Non-Hermitian tricriticality: a field theoretical approach

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Based on [AM, Takàcs, Lencsès, Mussardo; to appear]

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Motivations : Yang-Lee formalism

Basics

Consider the (analytic extension of the) partition function of a system at volume V :

$$\Omega_V(z) = \sum_{N=0}^{M} \frac{\mathcal{Z}_N(V)}{N!} z^N = \prod_{l=1}^{M} \left(1 - \frac{z}{z_l}\right) ,$$

The zeros play a role also in the thermodynamic limit: a density of zeros $\eta(z)$ can be studied!!

[Yang,Lee,'52; Lee-Yang,'52]

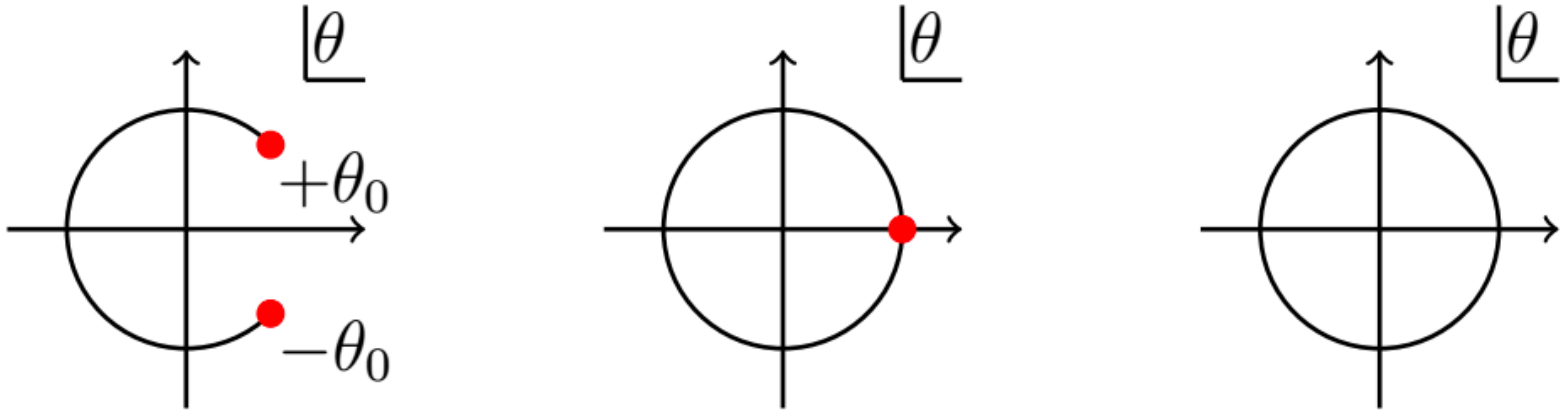
Motivations : Yang-Lee formalism

Ising model

Consider the Ising model: $$H = J\sum_{\langle i,j\rangle} s_i s_j + h\sum_i s_i\ , \qquad s_i \in \{-1,1\}$$

$$z = e^{-\beta h} = e^{i\theta}$$

Distribution of the zeros in the T.L. :



$T > T_c$ $\qquad$ $T = T_c$ $\qquad$ $T < T_c$

In the red points:

$$\eta(\theta) \overset{\theta\to\theta_0}{\sim} |\theta - \theta_0|^{-\mu}\ ,$$

$$m(h) \sim |h - ih_0|^{-\sigma}$$

Critical Point!!

[Yang,Lee,’52; Lee-Yang,’52]

Motivations : Yang-Lee formalism

The moral

- **There is a critical point when an imaginary magnetic field is switched on!!**

History of the critical point:

- The critical point was discussed from the lattice point of view;

[Kortman, Griffiths,'71]

- The Ginzburg-Landau Lagrangian is known [Fisher,'78]
- The CFT that controls the critical point is known; [Cardy,'85]
- Numerical checks are present in literature.

[Fonseca,Zamolodchikov,'01; Xu, Zamolodhikov,'22]

Our question

What about the Tricritical Ising ?

- The existence of a critical point was studied in the lattice formulation; [von Gehlen,'94]
- We <u>don't</u> know the Ginzburg-Landau Lagrangian
- We <u>don't</u> know which is the CFT that controls the critical point

The Plan

- Review of main results in 2D CFTs
- Review of the Ising case (but using our tools)
- Some results on the Tricritical Ising
- Conjetures on non-Hermitian multicritical points

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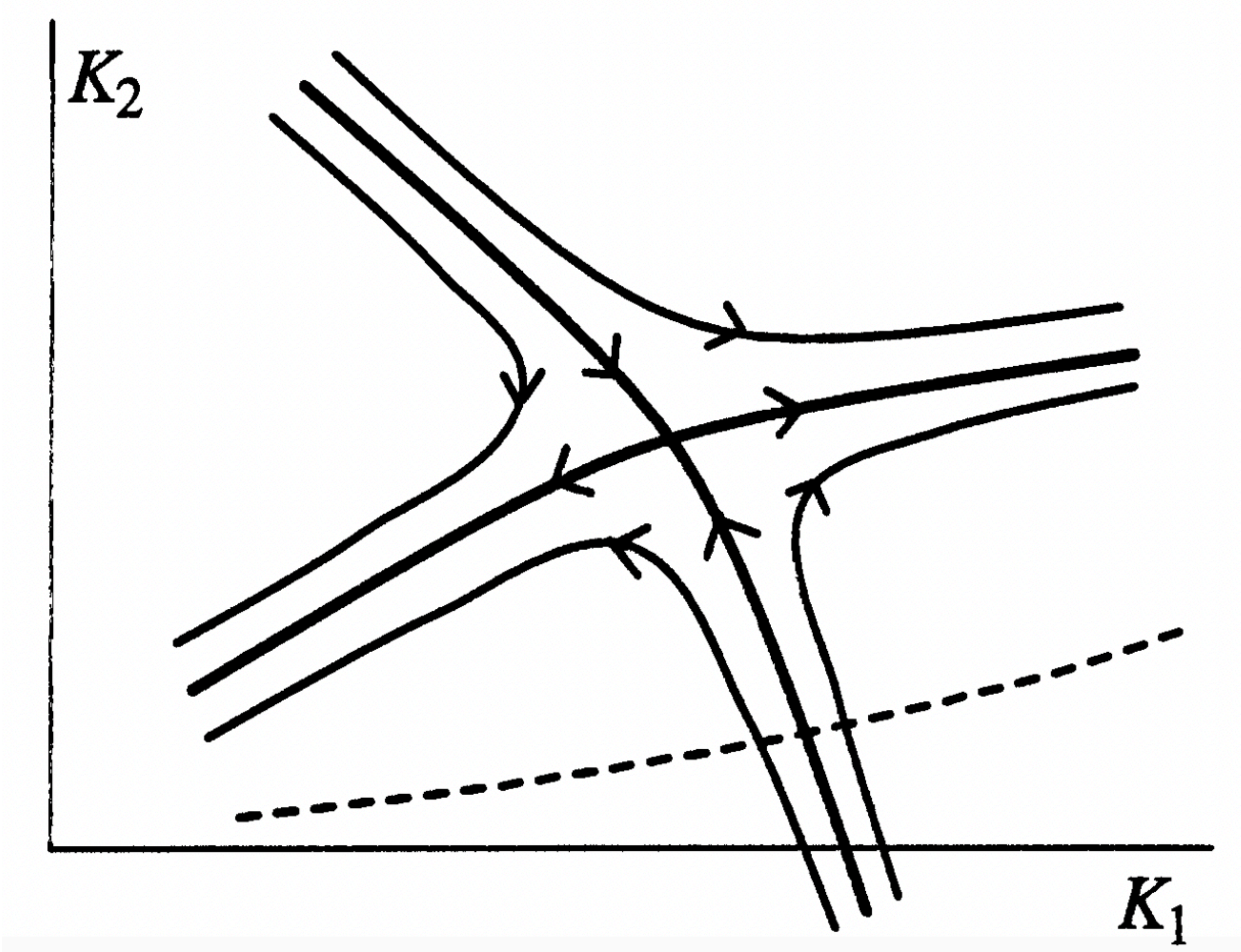
2D CFTs

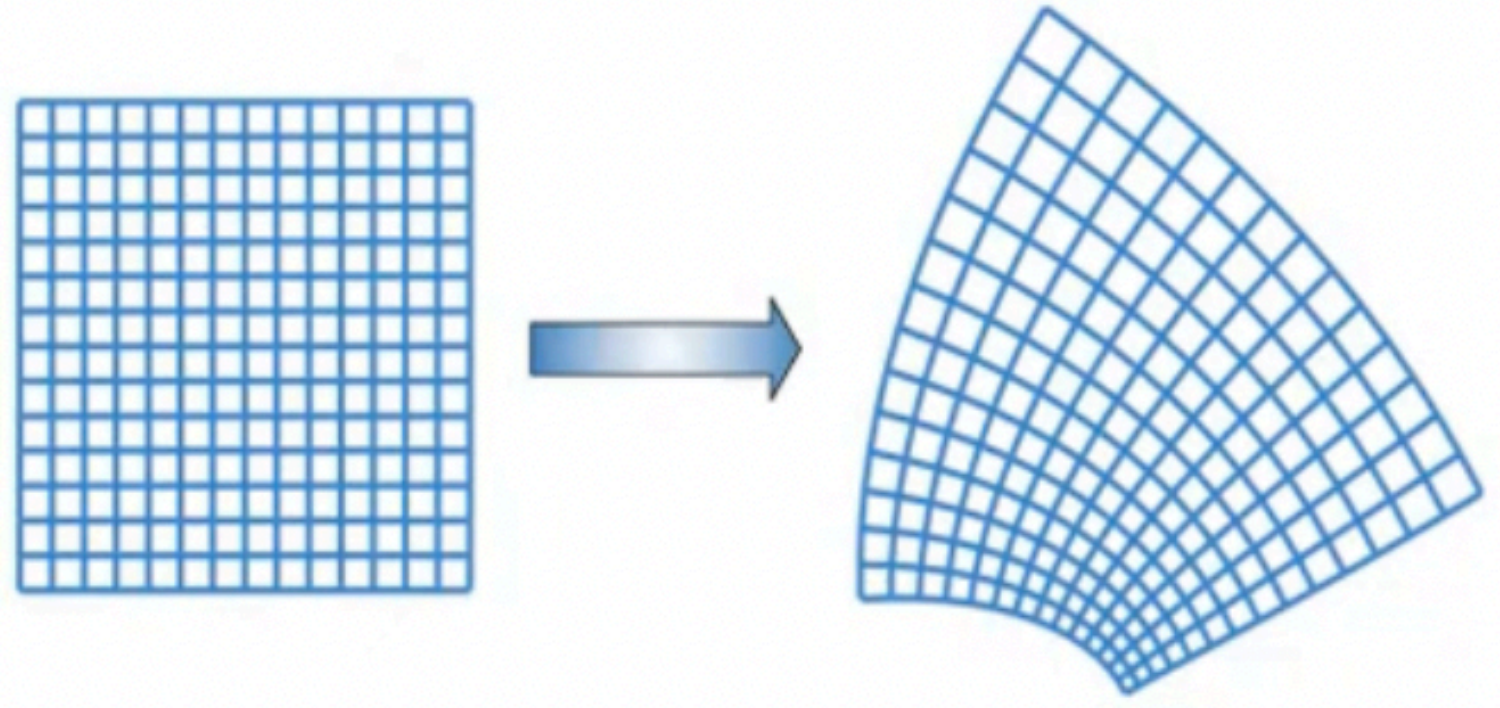
Renormalization group and CFTs



In the space of quantum field theories, the fixed points of the renormalization group flow are either conformal field theories (CFTs) or trivial theories

[Zamolodchikov,’86]

CFTs are QFTs invariant under angle-preserving transformations of spacetime



2D CFTs
Virasoro algebra

In 2D a CFT is invariant under the Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n\left(n^2-1\right)\delta_{n+m,0}\ , \qquad [\overline{L}_n, \overline{L}_m] = (n-m)\overline{L}_{n+m} + \frac{c}{12}n\left(n^2-1\right)\delta_{n+m,0}\ , \qquad [L_m, \overline{L}_m] = 0$$

States in a representation of the Virasoro algebra (analytic sector) can be written as

$$|\phi; n_1, n_2, \ldots, n_k\rangle = L_{-n_1}L_{-n_2}\ldots L_{-n_k}|\phi\rangle\ , \quad n_1 \leq n_2 \leq \ldots \leq n_k\ .$$

And the Hilbert space

$$\mathscr{H} = \bigoplus_{\phi,\varphi} V(\phi) \otimes \overline{V}(\varphi)$$

[Beliavin, Polyakov, Zamolodchikov,'84]

2D CFTs

Minimal models

Some CTFs have a finite number of modules: these are the **minimal models.**

Some facts:

- *The minimal models are identified by M(p,q) where p and q are co-prime integers.*
- *Some m.m. are unitary and these are classified as M(p,p+1); the others are non-unitary.*

$$\mathscr{H} = \bigoplus_{\phi} V(\phi) \otimes \overline{V}(\phi)$$

$$\Delta_{r,s} = \frac{(pr-qs)^2-(p-q)^2}{4pq}$$

$$1 \le r \le q-1 \,,\; 1 \le s \le p-1$$

$$c = 1 - \frac{(p-q)^2}{pq}$$

[Beliavin, Polyakov, Zamolodchikov,’84; Cardy; Friedan, Qiu, Shenker,’85]

2D CFTs

Unitary minimal models and Ginzburg-Landau

M(p,p+1) $\Longleftrightarrow$ $$\mathscr{L}_{G.L.} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + a_1\varphi + a_2\varphi^2 + \ldots + g_{2p-4}\varphi^{2p-4} + \varphi^{2p-2}$$

$$a_1 = a_2 = \ldots = a_{2p-4} = 0$$

Strategy sketch:

- *Use the OPEs to relate normal ordered powers of the most relevant field with the other primaries:*

$$\varphi \times \varphi = [\cancel{\mathbb{1}}] + \overbrace{[\psi]}^{:\varphi^2:} + [\sigma] + \ldots \quad \Rightarrow \quad \psi \sim :\varphi^2:$$

- *Iterate the process until:*

$$L_{-1}\bar{L}_{-1}\varphi \sim :\varphi^{2p-3}: \Rightarrow \partial\bar{\partial}\varphi \sim :\varphi^{2p-3}:$$

EoM of a Ginzburg-Landau

[Zamolodchikov,'86]

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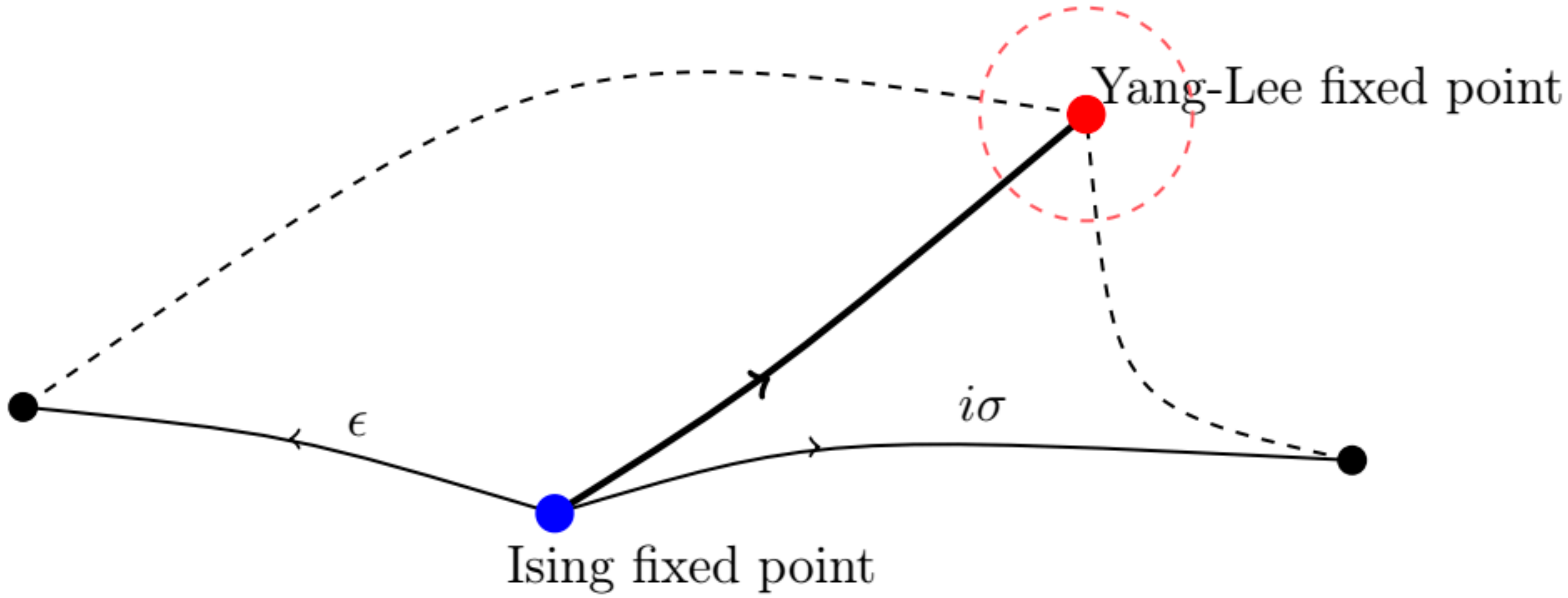
Ising and Y.L. edge singularity

The field theoretical approach



We need a thermal deformation combined with an imaginary magnetic deformation of the minimal model M(3,4) (Ising).

$$\mathscr{L}_{Y.L.} = \underbrace{\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}}_{\text{M(3,4)}} + \underbrace{im\bar{\psi}\psi}_{\text{Thermal deformation}} + \underbrace{ih\sigma}_{\text{(Imaginary) magnetic deformation}}$$

Ising and Y.L. edge singularity

Comments on PT symmetry

The PT symmetry depends on two conditions:

$$i)\ \ [\boldsymbol{H}, \boldsymbol{PT}] = 0\ ,\ \ ii)\ \ \boldsymbol{PT}\,|\psi\rangle = e^{i\varphi}\,|\psi\rangle \qquad (\boldsymbol{H}\,|\psi\rangle = E_{\psi}\,|\psi\rangle)\ .$$

- *<u>Non PT-symmetric phase</u>*: If i) and ii) do not holds the energy spectrum is complex
- *<u>PT-symmetric phase</u>:* If i) and ii) hold the energy spectrum is real
- *<u>Spontaneously broken PT phase</u>:* If i) holds but ii) does not hold the energies appear either as real values or in complex conjugate pairs

Ising and Y.L. edge singularity

Comments on PT symmetry

$$\mathscr{L}_{Y.L.} = \psi\bar{\partial}\psi + \overline{\psi}\partial\overline{\psi} + im\overline{\psi}\psi + ih\sigma$$

In our case the Lagrangian is PT invariant, indeed the PT transformations are

$$x \to -x\ , \quad i \to -i\ , \quad \psi \to i\psi\ , \quad \overline{\psi} \to i\overline{\psi}\ , \quad \sigma \to -\sigma\ .$$

So i) holds, but we don't know, a priori, if we are in the PT-symmetric phase or in the spontaneously broken symmetric phase.

The Yang-Lee fixed point is believed to be the critical point that separates a PT-symmetric regime from a spontaneously broken PT regime.

Ising and Y.L. edge singularity

ceff-theorem

The usual c-theorem can be extended for non-unitary models with the following differences:

[Castro-Alvaredo, Doyon, Ravanini,'17]

- We have to be in the unbroken PT-phase (real spectrum);
- The c-function interpolates between <u>effective</u> central charges;

$$c_{eff} = c - 24\Delta_{min}$$

In our case this theorem provides a bound on the effective central charge:

$$c_{eff}^{ir} < c_{eff}^{uv} = \frac{1}{2}$$

$\Rightarrow$ The only possibility is M(2,5)

Ising and Y.L. edge singularity

The Fisher's and Cardy's arguments (revisited)

- Fisher proved that the Ginzburg Landau of the infrared theory is

$$\mathscr{L}_{Y.L.} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + i(h-h_0)\varphi + i\gamma\varphi^3 \ .$$

[Fisher,'78]

- Starting from the Fisher's result, Cardy proved that the CFT that controls the Yang-Lee edge singularity is the minimal model M(2,5). Indeed this is the only CFT (minimal model) that satisfy the following conditions:

• There is only one relevant field ϕ ;

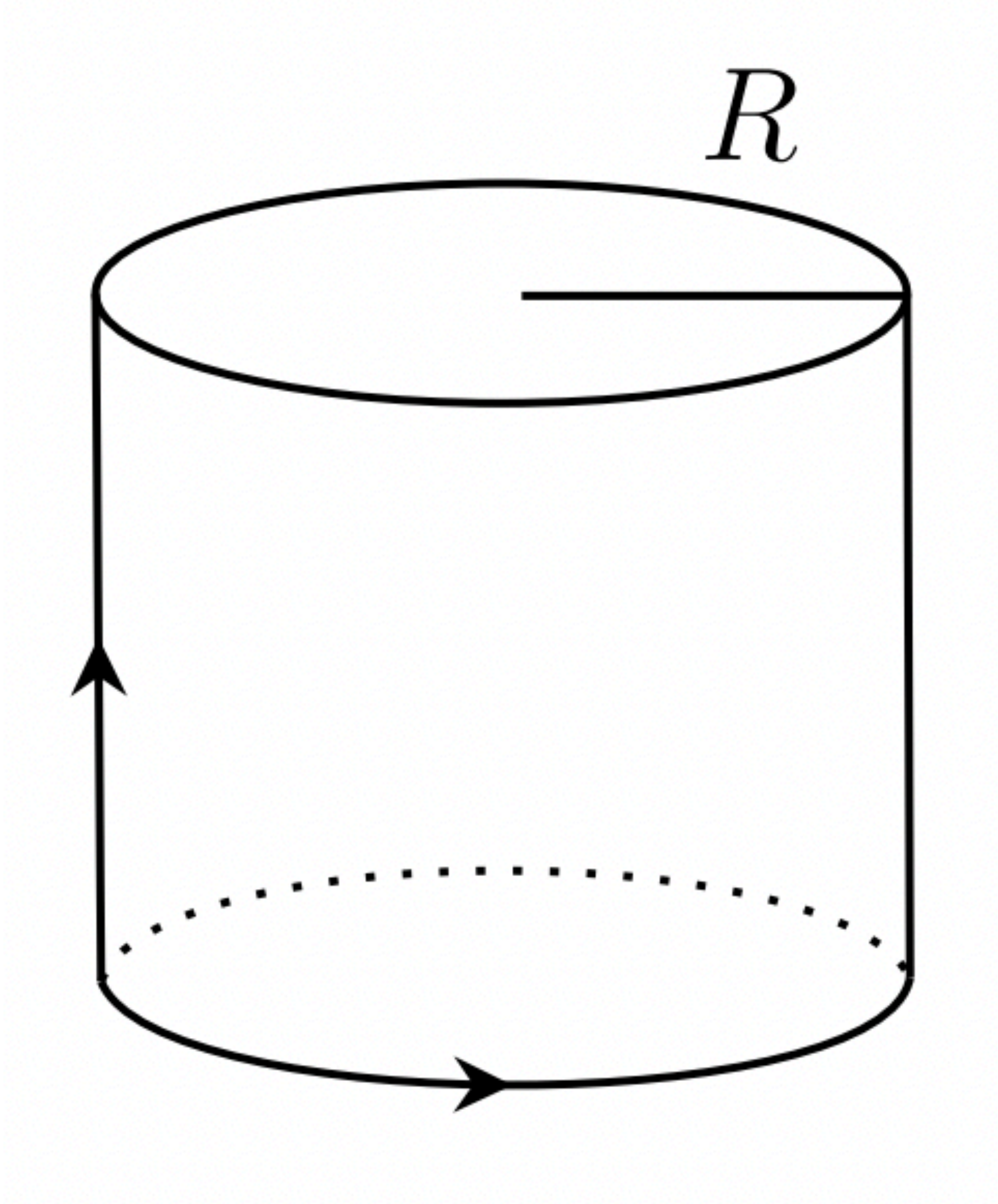
• The three point function $\langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle$ is nonzero.

[Cardy,'85]

Truncated conformal space approach (TCSA)

A numerical approach

Step 1: Compute the (finite volume) Hamiltonian in the conformal basis



$$\boldsymbol{H} = \boldsymbol{H}_{CFT} + \boldsymbol{V} = \frac{2\pi}{R}\left(\boldsymbol{L}_0 + \overline{\boldsymbol{L}}_0 - \frac{c}{12}\right) + \lambda \int \phi \ \mathrm{d}^2 z =$$

$$= \frac{2\pi}{R}\begin{pmatrix} \star & 0 & 0 & \cdots & \cdots \\ 0 & \star & 0 & \cdots & \cdots \\ 0 & 0 & \star & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} + R^{1-2\Delta}\begin{pmatrix} \star & \star & \star & \cdots & \cdots \\ \star & \star & \star & \cdots & \cdots \\ \star & \star & \star & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} .$$

TCSA

A numerical approach

Step 2: Evaluate the Hamiltonian of the theory truncated at a certain energy scale;

This is equivalent to truncate the Hilbert space:

$$L_0 | \phi; n_1, \dots, n_k \rangle = \Delta_\phi + n_1 + \dots + n_k$$

$$\Downarrow$$

$$\frac{2\pi}{R}(\Delta_\phi + \overline{\Delta}_\phi + n_1 + \overline{n}_1 + \dots + n_k + \overline{n}_k) \leq \Lambda \ .$$

Step 3: Diagonalize the resulting (finite) matrix to find the non-perturbative energy spectrum.

TCSA

The non-perturbative spectrum at finite volume

For $R \ll \xi$ ➡ UV theory $\longrightarrow$ $E_n \simeq \frac{2\pi}{R}\left(2\Delta^{uv} + 2N^{uv} - \frac{c^{uv}}{12}\right)$

For $R \gg \xi$ ➡ IR theory

$$E_n \simeq \frac{\epsilon_0}{\xi^2}R + \sum_i M_i$$

$$E_n \simeq FR + \frac{2\pi}{R}\left(2\Delta^{ir} + 2N^{ir} - \frac{c^{ir}}{12}\right)$$

TCSA

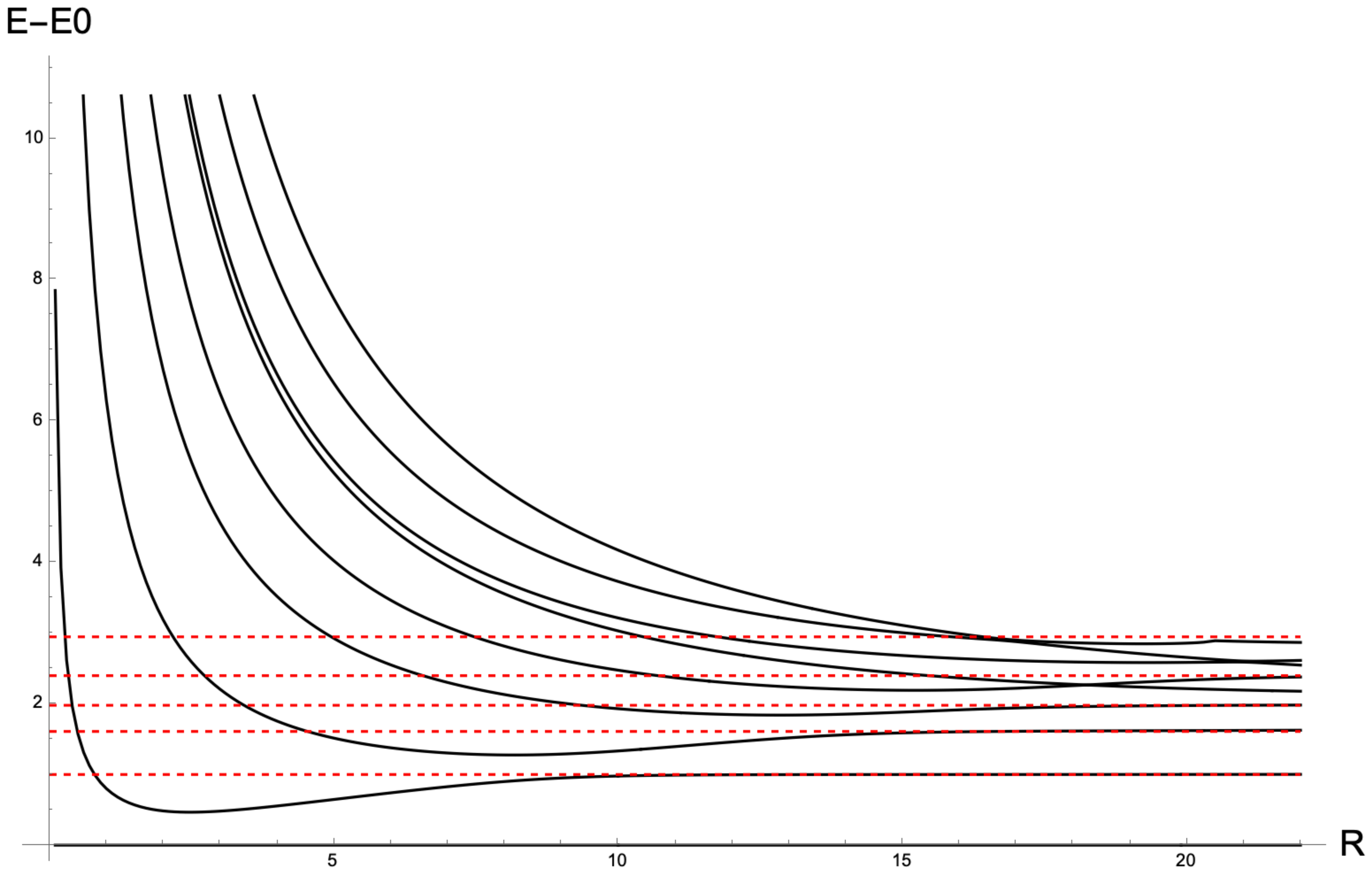
Massive RG flows: does it work?

Example : Magnetic deformation of M(3,4) (Ising).

From the integrable bootstrap program:

	Analytic	Numerical	TCSA
m_1	1	1	1.00222
m_2	$2\cos(\pi/5)$	1.6180	1.62724
m_3	$2\cos(\pi/30)$	1.9890	1.98353
m_4	$4\cos(\pi/5)\cos(7\pi/30)$	2.4029	2.38096
m_5	$4\cos(\pi/5)\cos(2\pi/15)$	2.9563	2.93015
m_6	$4\cos(\pi/5)\cos(\pi/30)$	3.2183	3.18481
m_7	$4\cos^2(\pi/5)\cos(7\pi/30)$	3.8911	3.90557
m_8	$4\cos^2(\pi/5)\cos(2\pi/15)$	4.7834	4.76382

[Zamolodchikov,'89]



TCSA

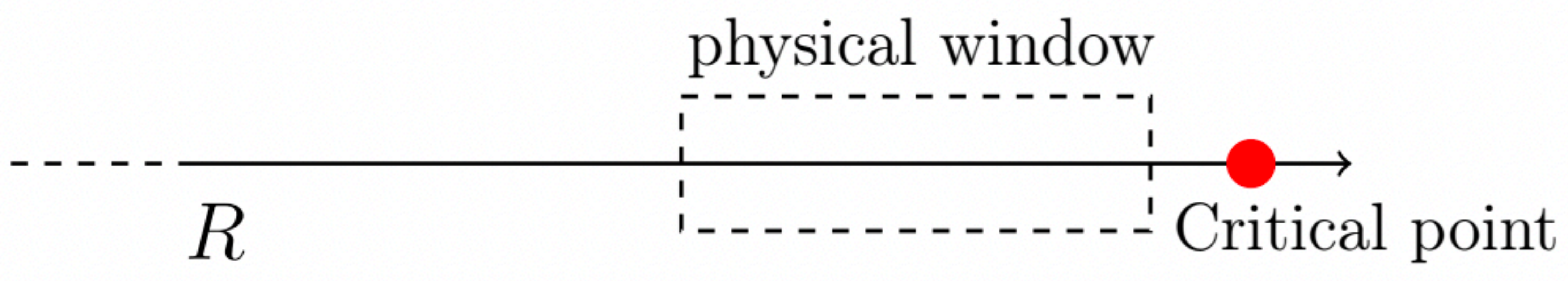
Massless RG flows

At finite volume it is impossible to reach exactly the critical point, so

- Localize the critical point is (the mass of the lightest particle is zero) in R-space;

- The position depends on the coupling constants: change the coupling constants to push the critical point at "infinite" volume;

- Choose a "physical window" in the R-space in which the spectrum approach the CFT spectrum (without really reach the CFT).

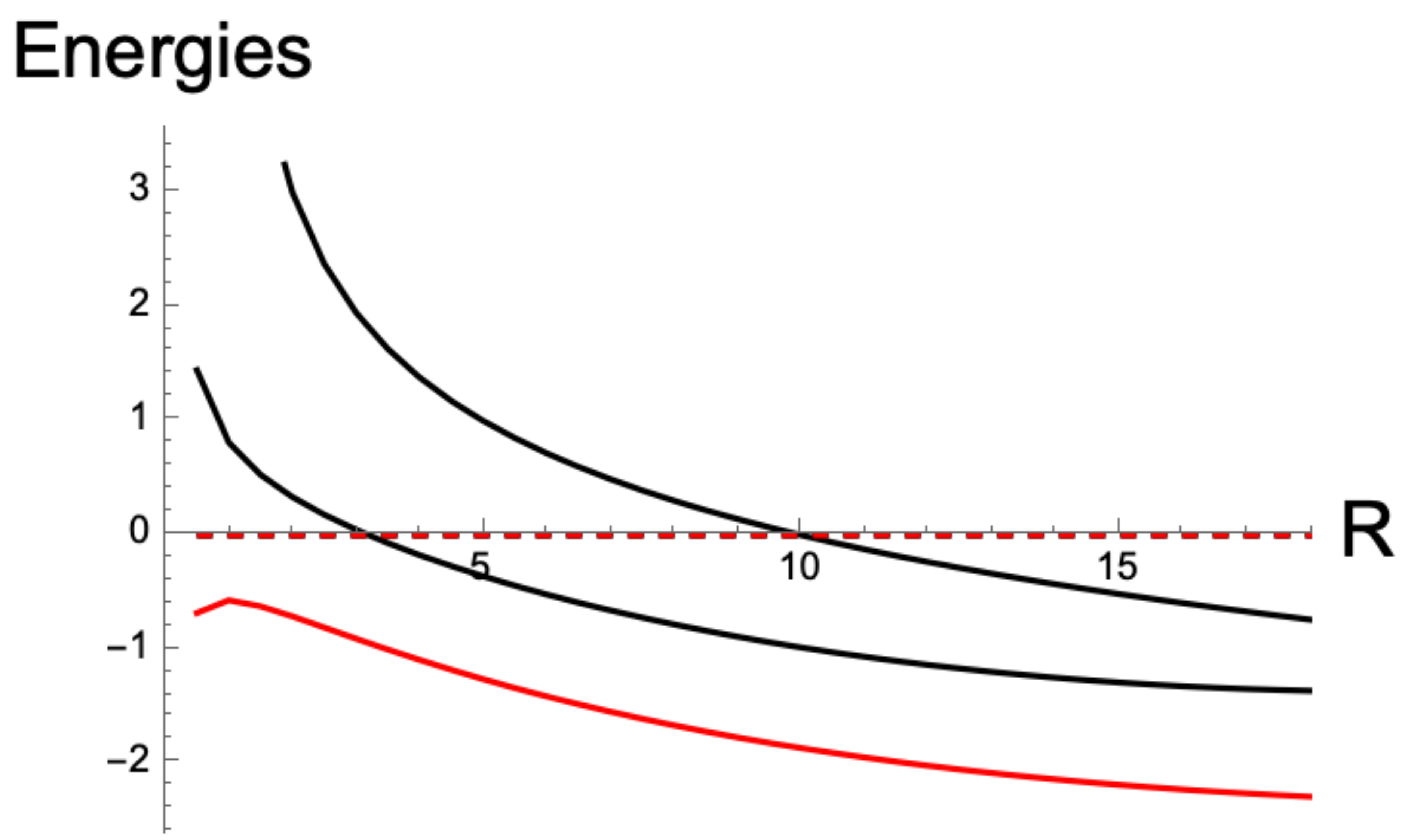


$$E_i - E_0 \simeq \frac{4\pi}{R}\left(\Delta^{ir} - \Delta^{ir}_{min} + n^{ir}\right) = \frac{4\pi}{R} C_i$$

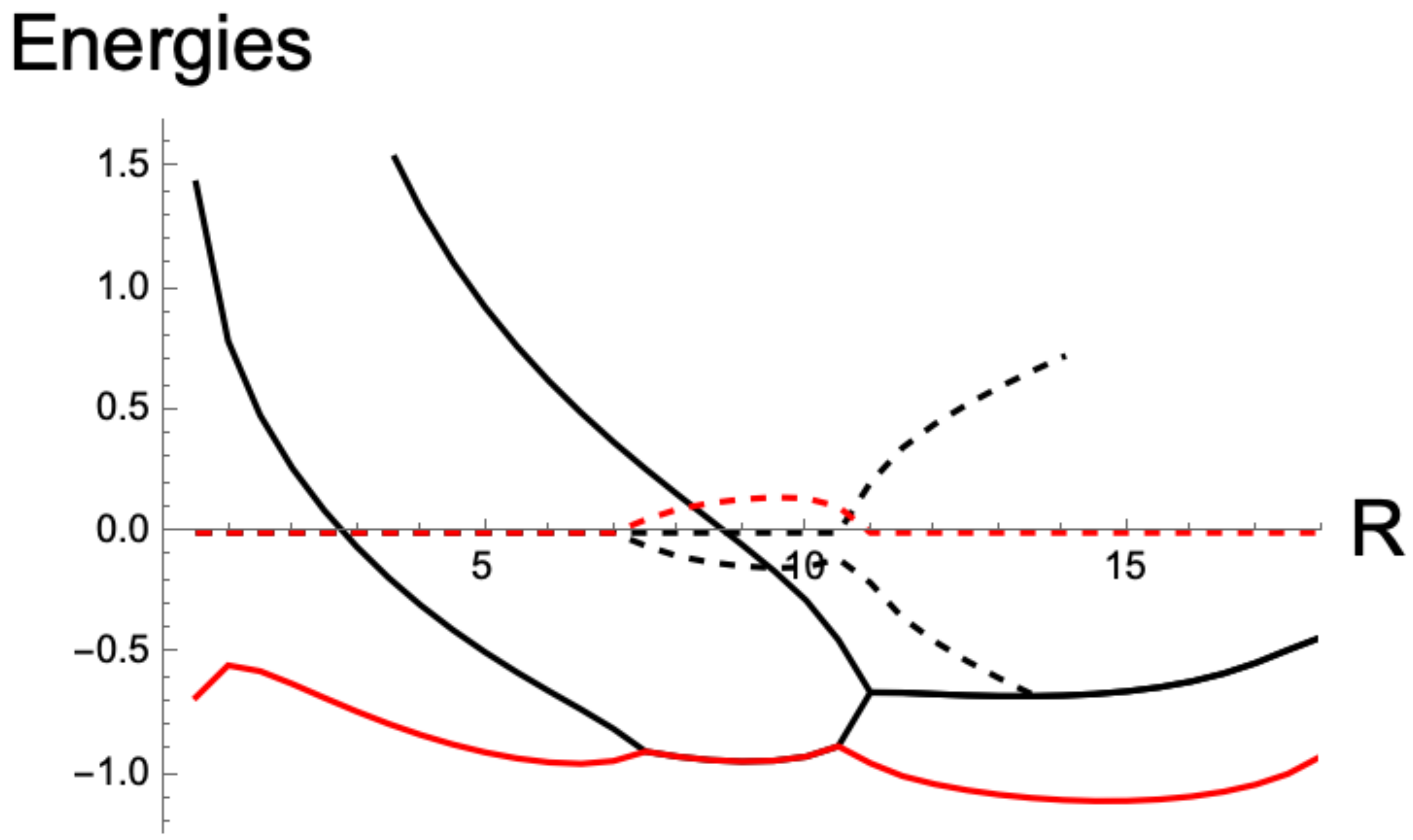
TCSA

Yang-Lee edge singularity: phenomenology

Before the critical point



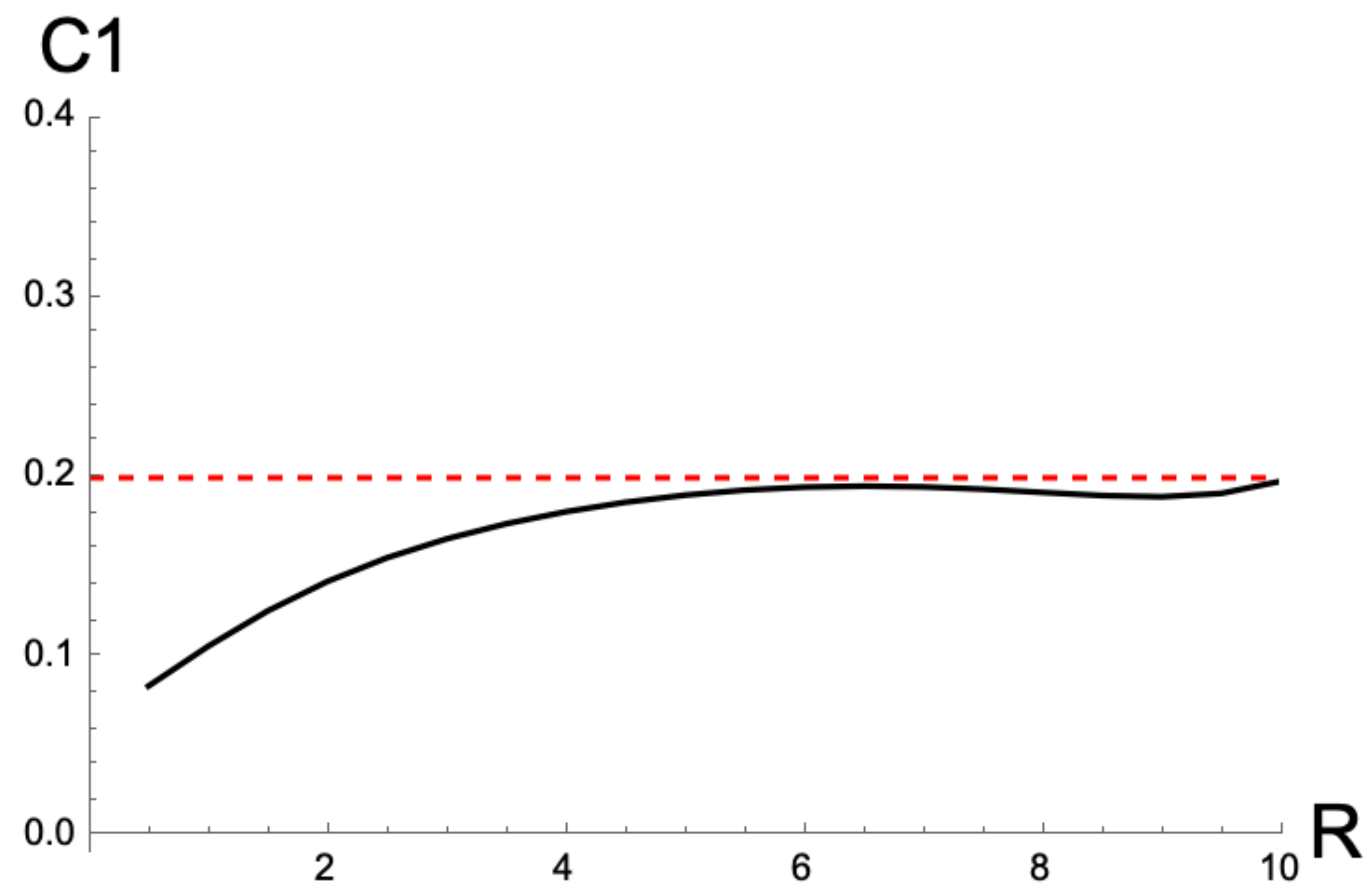
“After” the critical point



[Fonseca,Zamolodchikov,’01]

TCSA results

Yang-Lee edge singularity: CFT



The prediction from the minimal model M(2,5)

$$C_1^{M(2,5)} = \frac{R}{4\pi}\left(E_1 - E_0\right) \sim \Delta_{\mathbf{1}} - \Delta_\varphi + n = 0 + \frac{1}{5} + 0 = 0.2$$

An aside: Xu and Zamolodchikov proposed an effective field theory approach to the Yang-Lee edge singularity such as

$$\mathcal{A}_{\text{eff}} = \mathcal{A}_{\text{YLCFT}} + \lambda \int \phi(x)\, d^2x + \frac{\alpha}{\pi^2} \int T\bar{T}(x)\, d^2x + \frac{\beta}{2\pi} \int \Xi(x)\, d^2x\,,$$

$$\Xi(x) = \left(L_{-4} - \frac{625}{624} L_{-1}^4\right)\left(\bar{L}_{-4} - \frac{625}{624}\bar{L}_{-1}^4\right)\phi(x)\,.$$

[Xu ,Zamolodchikov,’21]

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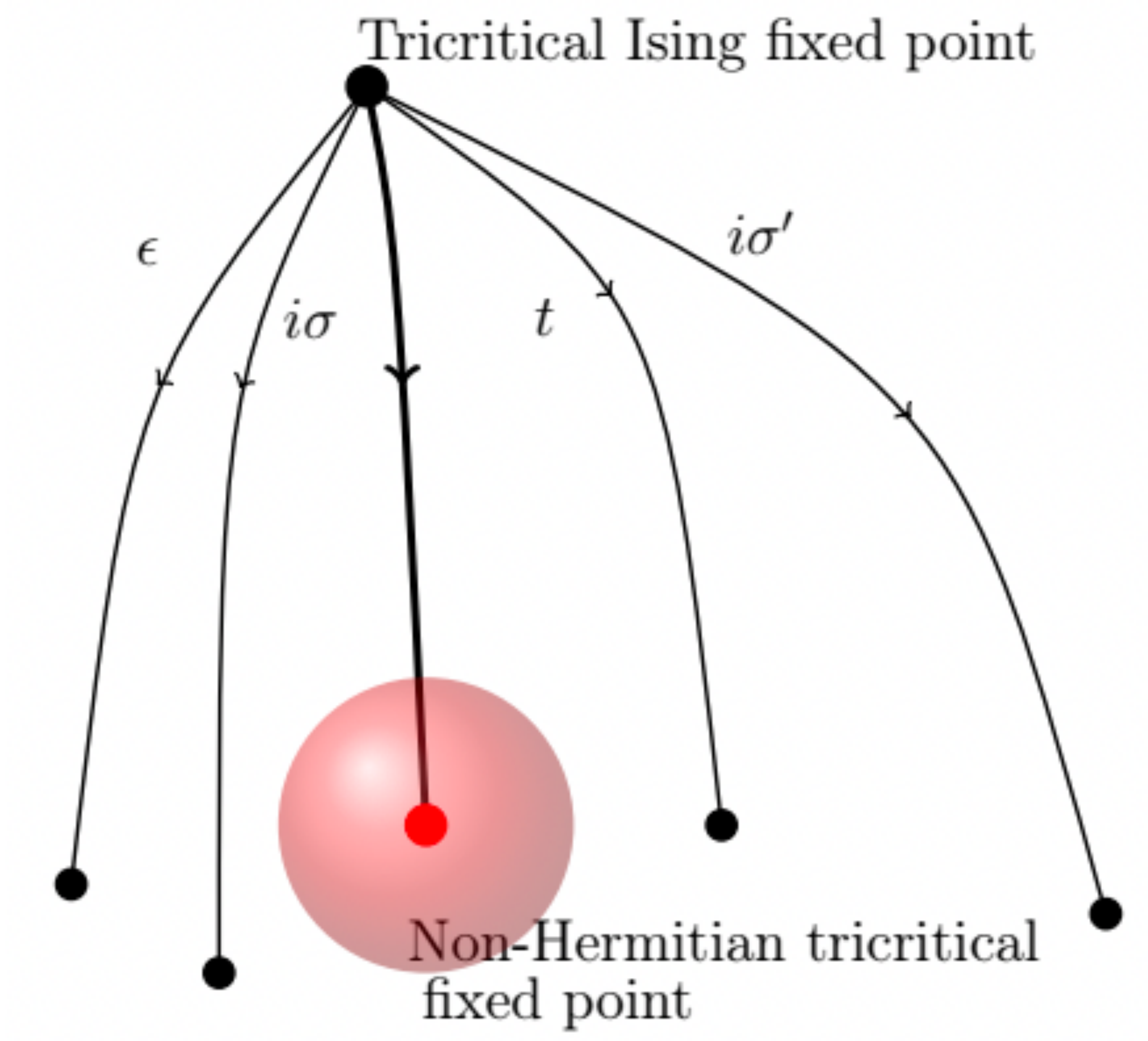
Tricritical Ising and non-Hermitian tricriticality

The field theoretical approach



We need all the scaling region with an <u>imaginary sub-magnetic and magnetic</u> deformations of the minimal model M(4,5) (tricritical Ising).

The considerations on PT-symmetry are the same we saw in the Ising case.

➡ M(2,5) M(2,7)

M(2,9) M(3,5)

Observe: the physical magnetic field is a combination of the magnetic field and the submagnetic field.

Tricritical Ising and non-Hermitian tricriticality

The Fisher's argument (revisited)

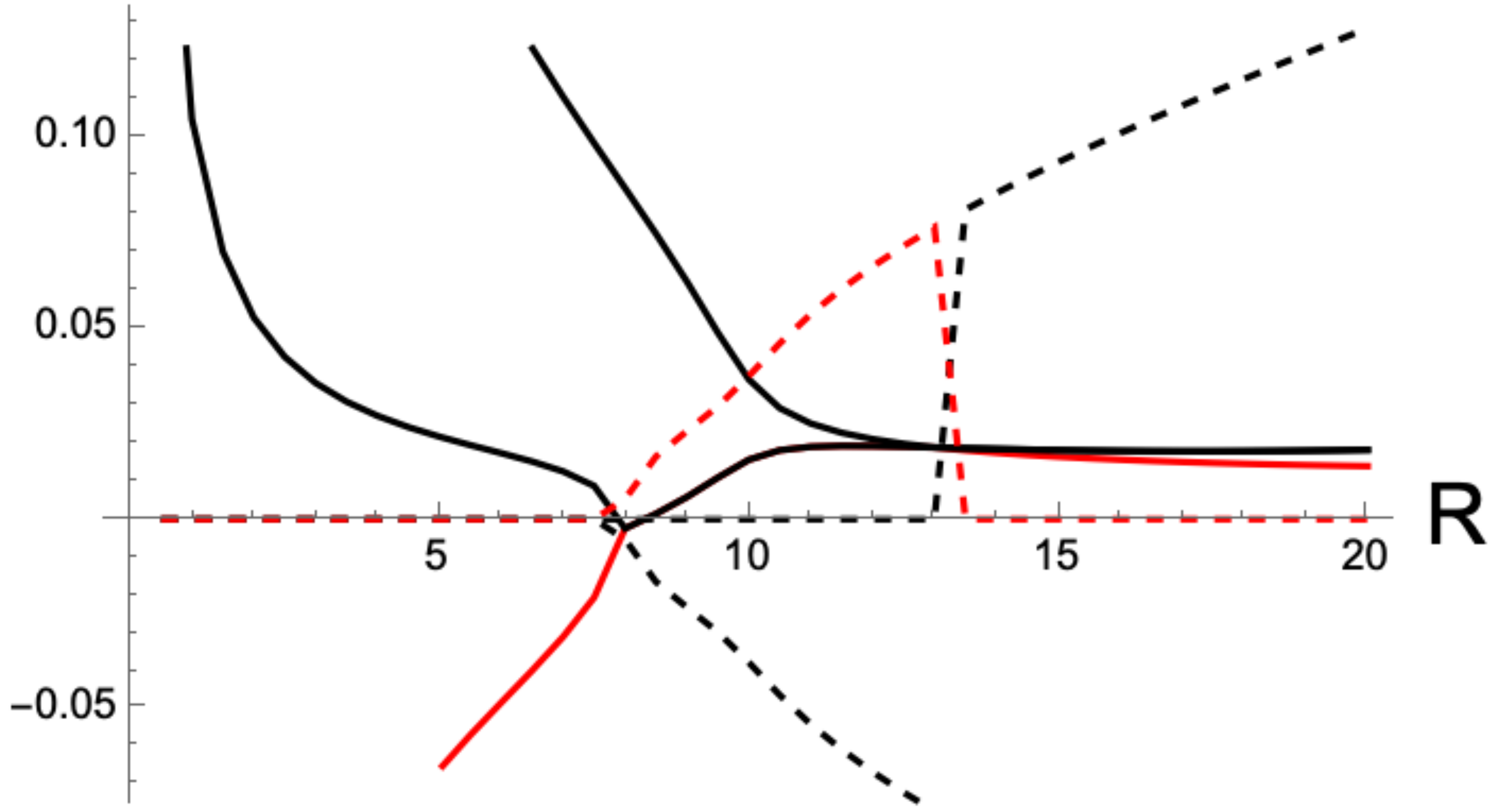
- The Fisher argument, adapted in our case gives:

$$\mathscr{L}_{Y.L.} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + i(h-h_0)\varphi + i\gamma\varphi^5 \ .$$

- This is not correct because the Fisher argument works when the couplings are independent, but in our case the couplings are not independent. A counting of expected relevant fields in the infrared theories gives 2.

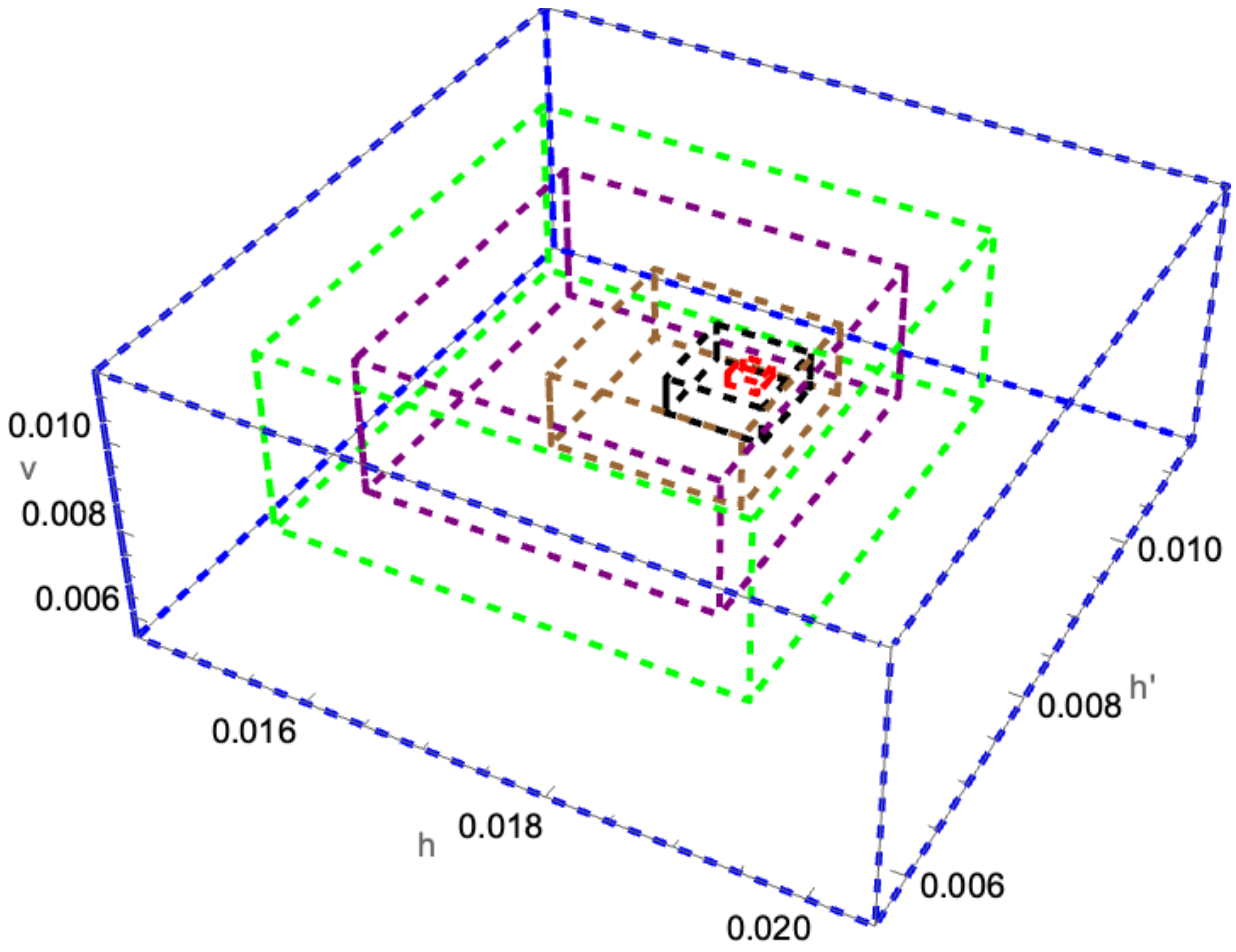
$\Rightarrow$ M(2,7)

Energies



Tricritical Ising and non-Hermitian tricriticality

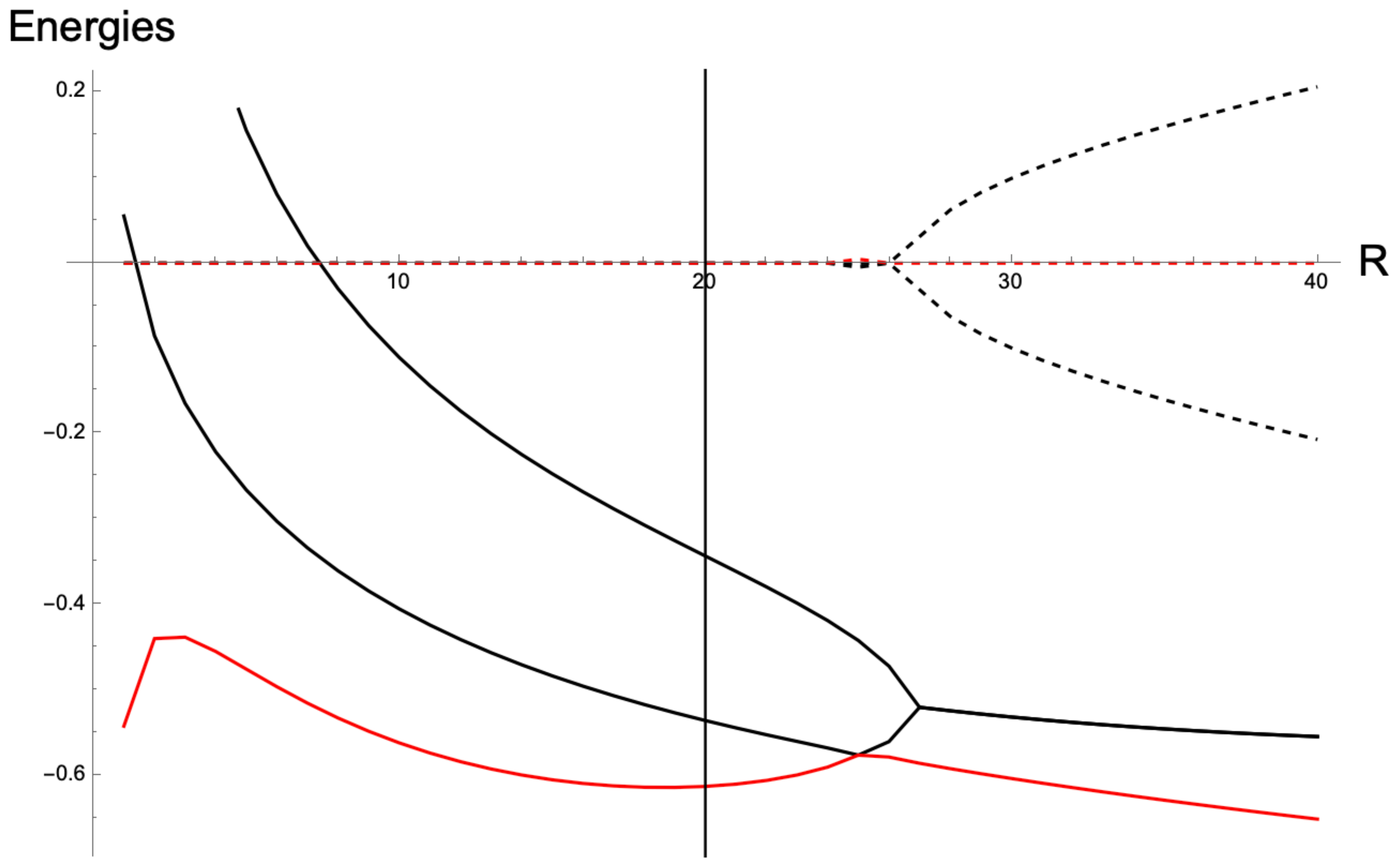
TCSA Results: the non-Hermitian tricritical point



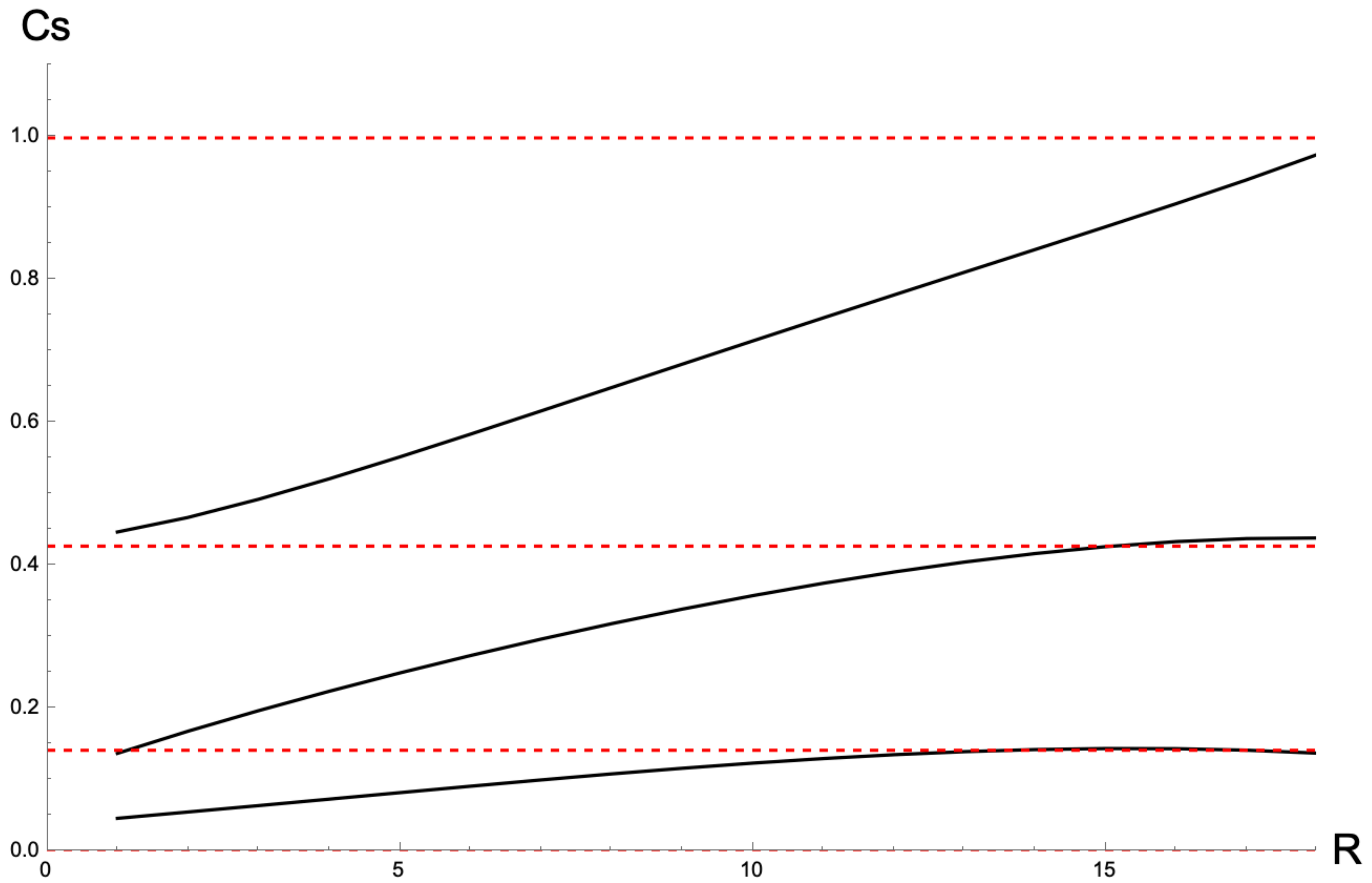
Tricritical Ising and non-Hermitian tricriticality

TCSA Results: a non-hermitian tricritical point

Raw Spectrum:



C1:



M(2,7)

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Non-Hermitian multicritical points

A conjecture

The natural generalization is that the non-Hermitian multicritical points are controlled by **M(2,2n+3) , n = 1,2, …**

Indeed:

- The number of expected relevant fields coincides;
- The ceff-theorem bound is satisfied;
- It is very hard to test the conjectures with TCSA;

Non-Hermitian multicritical points

A conjecture and new RG flows

Assuming the conjecture is true we expect new RG flows:

$$M(2,q) + i\lambda\phi_{n,m} \rightarrow M(2,q-2)$$

This flows are integrable and we can check if the infrared theory is M(2,q-2) by using the (massless) thermodynamic Bethe ansaz and TCSA.

[AM, Takàcs, Lencsès, Mussardo; in preparation]

THANK YOU
for your
ATTENTION!

The minimal model M(2,5)

Some proprieties

- It is a non-unitary minimal model;
- It contains only two primary fields: the identity and a field of weights (-1/5,-1/5) which is also the only relevant field in the theory;
- The only relevant OPE is

$$\phi(x)\phi(x') = |x-x'|^{4/5}(\mathbf{1} + \text{descendants}) + c^{\phi}_{\phi\phi}|x-x'|^{2/5}(\phi(x) + \text{descendants}) \ ,$$

$$c^{\phi}_{\phi\phi} = i\left(\frac{\Gamma(1/5)}{\Gamma(4/5)}\right)^{3/2}\left(\frac{\Gamma(2/5)}{\Gamma(3/5)}\right)^{1/2} \ .$$

$$\mathscr{L}_{Y.L.} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + i(h-h_0)\varphi + i\gamma\varphi^3 \ .$$

[Mussardo,Cardy,'89]

Integrability

Existence of an infinite tower of conserved charges

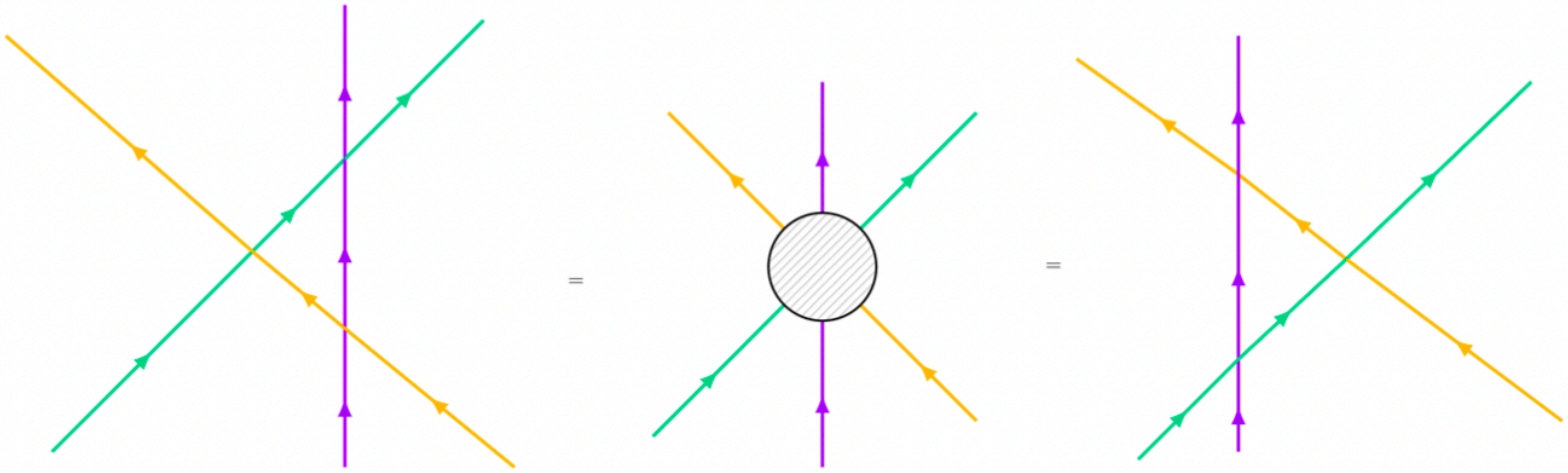
$$\mathbf{\mathfrak{Q}}_s \left|\theta^a\right\rangle = q_s^a e^{s\theta^a} \left|\theta^a\right\rangle$$

$$\mathbf{\mathfrak{Q}}_s \left|\theta_1^{a_1}, \ldots, \theta_n^{a_n}\right\rangle = \left(q_s^{a_1} e^{s\theta_1^{a_1}} + \ldots + q_s^{a_n} e^{s\theta_1^{a_n}}\right) \left|\theta_1^{a_1}, \ldots, \theta_n^{a_n}\right\rangle$$

The rapidity

$$p_0 + p_1 = m e^{\theta} \,, \qquad p_0 - p_1 = m e^{-\theta}$$

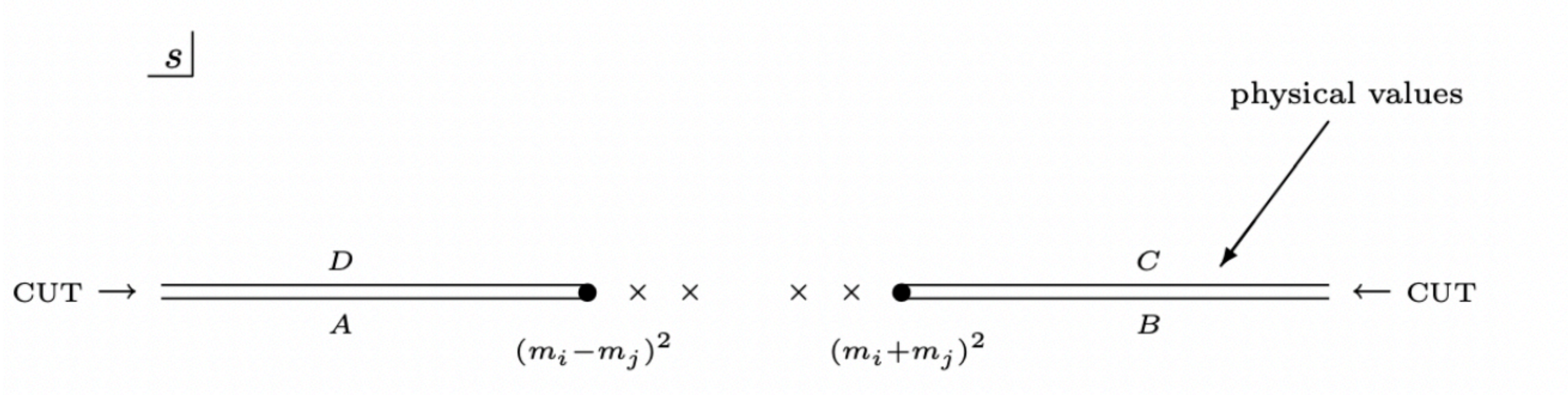
A fundamental propriety: the Yang-Baxter



[Dorey]

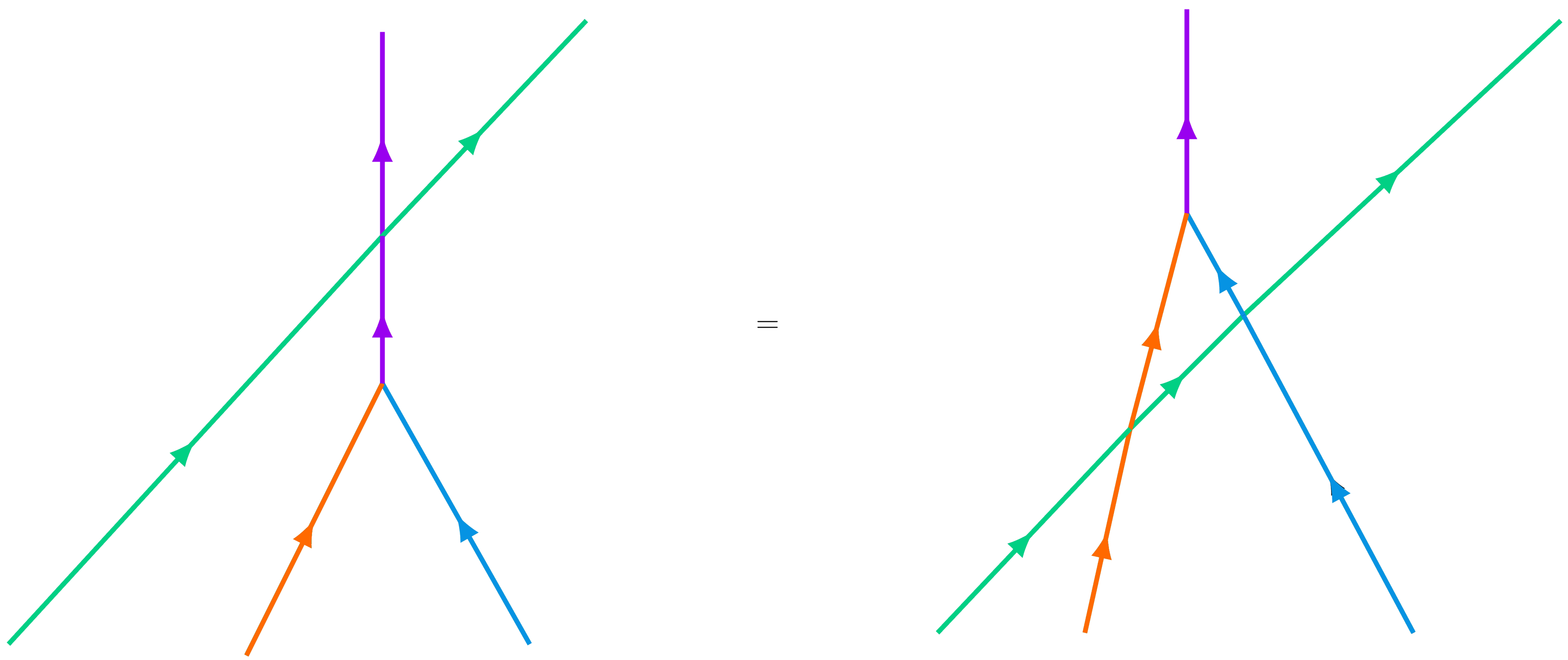
The integrable bootstrap program

Every pole of the S-matrix corresponds to a new particle



We can add particles in the theory (i.e. poles in the S-matrix) until the bootstrap equations are totally consistent

The bootstrap equation visualized:



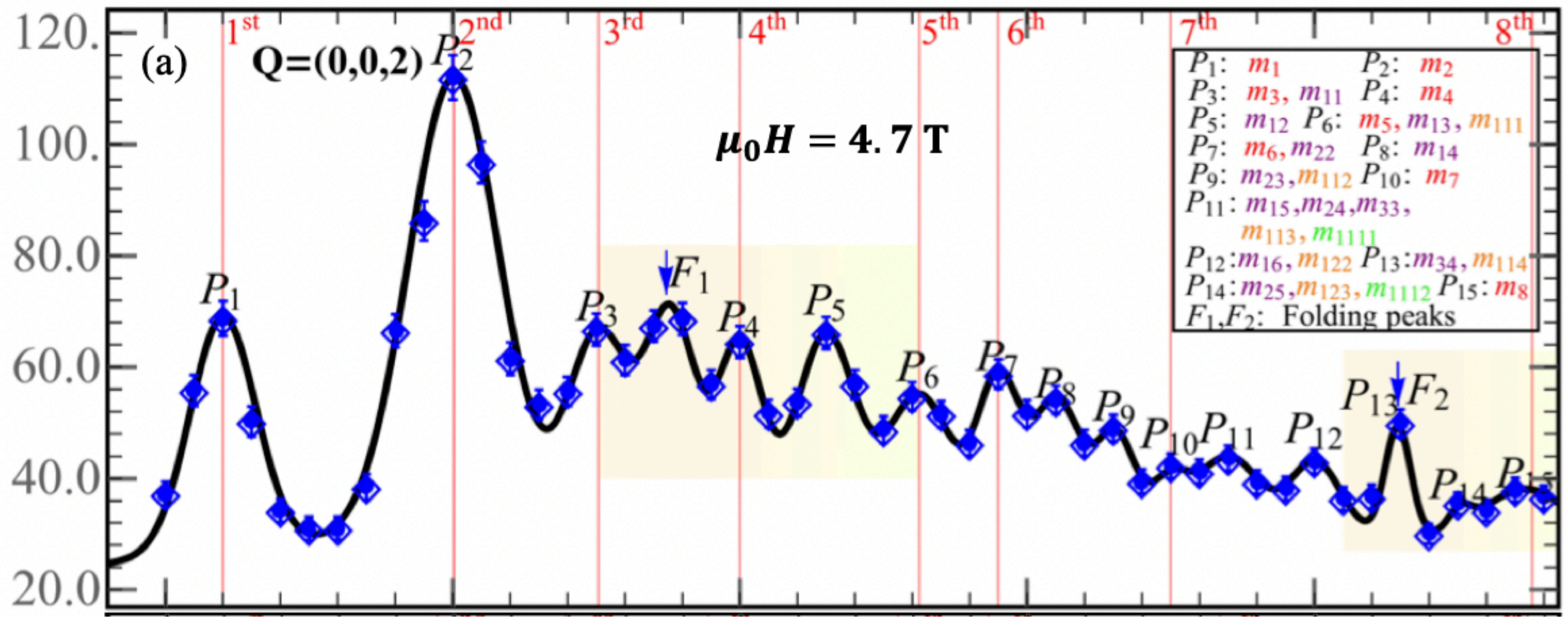
[Zamolodchikov, Mussardo]

A look into the experiments

Magnetic deformation of Ising

E_8 Spectra of Quasi-one-dimensional Antiferromagnet $BaCo_2V_2O_8$ under Transverse Field

Haiyuan Zou,[1,*] Yi Cui,[2,*] Xiao Wang,[1,*] Z. Zhang,[1] J. Yang,[1] G. Xu,[3] A. Okutani,[4] M. Hagiwara,[4] M. Matsuda,[5] G. Wang,[6] Giuseppe Mussardo,[7] K. Hódsági,[8] M. Kormos,[9] Zhangzhen He,[10] S. Kimura,[11] Rong Yu,[2] Weiqiang Yu,[2,†] Jie Ma,[12,‡] and Jianda Wu[1,13,§]



A look into the experiments

The Yang-Lee zeros

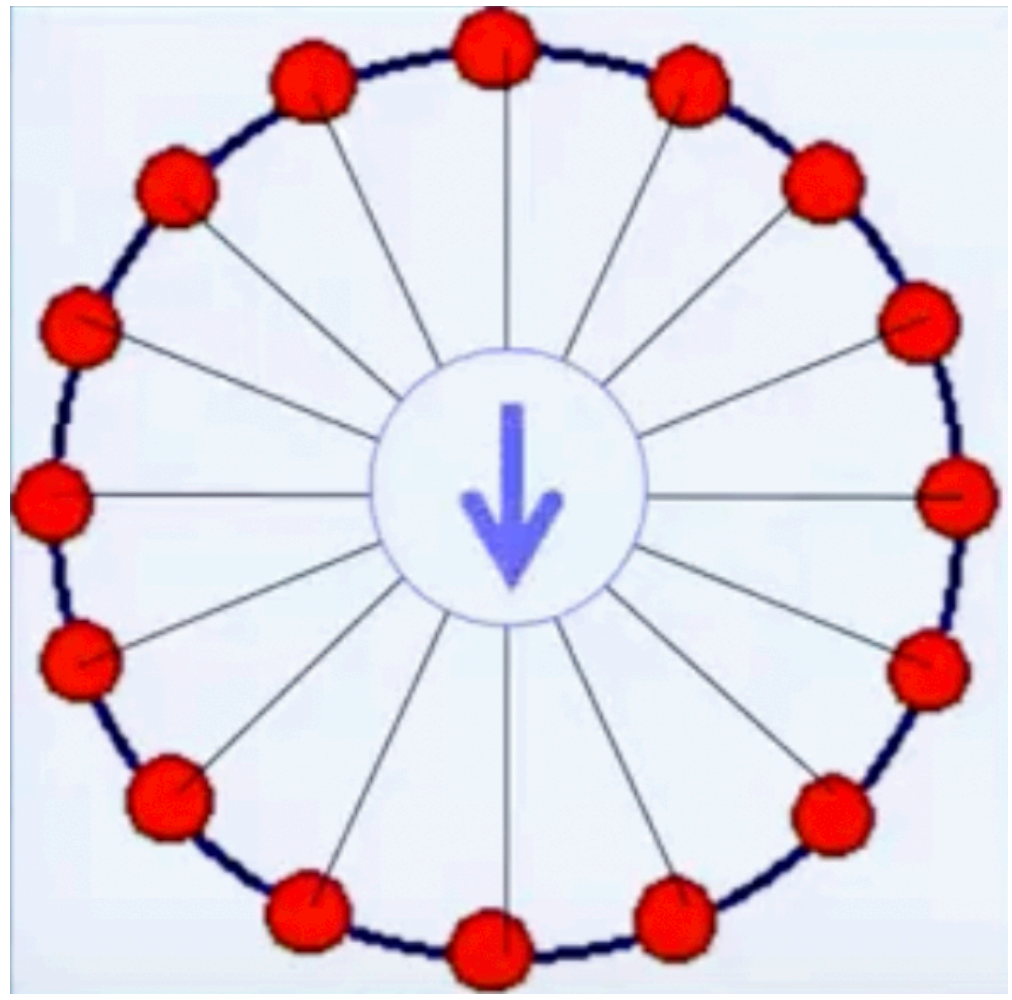
Ising

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j + H \sum_i \sigma_i$$

$+$

coupled with an external spin S

$$\lambda S \sum_i \sigma_i$$



The Spin S can fluctuate

$$|\psi(0)\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}} \otimes |E\rangle$$

$$\langle \psi(t) | S_x | \psi(t) \rangle = \prod_n \frac{\left(e^{-2\beta h + 4i\lambda t} - z_n\right)}{e^{-2\beta t} - z_n}$$

YANG-LEE ZEROS

A look into the experiments

The Yang-Lee zeros

Experimental Observation of Lee-Yang Zeros

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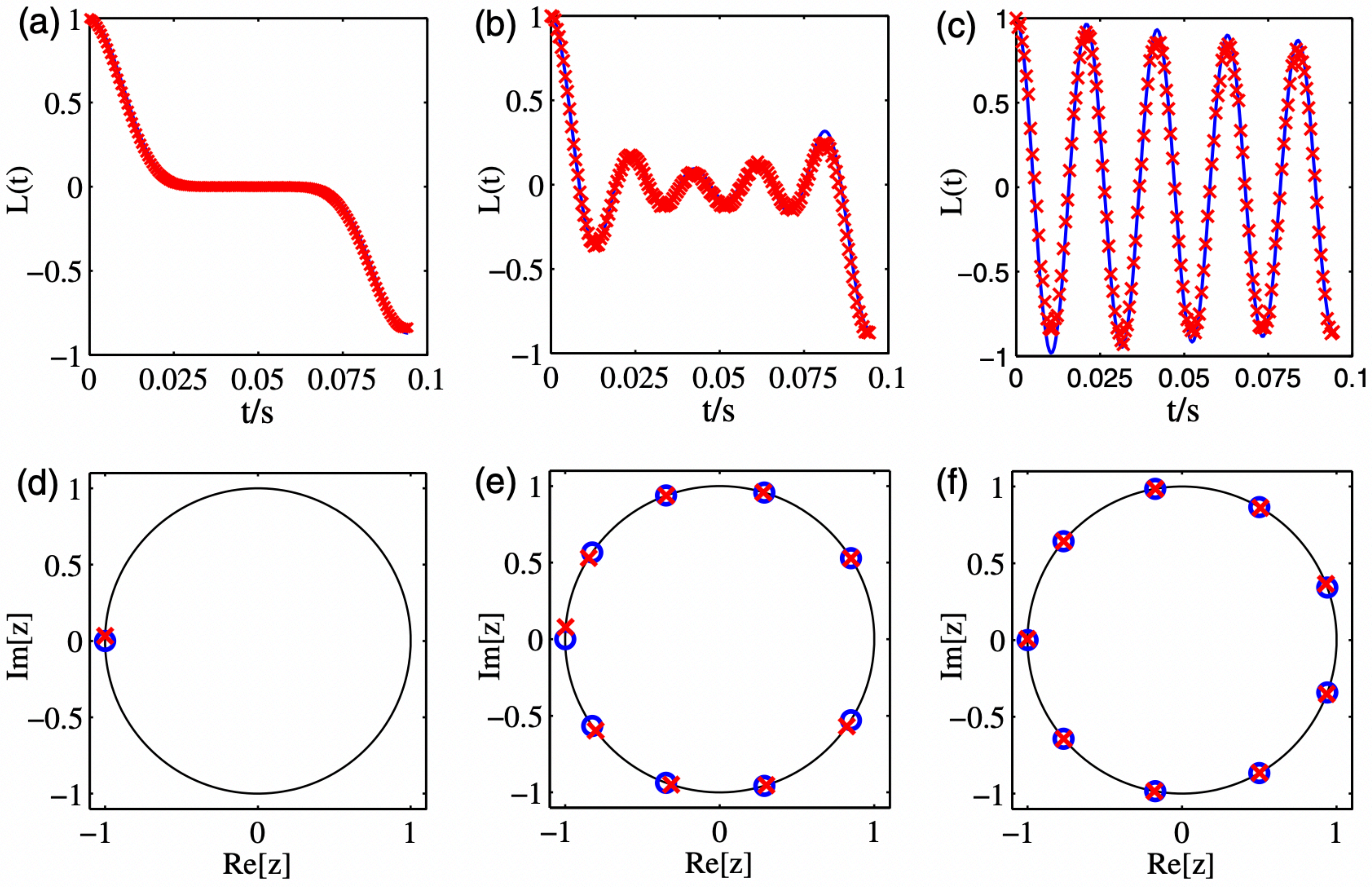

YANG-LEE ZEROS