Homogeneity + isotropy:

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

$$k = \begin{cases} 1 & \text{spherical} \\ 0 & \text{flat} \\ -1 & \text{hyperspherical} \end{cases} \xrightarrow{\text{Einstein}} \underbrace{H \equiv \frac{\dot{a}}{a} = \left(\frac{\rho}{3M_P^2}\right)^{1/2}}_{\text{Friedmann}} \quad \rho = \sum_{i=R,m,k,\Lambda} \rho_i$$

Assume perfect fluid: $T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & P & & \\ & & P & \\ & & & P \end{pmatrix}$ $\quad P = w\rho$ $\begin{cases} w = \frac{1}{3}: \text{radiation} \\ w = 0 \text{ matter} \end{cases}$

Continuity $\nabla_\mu T^{\mu 0} = 0 \rightarrow \dot{\rho} + 3H(\rho + P) = 0$

$$\dot{\rho} + 3H\rho(1+w) = 0$$

$\rightarrow$ 2nd Friedmann eq.: $\frac{\ddot{a}}{a} = -\frac{1}{6M_P^2}(\rho + 3P) = -\frac{1}{6M_P^2}\rho(1+3w)$

Hot Big Bang model: $w = \frac{1}{3}$ $\quad T_{max} \gtrsim T_{BBN} \sim \text{MeV}$

- **Flatness problem:** $\Omega_i \equiv \frac{\rho_i}{\rho_c}$ $\quad \rho_c = 3M_P^2 H \iff \sum_i \Omega_i = 1 \rightarrow k = 0$

$$|\Omega - 1| = \frac{-k}{(aH)^2} \equiv \Omega_k \quad \Omega_k(a_0) = 0.0007 \pm 0.0019 \text{ (Planck)}$$

$\rightarrow$ flat universe

$$\frac{d\Omega}{d\ln a} = (1+3w)\Omega(\Omega - 1) \begin{cases} > 0 \text{ for } \Omega \geq 1 \\ < 0 \text{ for } \Omega \leq 1 \end{cases} \text{if } \begin{matrix} w > -\frac{1}{3}: \underline{\text{unstable}} \\ w < -\frac{1}{3} \text{ stable (SEC)} \end{matrix}$$

Planck → $|1-\Omega(a_{GUT})| < 10^{-55}$: fine tuning?

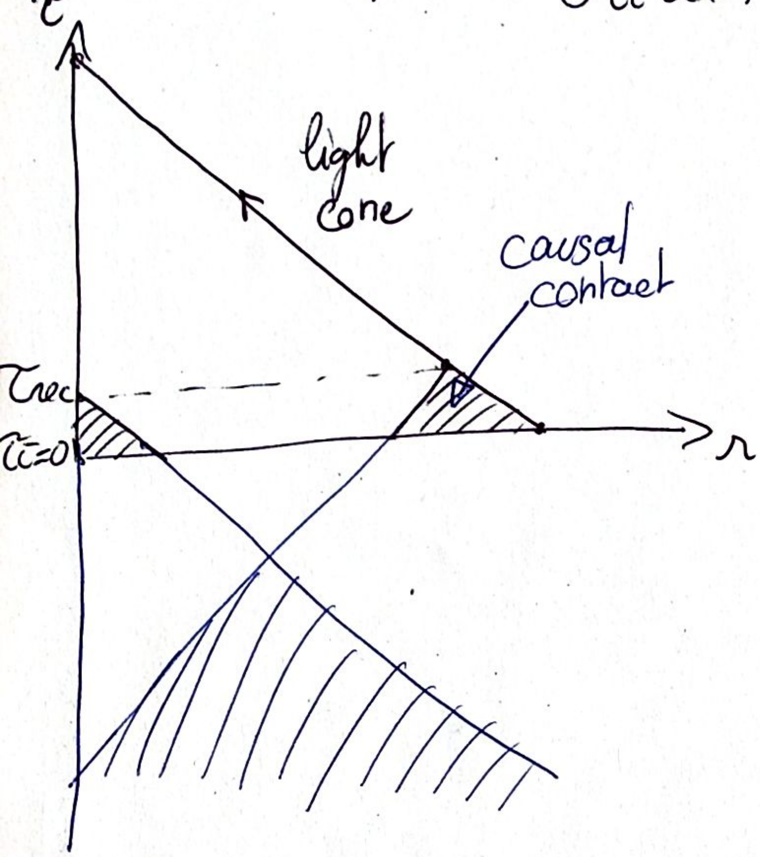
- Horizon problem : $ds^2 = a^2(\tau)(d\tau^2 - dx^2)$ $(k=0)$ $\boxed{d\tau = \frac{dt}{a}}$

$\left|\frac{\delta T(\hat{x})}{T}\right| \lesssim 10^{-5}$. $\theta \sim 2°$ horizon → 10^4 disconnected patches

$= a^2(\tau)\eta^{\mu\nu}dx^\mu dx^\nu$ → Minkowski

photon : $ds^2 = 0 \to r(\tau) = \pm\tau + \text{constant}$

max comoving distance: $\int_{t_i}^{t} \frac{dt'}{a(t')} = \int_{\tau_i}^{\tau} d\tau' = \tau - \tau_i$



$\tau = \int_{a_i}^{a} d\ln a' \left[\frac{1}{a'H}\right]$ = comoving Hubble radius

$\frac{1}{aH} \propto a^{\frac{1}{2}(1+3w)} \xrightarrow[w=\frac{1}{3}]{} a$

$\tau \sim \frac{2}{(1+3w)} a^{\frac{1}{2}(1+3w)}$

- $\lim_{a_i \to 0} \tau \to 0$ "Big Bang singularity" $w = 1/3$
- $\lim_{a_i \to 0} \tau \to -\infty$ for $w \leq -1/3$

→ when $a_i \simeq 0$ → quantum effects : classical approach breaks down!

- Unwanted relics in GUT $\rho \sim a^{-3}$ → overclose universe

Solution: inflation $\frac{d}{dt}\left(\frac{1}{aH}\right) < 0$ schrinking Hubble sphere

$$\Leftrightarrow w < -\frac{1}{3} \xrightarrow{\text{Friedmann}} \frac{\ddot{a}}{a} = -\frac{1}{6M_p^2}\rho(1+w) > 0$$

$\boxed{\ddot{a} > 0}$ accelerated expansion

Assume H = constant

$dN \equiv d\ln a \rightarrow a \sim e^{N}$

- $|\Omega - 1| \sim e^{-2N}$: flatness $10^{-55} \sim e^{-2N}$ $\rightarrow N \sim 60$
- $\rho \sim e^{-3N}$: relics

Slow-roll inflation: $S = \int d^4x \sqrt{-g}\left(\frac{M_p^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right)$

$\phi = \phi(t)$

$\xrightarrow{\text{EOM}} \ddot{\phi} + 3H\dot{\phi} + V_\phi = 0$ $\quad V_\phi = \frac{\partial V}{\partial \phi}$ $\quad \begin{cases} \rho = \frac{\dot{\phi}^2}{2} + V(\phi) \\ P = \frac{\dot{\phi}^2}{2} - V(\phi) \end{cases}$

$H^2 = \frac{1}{3M_p^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)$ $\quad \frac{\ddot{a}}{a} = -\frac{1}{3M_p^2}\left(\dot{\phi}^2 - V(\phi)\right)$

$\boxed{w < \frac{1}{3} \rightarrow \frac{\dot{\phi}^2}{2} < V(\phi)}$ $\quad \cancel{\rightarrow H^2 \simeq \frac{1}{3M_p^2}V(\phi)}$ $\quad \rightarrow w = \frac{P}{\rho} \simeq -1 \rightarrow$ cosmological constant

$\varepsilon \equiv \frac{\dot{\phi}^2}{2M_p^2H^2} < 1$
slow roll parameter

inflation persists if $\left|\delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}\right| \ll 1$

$\eta = 2(\varepsilon - \delta)$

consider $\varepsilon \ll 1 \rightarrow 3H\dot{\phi} \simeq -V_\phi \rightarrow \varepsilon \simeq \frac{M_p^2}{2}\left(\frac{V_\phi}{V}\right)^2 \equiv \varepsilon_V$

$\delta + \varepsilon \simeq M_p^2 \frac{V_{\phi\phi}}{V} \equiv \eta_V \rightarrow$ flat potential

$|\varepsilon_V|, |\eta_V| \ll 1$ $\rightarrow$ then $\ddot{a} = 0$: end of inflation
inflation

CMB $\left|\frac{\delta T(x)}{T}\right| \lesssim 10^{-5}$ → introduce fluctuations

gauge choice: $x'_\mu = \bar{x}_\mu + \xi_\mu(x,t)$

coordinates in frame

background space-time coordinates

map: define gauge

ξ_μ: small

$$\boxed{g'_{\mu\nu}(x',t) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(x,t)}$$

$$\phi(x',t) = \bar{\phi}(t) + \delta\phi(x,t)$$

$\delta g_{\mu\nu}$: 10 dof $- \underset{\text{gauge choice}}{4}$ = | 2 scalars, 2 vectors, 2 tensor | under 3D rotation

→ small perturbations → Taylor expansion

→ work with gauge invariant quantities

- Einstein to set $\delta\phi = 0$
- $\delta g_{ij} = a^2(1-2\mathcal{R})\delta_{ij} + a^2 h_{ij}$

curvature perturbation

tensor

$$\rightarrow S = \frac{1}{2}\int d^4x\, a^3 \frac{\dot{\phi}^2}{H^2}\left[\dot{\mathcal{R}}^2 - \frac{1}{a^2}(\partial_i \mathcal{R})^2\right] + \underbrace{\dots}_{\text{higher order in perturbation}}$$

Mukhanov: $v \equiv z\mathcal{R}$ with $z^2 = 2a^2\varepsilon$

$$\hookrightarrow S = \frac{1}{2}\int d\tau\, d^3x \left[(v')^2 - (\partial_i v)^2 + \underbrace{\frac{z''}{z}}_{\equiv -m_{eff}^2(\tau)} v^2\right]$$

: harmonic oscillator with τ-dependent mass

EOM in Fourier space: $v_k'' + \underbrace{\left(k^2 - \frac{z''}{z}\right)}_{\omega_k^2(\tau)} v_k = 0$ (Mukhanov Sasaki)

origin of fluctuations: quantum

$$v \to \hat{v} = \int \frac{d^3k}{(2\pi)^{3/2}} \left[\hat{a}_k v_k(\tau) e^{ikx} + \hat{a}_k^{\dagger} v_k^*(\tau) e^{-ikx} \right]$$

. $\hat{a}_k |0\rangle = 0$. $[\hat{a}_k, \hat{a}_{k'}^{\dagger}] = (2\pi)^3 \delta^{(3)}(k-k')$

in De Sitter (H = constant) $a = -(H\tau)^{-1} \to \omega_k^2(\tau) = k^2 - \frac{2}{\tau^2}$

→ ambiguity in GR vacuum state for expanding background

but $\lim_{\tau \to -\infty} \omega_k \to k$ $v_k'' + k^2 v_k = 0$ ⇔ Minkowski

this fixes $\lim_{\tau \to -\infty} v_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau}$

→ solution $\boxed{v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right)}$

modes inside horizon $\frac{k}{aH} \sim |k\tau| \gg 1$ ⇔ $\tau \gg \lambda = \frac{1}{k}$

quantum 0-point fluctuations:

$$\langle 0|\hat{v}_k \hat{v}_{k'}|0\rangle = |v_k|^2 \delta^{(3)}(k+k')$$
$$\equiv P_v(k)\, \delta^{(3)}(k+k') \qquad P_v = \frac{1}{2k^3\tau^2} = \frac{1}{2k^3}(aH)^2$$

$$\hookrightarrow \langle 0|\hat{R}_k \hat{R}_{k'}|0\rangle = P_R(k)\, \delta^{(3)}(k+k')$$

$$P_R = \frac{1}{4k^3}\frac{H^2}{\varepsilon} = \frac{1}{2k^3}\frac{H^4}{\dot{\phi}^2} \qquad P_R = \frac{1}{z^2} P_v$$

$\lim_{k \ll aH} \dot{R}_k = 0 \rightarrow R_k$ constant outside horizon

$$P_R(k) = \frac{1}{4k^3}\frac{H^2}{\varepsilon}\Big|_{k=aH} \qquad \Delta_s^2 \equiv \frac{k^3}{2\pi^2} P_R(k) = \frac{1}{8\pi^2}\frac{H^2}{\varepsilon}\Big|_{k=aH}$$

$$\frac{d\ln \Delta_s^2}{d\ln k} \equiv n_s - 1$$: scalar spectral index
quantify deviation from scale invariance

$$\frac{d\ln\Delta_s^2}{dN} = \frac{2\,d\ln H_*}{dN} - \frac{d\ln \varepsilon_*}{dN} = -2\varepsilon - \eta$$

$$\frac{dN}{d\ln k} \simeq 1+\varepsilon \rightarrow \boxed{n_s - 1 = -2\varepsilon - \eta}$$