Hexagonalization of Wilson Loops

History, Challenges and Perspectives

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DESY Theory Workshop 2022

Hamburg, 29.09.2022

Deutsche Forschungsgemeinschaft

$\mathcal{N} = 4$ SYM

Just a fast introduction to $\mathcal{N} = 4$ SYM:

$$\text{“}\mathcal{N} = 4 \text{ SYM”} \equiv \left(\overbrace{\mathfrak{g} \equiv \mathfrak{su}(N)}^{\text{Lie algebra}}, \overbrace{A, \phi, \lambda, \psi}^{\text{fields}}, \overbrace{g_{\text{YM}}, \tau_{\text{YM}}}^{\text{parameters}}, \overbrace{\mathscr{L}_{\mathcal{N}=4}}^{\text{Lagrangian Mink}_4 \text{ 4-form}} \right) \tag{1}$$

- $\mathfrak{su}(N)$ gauge theory with matter
- $\mathfrak{psu}(2,2|4)$ invariant
- superconformal at the quantum level [Sohnius and West, 1981] [Seiberg, 1988]

Solve the theory? $(\{\mathscr{G}^M(x)\})$

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$\mathcal{N} = 4$ SYM Dynamics

> Use the conformal algebra $\mathfrak{so}(2,4)$ to constrain the correlation functions

$$\langle \mathcal{O}_i(x)\,\mathcal{O}_j(y)\rangle \propto d_{xy}^{-2\Delta_i}, \quad \langle \mathcal{O}_i(x)\,\mathcal{O}_j(y)\,\mathcal{O}_k(z)\rangle \propto \frac{\mathtt{C}_{ijk}}{d_{xy}^{\Delta_i+\Delta_j-\Delta_k}\, d_{yz}^{\Delta_j+\Delta_k-\Delta_i}\, d_{zx}^{\Delta_k+\Delta_i-\Delta_j}} \tag{2}$$

Data of the theory

$$\text{“}\mathcal{N} = 4 \text{ SYM”} = \{\Delta_i, \mathtt{C}_{ijk}\} \tag{3}$$

where $\mathcal{D} \bullet \mathcal{O}_i = \Delta_i \mathcal{O}_i$

$\mathcal{N} = 4$ SYM Dynamics

> Consider the planar limit ['t Hooft, 1974]

Definition

Let $\mathcal{N} = 4$ SYM be the theory defined above. We call *planar limit* of the theory the following:

$$N \to \infty, \; g_{\text{YM}} \to 0$$
$$\lambda \equiv g_{\text{YM}}^2 N \text{ finite} \tag{4}$$

$$\mathcal{G}^M(x) = \sum_m \frac{1}{N^m} \mathcal{G}_m(\lambda) = \text{“genus } m\text{” diagrams}$$

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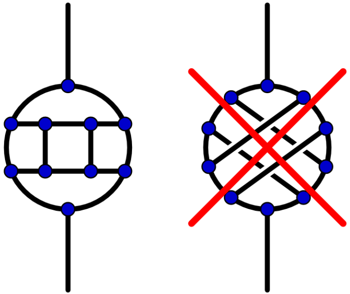
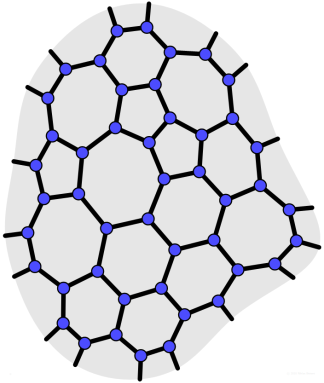
$\mathcal{N} = 4$ SYM Dynamics

We focus on $m = 0$ (strictly "planar") [Beisert *et al.*, 2010]





Integrable 2PFs

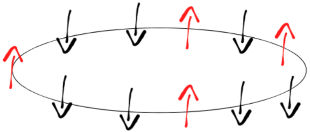
For $\triangle_i$ we have **nice news** [Beisert, 2005]:

$$\mathcal{D} \bullet \mathcal{O} \equiv \left(\overbrace{\Gamma^{(0)}}^{\text{trivial}} + \overbrace{\Gamma^{(1)}}^{\text{“one-loop” anom. dimensions}} + \ldots \right) \bullet \mathcal{O} =$$

$$\triangle^{(0)} \mathcal{O} + \mathbf{H}_{\mathfrak{psu}} \odot \mathcal{O} + \ldots$$

The dynamics of the spin chain is encoded in a 2-particle S-matrix, S_{Beisert}

$$\langle ZZWZWZZWZZW \rangle \Rightarrow$$



$W =$ “magnon”

Integrable 3PFs

What about c_{ijk}?

Thanks to the AdS/CFT correspondence [Maldacena, 1997], [Gubser, Klebanov, Polyakov, 1998], [Witten, 1998] we have naturally a string dual to three-point functions:

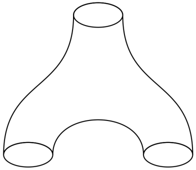


Figure: Classical “pair of pants”

Integrable 3PFs

In our setting (AdS/CFT/Spin chains) we can see the 3PF as the observable describing the scattering of three spin chains, each with its own excitations (“magnons”) that can propagate.

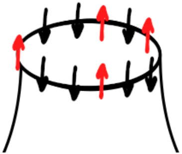


Figure: We attach a spin chain on every operator

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The so-called BKV prescription [Basso, Komatsu, Vieira, 2015] consists in the *cutting* of the world-sheet in two hexagonal patches

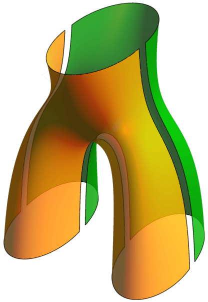

Figure: BKV cutting of the worldsheet

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$$\mathfrak{psu}(2,2)^2 \to \mathfrak{psu}(2,2)_{\text{diag}} \simeq \mathfrak{psu}(2,2) \qquad (5)$$

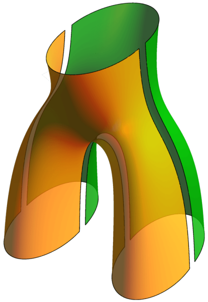


Figure: BKV cutting of the worldsheet

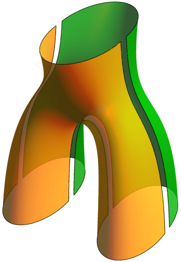
Integrable 3PFs

How can we reconstruct the structure constant?

- We have a “real”, physical set of excitations of the a single spin chain, that we can group as a bi-partition $(\alpha, \overline{\alpha})$

BKV Conjecture

$$\mathtt{C}_{ijk} \equiv \sum\!\!\!\!\!\!\int_{\beta} \sum_{\{\alpha\},\{\overline{\alpha}\}\ \text{partitions}} w(\alpha, \overline{\alpha}) \mathcal{H}^{\{\alpha,\beta\}} \mathcal{H}^{\{\overline{\alpha},\beta\}} \tag{6}$$



Integrable 3PFs

How can we reconstruct the structure constant?

- We have a set β of "mirror", virtual excitations arising from the cutting/gluing procedure

BKV Conjecture

$$C_{ijk} \equiv \sum\!\!\!\!\!\!\int_{\beta} \sum_{\{\alpha\},\{\overline{\alpha}\}\text{ partitions}} w(\alpha,\overline{\alpha})\,\mathcal{H}^{\{\alpha,\beta\}}\,\mathcal{H}^{\{\overline{\alpha},\beta\}} \qquad (7)$$

Integrable 3PFs

A new conjecture

The central takehome message here is that:

$$\mathtt{C}_{ijk} \Leftrightarrow \begin{array}{c}\text{“tesselation”}\\ \text{of a Riemann surface}\end{array} =$$

product of “bootstrapable” hexagonal form factors

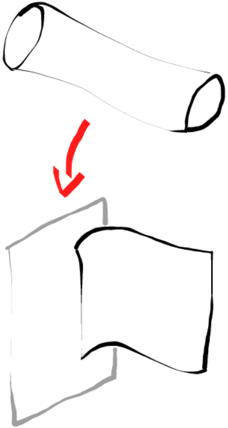
Definition

Let $\mathfrak{h}^{\mathtt{A}_1\dot{\mathtt{A}}_1\ldots\mathtt{A}_N\dot{\mathtt{A}}_N}$ be the creation amplitude for N magnons on a single hexagon edge.
We have:

$$\mathfrak{h}^{\mathtt{A}_1\ldots\mathtt{A}_N} = c_{\text{fermion}}\, c_{\text{dynamic}} \left\langle \dot{\mathtt{A}}_N \ldots \dot{\mathtt{A}}_1 \middle| \mathcal{S}_{\text{Beisert}} \middle| \mathtt{A}_1 \ldots \mathtt{A}_N \right\rangle$$

Wilson Loops

Question [Kim, Kiryu, 2018]: can we extend the hexagon procedure to an open string setting?



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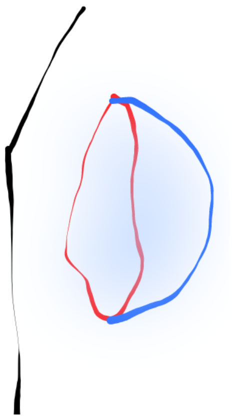
We can consider an open string worldsheet ending on $\partial \mathrm{AdS}_5 \simeq \mathrm{Mink}_4$ describing a loop $\mathcal{C}$ (dual to a Wilson loop, [Maldacena, 1998])



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Definition

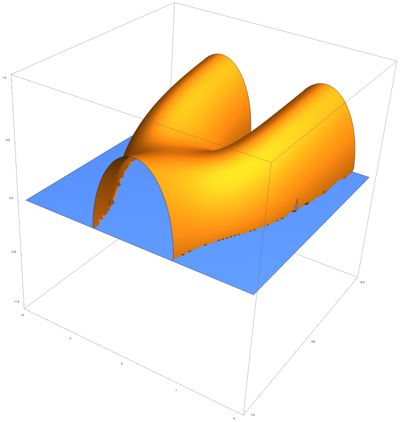
Let $\mathcal{C} \subset \mathrm{Mink}_4$ a closed path. Considering the space of fields of $\mathcal{N} = 4$ SYM, we define as $1/2$-BPS Wilson loop the following quantity:

$$W[\mathcal{C}] \equiv \mathrm{P}\exp\left(\oint_{\mathcal{C}} \imath A \cdot \mathrm{d}s + \vec{\phi} \cdot \vec{\mathbf{n}} |\mathrm{d}s|\right), \qquad (8)$$

$$\vec{\phi} \cdot \vec{\mathbf{n}} = \phi^j \delta_j{}^6 = \phi^6$$

Wilson Loops

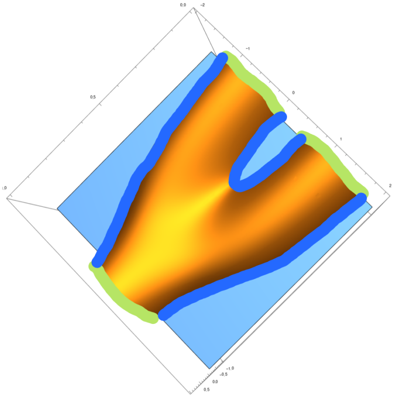
We know [Okamura, Takayama, Yoshida, 2005], [Drukker, Kawamoto, 2006] that Wilson loop's correlators admit an open **spin chain** representation!



Wilson Loops

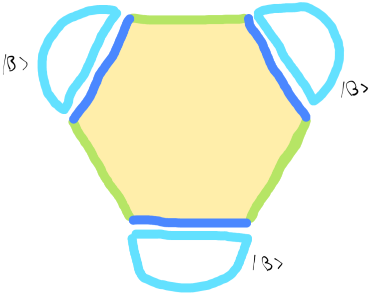
What is the translation of BKV cutting in the open string setting?

We have three edges that are physical (**end** of the OS) and three edges that are associated to the **propagation** of the OS.



Wilson Loops

(open) cutting $= |\mathcal{B}\rangle$-contraction



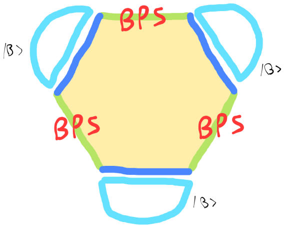
Wilson Loops

BKV Conjecture revisited [Kim, Kiryu, 2017], [Kiryu, Komatsu, 2018]

$$|\mathcal{B}\rangle \equiv \sum_k b_k \, |\psi_k\rangle \equiv \exp\left(\frac{1}{2}\int \mathbf{K}(q)\, a_q^\dagger \, a_{-q}^\dagger\right) |0\rangle$$

$$\mathsf{C}_{ijk}^{(\mathrm{BPS})} \equiv \sum\!\!\!\!\!\!\int_{\beta} b_{\beta_1} b_{\beta_2} b_{\beta_3} \mathcal{H}^{\{\beta\}}$$

where q is the mirror momentum and $\mathbf{K}(q)$ is the analitically continued reflection matrix (**R**)



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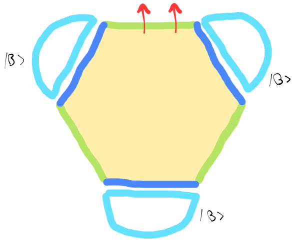
$$\mathcal{C}^{\bullet\circ\circ}_{ijk} \equiv \cancel{\sum_{\beta}} b_{\beta_1} b_{\beta_2} b_{\beta_3} \sum_{\{\alpha^+\},\{\alpha^-\}\ \text{partitions}} \overbrace{w^{\mathfrak{Re}}(\alpha^+,\alpha^-)}^{\textbf{R}\text{ here!}}\, \mathcal{H}^{\{\alpha^+,\,\alpha^-\ \beta\}}$$



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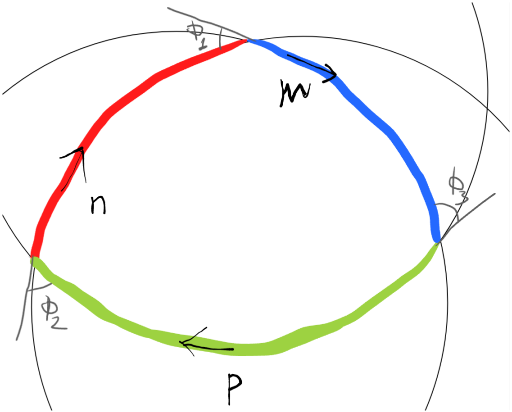
> [Cavaglià, Gromov, Levkovich-Maslyuk, 2018] analysed a three-cusped Wilson loop in the Q-functions framework (see [Gromov, Kazakov, Leurent, Volin, 2013]), encountering massive simplifications

Definition

Let $\{\theta_i\ ;\ \phi_i\}$ be the internal and physical angles defining the cusp and $g \equiv \frac{\sqrt{\lambda}}{4\pi}$ the 't Hooft coupling. We define as *ladder limit* the following:

$$\theta_i \to \imath\infty, \ g \to 0$$

$$\hat{g}_i \equiv \frac{g}{2} e^{-\imath\theta_i/2} \text{ finite}$$



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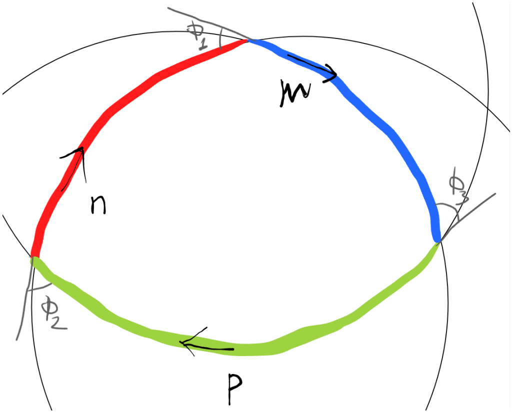
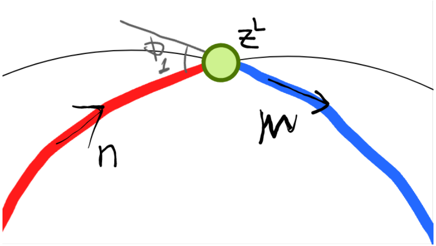


Wilson Loops

Learn from [Correa, Maldacena, Sever, 2012], [Drukker, 2012]

- Consider a Wilson loop with one insertion $W[\text{Tr}\left(Z^L\right)(0)]$, in the large L limit: this form an **open spin chain vacuum**
- Fix the **R** matrix
- "Open-closed string duality" (space-time flip to the mirror theory):

$$\text{boundary condition} \to |\mathcal{B}\rangle$$



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$$\mathbf{R} = R_0(p)\hat{\mathcal{S}}_{\text{Beisert}}$$

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$$\mathbf{R} = \frac{1}{\sigma_B(p)\sigma(p,-p)} \frac{1+(x^-)^2}{1+(x^+)^2} \hat{S}_{\text{Beisert}}$$

$$\sigma_B(p) = e^{\imath\chi(x^+)-\imath\chi(x^-)}$$

$$\chi(x) = \oint \frac{\mathrm{d}z}{2\pi\imath} \frac{1}{x-z} \log\left(\frac{\sinh\left(2\pi g(z+\frac{1}{z})\right)}{2\pi g(z+\frac{1}{z})}\right)$$

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$$\mathbf{K}^{A\dot{B}C\dot{D}}(q) = \left(\mathbf{R}^{-1}(z^{\pm})\right)^{A\dot{B}}{}_{\mathrm{E}\dot{\mathrm{F}}}\, \mathcal{C}^{\mathrm{E}\dot{\mathrm{F}}C\dot{D}}$$

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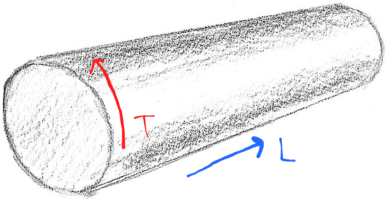

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- At this point introduce two **cusp** angles $\{\theta, \phi\}$ that rotate one of the two boundary
- Compute the ground state energy trough TBA ansatz ([Yang, Yang, 1969], [Zamolodchikov, 1990], [Dorey, Tateo, 1996], ...)
- Take the $L \to 0$ limit to get the cusp anomalous dimension

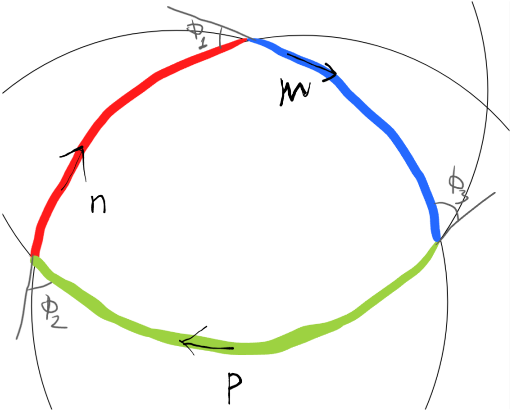
$$\mathbf{m} = \text{Diag}\left(e^{\imath\theta}, e^{-\imath\theta}, e^{\imath\phi}, e^{-\imath\phi}\right)$$

$$\mathbf{R}^{A\dot{B}}{}_{C\dot{D}} \mapsto \left(\mathbf{m}^{-1}\right)^{A}{}_{\mathtt{E}}\mathbf{m}^{\mathtt{F}}{}_{C}(\mathbf{R})^{\mathtt{E}\dot{B}}{}_{\mathtt{F}\dot{D}}$$

Wilson Loops

In the three-cusped Wilson loop, we have **three** different boundary states $|\mathcal{B}\rangle_i \equiv \mathcal{B}(\theta_i, \phi_i)$, each one specified by $\mathbf{K}(q, \theta_i, \phi_i)$ of [Correa, Maldacena, Sever, 2012], [Drukker, 2012]

Point of view of [Kim, Kiryu, Komatsu, Nishimura, 2017]: **no** local operators, structure constant of boundary changing operators



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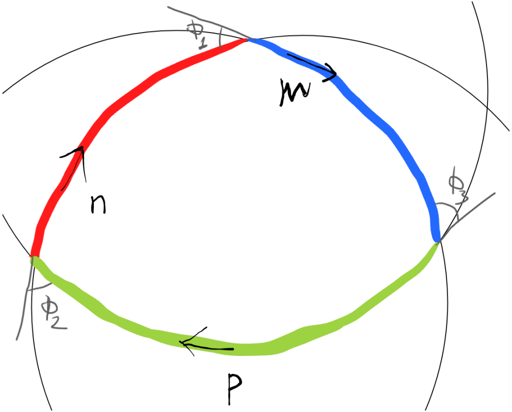

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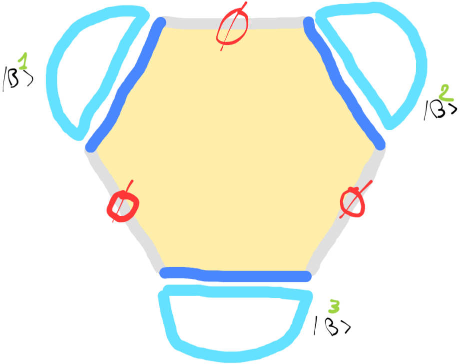
PURE **MIRROR** THEORY!



Perspectives

Some **starting point**:

- The [Cavaglià, Gromov, Levkovich-Maslyuk, 2018] results are obtained in the ladder limit
- The QSC perspectives underlined a profound difference between $C^{\bullet\circ\circ}$ (and its generalization $C^{\bullet\bullet\circ}$) and $C^{\bullet\bullet\bullet}$

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Some interesting **features** of the hexagon approach:

- ☐ The hexagon computations does not rely from the beginning on this limit
- ☐ From the hexagon point of view, turning one, two or three $\hat{g}_i$ is not a profound change (some effects have (?) to appear in the resummation)

Thank you!

Contact

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SYM (i)

> Gauge theory

- $(\mathfrak{g} \equiv \text{Lie}\,(G)\,, \langle\cdot,\cdot\rangle)$, “(Gauge) fields” $A \equiv \sigma\,(\mathcal{P})$, with $\pi_{(A)}\ :\ \mathcal{P} \to \mathcal{M}$

> ... with some matter

- $X \mid \text{Iso}\,(X) = G$, “Fields” $\phi \equiv \sigma\,(\mathcal{P} \times_G X)$, with $\pi_{(\phi)}\ :\ \mathcal{P} \times_G X \to \mathcal{M}$
- “Potential” $\equiv V\ :\ X \to \mathbb{R}$, G-invariant map

SYM (ii)

... that is also *Super*!

> We can introduce "dual fields":

- Dual $(A) \equiv \lambda \equiv \sigma \left(P \times_G \left(\mathfrak{g} \otimes \Pi\mathbb{F}^{2*}\right)\right)$
- Dual $(\phi) \equiv \sigma \left(\phi^* \left(P \times_G \left(TX \otimes_{\mathbb{F}} \Pi\mathbb{F}^{2*}\right)\right)\right)$

SYM (iii)

With these ingredients we can construct a *supersymmetric* lagrangian:

$$\mathscr{L} = \left\{ -\frac{1}{2}|F_A|^2 + \langle \mathrm{d}_A\overline{\phi}, \mathrm{d}_A\phi \rangle - \phi^* \|\mathrm{grad} W\|^2 - 2\phi^*|\mu|^2 + \textbf{fermions} \right\}, \qquad V = \|W\|^2 + 2|\mu|^2 \tag{9}$$

$\mathcal{N} = 4$ SYM (i)

We specialize to:

$$\text{“}\mathcal{N} = 4 \text{ SYM”} \equiv \left(\overbrace{\mathfrak{g} \equiv \mathfrak{su}(N)}^{\text{Lie algebra}}, \overbrace{A, \phi, \lambda, \psi}^{\text{fields}}, \overbrace{g_{\text{YM}}, \tau_{\text{YM}}}^{\text{parameters}}, \overbrace{\mathscr{L}_{\mathcal{N}=4}}^{\text{Lagrangian Mink}_4 \text{ 4-form}} \right) \tag{10}$$

Theorem

$\text{Lie}_\xi \mathscr{L}_{\mathcal{N}=4}$ (with $\xi \in \mathfrak{psu}(2,2|4)$) is d-exact

$\mathcal{N} = 4$ SYM (ii)

$$
\text{“}\mathcal{N} = 4 \text{ SYM”} \equiv \left(\overbrace{\mathfrak{g} \equiv \mathfrak{su}(N)}^{\text{Lie algebra}}, \overbrace{A, \phi, \lambda, \psi,}^{\text{fields}} \quad \overbrace{\mu_{\mathcal{D}} \equiv \prod_{\text{fields}} \mathcal{D}\text{field}}^{\text{integral measure over the field space}} \right.
$$
$$
\left. \overbrace{\text{b}, \ldots}^{\text{ghost fields}} , \overbrace{g_{\text{YM}}, \tau_{\text{YM}},}^{\text{parameters}} \quad \overbrace{\hat{\mathscr{L}}_{\mathcal{N}=4}}^{\text{(quantum) Lagrangian Mink}_4 \text{ 4-form}} \right) \tag{11}
$$

Theorem

$\text{Lie}_{\xi} \mu_{\mathcal{D}} = 0$ (with $\xi \in \mathfrak{psu}(2, 2|4)$)

Planar Limit

> Consider the planar limit ['t Hooft, 1974]

- $\mathscr{G}^M(x) \equiv \sum_\ell \frac{1}{N^\ell} \sum_I f_{I\ell}(x)\lambda^I \in \mathbb{R}\left(\!\left(\lambda, \frac{1}{N}\right)\!\right)$
- $\mathscr{G}^M(x) \equiv \sum_\ell \frac{1}{N^\ell} \mathscr{G}_\ell(\lambda)$ where we interpret $\mathscr{G}_\ell(\lambda)$ as the sum of all the correlation functions associated the Feynman diagrams that can be drawn on a genus ℓ surface

Planar Limit

If we identify $\mathsf{g}_{\text{string}} = N^{-1}$, we have

$$\mathscr{G}^{M}(x) \equiv \sum_{m} \mathsf{g}_{\text{string}}^{m} \mathscr{G}_{m}(x) \qquad (12)$$

We are dealing with free strings in a curved background!

This is just one example of the celebrated AdS/CFT duality [Maldacena, 1997], [Gubser, Klebanov and Polyakov, 1998], [Witten, 1998]

$$\overbrace{\mathscr{Z}_{\text{strings}}\left[\phi, \phi|_{\partial\mathcal{M}} \equiv \phi_0\right]}^{\text{string theory partition function}} = \overbrace{\langle \exp(-\phi_0 \cdot \mathcal{O}) \rangle_{\text{CFT},\partial\mathcal{M}}}^{\text{generator of connected CFT correlators}}$$

Planar $\mathcal{N} = 4$ SYM and Integrability

For $\ell = 0$, $M = 2$ we have:

$$\mathcal{O} = \prod_{i=1}^{n} \text{Tr}\left(\prod_j \mathcal{D}^j \text{Field}_j\right) \tag{13}$$

$$\mathcal{D} \bullet \mathcal{O} \equiv \left(\overbrace{\boldsymbol{\Gamma}^{(0)}}^{\text{trivial, from dimensional analysis}} + \overbrace{\boldsymbol{\Gamma}^{(1)}}^{\text{“one-loop” anomalous dimensions}} + \ldots \right) \bullet \mathcal{O} = \tag{14}$$

$$\Delta^{(0)}\mathcal{O} + \mathbf{H}_{\mathfrak{psu}} \odot \mathcal{O} + \ldots$$

Planar $\mathcal{N} = 4$ SYM and Integrability

Theorem [Minahan and Zarembo, 2002]

Let $\mathcal{O}$ be a single-trace operator of the following form:

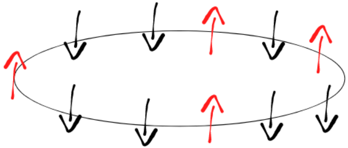
$$\mathcal{O}_{i_1,\ldots,i_m} \propto \mathrm{Tr}\left(\Phi_{i_1}\ldots\Phi_{i_m}\right) \tag{15}$$

where each Φ is a LC of scalar fields and let $\Gamma^{(1)}$ the one-loop anomalous dimension matrix. We have:

$$\Gamma^{(1)} = \mathbf{H}_{\mathfrak{so}(6)} \tag{16}$$

where $\mathbf{H}_{\mathfrak{so}(6)}$ is the hamiltonian of a SO(6) spin-chain with m sites in one dimension.

$$\langle ZZWZWZZWZZW\rangle \quad \Rightarrow$$



W = "magnon"

Planar $\mathcal{N} = 4$ SYM and Integrability

Theorem [Minahan and Zarembo, 2002]

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where $\mathbf{H}_{\mathfrak{so}(6)}$ is the hamiltonian of a $\mathrm{SO}(6)$ spin-chain with m sites in one dimension.

The model is integrable, we have that the whole dynamics is encoded in a **two**-particle S-matrix, *i.e.*:

$$\begin{aligned} \mathcal{S}_{\mathfrak{so}(6)} \; &: \; \mathrm{Mod}_{\mathfrak{so}(6)} \otimes \mathrm{Mod}_{\mathfrak{so}(6)} \\ &\to \mathrm{Mod}_{\mathfrak{so}(6)} \otimes \mathrm{Mod}_{\mathfrak{so}(6)} \end{aligned} \tag{17}$$

i.e. an intertwiner operator between fundamental modules

Planar $\mathcal{N} = 4$ SYM and Integrability

Theorem

Let $\mathcal{O}$ be a single-trace operator of the following form:

$$\mathcal{O}_{i_1,\dots,i_m} \propto \text{Tr}\left(\Phi_{i_1} \dots \Phi_{i_m}\right) \tag{18}$$

where each Φ is a LC of **the whole $\mathfrak{psu}(2,2|4)$ multiplet** and let $\mathbf{\Gamma}^{(1)}$ the one-loop anomalous dimension matrix. We have:

$$\mathbf{\Gamma}^{(1)} = \mathbf{H}_{\overline{\mathfrak{psu}}(2,2)^2} \tag{19}$$

where $\mathbf{H}_{\overline{\mathfrak{psu}}(2,2)^2}$ is the hamiltonian of a spin-chain with centrally extended $\mathfrak{psu}(2,2)^2$ symmetry with m sites in one dimension.

[Beisert, 2005]

$$\mathfrak{psu}(2,2|4) \to$$
$$(\mathfrak{psu}(2,2))^2 \ltimes \mathfrak{u}(1) \hookrightarrow$$
$$(\mathfrak{psu}(2,2))^2 \ltimes \mathbb{R}^3$$

Planar $\mathcal{N} = 4$ SYM and Integrability

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The centrally extended algebra knows about E and g!

Planar $\mathcal{N} = 4$ SYM and Integrability

Theorem

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[Beisert, 2005]

$$\mathcal{S}_{\text{Beisert}} : \text{Mod}_{\mathfrak{psu}} \otimes \text{Mod}_{\mathfrak{psu}} \to \text{Mod}_{\mathfrak{psu}} \otimes \text{Mod}_{\mathfrak{psu}}$$

Wilson Loops

Fact [Drukker, Kawamoto, 2006]: computations of Wilson loop deformations (or insertions) can be mapped to a spin chain!

This fact was an evidence of previous physical situations, when some dofs were added to $\mathcal{N}=4$ SYM ...

Theorem [Okamura, Takayama, Yoshida, 2005]

Let $\mathscr{L}_{\mathcal{N}=4}+\mathscr{L}_{\text{defect}}$ be the total lagrangian form for the dCFT coupled to $\mathcal{N}=4$ SYM that is dual to $\text{AdS}_5\times S^5$ bisected by an $\text{AdS}_4\times S^2$ brane. It follows that:

$$\mathbf{\Gamma}^{(1)}_{\text{SU}(2)}=\mathbf{H}^{(\text{open})}_{\text{Heis}}+2\left(1-\hat{\mathbf{Q}}^W_1-\hat{\mathbf{Q}}^W_L\right) \tag{20}$$

Wilson Loops

Fact [Drukker, Kawamoto, 2006]: computations of Wilson loop deformations (or insertions) can be mapped to a spin chain!

... but in $\mathcal{N} = 4$ we naturally have Wilson loops, so nothing more is needed:

Theorem [Drukker, Kawamoto, 2006]

Let $\mathscr{L}_{\mathcal{N}=4}$ be the total lagrangian form for $\mathcal{N} = 4$ SYM and let $W[\mathcal{O}_1 \dots \mathcal{O}_m]$ be the insertion of m $\mathsf{SU}(2)$ local operators on a $1/2$-BPS Wilson loop. It follows that:

$$\mathbf{\Gamma}^{(1)}_{\mathsf{SU}(2)} = \mathbf{H}^{(\mathsf{open})}_{\mathsf{Heis}} + 2\left(1 - \hat{\mathbf{Q}}^{\phi^6}_1 - \hat{\mathbf{Q}}^{\phi^6}_L\right) \tag{21}$$