Recent developments in Argyres–Douglas theories

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This talk is based on

[arXiv:2012.12852]

in collaboration with

Simone Giacomelli and **Matteo Sacchi**;

[arXiv:2105.08064], [arXiv:2110.06940], [arXiv:2203.16550]

in collaboration with

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Introduction

- An infinite class of 4d $\mathcal{N}=2$ superconformal field theories (SCFTs), known as **Argyres–Douglas (AD)** theories.
- **Feature:** The presence of Coulomb branch (CB) operators with fractional scaling dimension.
- Such a theory was discovered in the $\mathcal{N}=2$ $SU(3)$ SYM [Argyres, Douglas '95] and then in the $SU(2)$ theory with 1 flavour [Argyres, Plesser, Seiberg, Witten '95].
  - It's characterised by the presence of massless dyons which are relatively non-local, making the theory intrinsically interacting.
  - Nowadays, this AD theory is often referred to as the (A_1, A_2) theory in the notation of [Cecotti, Neitzke, Vafa '10].
  - This theory has 1 dimensional Coulomb branch (rank 1), with the single CB operator of dimension $\frac{6}{5}$.
  - Due to the lack of $\mathcal{N}=2$ marginal deformation, this theory does not admit an $\mathcal{N}=2$ Lagrangian description.
  - However, there is some 4d $\mathcal{N}=1$ gauge theory that flows to the (A_1, A_2) theory in the IR. SUSY gets enhanced in the IR. [Maruyoshi, Song '16].

Introduction

- More recently a vast number of AD theories can be realised using
  - the class $\mathcal{S}$ construction: 6d $\mathcal{N} = (2, 0)$ theory compactified on Riemann surfaces with punctures [Gaiotto, Moore, Neitzke '09; Xie '12; Wang, Xie '15, '18]
  - the geometric engineering in Type IIB via compactification on singular Calabi-Yau threefolds [Shapere, Vafa '99; Cecotti, Neitzke, Vafa '10; Xie, Yau '15]
  - **Note:** *Some* AD theories can be realised in both ways, the others cannot.
- In this talk, we focus on some infinite classes of models, known as
  1. $(A_m, A_n) = (A_n, A_m) = I_{n+1,m+1}$ [Cecotti, Neitzke, Vafa '10; Xie '12]
  2. $D_p(SU(N))$ [Cecotti, del Zotto '12; Cecotti, del Zotto, Giacomelli '13]

  We'll also discuss (A_m, D_n), (D_m, D_n), $D_p(SO(2n))$ but not in detail.
- **Main points:** Study many aspects of these theories, including conformal manifolds, dimensional reduction to 3d, Higgs branch structures, and dynamical consequences of 1-form symmetries.

Introduction

- They are characterised by Seiberg-Witten (SW) curves and SW differentials:

$$
\begin{aligned}
&(A_{N-1}, A_{k-1}): && x^N + z^k + \sum_{\substack{i,j\geq 0 \\ j\neq N-1,\, i\neq k-1}} u_{ij} z^i x^j = 0 \,, && \lambda_{\mathsf{SW}} = x dz \\
&D_p(SU(N)): && x^N + z^p + \sum_{\substack{i,j\geq 0 \\ j\neq N-1}} u_{ij} z^i x^j = 0 \,, && \lambda_{\mathsf{SW}} = x dz/z
\end{aligned}
$$

- Can assign the scaling dimension $[\lambda_{\mathsf{SW}}] = 1$:

$$
\begin{aligned}
&(A_{N-1}, A_{k-1}): && [x] = \tfrac{k}{N+k} \,, && [z] = \tfrac{N}{N+k} \,, && [u_{ij}] = \tfrac{Nk - iN - jk}{N+k} \\
&D_p(SU(N)): && [x] = 1 \,, && [z] = \tfrac{N}{p} \,, && [u_{ij}] = N - j - \tfrac{N}{p} i
\end{aligned}
$$

- The sum is restricted to the terms such that $[u_{ij}] \geq 0$.
- u_{ij}, such that $[u_{ij}] \geq 0$, are called the **deformation parameters**.

$$
\begin{aligned}
&(A_{N-1}, A_{k-1}): && [u_{ij}] = \tfrac{Nk - iN - jk}{N+k} \\
&D_p(SU(N)): && [u_{ij}] = N - j - \tfrac{N}{p} i
\end{aligned}
$$

Introduction

- u_{ij} such that $[u_{ij}] > 1 \quad \longleftrightarrow \quad$ Coulomb branch (CB) operators

  The rank of the 4d theory $=$ # CB operators

- u_{ij} such that $[u_{ij}] = 1 \quad \longleftrightarrow \quad$ mass deformations

  The rank of the global symmetry $=$ # mass deformations

$$(A_{N-1}, A_{k-1}): \quad \mathrm{GCD}(k, N) - 1$$

- $D_p(SU(N))$ has *at least* $SU(N)$ global symmetry. The set of mass deformations includes

  1. u_{0j} $(0 \leq j \leq N-2) \quad \longleftrightarrow \quad$ the mass Casimirs of the $SU(N)$ global sym.
  2. u_{ij} such that $[u_{ij}] = 1$. There are $\mathrm{GCD}(p, N) - 1$ of these. These give rise to an extra symmetry, in additional to $SU(N)$.

- u_{ij} such that $[u_{ij}] = 0 \quad \longleftrightarrow \quad$ exactly marginal deformations

Introduction

- **Example:** The $D_2(SU(N))$ theory [Cecotti, del Zotto, Giacomelli '13]
  - For $N = 2n$ even, this is a Lagrangian theory: the 4d $\mathcal{N} = 2$ $SU(n)$ gauge theory with $2n$ flavours. The flavour symmetry is $SU(2n) \times U(1)$.
    (In general, if p divides N, $D_p(SU(N))$ admits a 4d $\mathcal{N} = 2$ Lagrangian description.)
  - For $N = 2n - 1$ odd, the scaling dim. of the CB operators are

$$\frac{3}{2}, \ \frac{5}{2}, \ \dots, \ \frac{2n-1}{2}$$

    There are no $\mathcal{N} = 2$ marginal deformations. The theory is said to be an **isolated SCFT**. No $\mathcal{N} = 2$ Lagrangian description.

- In general, $\#\,\mathcal{N} = 2$ marginal deformations of (A_{N-1}, A_{k-1})

$$= \begin{cases} N-3 & \text{if } N = k \\ \mathrm{GCD}(N,k) - 2 & \text{if } k|N \text{ (or } N|k) \text{ and } N \neq k \\ \mathrm{GCD}(N,k) - 1 & \text{otherwise} \end{cases}$$

  [Giacomelli, NM, Sacchi '20]

- $\#\,\mathcal{N} = 2$ marginal deformations of $D_p(SU(N))$ is equal to $\mathrm{GCD}(p,N) - 1$ if $N \nmid p$, and $\mathrm{GCD}(p,N) - 2$ if $N \mid p$.

Introduction

- Those with $\mathcal{N}=2$ marginal deformations can be described by weakly gauging of various **isolated SCFTs**, *i.e.* those without $\mathcal{N}=2$ preserving marginal deformation. [Closset, Giacomelli, Schäfer-Nameki, Wang; Giacomelli, NM, Sacchi '20]

  Example: $D_4(SU(6)) = [D_2(SU(9)) \longleftarrow SU(3) \longrightarrow D_2(SU(3))]$

  Example: $(A_3, A_5) = [D_5(SU(3)) \longleftarrow SU(2) \longrightarrow \underbrace{D_5(SU(2))}_{=(A_1, D_5)}]$

  (The CB spectra and (a, c) central charges [Xie, Yan, Yau '16; Shapere, Tachikawa '08] of both sides match.)

- **Note:** $D_p(SU(2)) = (A_1, D_p)$, $(A_3, G) = (D_3, G)$, $D_p(SO(6)) = D_p(SU(4))$
- The $D_p(SU(N))$ (resp. $D_p(SO(2N))$) theory can be realised as a class $\mathcal{S}$ theory of the A-type (resp. D-type) assoc. with a Riemann sphere with an **irregular puncture** and a **regular full puncture**.
- For $p > N$ (resp. $p > 2N-2$), upon closing the full puncture,

$$D_p(SU(N)) \longrightarrow (A_{p-N-1}, A_{N-1})$$
$$D_p(SO(2N)) \longrightarrow (A_{p-(2N-2)-1}, D_N)$$

Main result 1: Dimensional reduction of AD theories to 3d

Main result 1: 3d mirror theories for AD theories

Motivations:

1. To determine the global symmetry of the AD theory, not just its rank.

   (E.g., (A_1, A_3) has an $SU(2)$ global sym. and (A_1, D_4) has an $SU(3)$ global sym.)

2. To understand the structure of the Higgs branch of the AD theory.

   (E.g., The Higgs branch of (A_1, A_3) is $\mathbb{C}^2/\mathbb{Z}_2$ and that of (A_1, D_4) is $\overline{\text{min}}_{SU(3)}$.)

3. Can we get interesting 3d $\mathcal{N} = 4$ SCFTs upon dimensional reduction? Do they admit gauge theory descriptions? (We will go into detail soon.)

Remark: AD theories do not have a 4d $\mathcal{N} = 2$ Lagrangian description. This renders the study of points 1 and 2 difficult from the 4d perspective.

Method: Reduce the 4d AD theory on a circle to 3d and use mirror symmetry

[Intriligator, Seiberg '96] (the result is a.k.a. *magnetic quivers*) [Hanany; Bourget, Grimminger, Sperling, Zafrir, Zhong, ...]

- The CB of the 3d mirror $=$ the HB of the 4d theory in question.
  The HB of the 3d mirror $=$ the CB of the dimensionally red. 4d theory.
- It turns out that 3d mirror theories admits a Lagrangian description, and so we can study points 1 and 2 from the CB of such 3d theories.

Main result 1: 3d mirror theories for AD theories

Main results: [Giacomelli, NM, Sacchi '20; (Carta, Giacomelli, NM, Mininno '21)[2]]

- The 3d mirrors for (A_m, A_n), (A_m, D_n), (D_m, D_n), $D_p(SU(N))$, $D_p(SO(2N))$ are 3d $\mathcal{N} = 4$ gauge theories (with quiver descriptions), plus a collection of free hypermultiplets. (True for general m, n, p, N.)
- The 3d mirrors for (A_m, A_n) and (A_m, D_n) are abelian quiver gauge theories, bearing the structure of a complete graph. The 3d mirrors for (D_m, D_n) are quivers that contain a non-abelian USp gauge group along with abelian gauge groups.
- **General prescriptions** for constructing the 3d mirrors for (A_m, A_n), (A_m, D_n), (D_m, D_n), $D_p(SU(N))$, $D_p(SO(2N))$, for general m, n, p, N, are given in **our papers**. I present only some examples in this talk.

Remarks:

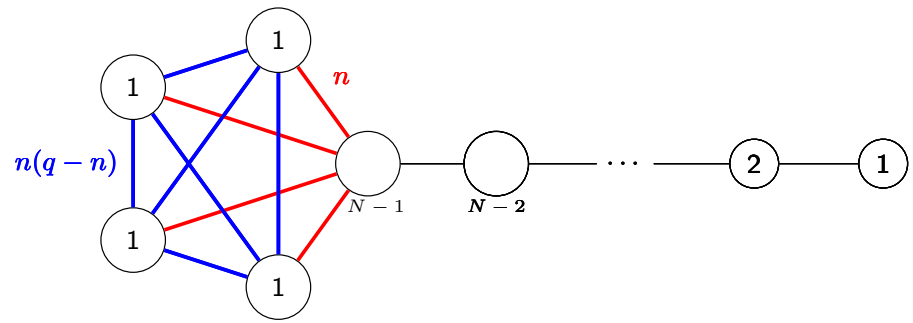
- The 3d mirrors for certain subclasses of (A_m, A_n), (A_m, D_n), $D_p(SU(N))$ are discussed by several authors, *e.g.* [Boalch '08; Xie '12; Xie, Zhao '13; del Zotto, Hanany '14; Buican, Nishinaka '15; Xie, Yau '17; Benvenuti, Giacomelli '17; Dey '20; Xie '21; Dey '21].

Main result 1: Mirror theories for $D_p(SU(N))$ and (A_{p-N-1}, A_{N-1})

[Giacomelli, NM, Sacchi, 20]

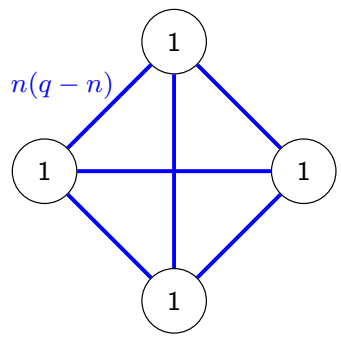
For $\mathrm{GCD}(p, N) = m$ (say $m = 4$), we define $n = N/m$ and $q = p/m$.

$D_p(SU(N))_{\text{3d mirr}}$:

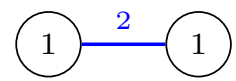


$$+ \frac{1}{2} m(n-1)(q-n-1) \text{ free hypers}$$

$(A_{p-N-1}, A_{N-1})_{\text{3d mirr}}$:



$$+ \frac{1}{2} m(n-1)(q-n-1) \text{ free hypers}$$

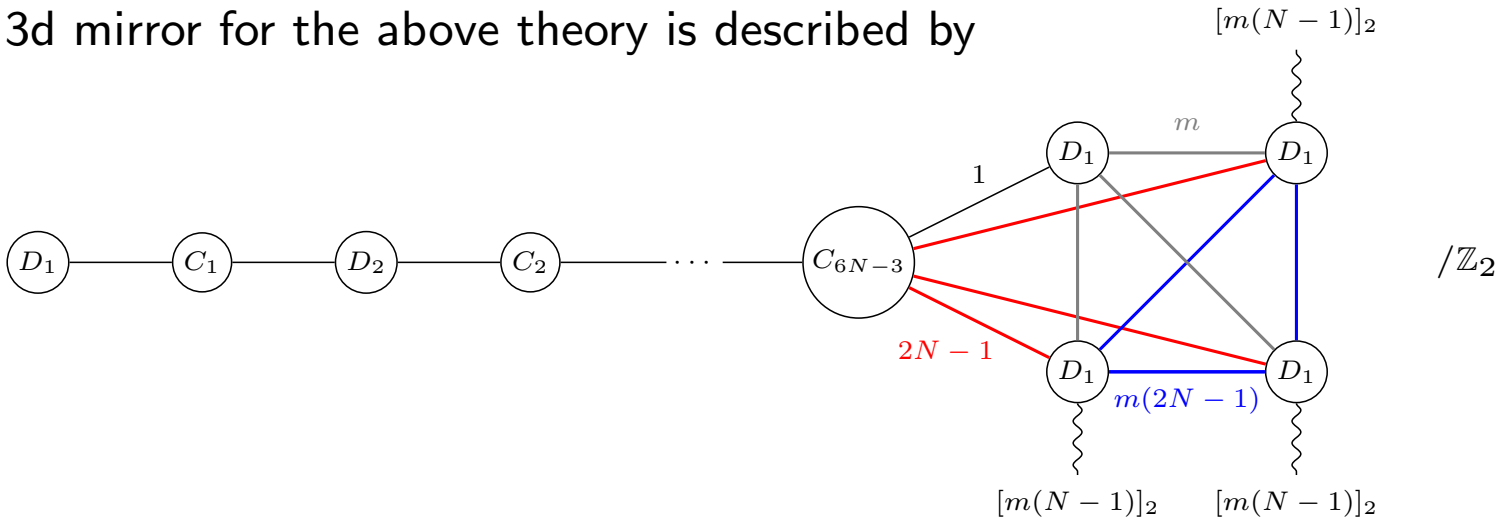
Example: (A_1, A_3). $p = 6$, $N = 4$. The mirror of (A_1, A_3) is 

SQED with 2 electrons, CB $= \mathbb{C}^2/\mathbb{Z}_2$, with $\mathrm{SU}(2)$ symmetry = HB of (A_1, A_3)

Example: $D_p(SO(12N-4))$ with $p=(12N-6)+6m$ and $\mathrm{GCD}(2N-1,m)=1$

[Carta, Giacomelli, NM, Mininno '21]

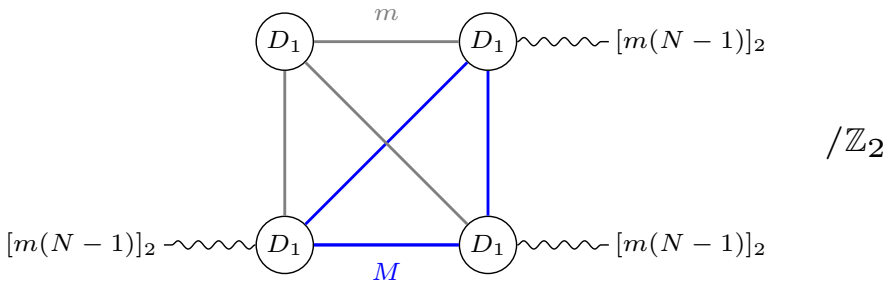
The 3d mirror for the above theory is described by



$+\ 3(N-1)(m-1)$ free hypers

The mirror theory for

(A_{6m-1}, D_{6N-2}) is described by



$+\ 3(N-1)(m-1)$ free hypers

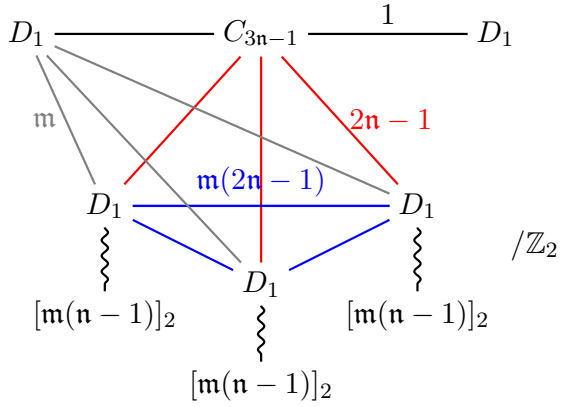
Notation: $C_n = USp(2n)$, $D_n = SO(2n)$,

$D_1 \sim\sim [F]_2$ denotes F hypers of charge 2 under $U(1) \cong D_1$, and $D_1 \overset{M}{\text{———}} D_1$ denotes M half-hypers in $[\mathbf{2};\mathbf{2}]$ of the gauge group $SO(2) \times SO(2)$.

Examples of 3d mirrors for (D_n, D_m) theories

(D_n, D_m) with **six** mass parameters are those with $n = 6\mathfrak{n} - 2$ and $m = 6\mathfrak{n} - 2 + 6\mathfrak{m}$ such that $\mathrm{GCD}(2\mathfrak{n} - 1, 2\mathfrak{m}) = 1$.

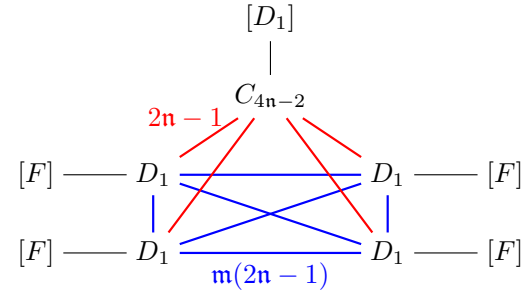
The 3d mirror is described by



$+\ 3(\mathfrak{m}-1)(\mathfrak{n}-1)$ free hypers

(D_n, D_m) with **five** mass parameters are those with $n = 8\mathfrak{n} - 3$ and $m = 8\mathfrak{n} - 3 + 8\mathfrak{m}$ such that $\mathrm{GCD}(2\mathfrak{n} - 1, 2\mathfrak{m}) = 1$.

The 3d mirror is described by



with $F = \mathfrak{m}(\mathfrak{n}+1) + (2\mathfrak{n}-1)$

$+\ 4(\mathfrak{m}-1)(\mathfrak{n}-1)$ free hypers

1. Each $D_1 \cong U(1)$ gauge node gives rise to a $U(1)$ global (topological) symmetry The **balanced** C_n gauge node (*i.e.* with $2n+1$ flavours) also gives rise to a $U(1)$ global symmetry in the IR [Gaiotto, Witten '08].
2. The CB symmetries are $U(1)^6$ and $U(1)^5$, respectively. Each of these is the global symmetry of the 4d theory in question.

Main result 2: non-Higgsable SCFTs

Main result 2: The 4d origin of 3d free hypermultiplets

[Giacomelli, NM, Sacchi '20]

- We have seen that 3d mirror theories of large class of AD theories contain free hypermultiplets. *What is their origin in 4d?*
- **Claim 1.** The free sector originates from the fact that at a generic point of the HB of the AD theory, there is some **SCFT with no HB but with a non-trivial CB** . We call the latter **non-Higgsable SCFTs** (nHSCFTs).
  Example. $(A_{\mathfrak{m}-1}, A_{\mathfrak{n}-1})$, with $\mathrm{GCD}(\mathfrak{m}, \mathfrak{n}) = 1$.
- **Claim 2.** Upon the reduction to 3d, a 4d **nHSCFT** of rank r becomes the 3d theory of r twisted hypermultiplets. (see also [Nanopoulos, Xie '10])
- **Justification.** Reducing to 3d and applying mirror symmetry, we obtain a theory with r quaternionic dimensional HB and no CB (rank-zero).
  Assumption. There is no interacting rank-zero 3d $\mathcal{N} = 4$ SCFTs with a non-trivial HB. $\longrightarrow$ Such a theory must be that of r free hypermultiplets.

Main result 2: Non-Higgsable SCFTs

- For a 4d $\mathcal{N} = 2$ SCFT whose generic point of of the HB consists of only a collection of H hypermultiplets,

$$24(c-a) = \dim_{\mathbb{H}} \text{Higgs branch} = H$$

('t Hooft anomaly matching between the SCFT point and a generic point of the HB.)

- For an nHSCFT, $24(c-a)$ can possibly be 0 or a non-integral number.

Examples.

$24(c-a)$	nHSCFT	rank
$1/5$	(A_1, A_2)	1
$2/7$	(A_1, A_4)	2
$3/7$	(A_2, A_3)	3
0	(A_2, D_4)	4
$1/5$	(A_2, D_7)	7
$10/13$	(A_4, D_5)	10
0	(A_4, D_6)	12

(See App. C of [arXiv:2105.08064] for an extensive list of nHSCFTs, and see [Kang, Lawrie, Song; Kang, Lawrie, Lee, Song '21] for more theories with $a = c$.)

Main result 2: Non-Higgsable SCFTs

- Let $\mathrm{GCD}(N,k)=m$, $n=N/m$, $\kappa=k/m$ (so that n and κ are coprime). We propose the following Higgs branch flow:

$$(A_{N-1},A_{k-1}) \longrightarrow m-1 \text{ hypers} + (A_{n-1},A_{\kappa-1})^{\otimes m}$$

  - **Test:** $24(c-a)$ are equal on both sides, as expected from the anomaly matching.

  The (A_{N-1},A_{k-1}) theory (and the $D_{N+k}(SU(N))$ theory) contains a non-Higgsable sector $(A_{n-1},A_{\kappa-1})^{\otimes m}$.
- For (A_m,D_n), (D_m,D_n) and $D_p(SO(2N))$, the pattern of nHSCFTs can be more complicated. [(Carta, Giacomelli, NM, Mininno '21)[2]]
  - **Example:** For $(D_{6\mathfrak{n}-2},D_{6\mathfrak{n}-2+6\mathfrak{m}})$ with $\mathrm{GCD}(2\mathfrak{n}-1,2\mathfrak{m})=1$ (*i.e.* those with six mass parameters), the nHSCFTs are $(A_{\mathfrak{m}-1},A_{2\mathfrak{n}-2})^{\otimes 3}$.
  - **Example:** For $(D_{2\mathfrak{n}+2},D_{4\mathfrak{n}+3})$, the nHSCFTs is $(A_{2\mathfrak{n}},D_{2\mathfrak{n}+2})$.

Main result 3: Dynamical consequences of 1-form symmetries

Main result 3: 1-form symmetries

[Gaiotto, Kapustin, Seiberg, Willett '14]

- A q-**form global symmetry** is generated by the symmetry generator associated with a codimension $q+1$ manifold M.

  The charged operators, acted by this symmetry, are q-dimensional objects. The correlation functions of charged operators are topological, *i.e.* they are unchanged under small changes of M, as long as these do not cross any charged operator.

  - If the symmetry is continuous, $\exists$ a conserved current $j(g)$. Charge $Q = \int_M *j(g)$. A small change of M does not change Q as long as this do not cross charged operators. Symmetry generator: $U_g(M) = \exp(\int *j(g))$.

- An ordinary symmetry $\leftrightarrow$ $q = 0$; the charged objects are points.

  We'll focus on 1-form symmetries; the charged objects are line operators.

- In gauge theories, the 1-form symmetry is generally determined by the centre of the gauge group, up to a screening due to matter in a given representation of the gauge group.

Main result 3: 1-form symmetries

[Gaiotto, Kapustin, Seiberg, Willett '14]

- **4d $U(1)$ gauge theory.** Two $U(1)$ one-form symmetries.

  Electric sym $\leftrightarrow$ closed form currents $*F$ (measures the elec. flux through M)

  Magnetic sym $\leftrightarrow$ closed form currents F (measures the mag. flux through M)

  The charged objects are constructed from Wilson lines and 't Hooft lines.

- **4d $U(1)$ gauge theory with charge N scalars.** The electric 1-form global $U(1)$ symmetry is explicitly broken to $\mathbb{Z}_N$. The Wilson lines are screened $\mathrm{mod}\, N$ by the charged N matter fields.

  No magnetic 1-form symmetry. (The 't Hooft line is not genuine – it needs a topological surface to attached to).

- **4d $SU(N)$ gauge theory.** Electric $\mathbb{Z}_N$ 1-form sym. No magnetic 1-form sym. (Confinement/no confinement $\leftrightarrow$ $\mathbb{Z}_N$ is unbroken/spon. broken.)

- **4d $SU(N)$ gauge theory with matter in $\mathbf{N}$.** The charged matter explicitly breaks the electric $\mathbb{Z}_N$ 1-form symmetry. No global 1-form symmetry. (Cannot distinguish between Higgs and confinement phases [Fradkin, Shenker; Banks, Rabinovici '79].)

Main result 3: 1-form symmetries

[Albertini, del Zotto, Garcia-Etxebarria, Hosseini '20; Closset, Schafer-Nameki, Wang; del Zotto, Garcia-Etxebarria, Hosseini '20; Hosseini, Moscrop '21]

- In general, AD theories do not admit $\mathcal{N} = 2$ Lagrangian descriptions. Nevertheless, their 1-form symmetries are computed from geometric engineering in Type IIB.
- A (G, G') theory can be realised by compatifying Type IIB on a CY_3 given by a hypersurface singularity X in $\mathbb{C}^4$.
- Let $\widehat{X}$ be the complex structure deformation of X. The D3-branes wrapping non-compact 3-cycles in $\widehat{X}$ correspond to line defects.
- The group of such line defects is given by $\mathrm{Tor}\, H_2(\partial\widehat{X}, \mathbb{Z})$. It encodes all (electric and magnetic) line operators of the (G, G') theory.
- $\mathrm{Tor}\, H_2(\partial\widehat{X}, \mathbb{Z})$ takes the form $\Gamma^{(1)} \oplus \Gamma^{(1)}$, where $\Gamma^{(1)}$ is the (electric or magnetic) 1-form symmetry of the (G, G') theory in question.

Main result 3: 1-form symmetries

[Albertini, del Zotto, Garcia-Etxebarria, Hosseini '20; Closset, Schafer-Nameki, Wang; del Zotto, Garcia-Etxebarria, Hosseini '20; Hosseini, Moscrop '21]

- **Examples.**

$$\Gamma^{(1)}(A_n, A_m) = 0$$

$$\Gamma^{(1)}(A_{n-1}, D_{m+1}) = \begin{cases} \mathbb{Z}_2^{\frac{1}{2}(\mathrm{GCD}(n,m)-1)} & \text{if} \quad 2 \nmid n; \\ \mathbb{Z}_2^{\frac{1}{2}(\mathrm{GCD}(n,m)-2)} & \text{if} \quad 2|n,m \text{ and } \mathrm{GCD}(n,2m)|m; \\ 0 & \text{otherwise} \end{cases}$$

- **Examples of some non-Higgsable SCFTs.**

$24(c-a)$	nHSCFT	rank	One-form symmetry
$1/5$	(A_1, A_2)	1	trivial
$2/7$	(A_1, A_4)	2	trivial
$3/7$	(A_2, A_3)	3	trivial
0	(A_2, D_4)	4	$\mathbb{Z}_2$
$1/5$	(A_2, D_7)	7	$\mathbb{Z}_2$
$10/13$	(A_4, D_5)	10	trivial
0	(A_4, D_6)	12	$\mathbb{Z}_2^2$

Main result 3: Dynamical consequences of 1-form symmetries

[Carta, Giacomelli, NM, Mininno '21]

- **Observation 1.** Any AD theory with a non-trivial 1-form sym always has a non-Higgsable sector, which has the same 1-form sym as the former.
  Consequence. The 3d mirror theory of an AD theory with a non-trivial 1-form sym always contains $r \geq 1$ **free** hypermultiplets, where r is the rank of the corresponding nHSCFT.
- **Observation 2.** If an AD theory has a trivial 1-form symmetry, then either it does not have a non-Higgsable sector or the non-Higgsable sector possesses a trivial 1-form symmetry.
- **Examples.**
  1. For $(D_{6\mathfrak{n}-2}, D_{6\mathfrak{n}-2+6\mathfrak{m}})$ with $\mathrm{GCD}(2\mathfrak{n}-1, 2\mathfrak{m}) = 1$ (*i.e.* six mass parameters), the nHSCFTs are $(A_{\mathfrak{m}-1}, A_{2\mathfrak{n}-2})^{\otimes 3}$. Trivial 1-form symmetry.
  2. For $(D_{2\mathfrak{n}+2}, D_{4\mathfrak{n}+3})$, the nHSCFTs is $(A_{2\mathfrak{n}}, D_{2\mathfrak{n}+2})$. Both theories have $\mathbb{Z}_2^{\mathfrak{n}}$ one-form symmetry.
- The 1-form symmetries are invariant under the Higgs branch flow.

Main result 4: Dualities between 3d $\mathcal{N} = 4$ gauge theories

Main result 4: Dualities between 3d $\mathcal{N}=4$ gauge theories

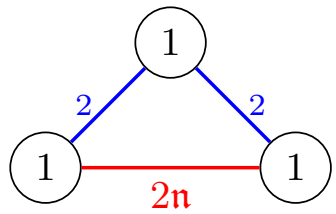
[(Carta, Giacomelli, NM, Mininno '21)[2]]

- **Duality 1:** The 3d mirror for $(D_3, D_{2\mathfrak{n}+2}) \cong (A_3, D_{2\mathfrak{n}+2})$ is $\mathfrak{n}$ free hypers + an interacting part that admits three dual descriptions:

I: $[C_\mathfrak{n}] - D_1 - C_1 - [D_2]$

$\longleftrightarrow$ II: $D_1 \overset{2}{\text{———}} D_1 \sim\sim [2\mathfrak{n}]_2$ $\quad /\mathbb{Z}_2$

$\longleftrightarrow$ III:



$/U(1)$

- $D_1 \overset{M}{\text{———}} D_1$ denotes M half-hypers in $[\mathbf{2};\mathbf{2}]$ of the gauge group $SO(2)\times SO(2)$. This gives rise to an $SU(M)^2 \times U(1)$ flavour symmetry. We view them as the half-hypers that transform in $\underbrace{\{U(1)\times U(1)\}}_{\text{gauge}} \times \underbrace{SU(M)\times SU(M)\times U(1)}_{\text{flavour}}$ as

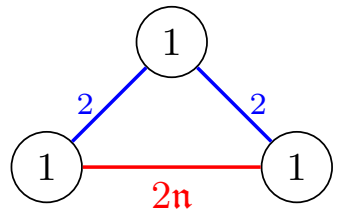
$$[+1;+1;\overline{\mathbf{M}};\mathbf{1};-1]\oplus[-1;-1;\mathbf{M};\mathbf{1};+1]$$
$$\oplus[+1;-1;\mathbf{1};\mathbf{M};+1]\oplus[-1;+1;\mathbf{1};\overline{\mathbf{M}};-1]$$

Main result 4: Dualities between 3d $\mathcal{N}=4$ gauge theories

I: $[C_{\mathfrak{n}}] - D_1 - C_1 - [D_2]$

$\longleftrightarrow$ II: $D_1 \overset{2}{\text{———}} D_1 \sim\sim [2\mathfrak{n}]_2$ $\quad /\mathbb{Z}_2$

$\longleftrightarrow$ III:

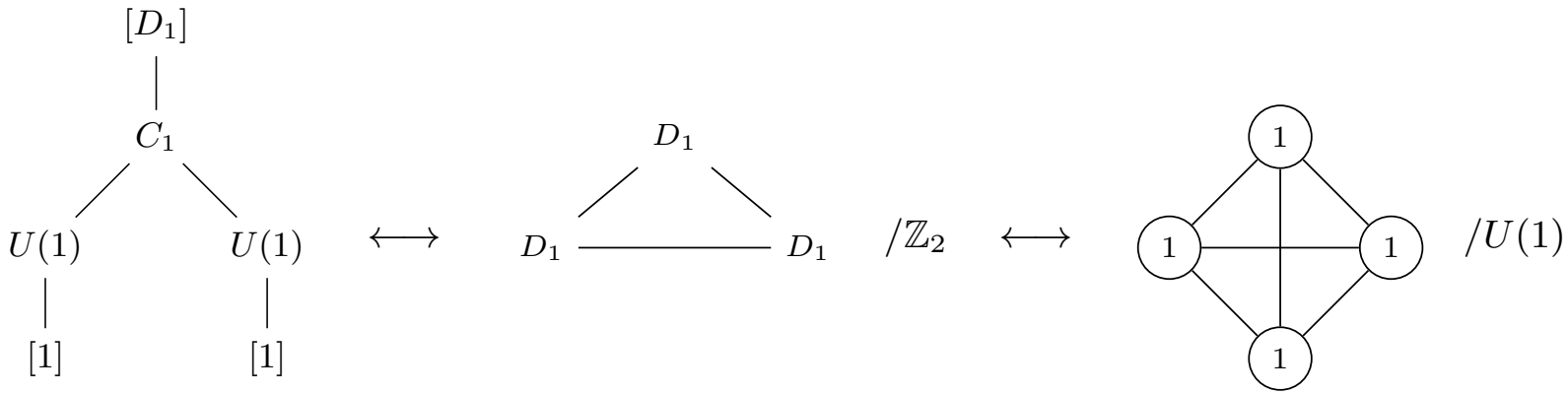


$/U(1)$

- This contains the duality between orthosymplectic and unitary quivers.
- The CB symmetry of each theory is $U(1)^2$.
  - This is manifest in II and III, as the topological sym. of the abelian nodes.
  - In theory I, the $C_1 = USp(2)$ gauge node is balanced, so it gives rise to a $U(1)$ topological symmetry. [Gaiotto, Witten '08]
- The HB symmetry of each theory is $U(2\mathfrak{n}) \times SO(4)$.
  - This is clear in theories II and III.
  - Theory I contains the HB operators of R-charge 1 in the rep. $\mathrm{Sym}^2\mathbf{2\mathfrak{n}} \oplus \wedge^2\mathbf{2\mathfrak{n}}$ of $C_{\mathfrak{n}} = USp(2\mathfrak{n})$ flavour symmetry. These correspond to the $U(2\mathfrak{n})$ flavour symmetry current.
- **Further tests:** Among these three theories, the HB/CB dimensions, the HB/CB Hilbert series and the indices can be matched.

Main result 4: Dualities between 3d $\mathcal{N} = 4$ gauge theories

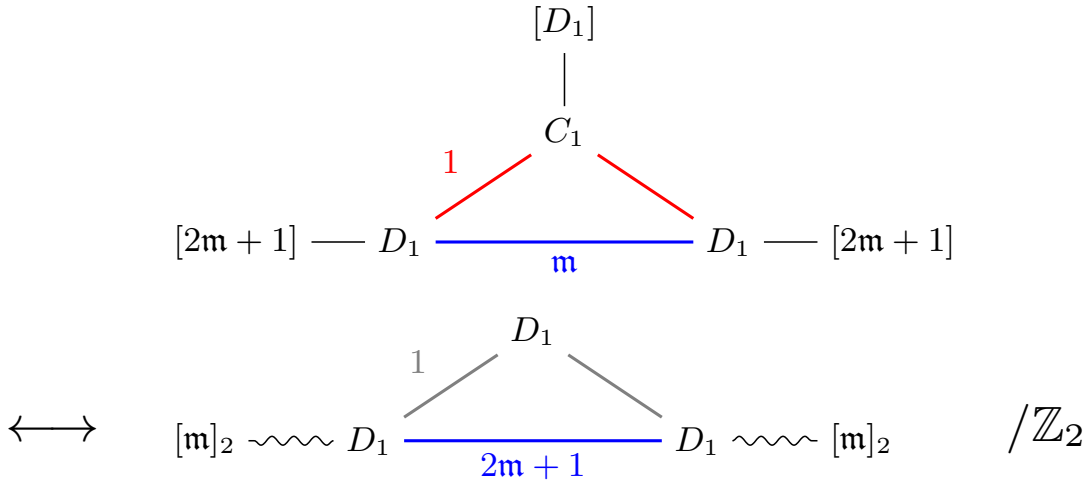
Duality 2: The 3d mirror for $(D_3, D_3) \cong (A_3, D_3) \cong (A_3, A_3)$ admits the following dual descriptions (denoted by I, II, III from left to right):



- Each theory is self-mirror. The HB (or CB) symmetry of each is $U(1)^3$.
- Theory I is the mirror theory of a class $\mathcal{S}$ theory assoc. with the A_2 twisted sphere with 2 minimal untwisted punctures $[2, 1]$ and 2 minimal twisted punctures $[2]_t$. [Beratto, Giacomelli, NM, Sacchi '20]
  This class $\mathcal{S}$ description realises the (D_3, D_3) theory.

Main result 4: Dualities between 3d $\mathcal{N}=4$ gauge theories

Duality 3: These are two dual descriptions of the 3d mirror for $(D_3, D_{4\mathfrak{m}+3}) \cong (A_3, D_{4\mathfrak{m}+3})$:



- The HB symmetry of each theory is $U(2\mathfrak{m}+1)^2 \times U(\mathfrak{m})^2 \times U(1)$, as it is manifest in each description.
- The CB symmetry of each theory is $U(1)^3$, which is also the global symmetry of $(D_3, D_{4\mathfrak{m}+3})$.

Conclusion

- Some interesting aspects of the AD theories
  - Reduction to 3d and their 3d mirror theories (magnetic quivers)
  - Non-Higgsable SCFTs at a generic point on the Higgs branch
  - Non-Higgsable SCFTs and one-form symmetries of the AD theories
  - Dualities between 3d $\mathcal{N} = 4$ gauge theories, coming from the reduction of AD theories that belong to different classes but are equivalent to each other
- **Open questions.**
  - Are there other dual descriptions for 3d reduction of these AD theories?
  - How about other classes of AD theories, *e.g.* $D_p(G)$ and (G, G') theories, with G or G' (or both) an exceptional group? (work in progress)
  - Can we say more about higher-form/higher-group symmetries of these models? (to appear)
  - Connection with 4d $\mathcal{N} = 1$ Lagrangians.

Backup slides

Dimensional reduction of AD theories to 3d

[Closset, Giacomelli, Schäfer-Nameki, Wang; Giacomelli, NM, Sacchi '20]

Remarks: For $D_p(SU(N)$ and (A_m, A_n), the 3d gauge theories obtained from *direct* dimensional reduction are known.

- For $D_p(SU(N))$, reduction to 3d gives a 3d $\mathcal{N} = 4$ quiver with a mixture of SU and U gauge groups, with a collection of twisted hypermultiplets. (**Note:** There is an alternative description in terms of quivers with unitary gauge groups with hypermultiplets carrying 'higher charges' [Dey '21].)
- **Example:** [Closset, Giacomelli, Schafer-Nameki, Wang '20]
  - Recall that $D_2(SU(2n))$ is a Lagrangian theory: 4d $\mathcal{N} = 2$ SCQCD $SU(n) - [2n]$. **Reduction to 3d** also gives the 3d $\mathcal{N} = 4$ $SU(n) - [2n]$.
  - However, the $D_2(SU(2n-1))$ is an isolated theory, which can be *embedded* into an asymptotically free 4d $\mathcal{N} = 2$ $SU(n) - [2n-1]$ (It is realised at a special point at the origin of the CB of this theory.) **Reduction to 3d** gives 3d $\mathcal{N} = 4$ $U(n-1) - [2n-1]$.
  - **Observation:** The $U(n-1)$ gauge group has the same rank as the $SU(n)$ gauge group of the 4d theory in which we embed the isolated SCFT.

Justification of the claim for $D_2(SU(2n-1))$

[Closset, Giacomelli, Schäfer-Nameki, Wang '20]

- Starting from the SW curve of the 4d $SU(n)$ SQCD with $2n-1$ flavours

$$y^2 = P(x)^2 - 4\Lambda \prod_{i=1}^{2n-1} \left(x - m_i + \frac{\Lambda}{n} \right) , \quad P(x) = x^n + \sum_{i=2}^{n} u_i x^{n-i} .$$

- There is a singular locus on which the effective theory is $U(n-1)$ with $2n-1$ flavours. This can be achieved by setting

$$m_i = \frac{\Lambda}{n} - m , \quad P(x) = (x - (n-1)m)\,(x+m)^{n-1} .$$

- There is a special point corresponding to $D_2(SU(2n-1))$ if we set $m = 0$.

$$m_i = \frac{\Lambda}{n} \text{ (all masses equal and prop. to } \Lambda) , \quad P(x) = x^n$$

This special point is located at the origin of the Coulomb branch (*i.e.* $P(x) = x^n$).

Justification of the claim for $D_2(SU(2n-1))$

[Closset, Giacomelli, Schäfer-Nameki, Wang '20]

- The $D_2(SU(2n-1))$ theory is well-approximated by the 4d SQCD Lagrangian theory at the origin of the Coulomb branch and at energies $E \ll \Lambda$, provided that $m_i = \Lambda/n$ for all i.
- Now we compactify the 4d theory on a circle with radius r and **keep the 4d scale Λ finite**.
- Since

$$\Lambda = \exp\left(-\frac{8\pi^2}{g_4^2}\right) = \exp\left(-\frac{4\pi}{r g_3^2}\right) \ , \quad g_4^2 = 2\pi r g_3^2 \ ,$$

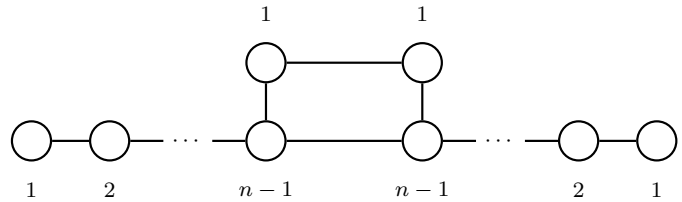
  the 3d gauge coupling goes to infinity like $g_3^2 \sim 1/r$ in the limit $r \to 0$.
- In particular, we should focus on the 3d $SU(n)$ SQCD with $2n-1$ flavours in the IR, at the origin of the Coulomb branch, but **keeping the flavour masses finite**.

Justification of the claim for $D_2(SU(2n-1))$

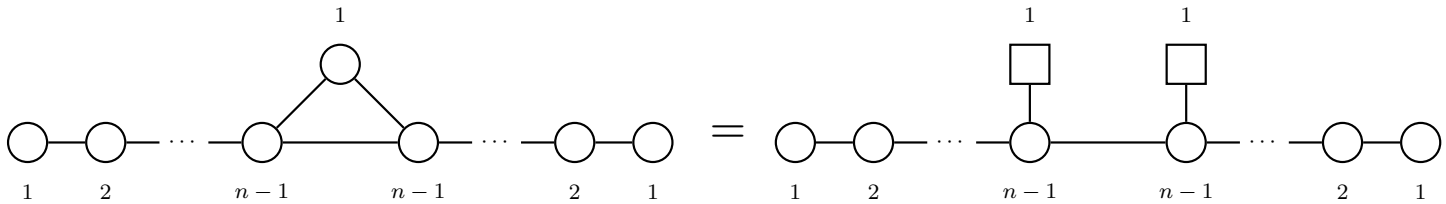
[Closset, Giacomelli, Schafer-Nameki, Wang '20]

- Study this in terms of the mirror theories:
  1. Mirror of the 3d $\mathcal{N}=4$ $SU(n)$ SQCD with $2n-1$ flavours

     [Bourget, Cabrera, Grimminger, Hanany, Zhong '19]:



  2. The flavour masses in the SQCD theory get mapped to the FI parameters in the mirror. Now we turn on the *common* FI parameters to all gauge nodes.



     Note: The long chain in the bottom does not get affected by such FI parameters.

     This is the mirror theory of the $U(n-1)$ SQCD with $2n-1$ flavours.

- **Summary:** The 3d reduction of the isolated $D_2(SU(2n-1))$ theory, which can be embedded into the $SU(n)$ SQCD with $2n-1$ flavours, gives the 3d $\mathcal{N}=4$ $U(n-1)$ SQCD with $2n-1$ flavours.

Example: $D_6(SU(4))$ and its 3d reduction

The fully deformed curve is

$$x^4 + u_{02}x^2 + u_{01}x + u_{00} + z(u_{10} + u_{11}x + u_{12}x^2 + u_{13}x^3) + z^2(u_{20} + u_{21}x + u_{22}x^2) + z^3(u_{30} + u_{31}x) + z^4(u_{40} + u_{41}x) + z^6 = 0 \ , \quad \lambda_{\rm SW} = xdz/z \ .$$

Consider the terms $u_{kl}z^k x^l$: $[x] = 1$, $[z] = \frac{2}{3}$, $[u_{kl}] = 4 - \frac{2}{3}k - l > 0$

- u_{0l} are the mass Casimirs of the $SU(4)$ global symmetry.
- If $\frac{2}{3}k \notin \mathbb{Z}$, then the CB operators u_{kl}, such that $[u_{kl}] > 1$, give rise to a unitary gauge group in 3d: n_k of such operators $\rightarrow$ a $U(n_k)$ factor.
  (**Above example:** $U(3)$, $U(2)$ and $U(1)$ factors, from the terms with z, z^2, z^4.)
- If $\frac{2}{3}k \in \mathbb{Z}$, then $[u_{kl}]$ are integral.
  - u_{kl} with $[u_{kl}] = m_k, m_k - 1, \ldots, 2$ are the CB parameters for a conformally-gauged $SU(m_k)$ factor. That with $[u_{kl}] = 1$ is mass term.
    (**Above example:** u_{30} is the CBO of the conf. $SU(2)$ factor; u_{31} is the mass term.)
  - This remains an $SU(m_k)$ gauge factor in the 3d quiver.
- The 3d reduction of $D_6(SU(4))$ gives

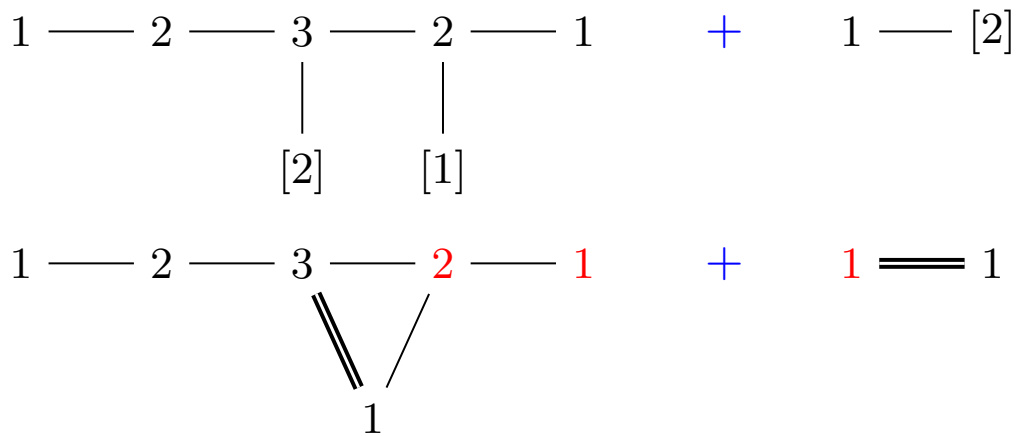
$$[4] - U(3) - U(2) - SU(2) - U(1)$$

Example: $D_6(SU(4))$, its 3d reduction, and the mirror theory

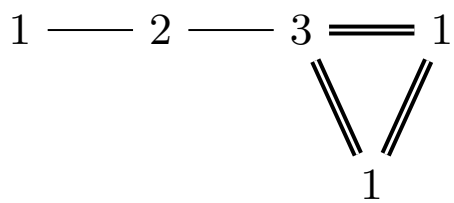
- The 3d reduction of $D_6(SU(4))$ gives

$$(D_6(SU(4)))_{3d}: \quad [4]-U(3)-U(2)-[2] \longleftarrow SU(2) \longrightarrow [2]-U(1)$$

- The mirror theory of $(D_6(SU(4)))_{3d}$ can be obtained by gluing the following theories (see *e.g.* [Xie, Yau '16]):



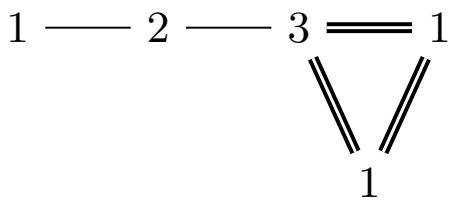
$(D_6(SU(4)))_{\text{3d mirr}}$:



- This is how the structure of a complete graph arises.

Example: The 3d mirror for (A_1, A_3)

$(D_6(SU(4)))_{\text{3d mirr}}$:



- **Recall:** $D_p(SU(N))_{p>N}$ is an A_{N-1} class $\mathcal{S}$ theory assoc. with a sphere with an irreg. puncture realising (A_{p-N-1}, A_{N-1}) and a full puncture $[1^N]$.
- Starting from $D_p(SU(N))$, $p > N$, we can close the full puncture and obtain the (A_{p-N-1}, A_{N-1}) theory.
- In terms of the 3d mirror, this corresponds to removing the quiver tail $1 - 2 - \cdots - (N-1)$
- From the above mirror theory, we obtain the mirror for (A_1, A_3):

$(A_1, A_3)_{\text{3d mirr}}$: $\quad 1 \Longrightarrow 1 \quad$ or $\quad U(1) - [2]$

- This confirms that the Higgs branch of the (A_1, A_3) theory is $\mathbb{C}^2/\mathbb{Z}_2$, and that (A_1, A_3) has an $SU(2)$ global symmetry.

Example: $D_7(SU(4))$ and its 3d reduction

- Using above procedure, we find that the 3d reduction of $D_7(SU(4))$ gives

$$(D_7(SU(4)))_{3d}: \quad [4] - U(3) - U(2) - U(2) - U(1) - U(1)$$

- However, some of these gauge groups can be dualised to free twisted hypermultiplets, according to the duality [Gaiotto-Witten '08]:
  $U(N)$ with $2N-1$ flavours
  $\leftrightarrow$ $U(N-1)$ with $2N-1$ flavours + one twisted hypermultiplet
- Thus, we rewrite the above theory as

$$(D_7(SU(4)))_{3d}: \quad [4] - U(3) - U(2) - U(1) \; + \; 3 \text{ twisted hypers}$$

  This theory flows to the $T(SU(4))$ theory with 3 twisted hypers in the IR.
  The 3d mirror for $D_7(SU(4))$ is $T(SU(4))$ plus 3 free hypers.
- Closing the full puncture in $D_7(SU(4))$ yields (A_2, A_3).
  The 3d mirror for (A_2, A_3) is 3 free hypermultiplets.

General prescription for the 3d mirrors for $D_p(SU(N))_{p>N}$ and (A_m, A_n)

[Giacomelli, NM, Sacchi '20]

$$\mathrm{GCD}(p,N) = m \ , \quad n = N/m, \quad q = p/m \ .$$

$(D_p(SU(N)))_{\text{3d mirr}}$, with $p > N$, can be constructed as follows:

1. Start with a complete graph with m $U(1)$ vertices such that each edge has multiplicity $n(q-n)$.
2. Construct the tail $(N-1)-(N-2)-\cdots-2-1$
3. Connect the tail to the complete graph such that the node $U(N-1)$ of the tail is connected to each $U(1)$ node in the complete graph by the edges, each with equal multiplicity n.
   Note that this quiver has an overall $U(1)$ that needs to be decoupled.
4. The number of free hypers is $H_{\text{free}} = \frac{1}{2}m(n-1)(q-n-1)$.

Removing the tail in step 2, we obtain $(A_{p-N-1}, A_{N-1})_{\text{3d mirr}}$:

- A complete graph with m $U(1)$ vertices such that each edge has multiplicity $n(q-n)$, together with H_{free} free hypers.

3d reduction of $D_p(SU(N))$, $p \geq N$

$$\mathrm{GCD}(p, N) = m \ , \quad n = N/m, \quad q = p/m \ .$$

The 3d reduction of $D_p(SU(N))$, with $p \geq N$, gives

$$[N] - (N-1) - (N-2) - \cdots - (N-n+1) - (N-n)^{q-n} - SU(N-n)$$
$$- (N-n-1) - (N-n-2) - \cdots - (N-2n+1) - (N-2n)^{q-n} - SU(N-2n) - \cdots$$
$$- SU(2n) - (2n-1) - \cdots - (n+1) - (n)^{q-n} - SU(n) - (n-1) - (n-2) - \cdots 2 - 1$$

with $H_{\text{free}} = \frac{1}{2} m(n-1)(q-n-1)$ twisted hypers

- $(x)^{q-n}$ means $\underbrace{(x) - \cdots - (x)}_{q-n \text{ nodes}}$
- There are in total $(m-1)$ special unitary gauge groups $SU(N-jn)$ with $1 \leq j \leq m-1$.
  - However, when $N \mid p$, *i.e.* $m = N$ and $n = 1$, the last SU group is $SU(1)$. The total number of non-trivial SU gauge groups is $m - 2 = N - 2$.
- The **conformal manifold** of the 4d $D_p(SU(N))$ SCFT is $m-1$ dimensional if $N \nmid p$ and is $m-2$ dimensional if $N \mid p$.

Examples: $D_p(SU(N))$ where N divides p

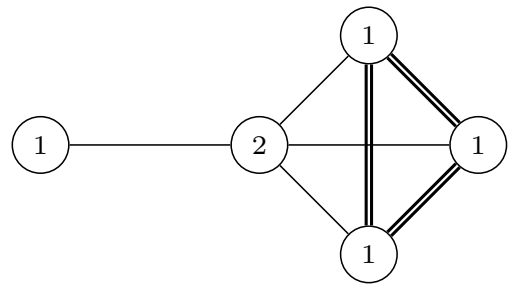
Take $p = N\mathfrak{m}$ for $\mathfrak{m} \in \mathbb{Z}_{\geq 1}$. ($m = \mathrm{GCD}(p, N) = N$, $n = 1$ and $q = \mathfrak{m}$.)

$$(D_{N\mathfrak{m}}(SU(N))_{3d}: \quad [N] - (N-1)^{\mathfrak{m}-1} - SU(N-1) - (N-2)^{\mathfrak{m}-1} - SU(N-2) - \cdots$$
$$- (2)^{\mathfrak{m}-1} - SU(2) - (1)^{\mathfrak{m}-1} - [1]$$

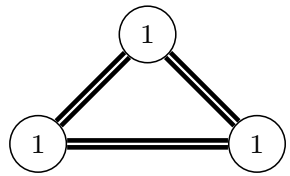
The mirror theory consists of

- a complete graph with N $U(1)$ nodes and edge multiplicity $\mathfrak{m} - 1$,
- the tail $(N-1) - \cdots - (2) - (1)$, and
- the edges (multiplicity 1) connecting the tail to each vertex in the complete graph.
- $H_{\text{free}} = 0$. Can decouple the tail to obtain $(A_{N(\mathfrak{m}-1)-1}, A_{N-1})_{\text{3d mirr}}$.

$(D_9(SU(3)))_{\text{3d mirr}}$:



$(A_5, A_2)_{\text{3d mirr}}$:



Global symmetry of the magnetic quiver for $D_p(SU(N))_{p\geq N}$

- In general, the CB symmetry of the magnetic quiver is $SU(N) \times U(1)^{m-1}$.
  - $SU(N)$ comes from the enhanced topological symmetry of the tail
  - $U(1)^{m-1}$ comes from the topological symmetries of m $U(1)$ nodes in the complete graph, with an overall $U(1)$ decoupled.
    (This could possibly get enhanced further in some special cases.)
  - This agrees with the global symmetry of the 4d $D_p(SU(N))$ theory.
- The HB symmetry (neglecting the contribution of the free hypers) is $SU(n)^m \times U(1)^m \times SU(nq-n^2)^{\frac{1}{2}m(m-1)}$ (for $m \geq 3$)
  - The 2nd and the 3rd factors come from the complete graph, and the 1st factor comes from the edges connecting the tail with the complete graph.
  - For $m = 1$ the HB symmetry is only $SU(n)$, and for $m = 2$ we have $U(1) \times SU(n)^2 \times SU(nq-n^2)$.
  - For general m, such a HB symmetry agrees with the sophisticated enhancement of the topological symmetry in the mixed U/SU quiver

$$
\begin{aligned}
&[N] - (N-1) - (N-2) - \cdots - (N-n+1) - (N-n)^{q-n} - SU(N-n) \\
&- (N-n-1) - (N-n-2) - \cdots - (N-2n+1) - (N-2n)^{q-n} - SU(N-2n) - \cdots \\
&- SU(2n) - (2n-1) - \cdots - (n+1) - (n)^{q-n} - SU(n) - (n-1) - (n-2) - \cdots 2 - 1
\end{aligned}
$$

3d reduction of $D_p(SU(N))$, $p < N$

$$x = \lfloor N/p \rfloor \ , \qquad m = \mathrm{GCD}(p, N) \ , \qquad n = N/m \ , \qquad q = p/m$$
$$m_A = q(1+x) - n \ , \qquad m_B = n - qx$$

- The 3d reduction gives the following mixed U/SU quiver:

$$[N] - (N - (x+1)) - \cdots - (N - m_B(x+1)) - C[N-n] - SU(N-n)$$
$$- (N - n - (x+1)) - \cdots - (N - n - m_B(x+1)) - C[N-2n] - SU(N-2n) - \cdots$$
$$- SU(2n) - (2n - (x+1)) - \cdots - (2n - m_B(x+1)) - C[n] - SU(n)$$
$$- (n - (x+1)) - \cdots - (n - m_B(x+1)) - C[0]$$

where we use the shorthand notation:

$$C[Y] = (Y + (m_A - 1)x) - (Y + (m_A - 2)x) - \ldots - (Y + 2x) - (Y + x) \ .$$

- There are $m - 1$ SU gauge groups.
- The conformal manifold is $m - 1$ dimensional for $p < N$.

Magnetic quivers for $D_p(SU(N))$ with $p < N$

$$x = \lfloor N/p \rfloor \ , \qquad m = \mathrm{GCD}(p, N) \ , \qquad n = N/m \ , \qquad q = p/m$$
$$m_A = q(1+x) - n \ , \qquad m_B = n - qx \qquad M = N - (x+1)$$

1. Start with a complete graph with m $U(1)$ nodes where each edge has multiplicity $m_G = m_A m_B$.
2. Construct two types of tails:

   Tail A : $(p-1) - 2(p-1) - \cdots - (p-1)(x-2) - (p-1)(x-1) - (p-1)x$

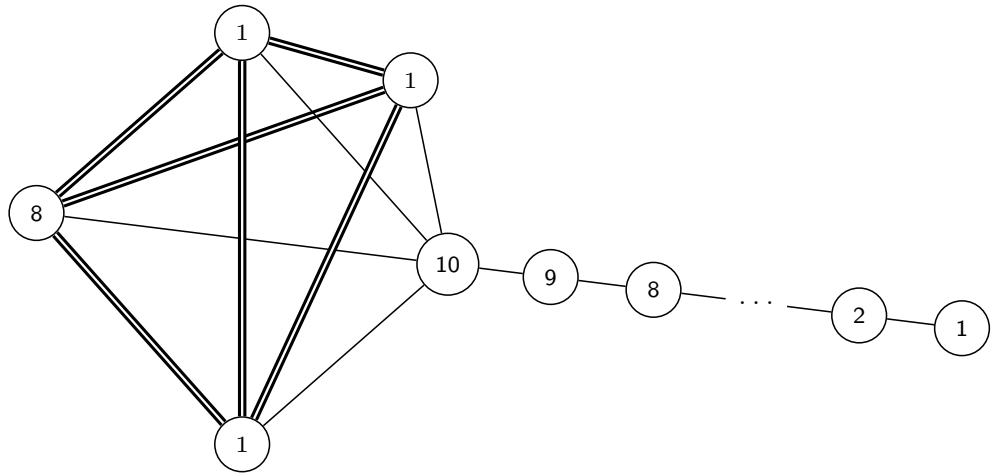
   Tail B : $M - (M-1) - (M-2) - \ldots - 2 - 1$
3. Attach the node $(p-1)x$ of tail A to each $U(1)$ node in the complete graph with edges with equal multiplicity m_A.
4. Attach the node M of tail B to each $U(1)$ node in the complete graph with edges with equal multiplicity m_B.
5. Connect the node $(p-1)x$ of tail A to the node M of tail B with an edge with multiplicity 1.
6. There is an overall $U(1)$ that needs to be decoupled.
7. There are $H_{\text{free}} = \frac{1}{2} m(n - qx - 1)(q(1+x) - n - 1)$ free hypers.

Example: $D_9(SU(12))$

- We have $m = 3$, $x = 1$, $m_A = 2$, $m_B = 1$ and $m_G = 2$.
- The reduction to 3d gives

$$(D_9(SU(12)))_{3d}: \quad [12] - 10 - 9 - SU(8) - 6 - 5 - SU(4) - 2 - 1$$

- The mirror theory is



with $H_{\text{free}} = 0$.

- The HB symmetry of the magnetic quiver is $U(1)^4 \times SU(2)^6$. The CB symmetry of the magnetic quiver is $SU(12) \times U(1)^2$, in agreement with the global symmetry of the 4d $D_9(SU(12))$ SCFT.

Global symmetry of the magnetic quiver for $D_p(SU(N))_{p<N}$

- In general, the CB symmetry of the magnetic quiver is $SU(N) \times U(1)^{m-1}$.
  - $SU(N)$ comes from the enhanced topological symmetry of tails A and B
  - $U(1)^{m-1}$ comes from the topological symmetries of m $U(1)$ nodes in the complete graph, with an overall $U(1)$ decoupled.
    (This could possibly get enhanced further in some special cases.)
  - This agrees with the global symmetry of the 4d $D_p(SU(N))$ theory.
- The HB symmetry (neglecting the contribution of the free hypers) is $SU(m_A)^m \times SU(m_B)^m \times U(1)^{m+1} \times SU(m_G)^{\frac{1}{2}m(m-1)}$, $m_G = m_A m_B$.
  - The 1st and the 2nd terms arise from the edges connecting tails A and B to the complete graph, respectively. The 3rd and the 4th terms arise from the complete graph.
  - Such a HB symmetry agrees with the sophisticated enhancement of the topological symmetry in the mixed U/SU quiver.