

Soft anomalous dimension matrices for $t\bar{t}j$ production

Based on [hep-ph/2206.10977](https://arxiv.org/abs/hep-ph/2206.10977) + work in progress

September 28, 2022

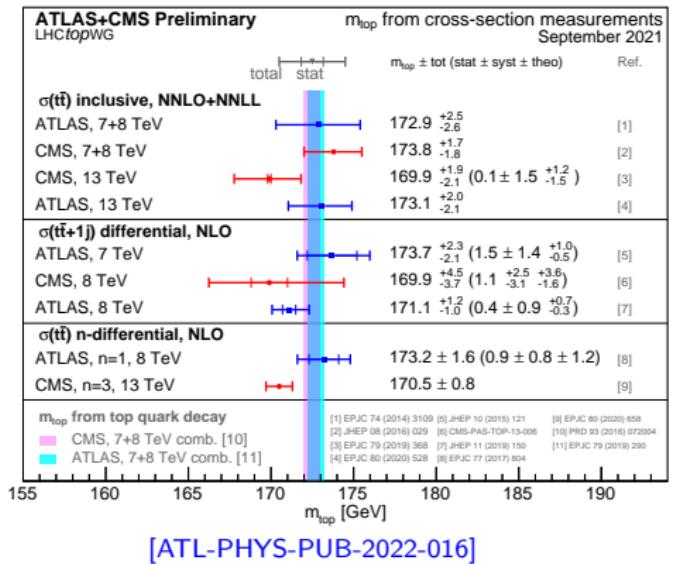
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Motivation

The indirect top-quark mass measurement are becoming a clear competitor of the direct methods.



- ▶ It has been shown that in case of $t\bar{t}+\text{jet}$, the observable $\rho_s = 2m_t/\sqrt{s_{t\bar{t}j}}$ shows very good sensitivity on the top quark mass [[hep-ph/1303.6415](https://arxiv.org/abs/hep-ph/1303.6415)]
- ▶ Latest studies using $t\bar{t}+\text{jet}$ cross-section use **NLO (NLO+PS)** predictions for the mass determination
- ▶ Elevating the accuracy of the theory predictions beyond NLO would improve the current m_t estimates and decrease their uncertainties



Fully accurate NNLO computation for $2 \rightarrow 3$ process with partons present in both initial and final states imposes a serious technical challenge:

- ▶ 2-loop master integrals for $2 \rightarrow 3$ kinematics with massive partons are mostly unknown.
- ▶ No easy way to deal with soft and collinear singularities at NNLO level. Multiple approaches are under development:
 - ▶ Antenna subtraction [[Gehrmann et al.](#)]
 - ▶ CoLoRFul subtraction [[Somogyi et al.](#)]
 - ▶ Sector-improved residue subtraction [[Czakon et al.](#)]
 - ▶ Local analytic sector subtraction [[Magnea et al.](#)]
 - ▶ qT-slicing [[Catani et al.](#)]
 - ▶ ...
- ▶ Automation of these procedures will take some time...

Meanwhile, it is possible to work in the **threshold limit** of the $t\bar{t}$ +jet production and by summing the **large logarithms** to all orders of the perturbation theory, improve the current NLO prediction.

- ▶ In the threshold limit the partonic cross-section can be factorized as [Collins, Soper, Sterman, 1983]:

$$\hat{\sigma} = \psi_i \otimes \psi_j \otimes H \otimes S \otimes J$$

- ▶ ψ_i - initial state jet functions, modeling the initial state collinear radiation
- ▶ S - soft gluon exchange
- ▶ H - hard amplitude matrix
- ▶ J - final state jet function

General strategy of the calculation:

- ▶ Evaluate each of these functions perturbatively at scales at which they are free of large logs
- ▶ By means of the RGEs evolve everything back to the common scale
- ▶ Expand the result to the desired fixed order
- ▶ The strategy has been successfully applied on $t\bar{t}$ production processes with associated bosons: $t\bar{t} + W/Z/H$ [Kulesza et al.], [Broggio et al.].

In case of $t\bar{t}$ +jet complications come from the:

- Involved calculations because of richer color structure
- Appearance of the final state jet, which has highly nontrivial soft singularity structure

In this talk we focus on calculation of the **soft function**.

- ▶ The RG equation for the soft function:

$$\mu \frac{d}{d\mu} S_{LI}^{(f)} = \left(\mu \frac{\partial}{\partial \mu} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) S_{LI}^{(f)} = - \left(\Gamma_S^{(f)} \right)_{LB}^\dagger S_{BI}^{(f)} - S_{LA}^{(f)} \left(\Gamma_S^{(f)} \right)_{AI}, \quad (1)$$

- ▶ where at 1-loop level the soft anomalous dimension is given as:

$$\left(\Gamma_S^{(f)} \right)_{LI} (g) = -\alpha_s \frac{\partial}{\partial \alpha_s} \text{Res}_{\epsilon \rightarrow 0} \left(Z_S^{(f)} \right)_{LI} (g, \epsilon) \quad (2)$$

- ▶ Z_S are UV-divergent parts of the eikonal amplitudes constructed using the Wilson lines.
- ▶ All the integrals which arise at 1-loop level in this calculation are known e.g. from [\[Kidonakis 1997\]](#).

Choosing the color basis



The strategy of choosing the color basis is explained in [Sjödahl 2008], here we quote the result:

gg-channel

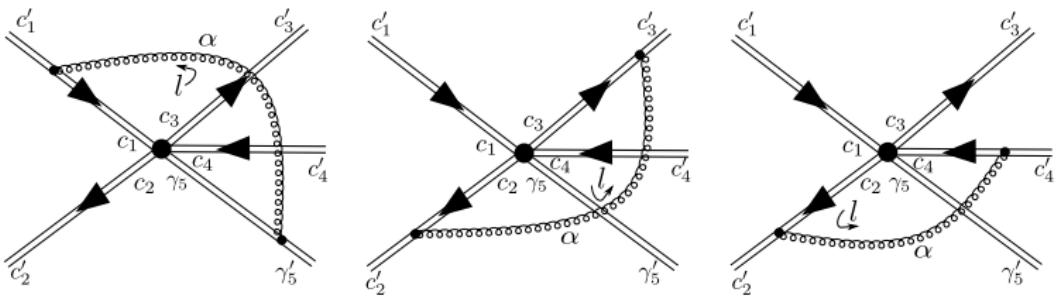
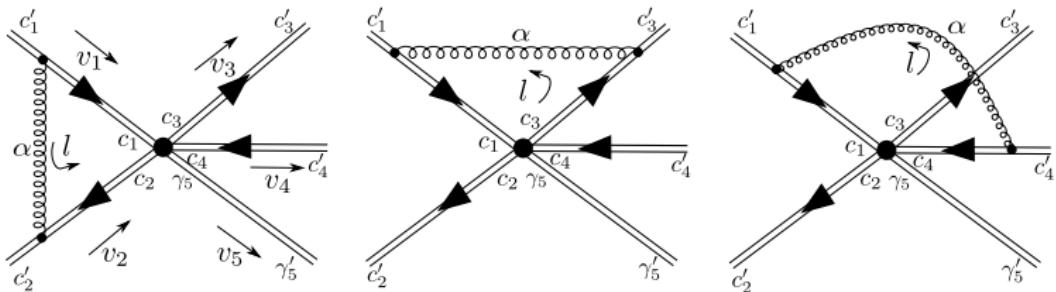
$$\begin{aligned}\mathbf{c}_{abcde}^1 &= t_{cd}^e \delta_{ab} \\ \mathbf{c}_{abcde}^2 &= if_{abe} \delta_{cd} \\ \mathbf{c}_{abcde}^3 &= id_{abe} \delta_{cd} \\ \mathbf{c}_{abcde}^4 &= if_{abn} if_{men} t_{cd}^m \\ \mathbf{c}_{abcde}^5 &= d_{abn} if_{men} t_{cd}^m \\ \mathbf{c}_{abcde}^6 &= if_{abn} d_{men} t_{cd}^m \\ \mathbf{c}_{abcde}^7 &= d_{abn} d_{men} t_{cd}^m \\ \mathbf{c}_{abcde}^8 &= P_{abme}^{10+\overline{10}} t_{cd}^m \\ \mathbf{c}_{abcde}^9 &= P_{abme}^{10-\overline{10}} t_{cd}^m \\ \mathbf{c}_{abcde}^{10} &= -P_{abme}^{27} t_{cd}^m \\ \mathbf{c}_{abcde}^{11} &= P_{abme}^0 t_{cd}^m\end{aligned}$$

q \bar{q} -channel

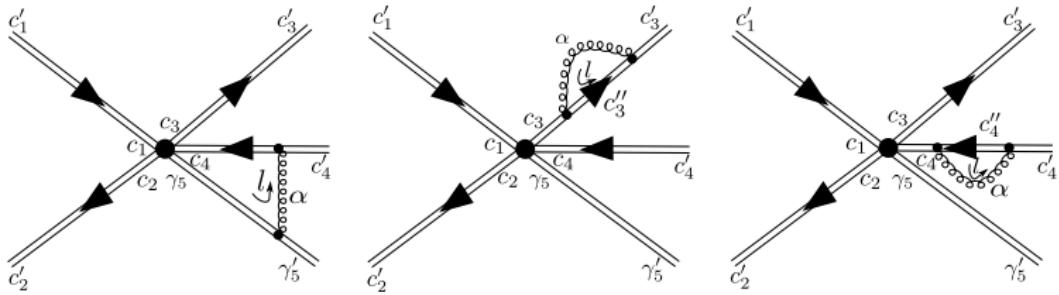
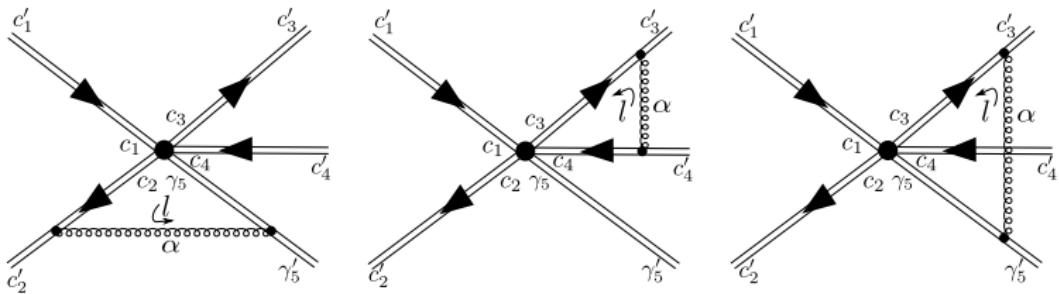
$$\begin{aligned}\mathbf{c}_{abcde}^1 &= t_{cd}^e \delta_{ab} \\ \mathbf{c}_{abcde}^2 &= t_{ab}^e \delta_{cd} \\ \mathbf{c}_{abcde}^3 &= t_{ba}^m t_{cd}^n if_{mne} \\ \mathbf{c}_{abcde}^4 &= t_{ba}^m t_{cd}^n d_{mne}\end{aligned}$$

P^i are projectors: $P_{ABmn}^i P_{mnCD}^j = \delta_{ij} P_{ABCD}^i$

Wilson webs



Wilson webs



Connection $(i - j)$	Kinematical part (κ_{ij}) before integration	Color part (\mathcal{F}_{ij})
1 – 2	$\frac{v_1^\mu}{-v_1 \cdot l + i\epsilon} \frac{-v_2^\nu}{v_2 \cdot l + i\epsilon} \frac{N_{\mu\nu}(l)}{l^2 + i\epsilon}$	$T_{c_1 c'_1}^\alpha T_{c'_2 c_2}^\alpha \delta_{c_3 c'_3} \delta_{c_4 c'_4} \delta_{\gamma_5 \gamma'_5}$
1 – 3	$\frac{v_1^\mu}{v_1 \cdot l + i\epsilon} \frac{v_3^\nu}{v_3 \cdot l + i\epsilon} \frac{N_{\mu\nu}(l)}{l^2 + i\epsilon}$	$T_{c_1 c'_1}^\alpha T_{c'_3 c_3}^\alpha \delta_{c_2 c'_2} \delta_{c_4 c'_4} \delta_{\gamma_5 \gamma'_5}$
1 – 4	$\frac{v_1^\mu}{v_1 \cdot l + i\epsilon} \frac{-v_4^\nu}{v_4 \cdot l + i\epsilon} \frac{N_{\mu\nu}(l)}{l^2 + i\epsilon}$	$T_{c_1 c'_1}^\alpha T_{c'_4 c_4}^\alpha \delta_{c_2 c'_2} \delta_{c_3 c'_3} \delta_{\gamma_5 \gamma'_5}$
1 – 5	$\frac{v_1^\mu}{v_1 \cdot l + i\epsilon} \frac{-v_5^\nu}{v_5 \cdot l + i\epsilon} \frac{N_{\mu\nu}(l)}{l^2 + i\epsilon}$	$T_{c_1 c'_1}^\alpha (-if^{\alpha \gamma_5 \gamma'_5}) \delta_{c_2 c'_2} \delta_{c_3 c'_3} \delta_{c_4 c'_4}$
2 – 3	$\frac{-v_2^\mu}{-v_2 \cdot l + i\epsilon} \frac{v_3^\nu}{-v_3 \cdot l + i\epsilon} \frac{N_{\mu\nu}(l)}{l^2 + i\epsilon}$	$T_{c'_2 c_2}^\alpha T_{c'_3 c_3}^\alpha \delta_{c_1 c'_1} \delta_{c_4 c'_4} \delta_{\gamma_5 \gamma'_5}$
2 – 4	$\frac{-v_2^\mu}{-v_2 \cdot l + i\epsilon} \frac{-v_4^\nu}{-v_4 \cdot l + i\epsilon} \frac{N_{\mu\nu}(l)}{l^2 + i\epsilon}$	$T_{c'_2 c_2}^\alpha T_{c_4 c'_4}^\alpha \delta_{c_1 c'_1} \delta_{c_3 c'_3} \delta_{\gamma_5 \gamma'_5}$
2 – 5	$\frac{-v_2^\mu}{-v_2 \cdot l + i\epsilon} \frac{v_5^\nu}{-v_5 \cdot l + i\epsilon} \frac{N_{\mu\nu}(l)}{l^2 + i\epsilon}$	$T_{c'_2 c_2}^\alpha (-if^{\alpha \gamma'_5 \gamma_5}) \delta_{c_1 c'_1} \delta_{c_3 c'_3} \delta_{c'_4 c_4}$
3 – 4	$\frac{v_3^\mu}{-v_3 \cdot l + i\epsilon} \frac{-v_4^\nu}{v_4 \cdot l + i\epsilon} \frac{N_{\mu\nu}(l)}{l^2 + i\epsilon}$	$T_{c'_3 c_3}^\alpha T_{c_4 c'_4}^\alpha \delta_{c_1 c'_1} \delta_{c_2 c'_2} \delta_{\gamma_5 \gamma'_5}$
3 – 5	$\frac{v_3^\mu}{-v_3 \cdot l + i\epsilon} \frac{-v_5^\nu}{v_5 \cdot l + i\epsilon} \frac{N_{\mu\nu}(l)}{l^2 + i\epsilon}$	$T_{c'_3 c_3}^\alpha (-if^{\alpha \gamma_5 \gamma'_5}) \delta_{c_1 c'_1} \delta_{c_2 c'_2} \delta_{c_4 c'_4}$
4 – 5	$\frac{v_4^\mu}{-v_4 \cdot l + i\epsilon} \frac{-v_5^\nu}{v_5 \cdot l + i\epsilon} \frac{N_{\mu\nu}(l)}{l^2 + i\epsilon}$	$T_{c_4 c'_4}^\alpha (-if^{\alpha \gamma_5 \gamma'_5}) \delta_{\gamma_1 \gamma'_1} \delta_{\gamma_2 \gamma'_2} \delta_{c_3 c'_3}$
3 – 3	$\frac{v_3^\mu}{v_3 \cdot l + i\epsilon} \frac{v_3^\nu}{-v_3 \cdot l + i\epsilon} \frac{N_{\mu\nu}(l)}{l^2 + i\epsilon}$	$T_{c''_3 c_3}^\alpha T_{c'_3 c''_3}^\alpha \delta_{c_1 c'_1} \delta_{c_2 c'_2} \delta_{c_4 c'_4} \delta_{\gamma_5 \gamma'_5}$
4 – 4	$\frac{-v_4^\mu}{v_4 \cdot l + i\epsilon} \frac{-v_4^\nu}{-v_4 \cdot l + i\epsilon} \frac{N_{\mu\nu}(l)}{l^2 + i\epsilon}$	$T_{c''_4 c_4}^\alpha T_{c'_4 c''_4}^\alpha \delta_{c_1 c'_1} \delta_{c_2 c'_2} \delta_{c_3 c'_3} \delta_{\gamma_5 \gamma'_5}$

Example results for $t\bar{t} + \text{jet}$

- ▶ Example components from $q\bar{q} \rightarrow t\bar{t}g$ -channel

$$\Gamma_{1,1}^{(1)} = \frac{1}{2N_c} \left[2L_\beta + N_c^2 \left(-2 \log(\nu_5) + 2 \log(\nu_{35}) + 2 \log(\nu_{45}) + 2 \log\left(\frac{s}{m_t^2}\right) - \log(16) + 2 \right) \right. \\ \left. + (N_c^2 - 1) (-\log(\nu_1) - \log(\nu_2) + 2i\pi) + \log(4) \right]$$

$$\Gamma_{1,2}^{(1)} = \frac{1}{N_c} \left[\log(\nu_{13}) - \log(\nu_{14}) - \log(\nu_{23}) + \log(\nu_{24}) \right]$$

- ▶ Example components from $gg \rightarrow t\bar{t}g$ -channel

$$\Gamma_{2,9}^{(1)} = 0$$

$$\Gamma_{2,10}^{(1)} = \frac{N_c + 3}{4N_c + 4} \left[\log(\nu_{13}) - \log(\nu_{14}) - \log(\nu_{23}) + \log(\nu_{24}) \right]$$

$$\Gamma_{11,8}^{(1)} = \frac{N_c(N_c + 1) - 2}{2N_c} \left[\log(\nu_{13}) + \log(\nu_{14}) - 2 \log(\nu_{15}) - \log(\nu_{23}) - \log(\nu_{24}) + 2 \log(\nu_{25}) \right]$$

$$\Gamma_{11,9}^{(1)} = \frac{N_c(N_c + 1) - 2}{2N_c} \left[-\log(\nu_{13}) + \log(\nu_{14}) + \log(\nu_{23}) - \log(\nu_{24}) \right]$$

- ▶ 2 tools have been developed to make the calculation of soft anomalous dimension matrices fully automatic:
 - ▶ **WilsonWebs** - Automatic generation of n-loop Wilson diagrams and amplitudes
 - ▶ **FORMSoft** - Color decomposition and master integral evaluation of eikonal amplitudes
- ▶ As for now, the color bases should be provided by the user
- ▶ If the analytical simplicity of the results is not a priority, it is possible to use trace bases and combine these tools with **ColorFull** [Sjödahl 2014]
- ▶ Automatic generation of multiplet bases needs further effort (work in progress...)

4-top production:

- ▶ The multiplet bases for this process are easy to derive and they are given e.g. in [Keppeler, Sjödahl 2012]
- ▶ A simple call:

```
./WilsonWebs -L 1 -i q qbar -f t tbar t tbar | ./FORMSSoft
```

- can generate 1-Loop soft anomalous dimension matrices in less than 1 millisecond.
- ▶ Example results:

$$\Gamma_{1,1}^{(1)} = \frac{N_c^2 - 1}{N_c} \left[-L_\beta^{(34)} - L_\beta^{(56)} - \frac{\log(\nu_1)}{2} - \frac{\log(\nu_2)}{2} + \log(-\nu_{12}) - 1 - \frac{\log(4)}{2} \right]$$

$$\begin{aligned} \Gamma_{5,5}^{(1)} = & \frac{1}{2N_c} \left[2L_\beta^{(34)} - 6L_\beta^{(35)} - 6L_\beta^{(46)} + 2L_\beta^{(56)} + N_c^2 \cdot \left(2 \log \left(\frac{s}{m_t^2} \right) - 2 \right) + \right. \\ & \left(N_c^2 - 6 \right) \left(-L_\beta^{(36)} - L_\beta^{(45)} + \log(\nu_{13}) + \log(\nu_{15}) + \log(\nu_{24}) + \log(\nu_{26}) \right) + \\ & \left(N_c^2 - 1 \right) \left(-\log(\nu_1) - \log(\nu_2) - \log(4) \right) - 2 \log(-\nu_{12}) + 6 \log(\nu_{14}) + 6 \log(\nu_{16}) + \\ & \left. 6 \log(\nu_{23}) + 6 \log(\nu_{25}) + 2 \right] \end{aligned}$$

Multi-jet production:

- Following command:

```
./WilsonWebs -L 1 -i g g -f t tbar g g g g g g g g g g g g g g g g
```

successfully constructs all the necessary ingredients for 1-Loop anomalous dimension calculation of $t\bar{t}$ in association with 10 jets in gg -channel within less than 1 millisecond (except the color basis vectors)

- ▶ The flag `-L` controls the loop count and can be set to an arbitrary high value (bearing in mind the computational time limitations)
 - ▶ Multiloop calculations have not been tested so far using these tools
 - ▶ The master integrals needed for the multiloop calculations are also not included in these packages

- ▶ Analytical results of the soft anomalous dimension matrices at 1-loop for all partonic channels of $t\bar{t}j$ production have been computed and released in an [arXiv paper](#).
- ▶ This allows evaluation of 1-loop soft function, an essential tool in the threshold resummation at NLL accuracy.
- ▶ Numerous consistency checks have been successfully passed.
- ▶ The software tool for performing this calculation are openly available and they can be applicable to other projects as well:
 - ▶ **WilsonWebs** - automatic generation of n-loop Wilson diagrams.
 - ▶ **FORMSoft** - Soft anomalous dimension matrices at 1-loop.
 - ▶ **FORMHard** - Hard matrix generator (with or without the color decomposition).
 - ▶ **pyDipole** - NLO pole structure generator using Catani-Seymour formalism and $\Gamma^{(1)}$ and $H^{(0)}$ matrices, used for the consistency check.

Thanks for your attention!

Backup

Consider the process $q_a \bar{q}_b \rightarrow t_c \bar{t}_d q_e$:

- ▶ Initial state singlet \Rightarrow final state is singlet \Rightarrow gluon octet must matched by $t\bar{t}$ octet
 $\Rightarrow \delta_{ab} t_{cd}^e$
- ▶ Initial state octet \Rightarrow final state is octet \Rightarrow
 - ▶ gluon matches the initial octet $\Rightarrow t\bar{t}$ is singlet $\Rightarrow t_{ab}^e \delta_{cd}$
 - ▶ if $t\bar{t}$ is octet $\Rightarrow 8 \otimes 8$ contains 2 octets to match the initial octet:
 - ▶ 8^a : $t_{ba}^m t_{cd}^n i f_{mne}$
 - ▶ 8^s : $t_{ba}^m t_{cd}^n d_{mne}$

Soft function

- ▶ For the process $a(\beta_a)b(\beta_b) \rightarrow 1(\beta_1)2(\beta_2)3(\beta_3)$ define a eikonal nonlocal operator ω_I :

$$\begin{aligned} \omega_I^{(f)}(x)_{\{c_k\}} = & \sum_{d_i} \Phi_{\beta_3}^{f_3}(\infty, 0; x)_{c_3, d_3} \Phi_{\beta_2}^{f_2}(\infty, 0; x)_{c_2, d_2} \Phi_{\beta_1}^{f_1}(\infty, 0; x)_{c_1, d_1} \left(c_I^{(f)} \right)_{d_3 d_2 d_1, d_b d_a} \\ & \times \Phi_{\beta_a}^{f_a}(0, -\infty; x)_{d_a, c_a} \Phi_{\beta_b}^{f_b}(0, -\infty; x)_{d_b, c_b} \end{aligned}$$

- ▶ Eikonal cross-section is given as:

$$\sigma_{LI}^{(f), \text{eik}}(\alpha_s, \epsilon) = \sum_{\xi} \delta(w - w(\xi)) \times \left\langle 0 \left| \overline{T} \left[\left(\omega_L^{(f)}(0) \right)_{\{b_i\}}^\dagger \right] \right| \xi \right\rangle \left\langle \xi \left| T \left[\omega_I^{(f)}(0)_{\{b_i\}} \right] \right| 0 \right\rangle$$

- ▶ Define the soft function as the part of this cross-section which is free of collinear divergences

$$\sigma_{LI}^{(f), \text{eik}^N} = S_{JL}^N j_a^N j_b^N j_1^N j_2^N j_3^N$$

Evolution equation for the soft function



- ▶ Because the soft matrix is defined as a product of two operators it has to be renormalized multiplicatively:

$$S_{LI}^{(f)(B)} = \left(Z_S^{(f)\dagger}\right)_{LB} S_{BA}^{(f)} \left(Z_S^{(f)}\right)_{AI} \quad (3)$$

- ▶ Derive RGE:

$$\mu \frac{d}{d\mu} S_{LI}^{(f)} = \left(\mu \frac{\partial}{\partial \mu} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s}\right) S_{LI}^{(f)} = - \left(\Gamma_S^{(f)}\right)_{LB}^\dagger S_{BI}^{(f)} - S_{LA}^{(f)} \left(\Gamma_S^{(f)}\right)_{AI}, \quad (4)$$

- ▶ where at 1-loop level:

$$\left(\Gamma_S^{(f)}\right)_{LI}(g) = -\alpha_s \frac{\partial}{\partial \alpha_s} \text{Res}_{\epsilon \rightarrow 0} \left(Z_S^{(f)}\right)_{LI}(g, \epsilon) \quad (5)$$

- ▶ Z_S are UV-divergent parts of the eikonal amplitudes. Solution of eq. (4):

$$S(\mu) = \bar{\mathcal{P}} \exp \left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \Gamma_S^\dagger(\alpha_s(\mu'^2)) \right] S(\mu_0) \mathcal{P} \exp \left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \Gamma_S(\alpha_s(\mu'^2)) \right] \quad (6)$$

The gluon propagator in a general axial gauge is given as:

$$N^{\mu\nu}(k) = g^{\mu\nu} - \frac{n^\mu k^\nu + n^\nu k^\mu}{n \cdot k} + n^2 \frac{k^\mu k^\nu}{(n \cdot k)^2} \quad (7)$$

To deal with the unphysical singularity introduced by the axial gauge we use the principal value prescription [Leibbrandt 1987]:

$$\frac{\mathcal{P}}{(l \cdot n)^\beta} = \frac{1}{2} \left(\frac{1}{(l \cdot n + i\epsilon)} + (-1)^\beta \frac{1}{(-l \cdot n + i\epsilon)} \right) \quad (8)$$

As a result, each integral over the kinematical part can be reduced to the following form:

$$\begin{aligned} \omega_{ij}(\delta_i v_i, \delta_j v_j, \Delta_i, \Delta_j) &= \Delta_i \Delta_j \delta_i \delta_j \left(I_1(\delta_i v_i, \delta_j v_j) - \frac{1}{2} I_2(\delta_i v_i, n) - \frac{1}{2} I_2(\delta_i v_i, -n) \right. \\ &\quad \left. - \frac{1}{2} I_3(\delta_j v_j, n) - \frac{1}{2} I_3(\delta_j v_j, -n) + I_4(n^2) \right) \end{aligned} \quad (9)$$

The integrals $I_1 - I_4$ are evaluated in [Kidonakis 1997].

For example, when both partons are massless:

$$I_1^{\text{UV pole}} = \frac{\alpha}{2\pi} \left\{ \frac{2}{\epsilon^2} - \frac{1}{\epsilon} \left[\gamma + \ln \left(\delta_i \delta_j \frac{\nu_{ij}}{2} \right) - \ln(4\pi) \right] \right\}$$

$$I_2^{\text{UV pole}} = \frac{\alpha}{2\pi} \left\{ \frac{2}{\epsilon^2} - \frac{1}{\epsilon} [\gamma + \ln(\nu_i) - \ln(4\pi)] \right\}$$

$$I_3^{\text{UV pole}} = \frac{\alpha}{2\pi} \left\{ \frac{2}{\epsilon^2} - \frac{1}{\epsilon} [\gamma + \ln(\nu_j) - \ln(4\pi)] \right\}$$

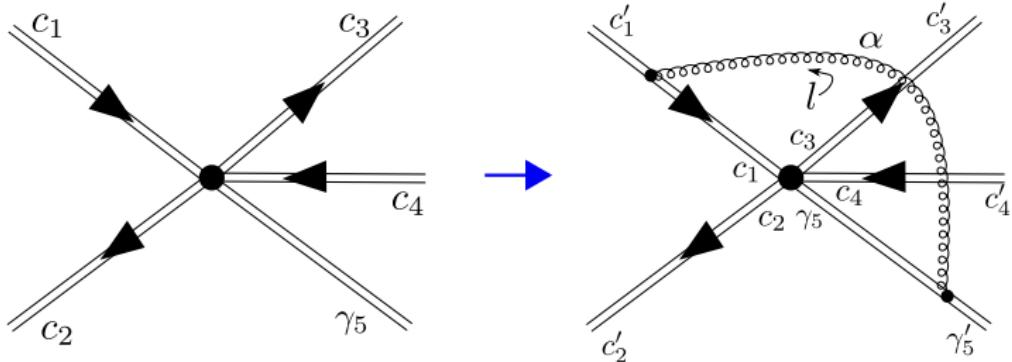
$$I_4^{\text{UV pole}} = -\frac{\alpha}{\pi} \frac{1}{\epsilon}.$$

where

$$\nu_a = \frac{(\nu_a \cdot n)^2}{|n|^2} \quad (10)$$

and $\nu_{ij} = \nu_i \cdot \nu_j = \frac{2p_i p_j}{s}$.

Color decomposition example



Before the gluon exchange:

$$\mathbf{c}_1 = \delta_{c_1 c_2} T_{c_3 c_4}^{\gamma_5}$$

$$\mathbf{c}_2 = \delta_{c_3 c_4} T_{c_2 c_1}^{\gamma_5}$$

$$\mathbf{c}_3 = T_{c_3 c_4}^\alpha T_{c_3 c_4}^\beta \text{ if } \alpha \neq \beta \neq \gamma_5$$

$$\mathbf{c}_4 = T_{c_3 c_4}^\alpha T_{c_3 c_4}^\beta d^{\alpha \beta \gamma_5}$$

After the gluon exchange:

$$\mathbf{c}'_1 = \delta_{c'_1 c'_2} T_{c'_3 c'_4}^{\gamma'_5}$$

$$\mathbf{c}'_2 = \delta_{c'_3 c'_4} T_{c'_2 c'_1}^{\gamma'_5}$$

$$\mathbf{c}'_3 = T_{c'_3 c'_4}^\alpha T_{c'_3 c'_4}^\beta \text{ if } \alpha \neq \beta \neq \gamma'_5$$

$$\mathbf{c}'_4 = T_{c'_3 c'_4}^\alpha T_{c'_3 c'_4}^\beta d^{\alpha \beta \gamma'_5}$$

Color decomposition example

The vertex correction $\mathcal{F}_{15} = T_{c_1 c'_1}^\alpha i f^{\alpha \gamma'_5 \gamma_5} \delta_{c_2 c'_2} \delta_{c_3 c'_3} \delta_{c_4 c'_4}$, modifies the Born basis in the following way:

$$\mathbf{c}_1 \mathcal{F}_{15} = -\mathbf{c}'_3, \quad (11)$$

$$\mathbf{c}_2 \mathcal{F}_{15} = -\frac{N_c}{2} \mathbf{c}'_2, \quad (12)$$

$$\mathbf{c}_3 \mathcal{F}_{15} = -\frac{1}{2} \mathbf{c}'_1 - \frac{N_c}{4} \mathbf{c}'_3 - \frac{N_c}{4} \mathbf{c}'_4, \quad (13)$$

$$\mathbf{c}_4 \mathcal{F}_{15} = \left(\frac{1}{N_c} - \frac{N_c}{4} \right) \mathbf{c}'_3 - \frac{N_c}{4} \mathbf{c}'_4. \quad (14)$$

The linear transformation, which describes the modification of the Born-color structure caused by the soft gluon exchange between two partons:

$$\mathcal{F}_{15} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{N_c}{2} & 0 & 0 \\ -1 & 0 & -\frac{N_c}{4} & \frac{1}{N_c} - \frac{N_c}{4} \\ 0 & 0 & -\frac{N_c}{4} & -\frac{N_c}{4}, \end{pmatrix} \quad (15)$$

Let us define the following tensor:

$$G_{AB} = \text{tr} \left(\mathbf{c}_A \mathbf{c}_B^\dagger \sum_{i,j; i \leq j} \omega_{ij}^{(\text{UV})} \mathcal{F}_{ij} \right). \quad (16)$$

The soft anomalous dimension matrix is then given as:

$$\Gamma_{IJ} = \left(S^{(0)} \right)^{-1}_{IK} G_{KJ}, \quad (17)$$

where $S^{(0)}$ is a Born-level soft matrix, defined as:

$$S_{LI}^0 = \langle \mathbf{c}_L | \mathbf{c}_I \rangle. \quad (18)$$

This procedure is fully automatized:

- ▶ Wilson diagrams and corresponding amplitudes are generated using a C-package **WilsonWebs**.
- ▶ The color decomposition is done using a FORM routine **FORMSoft**.

- ▶ The leading-order soft function:

$$S_{IJ}^{(0)} = \text{tr} \left(c_I^\dagger c_J \right) \quad (19)$$

- ▶ The lowest-order hard function:

$$H_{ij,IJ}^0 = h_{ij,I}^0 h_{ij,J}^{*0} \quad (20)$$

- ▶ $h_{ij,J}^{(0)}$ are color projected amplitudes (A):

$$h_I^{(0)} = \left(S^{(0)} \right)_{IK}^{-1} \text{tr} \left(c_K^\dagger A \right), \quad h_I^{*(0)} = \text{tr} \left(A^\dagger c_K \right) \left(S^{(0)} \right)_{IK}^{-1} \quad (21)$$

- ▶ Consistency check for the color decomposition procedure:

$$\text{tr} \left(H_{AB}^{(0)} S_{BC}^{(0)} \right) = AA^\dagger \quad (22)$$

All the results presented here satisfy the following checks:

(i) Correctness of the color decomposition procedure:

- ▶ $\text{Tr}(H^{(0)}S^{(0)})$ should evaluate back to the squared Born matrix element

(ii) Correctness of the 1-loop pole structure:

- ▶ The analytic pole structure at NLO is known, e.g. by evaluating the Catani-Seymour subtraction terms.
- ▶ The same structure can be generated using $H^{(0)}$ and $\Gamma^{(1)}$ matrices.