# Semileptonic $b$-decays with full charm mass dependence at NNLO 

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Manuel Egner | Hamburg, September 28, 2022
based on work with:
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## Outline

(1) Introduction
(2) Calculation
(3) Results
(4) Conclusion

## Motivation



- $\left|V_{c b}\right|$ is extracted from inclusive $B \rightarrow X_{c} \bar{\nu}$. Precise predictions for semileptonic $b$-decays are crucial input.
- Heavy Quark Expansion (HQE): Decay width of $B \rightarrow X_{c} / \bar{\nu}$ as sum of decay width of $b \rightarrow c / \bar{\nu}$ and corrections suppressed by the mass $m_{b}$.

$$
\Gamma\left(B \rightarrow X_{c} / \bar{\nu}\right)=\Gamma(b \rightarrow c \mid \bar{\nu})+\mathcal{O}\left(1 / m_{b}^{2}\right) .
$$

- In this talk: Semileptonic decay channel $b \rightarrow c \mid \bar{\nu}$ at $\mathcal{O}\left(\alpha_{s}^{2}\right)$.
- Next step: Hadronic decay channels $b \rightarrow c \bar{u} d$ and $b \rightarrow c \bar{c} s$ at $\mathcal{O}\left(\alpha_{s}^{2}\right)$.
- Input for the calculation of B-meson lifetimes in HQE.
- Knowing the $\mathcal{O}\left(\alpha_{s}^{2}\right)$ contributions will reduce the uncertainty induced by the renormalization scale variation.


Figure: Error contributions on B-meson lifetimes [Lenz, Piscopo, Rusov (2022)]

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## Previous calculations

Previous analytic calculations including finite charm quark mass:

Semileptonic decay channel:

- $\mathcal{O}\left(\alpha_{s}^{1}\right)[\operatorname{Nir}$ (1989)]
- $\mathcal{O}\left(\alpha_{s}^{2}\right)$ [Czarnecki, Pak (2008)], [Czarnecki, Dowling, Piclum (2008)]
- $\mathcal{O}\left(\alpha_{s}^{3}\right)$ [Fael, Schönwald, Steinhauser (2021)]
$\mathcal{O}\left(\alpha_{s}^{2}\right)$ and $\mathcal{O}\left(\alpha_{s}^{3}\right)$ calculations as expansions in mass ratios.

Hadronic decay channel:

- $b \rightarrow c \bar{u} d: \mathcal{O}\left(\alpha_{s}^{1}\right)$ [Bagan, Ball, Braun, Gosdzinsky (1994)]
- $b \rightarrow c \bar{c} s: \mathcal{O}\left(\alpha_{s}^{1}\right)$ [Bagan, Ball, Fiol, Gosdzinsky (1995)]
- $\mathcal{O}\left(\alpha_{s}^{2}\right)$, including only massless quarks in the final state, $b \rightarrow u$ [Czarnecki, Slusarczyk, Tkachov (2006)]


## Calculation setup

- Optical theorem:

$$
\Gamma=\frac{1}{m_{b}} \operatorname{Im}[\mathcal{M}(b \rightarrow b)]
$$

- Integrate out $W$-boson
- At $\mathcal{O}\left(\alpha_{s}^{2}\right)$ calculate imaginary part of 4-loop diagrams



## Calculation setup

(1) Generate diagrams with QGRAF [Nogueira (1993)].
(2) Mapping on integral families with TAPIR [Gerlach, Herren, Lang (2022)] and EXP [Harlander, Seidensticker, Steinhauser (1998-1999)].
(3) Reduction to master integrals with Kira [Klappert, Lange, Maierhö́rer, Usovitsch (2021)].

- Choose good basis of master integrals, where $\epsilon$ and $x=m_{c} / m_{b}$ factorize, with ImproveMasters.m [Smirnov, Smirnov (2020)].
(4) Calculating master integrals via differential equations:
- Determination of $\epsilon$-form with Canonica [Meyer (2018)] and Libra [Lee (2020)].
- Part of the differential equations which can not be brought to $\epsilon$-form with OreSys [Gerhold (2002)].
- Handling of iterated integrals with HarmonicSums [Ablinger (2010)].


## Master integrals

- After IBP reduction: 129 master integrals, calculate only imaginary part.
- 96 integrals with non-vanishing imaginary part (for expansion around $x=1$ ).
- Set up differential equation in $x=m_{c} / m_{b}$.
$\rightarrow$ Solution contains iterated integrals with square rooted letters.
$\rightarrow$ Perform variable transformation.

$$
x=\frac{1-t^{2}}{1+t^{2}} \leftrightarrow t=\sqrt{\frac{1-x}{1+x}}
$$

- Obtain $\epsilon$-form for 92 integrals with Canonica [Meyer (2018)]:

$$
\frac{\mathrm{d} \vec{l}}{\mathrm{~d} t}=A(t, \epsilon) \cdot \vec{l} \xrightarrow{\vec{l}=T \cdot \vec{J}} \frac{\mathrm{~d} \vec{J}}{\mathrm{~d} t}=\epsilon A^{\prime}(t) \cdot \vec{J}
$$

- Solve remaining 4 integrals with the help of OreSys [Gerhold (2002)].


## Boundary conditions

- Fix boundary condition in the limit $m_{c} \approx m_{b}$.
- Asymptotic expansion [Beneke, Smirnov (1997)] of master integrals in $\delta=1-m_{c}^{2} / m_{b}^{2}$.
- Two relevant momentum scalings:
- hard: $\left|k^{\mu}\right| \sim m_{b}$
- ultrasoft: $\left|k^{\mu}\right| \sim \delta \cdot m_{b}$
- Find momentum routing and scaling of the propagators for different momentum regions (check with ASY [Pak, Smirnov (2010)]), for example:

$$
\frac{1}{\left(m_{c}^{2}-\left(k+p_{b}\right)^{2}\right)} \sim \begin{cases}1 / \delta^{1}, & \text { if } \mathrm{k} \text { is ultrasoft } \\ 1 / \delta^{0}, & \text { if } \mathrm{k} \text { is hard }\end{cases}
$$

- For every region: $\{$ us, us, us, us $\},\{u s, ~ u s, ~ u s, ~ h\}, \ldots$.
- Expand integral in $\delta$.
- Expanded integrals can be calculated in terms of Gamma functions.
- Imaginary part only from ultrasoft integrals of the form

$$
\int \frac{\mathrm{d}^{d} k}{\left(-k^{2}\right)^{n_{1}}\left(-\delta-2 k \cdot p_{b}\right)^{n_{2}}} \sim(-\delta)^{-2 n_{1}-n_{2}+d},
$$

with ultrasoft loop momentum $k$ and hard external momentum $p_{b}$.

- Sum contributions from all regions.


## Master integrals

- Obtain master integrals in terms of iterated integrals over the alphabet

$$
\frac{1}{t}, \quad \frac{1}{1+t}, \quad \frac{1}{1-t}, \quad \frac{t}{1+t^{2}}, \quad \frac{t^{3}}{1+t^{4}}
$$

- Analytic expression $\rightarrow$ allows for expansion around different kinematic limits:
- Massless charm quark:

$$
x=\frac{m_{c}}{m_{b}}=0 \quad \leftrightarrow \quad t=1
$$

- Heavy charm quark:

$$
x=m_{c} / m_{b}=1 \quad \leftrightarrow \quad \delta=0 \quad \leftrightarrow \quad t=0
$$

## Results

$$
\Gamma(b \rightarrow c \mid \bar{\nu})=\frac{A_{e w} G_{F}^{2} m_{b}^{5}\left|V_{c b}\right|^{2}}{192 \pi^{3}}\left(x_{0}+\frac{\alpha_{s}}{\pi} C_{F} X_{1}+\left(\frac{\alpha_{s}}{\pi}\right)^{2} C_{F} X_{2}+\mathcal{O}\left(\alpha_{s}^{3}\right)\right)
$$

with

$$
X_{2}=C_{F} X_{F}+C_{A} X_{A}+T_{F}\left(n_{l} X_{l}+n_{c} X_{c}+n_{h} X_{h}\right)
$$

- $n_{l}=3, n_{c}=1, n_{h}=1$ number of light, charm and bottom quarks.
- All $\left\{X_{F}, X_{A}, X_{l}, X_{C}, X_{h}\right\}$ depending on the mass ratio $m_{c} / m_{b}$ as functions of $t$, for example:

$$
\begin{aligned}
X_{I}= & \frac{1}{(-1+t)^{2}(1+t)^{2}\left(1+t^{2}\right)^{12}}\left(-38-76 t^{2}+190 t^{4}+\ldots\right)+\ldots \\
& +\frac{H_{-1,-1}(t)}{24 t^{6}\left(1+t^{2}\right)^{4}}\left(405+100 t^{2}-770 t^{4}+3462 t^{6}+\ldots\right)+\ldots+H_{-1,0,0,\{4,1\}}(t)(\ldots)+\ldots
\end{aligned}
$$

## Comparing results

- Results as an expansion in $\delta=1-m_{c} / m_{b}$ are known up to $\mathcal{O}\left(\alpha_{s}^{3}\right)$ [Fael, Schönwald, Steinhauser (2021)]. Obtained by expansion in $\delta$ on diagram level.
- Expand amplitude in $\delta$ after insertion of analytic master integrals . $\rightarrow$ Reproduce same results for all color factors!



## Conclusion

- We obtained an analytic expression for the semileptonic decay width $b \rightarrow c / \bar{\nu}$ at $\mathcal{O}\left(\alpha_{s}^{2}\right)$.
- We compared our results with previous calculations [Fael, Schönwald, Steinhauser (2021)] and found agreement.

What's next?

- Hadronic decay channels $b \rightarrow c \bar{u} d$ and $b \rightarrow c \bar{c} s$ with finite charm mass at $\mathcal{O}\left(\alpha_{s}^{2}\right)$.



## Outlook: Hadronic decay channel

- Hadronic decay channels are more involved than semileptonic decays:
- Different effective operators in HQE:

$$
\mathcal{H}_{e f f}=C_{1} \cdot\left(\bar{d}_{\alpha} u_{\beta}\right)_{V-A}\left(\bar{c}_{\alpha} b_{\beta}\right)_{V-A}+C_{2} \cdot\left(\bar{d}_{\alpha} u_{\alpha}\right)_{V-A}\left(\bar{c}_{\beta} b_{\beta}\right)_{V-A}
$$

- MI's of semileptonic decays are subset of Ml's of hadronic decays $\rightarrow$ Use the obtained results here again
- NLO calcultaion for $b \rightarrow c \bar{u} d$ is done:

$$
\Gamma(b \rightarrow c \bar{u} d)=\frac{A_{e w} G_{F}^{2} m_{b}^{5}\left|V_{c b}\right|^{2}\left|V_{u d}\right|^{2}}{192 \pi^{3}}\left(X_{0}+\frac{\alpha_{s}}{\pi} C_{F}\left(C_{1}^{2} X_{11}+C_{2} X_{22}+C_{1} C_{2} X_{22}\right)+\mathcal{O}\left(\alpha_{s}^{2}\right)\right)
$$

- Next step: $b \rightarrow c \bar{u} d$ at $\mathcal{O}\left(\alpha_{s}^{2}\right)$
- Try to calculate as many master integrals as possible analytically! If no analytic expressions can be found: Construct expansion in $\delta$ with differential equations.


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Thank you for your attention!

## Backup

## Master integrals with no $\epsilon$-form

- Differential equation of the form

$$
\frac{\mathrm{d} \vec{I}}{\mathrm{~d} t}=A(t, \epsilon) \cdot \vec{I}+\vec{B}(t, \epsilon) .
$$

- Insert ansatz for master integrals $\vec{l}$

$$
I_{i}=\sum_{j=-3} \epsilon^{j} \cdot f[i, j](t)
$$

- For every order in $\epsilon$ :
- Decoupling of differential equations with OreSys
- Solve decoupled differential equations for $f[i, j](t) \rightarrow$ homogeneous and inhomogeneous solution!
- Determine coefficients in full solution by matching to boundary conditions.


## Master integrals: Expansions $\boldsymbol{\delta}$

Alternative approach to analytic solution: Construct expansion of master integrals with differential equations.
(1) Set up differential equation in expansion parameter $\delta$ :

$$
\frac{\mathrm{d} \vec{l}}{\mathrm{~d} \delta}=A(\epsilon, \delta) \cdot \vec{l} .
$$

(2) Make general ansatz for master integrals:

$$
I_{i}=\sum_{j=-3} \sum_{n=0} \sum_{m=0} c[i, j, n, m] \cdot \epsilon^{j} \cdot \delta^{n} \cdot \log ^{m} \delta .
$$

(3) Insert ansatz into differential equation and compare coefficients in $\epsilon, \delta$ and $\log \delta$.
(4) Reduce system of linear equations in terms of a small number of coefficients with Kira [Klappert, Lange, Maierhöfer, Usovitsch (2021)].
(5) Determine these coefficients from the asymptotic expansion of the master integrals.

## Comparing expansion and analytic results

- Expansion in $\delta$ shows good convergence.


