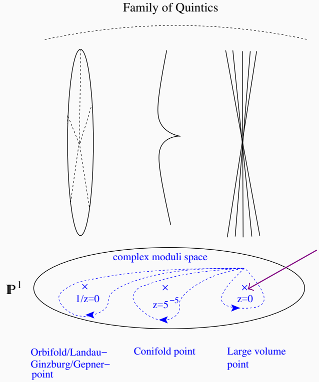

Identifies also the expansion point for the mirror map as point of maximal unipotent monodromy

Periods on 3-folds

Special geometry Bryant and Griffiths '83 implies that the periods can be expressed by a prepotential $\mathcal{F}$

triple logarithmic solution

double logarithmic solution

analytic solution

logarithmic solution

Periods on 3-folds

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and Candelas et al '91 identified near the MUM point $z = 0$

$$\boxed{\mathcal{F}(z) \equiv \mathcal{F}_0(t(z)), \qquad t = \frac{X^1}{X^0} = \log(z) + \mathcal{O}(z)}$$

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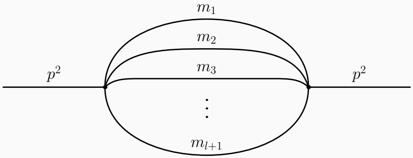
Hosono et. al '93 generalised to multiparameter CY and related the classical terms to the CTC Wall data $\kappa = D^3$, $\sigma = (\kappa \bmod 2)/2$ in

$$\mathcal{F} = -\frac{\kappa}{6}t^3 + \frac{\sigma}{2}t^2 + \frac{c_2 \cdot D}{24}t + \frac{\chi(M)}{2}\frac{\zeta(3)}{(2\pi i)^3} - \frac{1}{(2\pi i)^3}\sum_{\substack{\beta \in H_2(M,\mathbb{Z}) \\ \beta \neq 0}} n_0^\beta \mathrm{Li}_3(Q^\beta).$$

The mirror picture for the Banana geometry and the $\hat{\Gamma}$-class

The vertical quantum cohomology of W^{CI}_{l-1} relates natural to the banana graph



$$\longleftrightarrow W^{\mathrm{CI}}_{l-1} = \left(\begin{array}{c||cc} \mathbb{P}^1_1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ \mathbb{P}^1_{l+1} & 1 & 1 \end{array} \right) \subset \left(\begin{array}{c||c} \mathbb{P}^1_1 & 1 \\ \vdots & \vdots \\ \mathbb{P}^1_{l+1} & 1 \end{array} \right) = F_l \,.$$

In particular, in the high energy regime we get a one-to-one identification of the complexified (large volume) Kähler parameters t^k of the $l+1$ rational curves $\mathbb{P}^1_k$ with the physical parameters m_i^2/p^2

$$t^k \simeq \frac{1}{2\pi i} \int_{\mathbb{P}^1_k} (i\omega - b) + \mathcal{O}(e^{-t^k}) = \frac{\log}{2\pi i} \left(\frac{m_k^2}{p^2} \right) = \frac{\log(z_k)}{2\pi i}$$

for $k = 1, \ldots, l+1$.

The mirror picture for the Banana geometry and the $\hat{\Gamma}$-class

A powerful application of the geometric realization W^{CI}_{l-1} is the $\widehat{\Gamma}$-class formalism. It relates the Frobenius $\mathbb{Q}$-basis of solutions at the point of maximal unipotent monodromy (MUM) to an integral $\mathbb{Z}$-basis of solutions to the PFI.

Let I_p an index set of order $|I_p| = p$ and define the Frobenius basis at the MUM point:

$$S_{(p),k}(\underline{z}) = \frac{1}{(2\pi i)^p p!} \sum_{I_p} \kappa^{i_1,\ldots,i_p}_{(p),k} \varpi_0(\underline{z}) \log(z_{i_1}) \cdots \log(z_{i_p}) + \mathcal{O}(\underline{z}^{1+\alpha})\,.$$

Here $|S_{(p)}(\underline{z})|$ denotes the total number of solutions which are of leading order p in $\log(z_i)$ and $\kappa^{i_1,\ldots,i_p}_{(p),k}$ are intersection numbers of the mirror W^{CI}_{l-1}.

In particular, the Kähler parameters t^k are given by the mirror map

$$t^k(\underline{z}) = \frac{S_{(1),k}(\underline{z})}{S_{(0),0}(\underline{z})} = \frac{1}{2\pi i}\left(\log(z_k) + \frac{\Sigma_k(\underline{z})}{\varpi_0(\underline{z})}\right),$$

for $k = 1, \ldots, h^{11}(W_n) = h^{n-1,1}(M_n)$. Homological mirror symmetry predicts then the relevant maximal cut integral $(\mathbf{S} := S^{l-1}) \cap (\mathbf{T} := T^{l-1}) = 1$

$$\Pi_{\mathbf{S}}(\underline{t}(\underline{z})) = \int_{W_{l-1}} e^{\underline{\omega}\cdot\underline{t}}\,\widehat{\Gamma}(TW_{l-1}) + \mathcal{O}(e^{-\underline{t}}) \qquad (2)$$

The mirror picture for the Banana geometry and the $\hat{\Gamma}$-class

An extension also yields the full Feynman integral

$$J_{l,\underline{0}}(\underline{z},0) = \int_{F_l} e^{\underline{\omega}\cdot\underline{t}}\,\widehat{\Gamma}_{F_l}(TF_l) + \mathcal{O}(e^{-\underline{t}}) \; . \tag{3}$$

Here the extended $\widehat{\Gamma}$-class is given by

$$\widehat{\Gamma}_F(TF) = \frac{\widehat{A}(TF)}{\widehat{\Gamma}^2(TF)} = \frac{\Gamma(1-c_1)}{\Gamma(1+c_1)}\cos(\pi c_1) \; .$$

By comparing the powers of $t^k \sim \log(z_k)$ on both sides of (2),(3) using the mirror map these formulas determine uniquely the exact boundary conditions for the integrals in terms of topological intersection calculations on W^{CI}_{l-1} or the Fano variety F_l and the Frobenius basis for the banana graph [2].

The mirror picture for the Banana and the $\hat{\Gamma}$-class

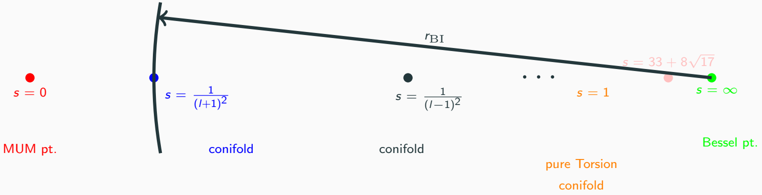
Let us give an example for $I_{l,1}(\underline{p}, \underline{m}, D = 2)$ up to five loops

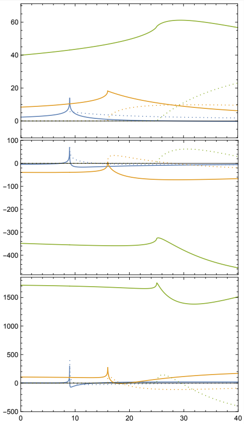
l	$S_{(0),1}$	$S_{(1),1}$	$S_{(2),1}$	$S_{(3),1}$	$S_{(4),1}$
1	$-2\pi i$				
2	$18\zeta(2)$	$6\pi i$			
3	$-16\zeta(3) + 24i\pi\zeta(2)$	$-72\zeta(2)$	$-12\pi i$		
4	$-450\zeta(4) - 80i\pi\zeta(3)$	$80\zeta(3) - 120\pi i\zeta(2)$	$180\zeta(2)$	$20\pi i$	
5	$-288\zeta(5) + 1440\zeta(2)\zeta(3) - 540i\pi\zeta(4)$	$2700\zeta(4) + 480i\pi\zeta(3)$	$-240\zeta(3) + 360\pi i\zeta(2)$	$-360\zeta(2)$	$-30\pi i$

The analytic structure of the Banana integral and results

Roadmap to the physical moduli space:

$$s = 1/t \in \mathcal{M}_{cs}(M_{l-1}) = \mathbb{P}^1 \setminus \left(\bigcup_{j=0}^{\lfloor \frac{l+1}{2} \rfloor} \left\{ \frac{1}{(l+1-2j)^2} \right\} \cup \{0\} \right)$$





The banana integrals $J_{l,1}^{(n)}$ for $l = 2, 3, 4$ (blue, orange, green) and ϵ order $n = 0, 1, 2$ (upper, middle and lower panels) against $1/t$.The solid: real part, dashed lines: imaginary part.

Master Integrals and integration by parts relations

Consider l-loop Feynman integrals in general dimensions $D \in \mathbb{R}_+$ of the form

$$I_{\underline{\nu}}(\underline{x}, D) := \int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \tag{4}$$

$D_j = q_j^2 - m_j^2 + i \cdot 0$ for $j = 1, \ldots, p$ are the propagators, q_j is the j^{th} momenta through D_j, $m_j^2 \in \mathbb{R}_+$ are masses, $i \cdot 0$ indicates the choice of contour/branchcut in $\mathbb{C}$. Subject to momentum conservation the q_j are linear in the external momenta $p_1, \ldots, p_E$, $\sum_{i=j}^{E} p_j = 0$ and the loop momenta k_r. We defined $\epsilon := \frac{D_0 - D}{2}$.

Master Integrals and integration by parts relations

The Feynman integral depends besides D on dot products of p_i and the masses m_j^2, written compactly in a vector $\underline{x} = (x_1, \ldots, N) = (p_{i_1} \cdot p_{i_2}, m_j^2)$ and dimensional analysis of $I_{\underline{\nu}}$ shows that it depends only on the ratios of two parameters x_i, we chose

$$z_k := \frac{x_k}{x_N} \qquad \text{for } 1 \leq k < N$$

and label now the parameters of the integrals $I_{\underline{\nu}}$ by the dimensionless parameters $\underline{z}$.

Master Integrals and integration by parts relations

The propagator exponents and $D \in \mathbb{Z}$ span a lattice $(\underline{\nu}, D) \in \mathbb{Z}^{p+1}$. The $I_{\underline{\nu}}(\underline{x}, D)$ are called master integrals.

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The integration by parts (IBP) identities

$$\int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial k_k^{\mu}} \left(q_l^{\mu} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \right) = 0 \ .$$

relate the master integrals with different exponents $\underline{\nu}$.

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relate the master integrals with different exponents $\underline{\nu}$.

There is a finite region in the lattice that contains all non-vanishing master integrals. In a basis of master integrals one can express derivatives w.r.t. the z_k as a linear combination rational coefficients by the IBP relations.

Master Integrals and integration by parts relations

- The basis of master integrals (graph cohomology) corresponds to the basis of the cohomology $H^{l-1}(M_l, \mathbb{Z})$.

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Among the elements in the lattice $\mathbb{Z}^p$ and, in particular, for the master integrals one can define sectors and a semi-ordering on the latter by defining a map

$$\underline{\nu} \mapsto \underline{\vartheta}(\underline{\nu}) =: (\theta(\nu_j))_{1\leq j\leq p} \,.$$

where θ is the Heaviside step function. The semi-ordering is then defined by $\underline{\vartheta}(\underline{\nu}) \leq \underline{\vartheta}(\underline{\tilde{\nu}})$, iff $\theta(\nu_j) \leq \theta(\tilde{\nu}_j)$, $\forall j$. This defines an inclusive order on subgraphs with less propagators and therefore simpler topology.

Detailed dictionary continued

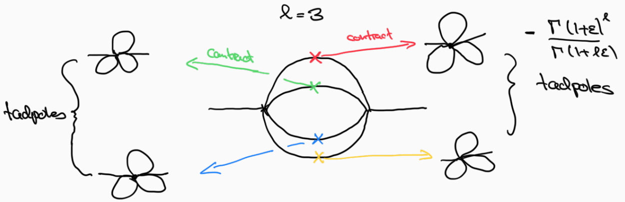
6	Contributions from subtopologies to the differential equations	Inhomogeneous extensions of the PFI or the GM connection
7	(Non-)maximal cut contours	(Relative) homology of CY geometry $H_n(M_n)$ ($H_{n+1}(F_{n+1}, \partial\sigma_{n+1})$)
8	Full banana integrals in $D = 2$ dimensions	Chain integrals in CY geometry or extensions of Calabi-Yau motive
9	Degenerate kinematics (e.g., $m_i^2 = 0$ or $p^2/m_i^2 \to 0$)	Critical divisors of the moduli space
10	Large-momentum regime $p^2 \gg m_i^2$	Point of maximal unipotent monodromy & $\widehat{\Gamma}$-classes of W_n
11	General logarithmic degenerations	Limiting mixed Hodge structure from monodromy weight filtration

The banana integrals as example for extensions

The banana graph has $2^{l+1} - 1$ master integrals in $l + 2$ sectors: $l + 1$ sectors correspond to $\vartheta(\underline{\nu}) = (1, \ldots, 1, 0, 1 \ldots 1)$. These sectors correspond all to l-loop tadpole integrals

$$J_{l,i}(\underline{z};\epsilon) = \frac{(-1)^{l+1}(p^2)^{l\epsilon}\epsilon^l}{\Gamma(1+l\epsilon)} I_{1..1,0,1..1}(\underline{x};D) = -\frac{\Gamma(1+\epsilon)^l}{\Gamma(1+l\epsilon)} \prod_{\substack{j=1\\ j\neq i}}^{l+1} z_j^{-\epsilon}.$$

These lower sectors are all tadpoles yielding already analytic expressions.



The banana integrals as example for extensions

Further $2^{l+1} - l - 2$ master integrals come from the sector $\vartheta(\underline{\nu}) = (1, \ldots, 1)$, $\underline{k} \in \{0,1\}^{l+1}$, $1 \leq |\underline{k}| \leq l - 1$,

$$J_{l,\underline{0}}(\underline{z};\epsilon) = \frac{(-1)^{l+1}}{\Gamma(1+l\epsilon)}(p^2)^{1+l\epsilon}\; I_{1,\ldots,1}(\underline{x}; 2-2\epsilon)\,,$$

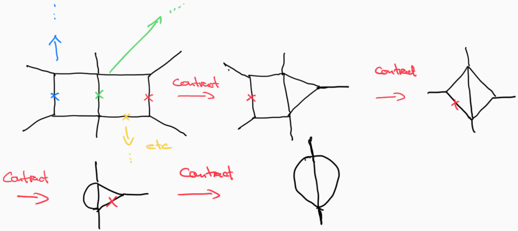
$$J_{l,\underline{k}}(\underline{z};\epsilon) = (1+2\epsilon)\cdots(1+|\underline{k}|\epsilon)\partial_{\underline{z}}^{\underline{k}} J_{l,\underline{0}}(\underline{z};\epsilon)\,.$$

Here $|\underline{k}| = \sum_{j=1}^{l+1} k_j$ and $\partial_{\underline{z}}^{\underline{k}} := \prod_{i=1}^{l+1} \partial_{z_i}^{k_i}$.

The latter correspond in the critical dimension, the leading order in $\epsilon \to 0$ the period integrals of families of Calabi-Yau $(n = l-1)$-folds.

The banana integrals as example for extensions

Banana integrals do occur in the iterative procedure within each more complicated Feynman diagram.

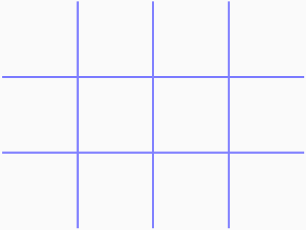


Detailed dictionary continued

12	Analytic structure and analytic continuation	Monodromy of the CY motive and its extension
13	Quadratic relations among maximal cut integrals	Quadratic relations from Griffiths transversality
14	Special values of the integrals for special values of the z_i	Reducibility of Galois action & L-function values
15	(Generalized?) modularity of Feynman integrals	Global $\mathrm{O}(\boldsymbol{\Sigma}, \mathbb{Z})$-monodromy, integrality of mirror map & instantons expansion

Net diagrams

Momentum space



$l = 2,\ d = 2$

$$I_{1,2} = \int \frac{d^2x_1 d^2x_2}{|x_1 - a_0||x_1 - a_1||x_1 - a_2||x_1 - x_2||x_2 - a_3||x_2 - a_4||x_2 - a_5|} = \int \frac{d^2x d\bar{x}^2}{\sqrt{P(z)\bar{P}(\bar{z})}}$$

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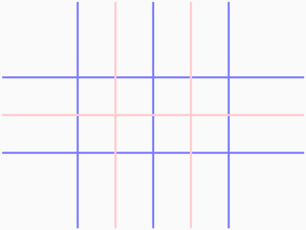

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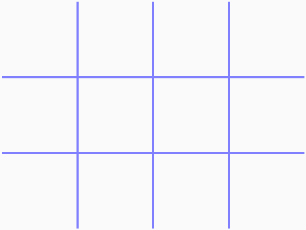
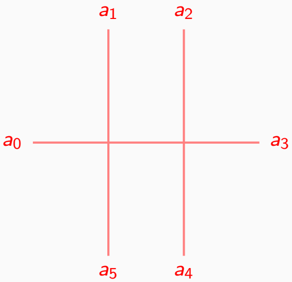


Position space

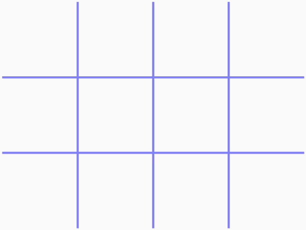


$l = 2,\ d = 2$

$$I_{1,2} = \int \frac{d^2x_1 d^2x_2}{|x_1 - a_0||x_1 - a_1||x_1 - a_2||x_1 - x_2||x_2 - a_3||x_2 - a_4||x_2 - a_5|} = \int \frac{d^2x d\bar{x}^2}{\sqrt{P(z)\bar{P}(\bar{z})}}$$

Net diagrams

Momentum space



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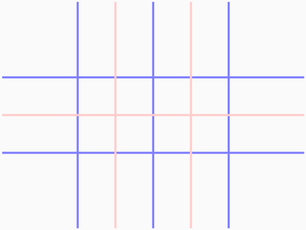

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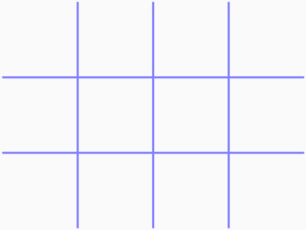
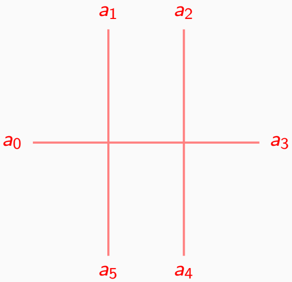


Position space



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Consider more generally in the momentum space of a QFT with a four vertex interaction an l-loop multibox graph $\hat{\Gamma}_{m,n}$ made of m rows and n columns of boxes together with its dual dual graph $\Gamma_{n,m}$ in the positions space

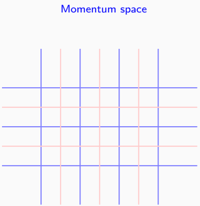

Figure 1: The 6-loop net graph $\hat{\Gamma}_{2,3}$ in the momentum space (blue) and the dual graph $\Gamma_{2,3}$ in position space (red) a $2n + 2m$ correlator.

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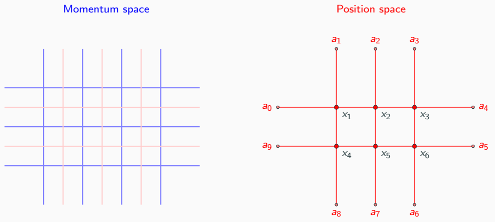

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where m_i is the propagator mass and λ is a propagator weight and we distinguished for latter use outer inner and outer propagators. For $D = 2$, $\lambda = 1/2$ and in this case one complex coordinates and write $x^2 = x\bar{x}$ so that the propagators become

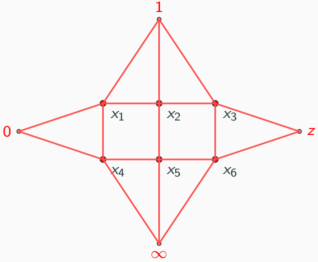
$$P_{ij} = 1/|x_i - x_j|, \qquad P_{i\alpha} = 1/|x_i - a_\alpha| \ .$$

The Feynman integral for a (m, n) net becomes a real quantity $|\mathrm{d}\mu|^2 = \wedge_{i=1}^{l} \mathrm{d}x_i \wedge \mathrm{d}\bar{x}_{\bar{\imath}}$

$$I_{n,m} = \int \Bigg(\prod_{\substack{\text{int} \\ \text{edges}}} P_{ij} \prod_{\substack{\text{ext} \\ \text{edges}}} P_{i\alpha} \Bigg) |\mathrm{d}\mu|^2,$$

Net diagrams

In $D = 2$ we can use the conformal symmetry $\mathrm{PSL}(2, \mathbb{C})$ to set 3 a_I to $0, 1, \infty$. We label the remaining $r = 2m + 2n - 3$ cross ratios by $z_i, \ldots, z_r$. A particular simple one parameter sub slice of the position space by the fishnet graphs



Net diagrams

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Indeed, let $l = mn$ and $\mathcal{P}_{ij}^{-2} = (x_i - x_j)$ and $\mathcal{P}_{i\alpha}^{-2} = (x_i - a_\alpha)$ inverses of a holomorphic version of the propagators. The l-fold $W^{(m,n)}$ is defined as the double covering of $B = (\mathbb{P}^1)^l$ branched at

$$y^2 = \prod_{\substack{\text{int}\\\text{edges}}} \mathcal{P}_{ij}^{-2} \prod_{\substack{\text{ext}\\\text{edges}}} \mathcal{P}_{i\alpha}^{-2} =: P(\underline{x}, \underline{u}) \ .$$

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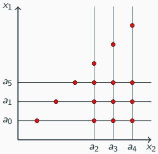

Claim 2: Each $W^{(m,n)}$ gives rise to a Calabi-Yau motive with integer symmetry (l even) or antisymmetric (l odd) intersection form Σ, a point of maximal unipotent monodromy and a period vector $\Pi(\underline{z}) = \int_{\Gamma_i} \Omega$ with $\Gamma_i \in H_l(W^{(m,n)}, \mathbb{Z})$.

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$$I_{m,n} = i^{l^2} \Pi^\dagger \Sigma \Pi = e^{-K(\underline{z},\bar{\underline{z}})} = Vol_q(M^{(m,n)})$$

and globally by analytic continuation of the periods. Here $M^{(m,n)}$ is the mirror of $W^{(m,n)}$.

Net diagrams

Claim 3: There exist an integrable conformal fishnet theories (CFNT) developed first (Gürdogan, Kazakov 2015) as deformation of $N = 4$ $SU(N_c)$ SYM theory. Let X, Z be $SU(N_c)$ matrix fields then the Lagrangian is

$$\mathcal{L}_{FN} = N_c \mathrm{tr}\left(-\partial_\mu X \partial^m u \bar{X} - \partial_\mu Z \partial^m u \bar{Z} + \xi^2 X Z \bar{X} \bar{Z}\right)$$

Each $I_{m,n}$ integral is an amplitude in the CFNT, i.e. $I_{m,n}(\underline{z})$ has to be single valued i.e. a Bloch Wigner dilogarithm or in the $D = 2$ case e^{-K}.

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The factorisation of the amplitudes of the integrable system subject to the Yang-Baxter relations imply many non-trivial relations for he periods of the $W^{(m,n)}$. E.g. we the one parameter specialisation the periods of $W^{(n,m)}$ are $(m \times m)$ minors of the periods $W_l^{(1,m+m)}$ etc.

Net diagrams

Claim 4: The conformal Yangian generated by the algebra

$$\begin{aligned} P_j^\mu &= -i\partial_{a_j}^\mu, & K_j^\mu &= -2ia_j^\mu(a_j^\nu\partial_{a_j,\nu} + \Delta_j) + ia_j^2\partial_{a_j}^\mu \\ L_j^{\mu\nu} &= i(a_j^\mu\partial_{a_j}^\nu - a_j^\nu\partial_{a_j}^\mu), & D_j &= -i(a_j^\mu\partial_{a_j,\mu}), \end{aligned}$$

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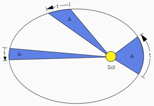
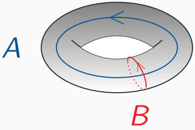
in differentials w.r.t. to the external position, generates together with the permutation symmetries of the latter a differential ideal that annihilates the $I_{m,n}(\underline{z})$ and is *equivalent* to the Picard-Fuchs differential ideal that describes the variation of the Hodge structure in the middle cohomology of $M_{\underline{z}}^{(m,n)}$ and annihilated the periods of Ω.

Conclusion and Outlook





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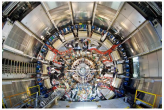
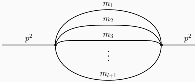
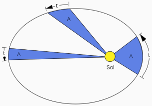
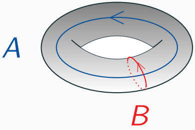




Physics

Mathematics





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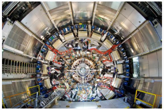
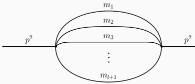


String Theory



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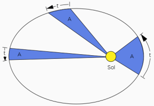
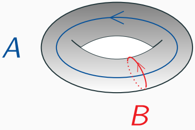
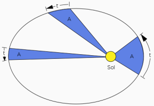
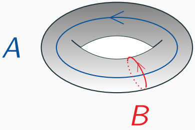




Enumerative Geometry

Conclusion and Outlook





String Theory

Black Hole Physics

Physics

Mathematics

Enumerative Geometry

Arithmetic Geometry