What is the correct definition of entropy for general relativistic field theory?

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Entropy

→ ”en” + “tropy”

[Clausius, 1865]

“energy” “τροπή”(Greek)

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Rudolf Julius Emmanuel Clausius
Germany, 1822-1888

→ Entropy allows us to describe the laws of thermodynamics most concisely.

I. (Energy conservation)

$$TdS = dU + pdV$$

II. (Monotonic increase of entropy)

$$dS \geq 0$$

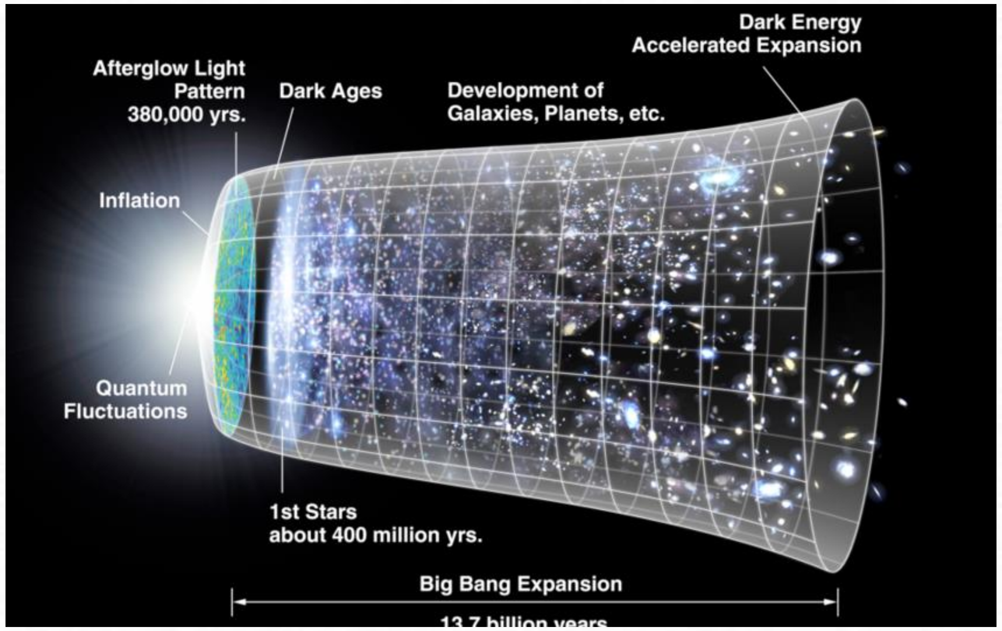
III. (Nernst-Planck’s theorem)

$$\lim_{T \to 0} S = 0$$

These are basic tools to study thermodynamic equilibrium system!

Local thermodynamic equilibrium (LTE)

There are systems in which thermodynamic equilibrium is achieved locally.



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Isotropic homogeneous expanding universe

Astronomical bodies

A precise analysis of these systems needs General Relativity.

Q1 What is the definition of entropy for a system in **curved spacetime**?

Q2 How are laws of thermodynamics modified in **curved spacetime**?

What is difficult in curved spacetime?

The definition of (total) energy for field theory on flat spacetime

$$E = \int_{R^3} d^3x \, T^{00}(t, \vec{x}) \qquad g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix} -1 & \vec{0} \\ \vec{0} & 1_3 \end{pmatrix}$$

$\partial_\mu T^{\mu\nu} = 0$ → Energy is conserved (time independent).

↓

Q. What is the definition of energy on curved spacetime?

$$E = ???\qquad g_{\mu\nu}(x) \neq \eta_{\mu\nu} = \begin{pmatrix} -1 & \vec{0} \\ \vec{0} & 1_3 \end{pmatrix}$$

→ The continuity equation changes into the 'covariant' conservation equation.

$$\Rightarrow \quad \nabla_\mu T^{\mu\nu} = 0 \quad \leftrightarrow \quad \partial_\mu T^{\mu\nu} = -\Gamma^\mu_{\mu\sigma} T^{\sigma\nu} - \Gamma^\nu_{\mu\sigma} T^{\mu\sigma}$$

What is the correct guiding principle to define energy?

There is a long history on this issue and remain several proposals.

"pseud-tensor"

[Einstein '16]
[Landau-Lifshitz '47, '75]

"quasi-local energy"

[ADM '62] [Bondi '62] [Brown-York '92] [Hawking-Horowitz '95]
[Horowitz-Mayers '98] [Balasubramanian-Kraus '98][Ashtekar-Das '98]...

"Komar mass"

[Komar '62]

Plan

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Our proposal of definition of energy

[Aoki-Onogi-SY '20]

$$E = \int_{\Sigma_t} d^3\vec{x}\,\sqrt{|g|}\,T^0{}_\mu(t,\vec{x})\,n^\mu(t,\vec{x})$$

n^μ **Time evolution vector field**

Σ_t **Time slice at an arbitrary time x^0=t**

g **Determinant of the metric in the total spacetime**

Comments

- This expression was written in the old textbook of Fock.

The quantity $$I = \int T^{\mu 0}\varphi_\mu \sqrt{(-g)}\cdot dx_1\, dx_2\, dx_3 \qquad (49.07)$$
will be constant, i.e. will be independent of x_0, the coordinate that has the character of time, if the vector φ_μ satisfies the equations
$$\nabla_\nu \varphi_\mu + \nabla_\mu \varphi_\nu = 0 \qquad (49.08)$$

[V. Fock, *The Theory of Space, Time and Gravitation* 1959]

Cf. [Trautman 2002]

- This is **manifestly invariant under general coordinate transformation**.
- This **reduces to the original definition in the flat limit**.

$$E \xrightarrow{\;g_{\mu\nu}(x)\to\eta_{\mu\nu}\;} E = \int_{R^3} d^3x\, T^{00}(t,\vec{x})$$

- This **reproduces the masses of well-known black holes**.

Extension to a general charge

$$Q[v] = \int_{\Sigma_t} d^{d-1}\vec{x}\,\sqrt{|g|}\,T^0{}_\mu(t,\vec{x})v^\mu(t,\vec{x})$$

[Aoki-Onogi-SY '20]

$v^\mu = n^\mu$ Time evolution ➡ $Q[n] = E$ **Energy**

$v^\mu = \delta^\mu_{(i)}$ Translation for i-th direction ➡ $Q[\delta_{(i)}] = P^i$ **Momentum**

This charge **conserves** when v=ξ is a **Killing vector field.**

$\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0$ → Q[ξ] is **a Neother charge.**

Q. Is there any case for Q[v] to conserve unless v is a Killing vector?

A. **YES** if EM tensor is covariantly conserved $\nabla_\mu T^{\mu\nu} = 0$

and ∃ a vector field to satisfy $T^\mu_\nu \nabla_\mu \zeta^\nu = 0$

→ $\partial_\nu s^\nu = 0$ where $s^\nu = \sqrt{|g|}T^\nu{}_\mu\zeta^\mu$

A wider class of conserved charges including Neother charge! Cf. [Kodama '80]

A new conserved charge

$$Q[\zeta] = \int_{\Sigma_t} d^{d-1}\vec{x}\,\sqrt{|g|}\,T^0{}_\mu(t,\vec{x})\,\zeta^\mu(t,\vec{x}) \qquad T^\mu{}_\nu \nabla_\mu \zeta^\nu = 0$$

[Aoki-Onogi-SY '20]

Q. Is there any physical meaning of the new conserved charge?

Claim

$Q[\zeta]$: entropy, $s^\nu = \sqrt{|g|}\,T^\nu{}_\mu \zeta^\mu$: entropy current

by finding the vector field ζ suitably.

Intuitive argument

Theory of gravity is fundamental and reversible. Entropy must be conserved.

(If this interpretation is not correct, then what is the physical meaning of this conserved quantity?)

Evidence

This interpretation leads to the **local Euler's relation** and the **1st law of thermodynamics** for several well-known gravitational systems.

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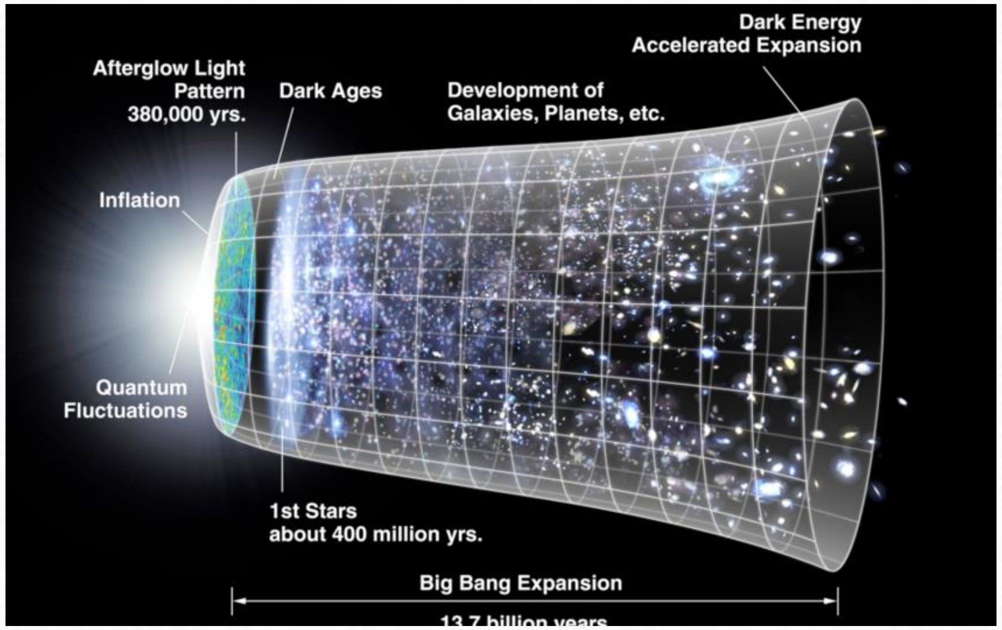
Application 1: FLRW model

[Aoki-Onogi-SY Int.J.Mod.Phys.A 36 (2021) 29, 2150201]



Isotropic homogeneous expanding universe

FLRW model

1 dynamical variable

Homogeneous & isotropic system $$ds^2 = -(dt)^2 + a(t)^2\, \tilde{g}_{ij} dx^i dx^j$$

LTE → Perfect fluid $$T^{\mu}{}_{\nu} = (\rho + P) u^{\mu} u_{\nu} + P \delta^{\mu}_{\nu}$$

Einstein eq: $$\rho = \frac{1}{8\pi G_N}\left(\frac{(d-1)(d-2)}{2}\frac{k + \dot{a}^2}{a^2} - \Lambda\right) \qquad P = \frac{1}{8\pi G_N}\left((2-d)\left(\frac{\ddot{a}}{a} + \frac{d-3}{2}\frac{k+\dot{a}^2}{a^2}\right) + \Lambda\right)$$

$\zeta^{\nu} = -\beta u^{\nu}$

$$T^{\mu}{}_{\nu} \nabla_{\mu} \zeta^{\nu} = 0 \;\Rightarrow\; \rho u^{\mu} \nabla_{\mu} \beta = P \theta \beta \;\Rightarrow\; \beta = \beta_0\, e^{-\int_{t_0}^{t} dt \left(\frac{PK}{\rho}\right)}$$

Comoving frame

Expansion

$\rightarrow \theta = d \times H$

entropy density: $s = \sqrt{|g|}\, T^{0}{}_{\mu} \zeta^{\mu} = \sqrt{\tilde{g}}\, a^{d-1} \rho \beta = u\beta$

$s := s^0$

$\beta = 1/T$ $\Rightarrow$ $Ts = u$

Local Euler's relation

$u := \rho v$ internal energy density

$v := \sqrt{\tilde{g}}\, a^{d-1}$ Volume element

Direct calculation

$$\Rightarrow \quad T\frac{ds}{dt} = \frac{du}{dt} + P\frac{dv}{dt}$$

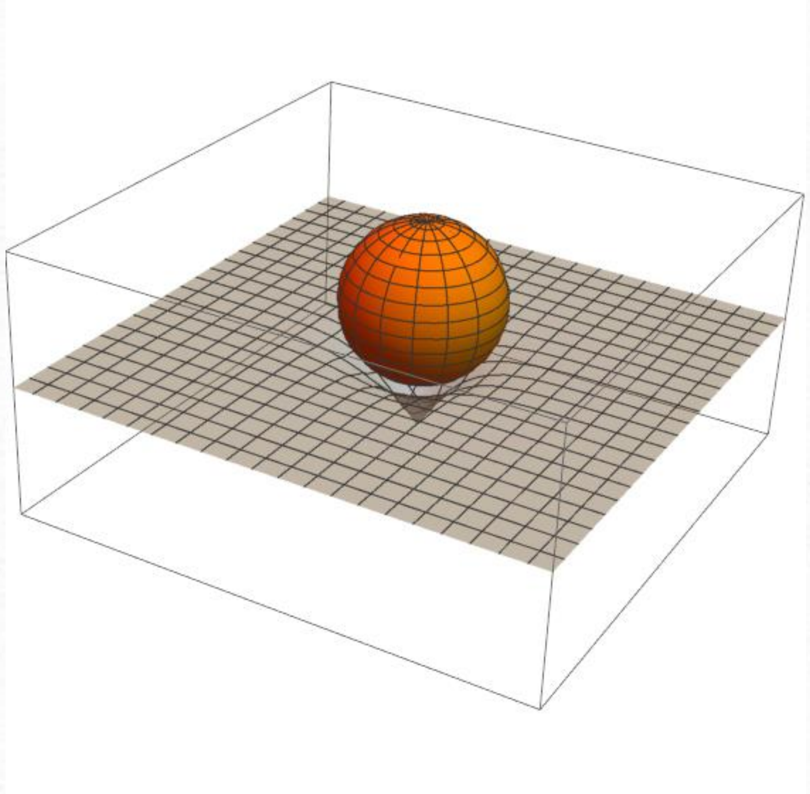
1st law of thermodynamics

<u>Comments</u>

- Energy does not conserve, but entropy does conserves. $T\frac{ds}{dt} = \frac{du}{dt} + P\frac{dv}{dt} = 0$
- The "Big-bang nature" of the universe is inevitable and easily seen.
- These properties hold regardless of any equation of state. Cf. [Kolb-Turner]

Application 2: Spherically symmetric hydrostatic equilibrium

[SY arXiv:2304.06196]



Spherically symmetric hydrostatic equilibrium

2 dynamical variables

Spherically symmetric system $$ds^2 = -f(dt)^2 + h(dr)^2 + r^2\,\tilde{g}_{ij}dx^i dx^j$$

LTE → Perfect fluid $$T^\mu{}_\nu = (\rho + p)u^\mu u_\nu + p\delta^\mu_\nu$$

Comoving frame

Einstein eq ➡ $$p' = -\frac{p+\rho}{2}(\log f)', \quad (\log h)' = \frac{2rh}{d-2}(8\pi G_N\rho + \Lambda) - \frac{(d-3)(h-1)}{r},$$
$$(\log f)' = \frac{2rh}{d-2}(8\pi G_N\rho - \Lambda) + \frac{(d-3)(h-1)}{r}$$

➡ **TOV equation** [Oppenheimer-Volkov '39]

$$T^\mu{}_\nu \nabla_\mu \zeta^\nu = 0 \quad \zeta^\nu = -\zeta u^\nu \quad ➡ \quad \zeta' = -\frac{p}{\rho}\frac{\rho'}{p+\rho}\zeta \quad ➡ \quad \zeta = \beta_0 u^t f\left(1 + \frac{p}{\rho}\right)$$

$$\beta = \frac{1}{T} := \beta_0\sqrt{f}$$

➡ $$s = \sqrt{|g|}T^0{}_\mu\zeta^\mu = \sqrt{f}v\rho\zeta = \sqrt{f}\beta_0(u + vp)$$ ➡ $$Ts = u + pv$$

$$u^t \to 1/\sqrt{f} \quad v := \sqrt{\tilde{g}}hr^{d-1}$$

Local Euler's relation

Direct calculation ➡ $$T\frac{ds}{dr} = \frac{du}{dr} + p\frac{dv}{dr}$$ **1st law of thermodynamics**

<u>Comments</u>

- Entropy density is a constant. $T\frac{ds}{dr} = \frac{du}{dr} + p\frac{dv}{dr} = 0$
- The local temperature $T \propto 1/\sqrt{f}$ is exactly coincident with that derived by Tolman. [Tolman '30] Cf. [MTW][Zel'dovich]
- These hold non-perturbatively in the Newton constant.

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Summary

・**A definition of charges whose form was introduced by Fock** in the past was **proposed** as the **precise** one for general relativistic field theory **on curved spacetime**.

・There was found **a new conserved charge different from the Noether one** for GR field theory with energy-momentum coveriantly conserved.

・ The newly found conserved charge was proposed as **entropy**.

・ The proposed interpretation leads to the **local Euler's relation** and the **1st law of thermodynamics** exactly holding in several well-known gravitational system such as FLRW model and a spherically symmetric hydrostatic equilibrium one.

・ For **FLRW model** for the isotropic homogeneous universe, **the energy does not conserve, but the entropy conserves**.

・ For the case of **LTE with spherical symmetry**, **the local temperature** satisfying the laws of thermodynamics is exactly coincident with the **Tolman temperature**.

Thank you!