(Single-field) Slow-roll inflation

– large-field & small field models

Starting point: observational constraints on P_S, P_T via n_S and r

$\rightarrow$ $n_S - 1 \simeq 0.96$ at 5σ (e.g. Planck 2018)

$\rightarrow$ BK15 (150 & 95 GHz): $r < 0.07$ (95%)

$r = 0.02 \pm^{0.021}_{0.018}$

$\downarrow$

arXiv: 2203.16556 BK18 (+ 220 GHz): $r < 0.036$ (95%)

$r = 0.014 \pm^{0.010}_{0.011}$

(maybe evolution of best fit vs error bars vs upper limit shows first non-zero hint of r? $\rightarrow$ wait & see)

following now arXiv: 1610.02743, section 2.5 ...

$\rightarrow$ can parametrize n_S observation as:

$$(*) \quad n_S - 1 = -\frac{q+1}{N} \simeq -\frac{\mathcal{O}(1)}{60}$$

N: e-folds before end of inflation

we have per definition: $k = aH = e^{N_{tot} - N} H$

$$n_S - 1 = \frac{d\ln P_S}{d\ln k} = -\frac{d\ln P_S}{dN}$$

$$P_S = \frac{1}{8\pi^2} \cdot \frac{H^2}{\epsilon}$$

$$= +\frac{d\ln\epsilon}{dN} - 2\frac{d\ln H}{dN}$$

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{d\ln H}{dN}, \quad dN = -H\,dt$$

$$= \frac{d\ln\epsilon}{dN} - 2\epsilon$$

ODE:

$$\frac{d\ln\epsilon}{dN} - (n_s - 1) - 2\epsilon = 0$$

(*)

solution:

$$\epsilon(N) = \frac{q}{2N}\,\frac{1}{1+\frac{e\cdot q}{2}N^q}$$

$$= \frac{q}{2N}\cdot\frac{1}{1+\left(\frac{N}{N_e}\right)^q}$$

$$\frac{e\cdot q}{2} \equiv N_e^{-q}$$

→ 4 regimes of which 2 have important:

I) $q>0,\ N_e \gg N$: $\epsilon = \frac{q}{2N}$

II) $q>0,\ N \gg N_e$: $\epsilon = \frac{q}{2N}\cdot\left(\frac{N_e}{N}\right)^q$

Case I: use slow-roll

$$3H\dot\phi = -3H^2\frac{d\phi}{dN} = -V'$$

$$\Leftrightarrow \frac{d\phi}{dN} = \frac{V'}{V},\ \left(\frac{d\phi}{dN}\right)^2 = 2\epsilon$$

$$\Rightarrow \frac{d\phi}{dN} = \sqrt{\frac{q}{N}} \Rightarrow \phi = 2\sqrt{qN}$$

$$\Rightarrow \frac{V'}{V} = \sqrt{\frac{q}{N}} = \frac{2q}{\phi}$$

$$\Rightarrow V(\phi) \sim \phi^p,\ p = 2q$$

monomial ("chaotic") potentials

→ Linde 1982

for powers $p \gtrsim 0.1$:

$$r \gtrsim 0.01$$

e.g. $p = 2$: $r = \frac{8}{N} \simeq 0.13$

$$n_s = 1 - \frac{2}{N} \simeq 0.967$$

at $N \simeq 60$

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case II :

$$q \neq 1 \rightarrow V(\phi) = V_0 \cdot e^{-\left(\frac{\phi}{\Lambda}\right)^{\frac{2q}{q-1}}}$$

$$\Lambda \equiv \sqrt{\frac{4q}{(1-q)^2} N_e} \cdot M_p$$

Λ defines <u>characteristic</u> mass scale $M \equiv \Lambda \cdot \frac{|1-q|}{q} = \frac{2}{\sqrt{q}} \sqrt{N_e} \cdot M_p$

$\mathcal{O}(1)$ here

if $q > 1$:

$$V(\phi) = V_0 \cdot \left[1 - \left(\frac{\phi}{\Lambda}\right)^n\right]$$

$$n = \frac{2q}{q-1} > 2$$

<u>hill-top</u> : <u>small-field</u> inflation

$$\epsilon \sim \left(\frac{n\left(\frac{\phi}{\Lambda}\right)^{n-1}}{1 - \left(\frac{\phi}{\Lambda}\right)^n}\right)^2 \rightarrow 0$$

for $\underline{\phi \ll \Lambda} = \mathcal{O}(M_p)$

special case : $q = 1$

$$\Rightarrow V \simeq V_0 \left(1 - e^{-\phi/M}\right)$$

at $\phi > M$ "exponential plateau"

we get:

$$\epsilon = \frac{1}{2M^2} \frac{e^{-2\phi/M}}{(1-e^{-\phi/M})^2}$$

$$\eta = -\frac{1}{M^2} \cdot \frac{e^{-\phi/M}}{1-e^{-\phi/M}}$$

$$N = \int H dt = \int \frac{d\phi}{\sqrt{2\epsilon}} = M \int d\phi \left(e^{\phi/M} - 1\right) \simeq M^2 e^{\phi/M} \text{ if } N \gg 1, \; M \simeq 1$$

$$\Rightarrow \epsilon = \frac{M^2}{2N^2}, \quad \eta = -\frac{1}{N}$$

$$\Rightarrow n_s \simeq 1 - \frac{2}{N}, \quad r = 16\epsilon = \frac{8M^2}{N^2}$$

$$N \simeq 55 \Rightarrow n_s \simeq 0.97$$

regardless of M

$$\left.\begin{array}{l} \text{but for } M \gtrsim M_p \\ \Rightarrow r \gtrsim 0.001 \end{array}\right\} (**)$$

$q \simeq 1$ needed because of n_s constraint.

so $(**)$ is general!

compare with 'Lyth-bound':

→ assume $V(\phi)$ such that ϵ increases monotonically between inflationary $\phi = \phi_N$ and $\phi = \phi_0$ the minimum of $V(\phi)$

$$\rightarrow \quad N = \int \frac{d\phi}{\sqrt{2\epsilon}} \leq \frac{\Delta\phi|_0^N}{\sqrt{2\cdot\epsilon(N)}}$$

$\epsilon(N) < \epsilon(0)$

$r = 16\epsilon$

$$\rightsquigarrow \quad r \lesssim 0.003 \cdot \left(\frac{\Delta\phi}{M_p}\right)^2 \cdot \left(\frac{N}{50}\right)^2$$

Trans-planckian field motion during observable 60 e-folds are needed to generate detectable r.

Why is $\Delta\phi \sim M_p$ relevant?

Look at effective QFT treatment of inflaton sector of a putative complete theory including the SM & quantum gravity:

⇝ couplings of inflaton ϕ to heavy fields

⇝ integrating out heavy fields generically generates higher-dim. operators correcting inflaton scalar potential:

$$V_0(\phi) \rightarrow V(\phi) = V_0(\phi) + \Delta V_{\substack{\text{high.}\\ \text{dim.}}}(\phi)$$

tree-level "designed" inflaton potential

$$\Delta V_{\substack{\text{high.}\\ \text{dim.}}}(\phi) \supset V_0(\phi) \cdot \sum_{n\geq 2} c_n \left(\frac{\phi}{M_p}\right)^n$$

dim-6 & higher corrections

insert this into the slow-roll parameters

$$\Rightarrow \Delta\epsilon \sim \left(\frac{\Delta V'_{\text{high. dim.}}}{V_0}\right)^2 \sim \sum_{n\geq 2} c_n \left(\frac{\phi}{M_p}\right)^{2n-2}$$

$$\Delta\eta \sim \frac{\Delta V''_{\text{high. dim.}}}{V_0} \sim \sum_{n\geq 2} c_n \left(\frac{\phi}{M_p}\right)^{n-2}$$

$\Rightarrow$ for $\Delta\phi \gtrsim M_p$ <u>all</u> corrections with $\dim = 4+n \geq 6$ <u>contribute $\mathcal{O}(1)$ to η</u>.

$\rightsquigarrow$ inflation fails, unless symmetry suppresses all c_n. $\rightarrow$ <u>shift symmetry!</u>

Exponential plateau potentials straddle border between truly large-field & truly small-field models ...

explicit models ?

① Historically: The "First Avenger" of all inflation – Starobinsky 1980

$$S_{R^2} = \int d^4x \sqrt{-g}\, \frac{M_p^2}{2} \cdot \left(R + \frac{R^2}{M^2}\right)$$

How is this inflation?

step 1: introduce auxiliary field ϕ

$$S_{\phi,R} = \int d^4x \sqrt{-g} \left[\frac{M_p^2}{2}\left(1 + \frac{2\phi}{M_p}\right)R - \frac{1}{2}M^2\phi^2\right]$$

integrate out ϕ : $\frac{\delta S}{\delta \phi} = 0 \Rightarrow \phi = M_p \frac{R}{M}$

$$\Rightarrow \quad S_{\phi,R} = S_{R^2}$$

now conformal transformation :

$$g_{\mu\nu} = \Omega^{-2} \tilde{g}_{\mu\nu}, \quad \omega \equiv \ln \Omega$$

$$\Rightarrow \quad R = \Omega^2 \cdot \left[\tilde{R} + 6 \tilde{\Box} \omega - 6 (\partial_\mu \omega)^2 \right]$$

plug into $S_{\phi,R}$:

$$S_{\phi,R} = \int d^4x \sqrt{-\tilde{g}}\, \Omega^{-2} \frac{M_p^2}{2} \left(1 + \frac{2\phi}{M_p}\right) \cdot \left[\tilde{R} + 6 \tilde{\Box} \omega - 6 \cdot (\partial_\mu \omega)^2 - \frac{1}{2} M^2 \phi^2 \right]$$

choose: $\Omega^2 = 1 + \frac{2\phi}{M_p}$

drop $\sim$ from $\tilde{g}, \tilde{R}, \tilde{\Box} \ldots$

"Einstein frame"

$$S^{(E)}_{\phi,R} = \int d^4x \sqrt{-g}\, \frac{M_p^2}{2} \cdot \left[R - \frac{6}{M_p^2 \left(1 + \frac{2\phi}{M_p}\right)^2} (\partial_\mu \phi)^2 - \Omega^{-4} \cdot \frac{1}{2} M^2 \phi^2 \right]$$

$$= \int d^4x \sqrt{-g} \left[\frac{M_p^2}{2} R - \frac{3}{\left(1 + \frac{2\phi}{M_p}\right)^2} \cdot (\partial_\mu \phi)^2 - V(\phi) \right]$$

$$V(\phi) = \Omega^{-4} \cdot \frac{1}{2} M^2 \phi^2 = \frac{M^2 \phi^2}{2 \left(1 + \frac{2\phi}{M_p}\right)^2} = \frac{M^2 M_p^2}{8} \cdot \left(1 - \frac{M_p}{2\phi} + \ldots\right)$$

at $\phi \gg M_p$

non-canonical kinetic term $\mathcal{L}_{kin}$ at $\phi \to \infty$, and constant potential:

$$\mathcal{L}_{kin} \to \frac{3 M_p^2}{4 \phi^2} \cdot (\partial_\mu \phi)^2, \quad V \to \frac{M^2 M_p^2}{8}$$

canonically normalize:

$$\frac{\phi}{M_P} = e^{\sqrt{\frac{2}{3}}\frac{\varphi}{M_P}}$$

$$P_S \sim \frac{H^2}{\epsilon} \sim 10^{-9}$$

$$\Downarrow$$

$$M \sim 10^{-5} M_P$$

$$\Rightarrow V(\varphi) = \frac{M^2 M_P^2}{8}\left(1 - \frac{1}{2} e^{-\sqrt{\frac{2}{3}}\frac{\varphi}{M_P}} + \dots\right)$$

comparing with earlier "plateau" results:

$$M = \sqrt{\frac{3}{2}}\, M_P \quad \text{for } S_{R^2}$$

$$\Rightarrow r = \frac{12}{N^2} \simeq 0.003 \quad \text{for } N \simeq 60.$$

(11) Higgs inflation: arXiv: 0710.3755

observe how conformal transf. of $S_{\phi,R}$ produced kinetic term for ϕ

$\Downarrow$

can start directly with ϕ having a kinetic term as well & dim.-less coupling except for EH-term – "Jordan frame":

$$S_h = \int d^4x \sqrt{-g}\left[\left(\frac{M_P^2}{2} + \xi h^2\right)R - \frac{1}{2}(\partial_\mu h)^2 - \frac{\lambda}{4} h^4\right]$$

ξ: non-minimal coupling > 0 here

$\left(M_P = 0,\ \xi = -\frac{1}{6} \rightarrow \text{conformally coupled}\right)$

similar conformal transf. as in (1) – compare to starting $S_{\phi,R}$ when replacing in S_h $h^2 \to M\phi$ – will bring S_h into the form $S^{(E)}_{\phi,R}$ but with:

$$\Omega^2 = 1 + \frac{2\xi h^2}{M_P^2}, \quad V(h) = \Omega^{-4} \cdot \frac{\lambda}{4} h^4$$

again, at $h \to \infty$:

$$\mathcal{L}_{kin} \to \frac{M_P^2}{12 h^2}(\partial_\mu h)^2.$$

$$V(h) \to \frac{\lambda M_P^4}{16 \xi^2} \cdot \left(1 - \frac{M_P^2}{2\xi h^2} + \dots\right)$$

canonically normalize: $h = \frac{M_P}{\sqrt{\xi}} e^{\frac{1}{\sqrt{6}}\frac{\varphi}{M_P}}$

$$\Rightarrow V(h(\varphi)) = \frac{\lambda M_P^4}{16 \xi^2} \cdot \left(1 - \frac{1}{2} e^{-\sqrt{\frac{2}{3}}\frac{\varphi}{M_P}} + \dots\right)$$

exponential plateau at large φ, h

now replace:

$$\frac{\lambda}{4} h^4 \to \frac{\lambda}{4}\left(h^2 - v^2\right)^2, \quad v \ll M_p$$

$\to$ still exponential plateau at large h, SSB at $h = v$

"Higgs inflation"

(III) common feature of (I) and (II) in Einstein frame:

$$\mathcal{L}_\phi = \frac{3\alpha}{4\phi^2}\left(\partial_\mu \phi\right)^2 - V(\phi)$$

with: $V(\phi) = A - \frac{B}{\phi} + \dots$

at $\phi \to \infty$

canonical normalization gives:

$$\phi = \frac{B}{A} \cdot e^{\sqrt{\frac{2}{3\alpha}} \frac{\varphi}{M_p}}$$

$$V(\varphi) = A \cdot \left(1 - e^{-\sqrt{\frac{2}{3\alpha}} \frac{\varphi}{M_p}} + \dots\right)$$

can flip $\phi \to \frac{1}{\phi}$

again: $\mathcal{L}_{kin} = \frac{3\alpha}{4\phi^2}\left(\partial_\mu \phi\right)^2$

$$V(\phi) = A - B \cdot \phi + \mathcal{O}(\phi^2)$$

near $\phi = 0$

$$\to \phi = \frac{A}{B} \cdot e^{-\sqrt{\frac{2}{3\alpha}} \frac{\varphi}{M_p}}$$

$\varphi \to \infty$ corresponds to $\phi \to 0$

<u>Exponential plateau near a pole of kinetic term</u> with dialable approach rate $\underline{\alpha}$.

choice of $V(\phi)$ <u>irrelevant</u>, as long as $V(0) \neq 0$, $V'(0) < 0$ at pole $\phi = 0$

$\rightarrow$ Exponential plateaus for nearly arbitrary potential from kinetic terms with a pole at $\phi = 0$ or $\phi = \infty$:

arXiv: 1311.0472 "α - attractors"

embeddings in supergravity exist...

$\leadsto$ α - attractor near $\phi = \infty$:

$$\mathcal{L} = \frac{3\kappa}{4\phi^2}(\partial_\mu \phi)^2 - V_0\left(1 - \frac{C}{\phi} + \frac{D}{\phi^2} + \dots\right)$$

ϕ : volume modulus

arXiv: 1707.05830 $\rightarrow$ realized as "fibre inflation" of CY volume moduli, IIB string theory

<u>Most "minimal" way of going from 1 field to 2 fields:</u>

"hybrid inflation" astro-ph/9307002 Linde

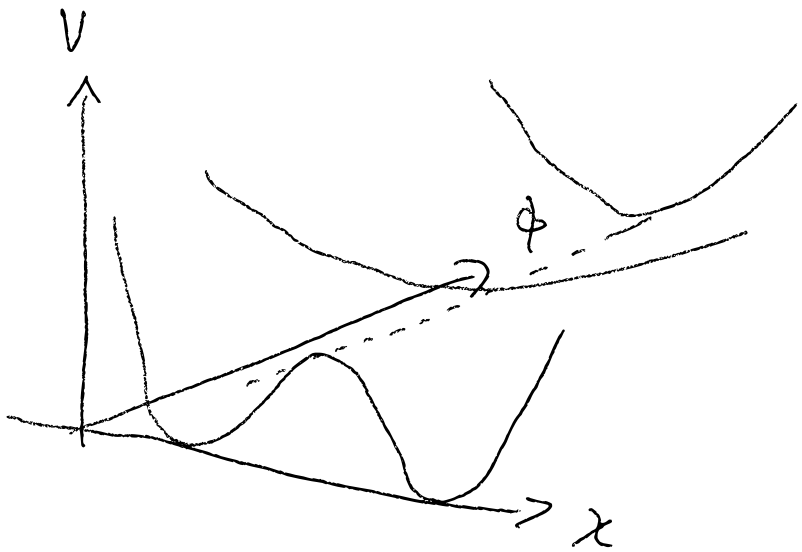
1st field χ $\rightarrow$ false vacuum, positive c.c. as in Guth's original model

2nd field ϕ $\rightarrow$ slow-rolling & <u>lowering</u> potential barriers stabilizing χ at positive c.c.

$\rightarrow$ once barrier gone: "waterfall" – χ rolls <u>fast</u> to <u>zero</u> c.c.

$$V(\phi, \chi) = \frac{\lambda}{4}(\chi^2 - v^2)^2 + g\phi^2\chi^2 + \Delta V_{\text{slow-roll}}(\phi)$$

for $\phi > \phi_c = \sqrt{\frac{\lambda v^2}{2g}}$: $m_\chi^2\big|_{\chi=0} > 0$

at $\phi < \phi_c$: $m_\chi^2 < 0$

"tachyonic water-fall"
of χ to $\chi = v$
ends inflation