# Effective field theory for cosmological phase transitions

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University of Nottingham

DESY Theory Seminar
28 November 2022

# Hot Big Bang

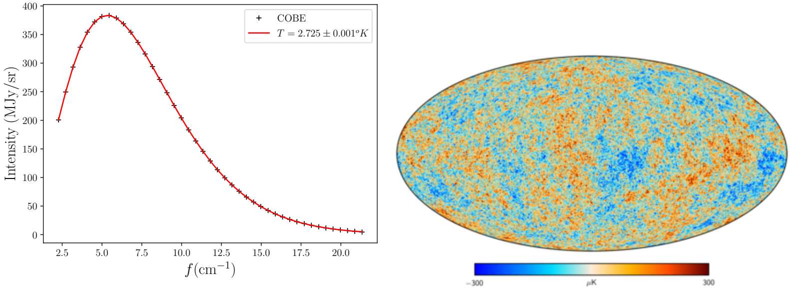
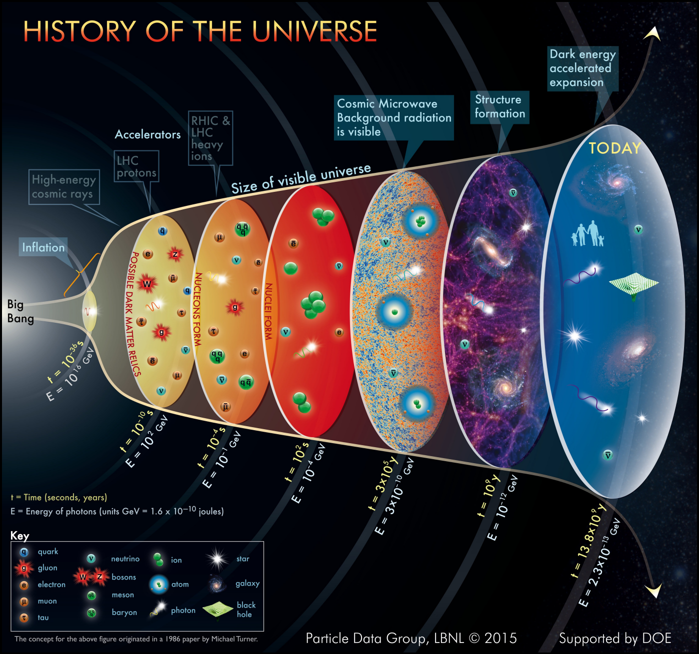



Figure: Blackbody spectrum of cosmic microwave background (COBE), and temperature anisotropies (Planck).

- Matter was very close to thermal in the early universe.
- Lots of interesting thermal physics.


HISTORY OF THE UNIVERSE
Dark energy accelerated expansion
Structure formation
Cosmic Microwave Background radiation is visible
RHIC & LHC heavy ions
Accelerators
LHC protons
High-energy cosmic rays
Size of visible universe
Inflation
Big Bang
TODAY
POSSIBLE DARK MATTER RELICS
NUCLEONS FORM
NUCLEI FORM
$t = 10^{-36}$ s
$E = 10^{16}$ GeV
$t = 10^{-10}$ s
$E = 10^{2}$ GeV
$t = 10^{-4}$ s
$E = 10^{-1}$ GeV
$t = 10^{2}$ s
$E = 10^{-4}$ GeV
$t = 3 \times 10^{5}$ y
$E = 3 \times 10^{-10}$ GeV
$t = 10^{9}$ y
$E = 10^{-12}$ GeV
$t = 13.8 \times 10^{9}$ y
$E = 2.3 \times 10^{-13}$ GeV
t = Time (seconds, years)
E = Energy of photons (units GeV = $1.6 \times 10^{-10}$ joules)
Key
quark
gluon
electron
muon
tau
neutrino
bosons
meson
baryon
ion
atom
photon
star
galaxy
black hole
The concept for the above figure originated in a 1986 paper by Michael Turner.
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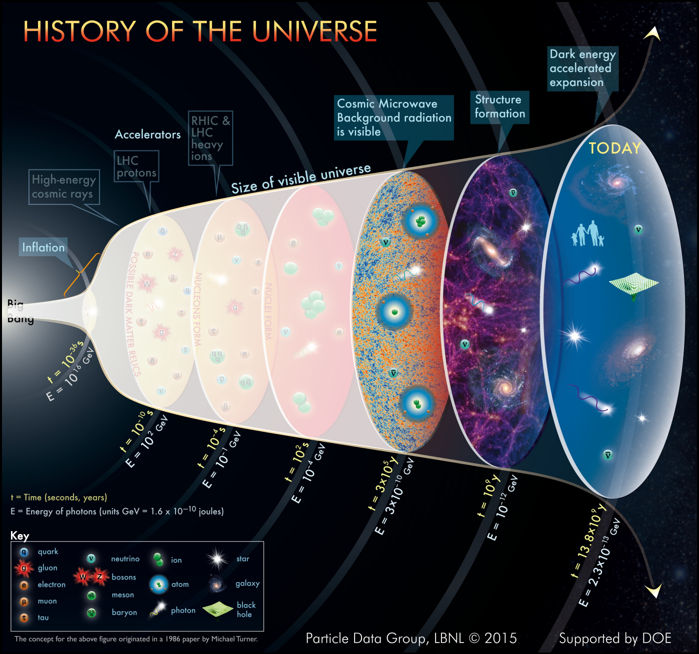
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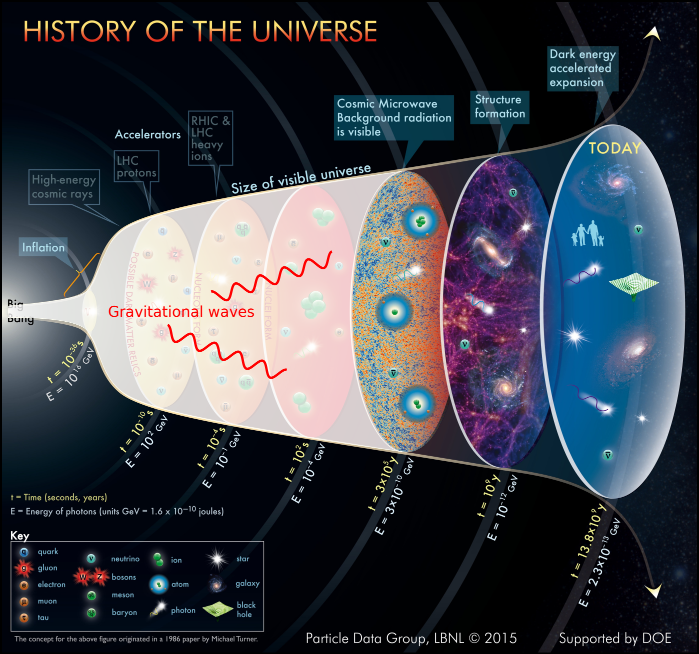

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# Gravitational waves

- Gravitational waves directly observed by LIGO/Virgo →
- Future experiments will extend sensitivity ↓

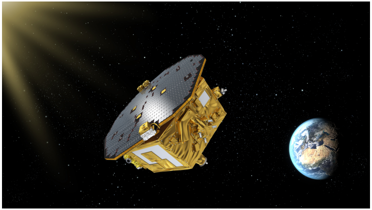


Figure: LISA Pathfinder

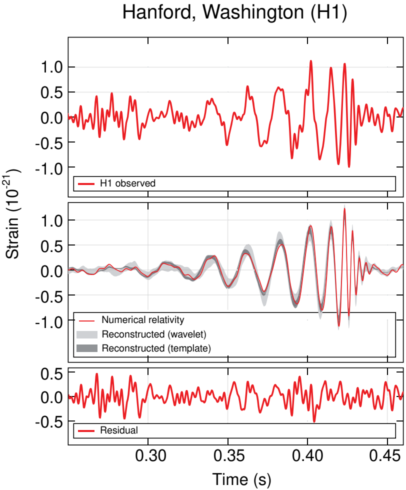
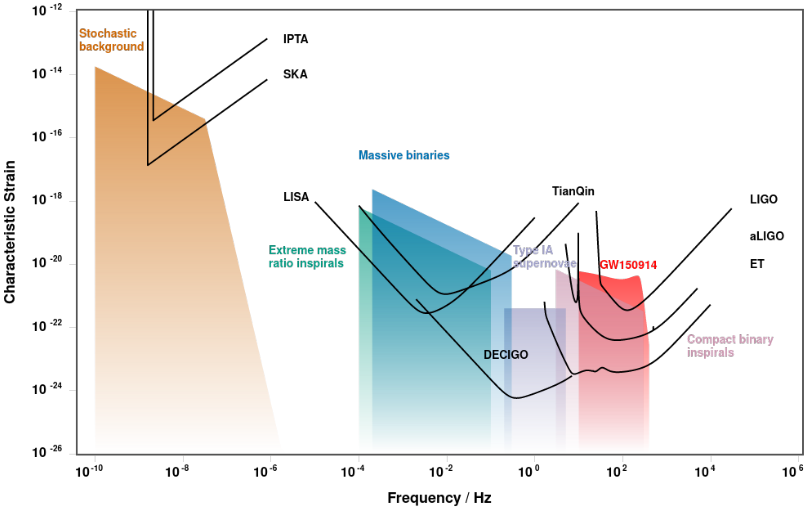



Figure: GW150914 1602.03837

# The gravitational wave spectrum




gwplotter.com

# Cosmological 1$^{st}$-order phase transitions

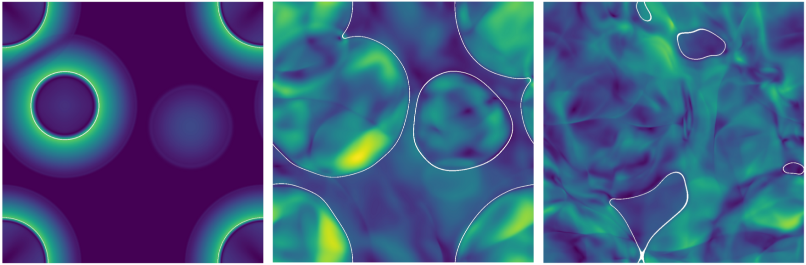


Figure: Cutting et al. arXiv:1906.00480.

- Bubbles nucleate, expand and collide
- This creates long-lived fluid flows
- And creates gravitational waves

# Gravitational waves from phase transitions: the pipeline

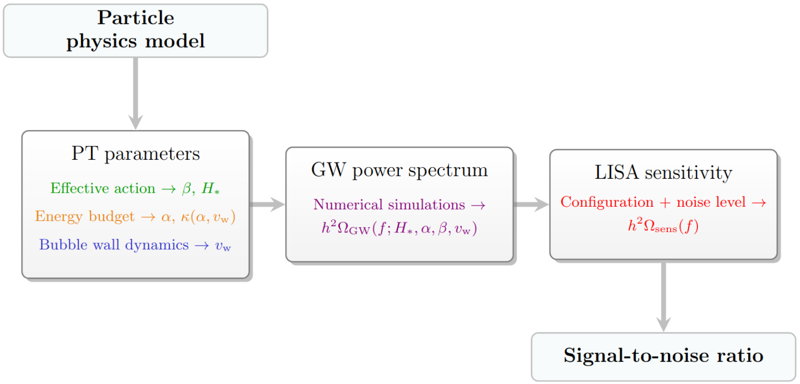



Figure: The Light Interferometer Space Antenna (LISA) pipeline $\mathscr{L} \to \mathrm{SNR}(f)$, Caprini et al. 1910.13125.

# Gravitational waves from phase transitions: the pipeline

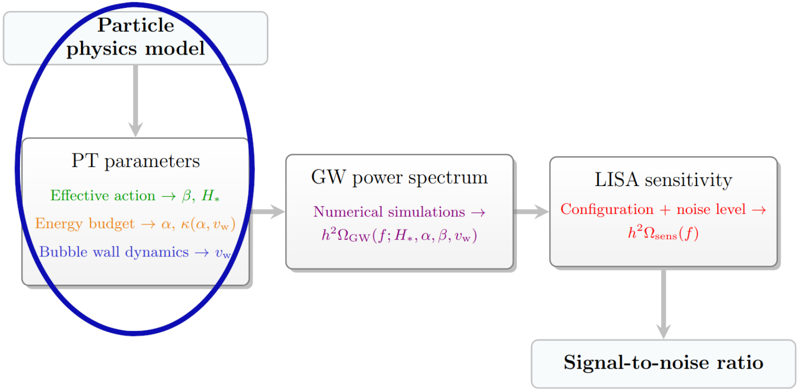
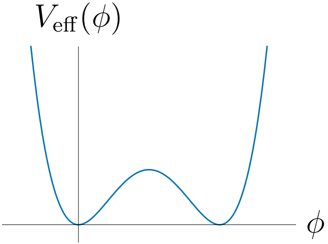
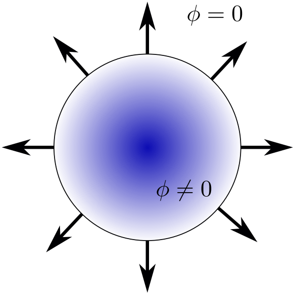



Figure: The Light Interferometer Space Antenna (LISA) pipeline $\mathscr{L} \to \mathrm{SNR}(f)$, Caprini et al. 1910.13125.

# Phase transition parameters




## Equilibrium (hom.)

- order of transition
- $T_c$, critical temperature
- $\Delta\theta_c$, latent heat
- $c_s^2$, sound speed

## Near-equilibrium

- $\Gamma$, bubble nucleation rate $\Rightarrow T_*$, $\Delta\theta_*$, $\alpha_*$, $\beta/H_*$

## Nonequilibrium

- $v_w$, bubble wall speed

# Standard approach to computing parameters

1-loop resummed approximation is based on

$$V_{\text{eff}}(\phi, T) = V_{\text{tree}}(\phi) + \underbrace{\frac{1}{2}\SumInt_P \log(P^2 + V''_{\text{tree}})}_{\text{1-loop}}$$

$$\underbrace{-\frac{T}{12\pi}\left((V''_{\text{tree}} + \Pi_T)^{3/2} - (V''_{\text{tree}})^{3/2}\right)}_{\text{daisy correction}}.$$

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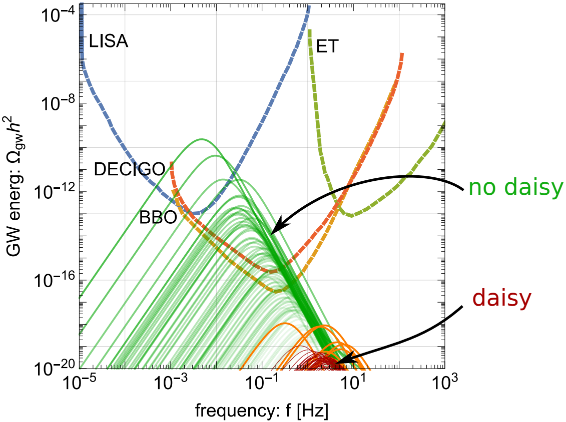
Solve:

- $\Re V'_{\text{eff}}(\phi, T) = 0 \Rightarrow$ phases
- $-\partial_r^2\phi - 2\partial_r\phi + \Re V'_{\text{eff}}(\phi, T) = 0 \Rightarrow$ critical bubble
- $\partial_t^2\phi - \partial_z^2\phi + \Re V'_{\text{eff}}(\phi, T) + \sum_i (m_i^2)'(\phi)\int_p \delta f_i(p, z) = 0 \Rightarrow v_w$

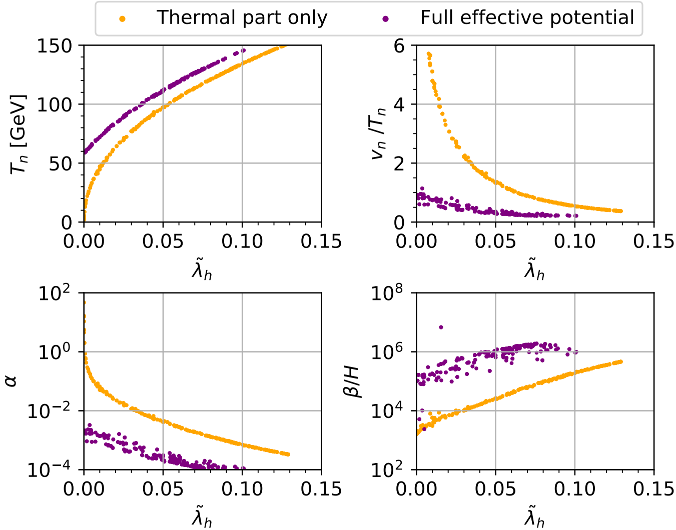
# Theoretical uncertainties




GW signals in two different 1-loop approximations for

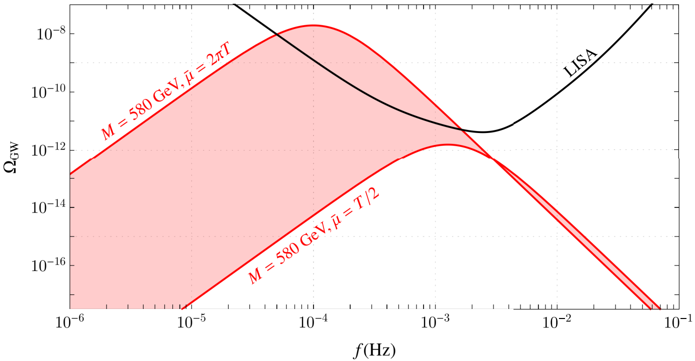
$$\mathscr{L} = \mathscr{L}_{SM} + \frac{a_2}{2}(\Phi^\dagger\Phi)\sigma^2 + \frac{1}{2}(\partial\sigma)^2 + \frac{m_\sigma^2}{2}\sigma^2 + \frac{b_4}{4}\sigma^4$$

Carena, Liu & Wang 1911.10206




Ellis et al. arXiv:2210.16305

# Theoretical uncertainties

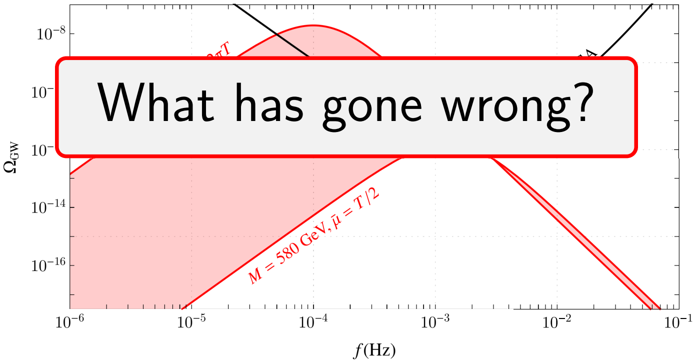



Renormalisation scale dependence of GW spectrum at one physical parameter point for

$$\mathscr{L} = \mathscr{L}_{\rm SM} + \frac{1}{M^2}(\Phi^\dagger\Phi)^3.$$

Croon, OG, Schicho, Tenkanen & White 2009.10080

# Theoretical uncertainties

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# What has gone wrong?

Possible sources of theoretical uncertainties:

- nonperturbativity? Linde '80
- inconsistencies? E. Weinberg & Wu '87, E. Weinberg '92
- higher order perturbative corrections? Arnold & Espinosa '92
- gauge dependence or infrared divergences? Laine '94
- renormalisation scale dependence? Farakos et al. '94
- . . .

# Overview

1. Motivation
2. Scale hierarchies in phase transitions
3. EFT for equilibrium physics
4. EFT for bubble nucleation
5. Conclusions

# Scale hierarchies in phase transitions

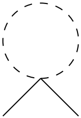
# A hierarchy problem

Let's assume there is some very massive particle $\chi$, $M_\chi \gg m_H$, coupled to the Standard Model Higgs $\Phi$ like

$$\mathscr{L} = \mathscr{L}_{\mathrm{SM}} + g^2 \Phi^\dagger \Phi \chi^\dagger \chi + \mathscr{L}_\chi .$$

If we integrate out $\chi$, we find that the Higgs mass parameter gets a correction of the form

$$(\Delta m_H^2)\Phi^\dagger\Phi =$$



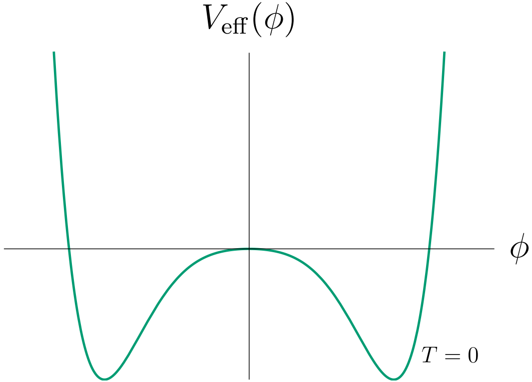
$$,$$

$$\sim g^2 M_\chi^2 \Phi^\dagger \Phi \, .$$

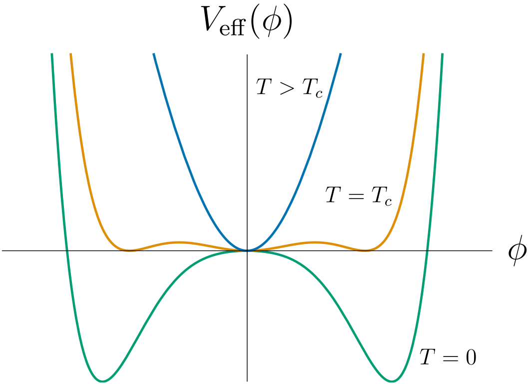
Relevant operators in the IR get large contributions from the UV,

$$\frac{\Delta m_H^2}{m_H^2} \sim g^2 \left( \frac{M_\chi}{m_H} \right)^2 .$$

# Phase transitions



# Phase transitions




For there to be a phase transition, thermal/quantum fluctuations should modify the potential at leading order,

$$V_{\text{eff}} = V_{\text{tree}} + \Delta V_{\text{fluct}}.$$

# Hierarchies in phase transitions

So, for there to be a phase transition, we need

$$\frac{\Delta V_{\mathrm{fluct}}}{V_{\mathrm{tree}}} \sim g^2 N \left(\frac{\Lambda_{\mathrm{fluct}}}{\Lambda_{\mathrm{tree}}}\right)^{\sigma} \overset{!}{\sim} 1,$$

where $\sigma > 0$ for relevant operators.

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$\Rightarrow$ either:

(i) $g^2 N \gtrsim 1$, i.e. strong coupling

(ii) $\Lambda_{\text{fluct}} \gg \Lambda_{\text{tree}}$, i.e. scale hierarchy

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Perturbative phase transitions require scale hierarchies!

# Infrared strong coupling

Infrared bosons are highly occupied; the effective expansion parameter $\alpha_{\rm eff}$ grows

$$\alpha_{\rm eff} \sim g^2 \frac{1}{1 - e^{E/T}} \approx g^2 \frac{T}{E}$$

Softer modes are classically occupied and more strongly coupled:

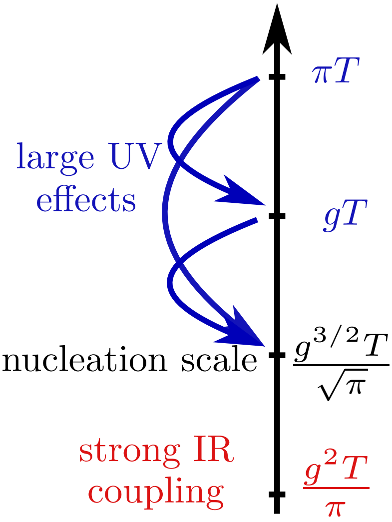
$$\begin{aligned}
\text{hard}: &\quad E \sim \pi T \Rightarrow \alpha_{\rm eff} \sim g^2 \sim 0.03, \\
\text{soft}: &\quad E \sim gT \Rightarrow \alpha_{\rm eff} \sim g \sim 0.18, \\
\text{supersoft}: &\quad E \sim g^{3/2} T \Rightarrow \alpha_{\rm eff} \sim g^{1/2} \sim 0.42, \\
\text{ultrasoft}: &\quad E \sim g^2 T \Rightarrow \alpha_{\rm eff} \sim g^0 \sim 1.
\end{aligned}$$

# UV and IR problems

There are two main difficulties

- large UV effects break loop expansion $\leftarrow$ EFT
- IR becomes strongly coupled $\leftarrow$ higher orders or lattice

$$\frac{\Delta V_{\text{fluct}}}{V_{\text{tree}}} \sim \alpha_{\text{eff}} \left( \frac{\Lambda_{\text{fluct}}}{\Lambda_{\text{tree}}} \right)^{\sigma}$$



# UV and IR in concert

For some observable $\mathcal{O}$ at $T = 0$

$$\mathcal{O}_0 = \underbrace{A}_{\text{0-loop}} + \underbrace{Bg^2}_{\text{1-loop}} + \underbrace{Cg^4}_{\text{2-loop}} + \underbrace{Dg^6}_{\text{3-loop}} + \underbrace{Eg^8}_{\text{4-loop}} + \dots$$

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At a Higgs-like 1st-order phase transition, instead

$$\mathcal{O}_T = \underbrace{a}_{\text{1-loop}^+} + \underbrace{bg}_{\text{2-loop}^+} + \underbrace{cg^{3/2}}_{\text{1-loop}^*} + \underbrace{dg^2}_{\text{3-loop}^+} + \underbrace{eg^{5/2}}_{\text{3-loop}^*} + \underbrace{fg^3}_{\infty\text{-loop}} + \dots$$

where $^+$ and $^*$ refer to different resummations of infinite classes of diagrams.

Ekstedt, OG & Löfgren 2205.07241

# EFT for equilibrium physics

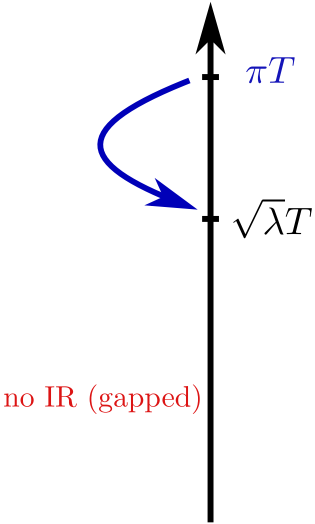
# Real scalar model

A simple model,

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \sigma\phi + \frac{m^2}{2}\phi^2 + \frac{\gamma}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4 + J_1\phi + J_2\phi^2,$$

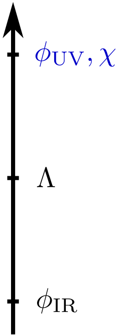
with only two scales: $\pi T$, $m_{\text{eff}} \sim \sqrt{\lambda}T$.

- large UV effects
- IR coupling $\alpha_{\text{eff}} \sim \sqrt{\lambda}$



# Wilsonian EFT

- Split degrees of freedom $\{\phi, \chi\}$ based on energy $\rightarrow$




- Integrate out the UV modes:

$$\int \mathcal{D}\phi \int \mathcal{D}\chi \; e^{-S[\phi,\chi]} = \int \mathcal{D}\phi_{\mathsf{IR}} \left( \int \mathcal{D}\phi_{\mathsf{UV}} \mathcal{D}\chi \; e^{-S[\phi,\chi]} \right)$$
$$= \int \mathcal{D}\phi_{\mathsf{IR}} \; e^{-S_{\mathsf{eff}}[\phi_{\mathsf{IR}}]}$$

- Careful power-counting cancels dependence on $\Lambda$.

Burgess '21, Hirvonen '22

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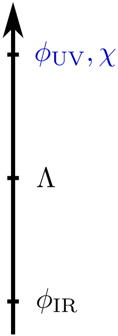


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Burgess '21, Hirvonen '22

This has now been automated!

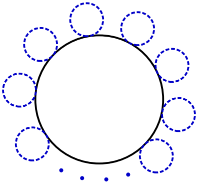
Ekstedt et al. '22

# Resummations with EFT

By first integrating out the UV modes

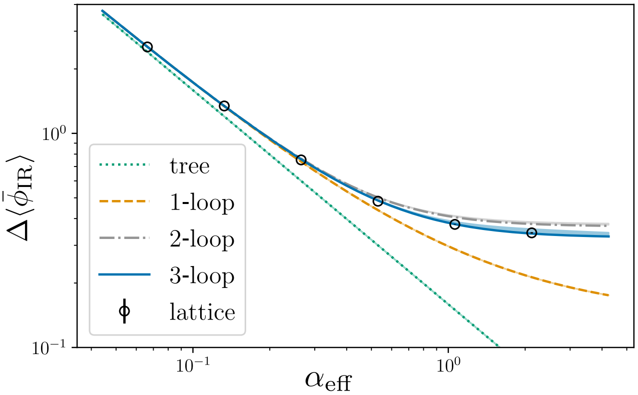
$$S_{\text{eff}}[\phi_{\text{IR}}] = S_\phi[\phi_{\text{IR}}] - \log \int \mathcal{D}\phi_{\text{UV}} \mathcal{D}\chi \; e^{-S[\phi_{\text{IR}}+\phi_{\text{UV}},\chi]+S_\phi[\phi_{\text{IR}}]},$$
$$\approx S_\phi[\phi_{\text{IR}}] + \int_x \left[ (\sigma_{\text{eff}} - \sigma)\phi_{\text{IR}} + \frac{1}{2}(m^2_{\text{eff}} - m^2)\phi^2_{\text{IR}} \right],$$

the daisy resummations arise naturally.



So do all other necessary resummations, order by order.

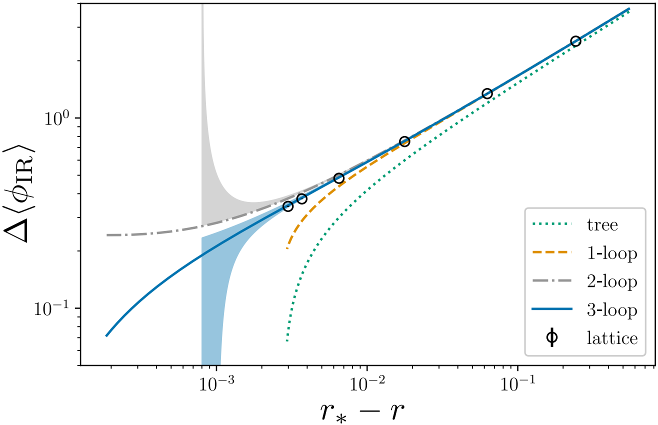
# Lattice vs perturbation theory: real scalar model




$$\Delta\langle\bar{\phi}_{\rm IR}\rangle = \frac{1}{4\pi\alpha_{\rm eff}}\Bigg[\, 2 + \sqrt{3}\,\alpha_{\rm eff} + \frac{1}{2}\left(1 + 2\log\tilde{\mu}_3\right)\alpha_{\rm eff}^2 + \sqrt{3}\left[-\frac{3}{8\sqrt{2}}\xi + \frac{21}{32}{\rm Li}_2\frac{1}{4} - \frac{7\pi^2}{128} - \frac{1}{2} + \frac{21}{64}\log^2\frac{4}{3} + \frac{5}{8}\log\frac{4}{3}\right]\alpha_{\rm eff}^3 + O\left(\alpha_{\rm eff}^4\right)\Bigg]$$

OG 2101.05528

# IR problems: real scalar model




As we approach the 2nd-order phase transition $m^2_{\rm eff} \to 0$ and $\alpha_{\rm eff} \to \infty$.

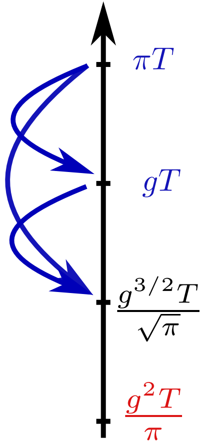
# SU(2) Higgs model

A more complicated model with all the scales

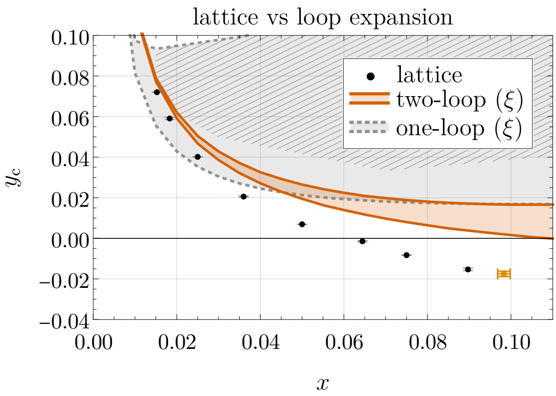
$$\mathscr{L} = \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + (D_\mu\Phi)^\dagger D_\mu\Phi$$
$$+ m^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2$$

where $m_{\text{eff}} \sim g^{3/2}T/\sqrt{\pi}$.

- large UV effects
- strongly coupled IR



# Problem: gauge dependence




- Loop-expanded $V_{\text{eff}}$ is strongly gauge dependent at high $T$.
- Naive solutions lead to infrared divergences or inconsistencies.

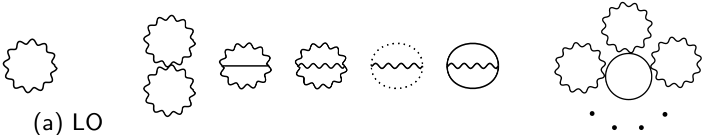
Laine hep-ph/9411252, Patel & Ramsey-Musolf 1101.4665

# Nucleation scale EFT

Integrating out the scales $\pi T$ and $gT$ gives

$$\mathscr{L}_{\text{eff}} = \frac{1}{2}\partial_i\Phi^\dagger\partial_i\Phi + \frac{m_{\text{eff}}^2}{2}\Phi^\dagger\Phi - \frac{g_{\text{eff}}^3(\Phi^\dagger\Phi)^{3/2}}{4(4\pi)} + \frac{\lambda_{\text{eff}}}{4}(\Phi^\dagger\Phi)^2 - \frac{11 g_{\text{eff}}\partial_i\Phi^\dagger\partial_i\Phi}{8(4\pi)(\Phi^\dagger\Phi)^{1/2}} - \frac{51}{64}\frac{g_{\text{eff}}^4\Phi^\dagger\Phi}{(4\pi)^2}\log\frac{g_{\text{eff}}^2\Phi^\dagger\Phi}{\tilde{\mu}_{\text{eff}}^2} + O(m_{\text{eff}}^3)$$

After integrating out the scale $\pi T$, the relevant diagrams are

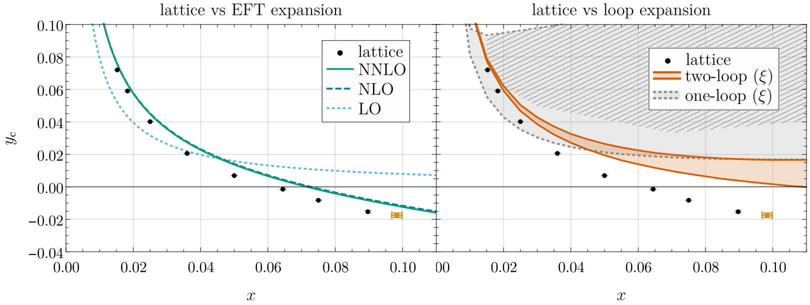


(a) LO

(b) NLO

(c) NNLO

# EFT solution: gauge independence

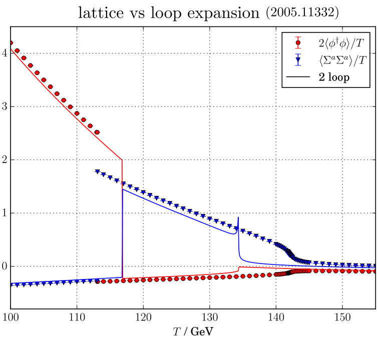


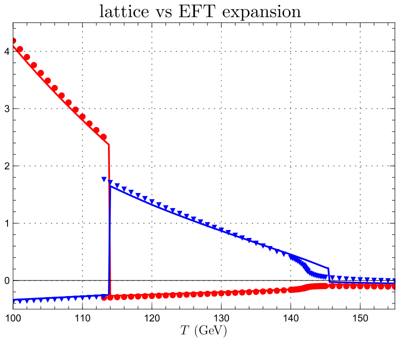

EFT approach provides exact order-by-order gauge invariance.

Ekstedt, OG & Löfgren 2205.07241

(see also Löfgren et al. 2112.05472, Hirvonen et al. 2112.08912)

# Triplet extension of the Standard Model




$$\mathscr{L} = \mathscr{L}_{\mathsf{SM}} + \frac{a_2}{2}\Phi^\dagger\Phi\Sigma^a\Sigma^a + \frac{1}{2}D_\mu\Sigma^a D_\mu\Sigma^a + \frac{m_\Sigma^2}{2}\Sigma^a\Sigma^a + \frac{b_4}{4}(\Sigma^a\Sigma^a)^2$$

Niemi et al. 2005.11332, OG & Tenkanen forthcoming

# EFT for bubble nucleation

# A potted history of nucleation theory

- '69 Langer's classical theory of nucleation
- '77 Callan and Coleman's bounce formalism for QFT at $T = 0$
- '81 Affleck and Linde give (conflicting) proposals for $T \neq 0$
- ...
- ...
- ...
- ...

# A potted history of nucleation theory

- '69 Lang
- '77 Calla FT at $T = 0$
- '81 Affle for $T \neq 0$
- ...
- ...
- ...
- ...
- ... this discrepancy was never resolved.

# High temperature proposals

## Linde '81

Vacuum energy replaced with free energy $-2\mathrm{Im}E_0 \to -2\mathrm{Im}F$,

$$\Gamma_{\text{Linde}} \equiv \mathcal{V}T \left(\frac{S_3[\phi_\mathrm{B}]}{2\pi T}\right)^{3/2} \left|\frac{\det{}' S_3''[\phi_\mathrm{B}]}{\det S_3''[\phi_\mathrm{F}]}\right|^{-1/2} e^{-\frac{1}{T}\int d^3x\left[\frac{1}{2}(\nabla\phi_\mathrm{B})^2+V_T(\phi_\mathrm{B})\right]},$$

where $V_T$ is a thermal effective potential.

## Affleck '81

Decay rate (in QM) of states with energy $E$, summed with Boltzmann weight,

$$\Gamma_{\text{Affleck}} \equiv \frac{\mathcal{V}}{2\pi} \left(\frac{S_3[\phi_\mathrm{B}]}{2\pi T}\right)^{3/2} \left(\frac{\det{}^{+} S_3''[\phi_\mathrm{B}]}{\det S_3''[\phi_\mathrm{F}]}\right)^{-1/2} e^{-\frac{1}{T}\int d^3x\left[\frac{1}{2}(\nabla\phi_\mathrm{B})^2+V(\phi_\mathrm{B})\right]},$$

where $V$ is the tree-level potential at $T=0$.

# Status of thermal nucleation theory

## Skeletons in the closet

1. Unclear what potential to use at high $T$
   - Nothing derived from first-principles (unlike at $T = 0$)
   - Obvious guesses are clearly wrong
     - $V_{\text{tree}}(\phi)$
     - $V_{\text{eff}}(\phi = \text{const})$
     - $\Re V_{\text{eff}}(\phi = \text{const})$ recognised by many authors

   see e.g. Langer '74, E. Weinberg '92

2. Sundry answers for the nucleation prefactor at high $T$
   - $\sqrt{|\lambda_-| + \eta^2/4} - \eta/2$ Langer '69
   - $\sqrt{|\lambda_-|}$ Affleck '81
   - $2\pi T$ Linde '81
   - $\sqrt{|\lambda_-|/(1+\xi^2)}$ Arnold & McLerran '87

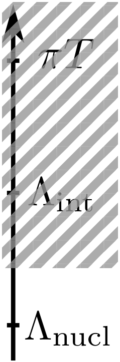
# EFT approach to thermal nucleation theory

First-principles definition of thermal nucleation rate:

1. Integrate out quantum and thermal fluctuations with energy scales $\Lambda \gg \Lambda_{\text{nucl}} \sim m_{\text{nucl}}$.
2. This yields a classical, statistical EFT,

$$S_{\text{nucl}}[\phi] = \int d^3x \left[ \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m_{\text{nucl}}^2\phi^2 + \dots \right].$$

3. Classical nucleation theory then gives the rate unambiguously.




OG & Hirvonen 2108.04377

# Classicalisation

- Bose enhancement of IR modes

$$n_{\mathrm{B}}(E) = \frac{1}{e^{E/T} - 1},$$
$$\approx \frac{T}{E} \gg 1.$$

- Dynamics of QFT at nucleation scale ($\Lambda_{\mathrm{nucl}} \ll T$) quasi-classical: quantum and thermal fluctuations give stochasticity.

$$\langle \phi(t,x)\phi(0,0) + \phi(0,0)\phi(t,x) \rangle$$

Greiner & Müller '97, Aarts & Smit '97,
Bödeker '97, Mueller & Son '02

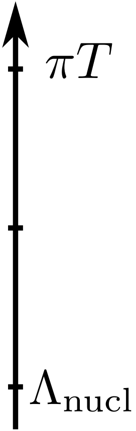



Figure: Nucleation scale much lower than thermal scale.

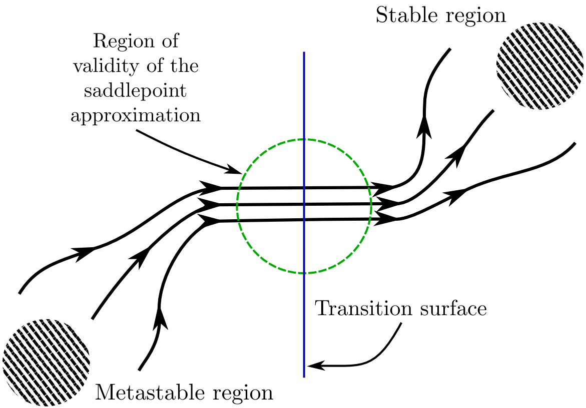
# Classical statistical field theory

Classicality means we can describe the system with a probability distribution $\rho$ on phase space $\phi_i, \pi_i \to \eta_i$. Conservation of probability implies

$$\frac{\partial \rho}{\partial t} + \sum_i \frac{\partial J_i}{\partial \eta_i} = 0$$

with probability flux $J_i$.

# Classical nucleation theory




$$\Gamma = \int_{\text{TS}} J \cdot dS_{\perp}$$

Langer '69

Kramers '40, Zel'dovich '42

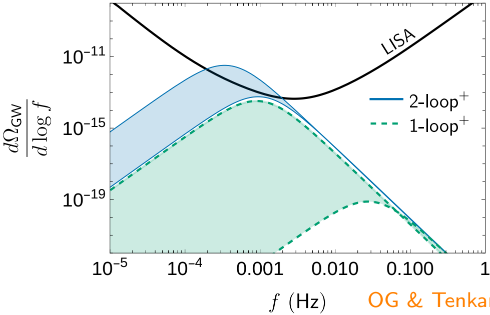
# Resolutions in EFT approach

- $S_{\rm nucl}[\phi]$ is real for all $\phi$
- Derivative expansion justified by $\Lambda_{\rm nucl} \ll \Lambda_{\rm UV}$
- Modes counted only once in path integral,

$$\Gamma = \underbrace{\frac{\kappa}{2\pi}\mathcal{V}\sqrt{\left|\frac{\det(S''_{\rm nucl}[\phi_{\rm meta}]/2\pi)}{\det'(S''_{\rm nucl}[\phi_{\rm cb}]/2\pi)}\right|}}_{\text{modes } E<\Lambda}\ \underbrace{e^{-S_{\rm nucl}[\phi_{\rm cb}]}}_{\text{modes } E>\Lambda}\ .$$

EFT methods ensure result is independent of $\Lambda$ order by order.

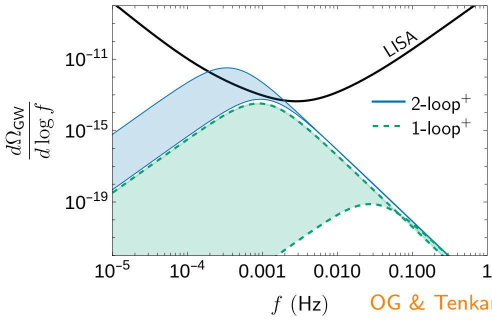
# Conclusions




OG & Tenkanen 2104.04399

- Phase transitions may produce observable gravitational waves
- Large theoretical uncertainties in standard computations
  - UV "hierarchy" problems
  - IR strong-coupling problems
  - Consistency problems for bubble nucleation
- EFT solves UV problems, and gives definition of nucleation rate
- Higher orders (or lattice) solves problems from IR

# Conclusions




OG & Tenkanen 2104.04399

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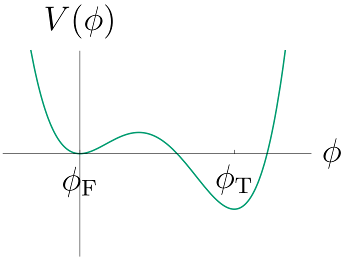
Thanks for listening!

# Backup slides

# Vacuum decay à la Callan & Coleman




Idea is that decay rate $\Gamma = -2\mathrm{Im}E_0$, Callan & Coleman '77

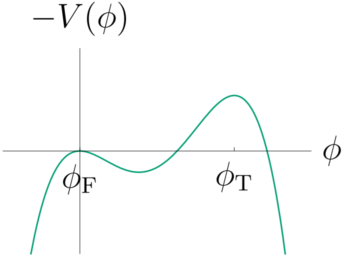
$$|\langle \phi_{\rm F}|e^{-i\hat{H}t}|\phi_{\rm F}\rangle|^2 \underset{t\to\infty}{=} e^{2\mathrm{Im}E_0 t},$$

and this can be calculated in Euclidean time $\tau = -it$:

$$\langle \phi_{\rm F}|e^{-\hat{H}\tau}|\phi_{\rm F}\rangle = \sum_n e^{-E_n\tau}\langle \phi_{\rm F}|n\rangle\langle n|\phi_{\rm F}\rangle \underset{\tau\to\infty}{=} e^{-E_0\tau}|\langle \phi_{\rm F}|0\rangle|^2,$$

$$= \int \mathcal{D}\phi e^{-S_{\rm E}[\phi]}.$$

# Euclidean saddlepoints




Altogether,

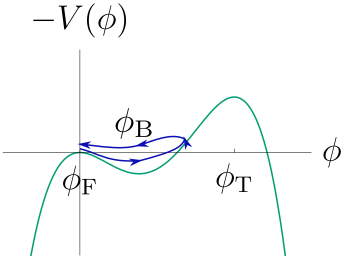
$$\Gamma \underset{\tau\to\infty}{=} \frac{2}{\tau}\mathrm{Im}\log\int \mathcal{D}\phi e^{-S_{\mathsf{E}}[\phi]}.$$

which can be expanded around saddlepoints of the action,

$$\int \mathcal{D}\phi e^{-S_{\mathsf{E}}[\phi]} \approx \sum_a \int \mathcal{D}\phi e^{-S_{\mathsf{E}}[\phi_a]-\frac{1}{2}S''_{\mathsf{E}}[\phi_a](\phi-\phi_a)^2},$$

$$\frac{\delta S_{\mathsf{E}}}{\delta\phi_a} = 0.$$

# Bounce result for vacuum decay




- Two saddlepoints satisfying the boundary conditions: the false vacuum $\phi_{\mathrm{F}}$ and the bounce $\phi_{\mathrm{B}}$.
- Evaluating the path integral around these gives the rate,

$$\Gamma \approx \underbrace{\mathcal{V}\left(\frac{S_{\mathsf{E}}[\phi_{\mathrm{B}}]}{2\pi}\right)^2}_{\text{zero modes}} \underbrace{\left|\frac{\det' S''_{\mathsf{E}}[\phi_{\mathrm{B}}]}{\det S''_{\mathsf{E}}[\phi_{\mathrm{F}}]}\right|^{-1/2}}_{\text{nonzero modes}} e^{-S_{\mathsf{E}}[\phi_{\mathrm{B}}]}.$$

# Rederiving vacuum decay

- Direct approach:

$$\Gamma \equiv - \lim_{\substack{t/t_{\mathrm{NL}} \to 0 \\ t/t_{\mathrm{slosh}} \to \infty}} \frac{1}{P_{\mathrm{F}}(t)} \frac{d}{dt} P_{\mathrm{F}}(t),$$

  physical definition reproduces Callan-Coleman result in limit.

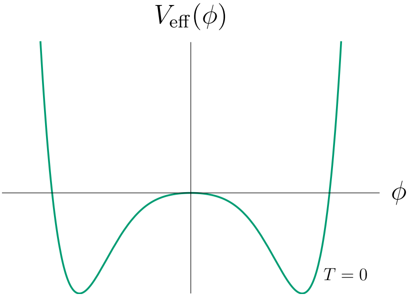
  Andreassen et al. arXiv:1602.01102

- Real-time approach: starting from Optical Theorem and using Picard-Lefschetz theory, reproduces Callan-Coleman result.

  Ai, Garbrecht & Tamarit arXiv:1905.04236

# Should we use the tree-level potential?

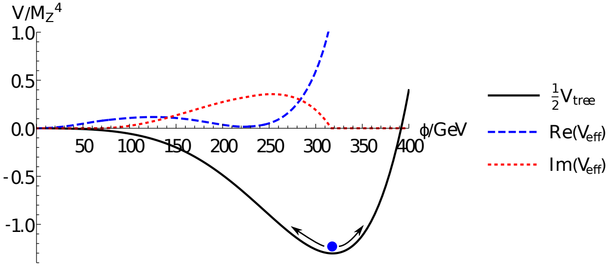
- $V_{\text{tree}}$ independent of $T$
  $\Rightarrow$ bounce independent of $T$
- Typically no phase coexistence at $T = 0$ $\rightarrow$
  $\Rightarrow \nexists$ bounce solution




- If thermal effects are of leading order in magnitude, then $V_{\text{tree}}$ is not a good leading order approximation

# What about the effective potential?

The (perturbative) effective potential answers the question:
*"What is the contribution to the energy from $P \neq 0$ fluctuations in the tree-level potential about a given homogeneous $\bar{\phi}$?"*




$$V_{\text{1-loop}} \sim \sum\!\!\!\!\!\!\int_P \log(P^2 + V''_{\text{tree}}(\bar{\phi}))$$

Modes with $P^2 + V''_{\text{tree}}(\bar{\phi}) < 0 \Rightarrow$ imaginary part, corresponds to decay of homogeneous $\bar{\phi}$ state. E. Weinberg & Wu '87

# What about the real part of the effective potential?

$V_{\text{eff}}$ is given by integration over all modes except the constant mode $\bar{\phi} = \text{const}$, Fukuda & Kyriakopoulos, '75

$$\int \mathcal{D}\phi \; e^{-S[\phi]} = \int d\bar{\phi} \left( \int_{P\neq 0} \mathcal{D}\phi' \; e^{-S[\phi=\bar{\phi}+\phi']} \right),$$
$$= \int d\bar{\phi} \; e^{-\frac{\mathcal{V}}{T} V_{\text{eff}}(\bar{\phi})}.$$

So, if we want $\Re V_{\text{eff}}$ in the exponent,

$$\Gamma \leftarrow \int \mathcal{D}\phi e^{-\frac{1}{T}\int_x \left[\frac{1}{2}(\nabla\phi)^2 + \Re V_{\text{eff}}(\phi)\right]}$$
$$= \int \mathcal{D}\phi e^{-\frac{1}{T}\int_x \left[\frac{1}{2}(\nabla\phi)^2 - \frac{T}{\mathcal{V}}\Re \log \int_{P\neq 0} \mathcal{D}\phi' \; e^{-S[\phi+\phi']}\right]}$$

then we are:

- double counting $\phi'$
- making an uncontrolled derivative expansion

# Factorisation

- Current factorises:

$$J = \underbrace{\sigma(u)}_{\text{nonequilibrium}} \cdot \underbrace{\frac{e^{-F_{\text{eff}}[\phi]/T}}{Z_{\text{meta}}}}_{\text{equilibrium}} .$$

- $\Rightarrow$ rate factorises:

$$\Gamma = \frac{\kappa_{\text{dyn}}}{2\pi} \cdot \Sigma_{\text{stat}} ,$$

$$\Sigma = \frac{\mathcal{N}}{Z_{\text{meta}}} \int \mathcal{D}\phi \; \delta(\phi_-) e^{-F_{\text{eff}}[\phi]/T} ,$$

  and $\Sigma_{\text{stat}}$ is calculable in equilibrium.

- Factorisation holds at all loop orders!

  Ekstedt arXiv:2201.07331

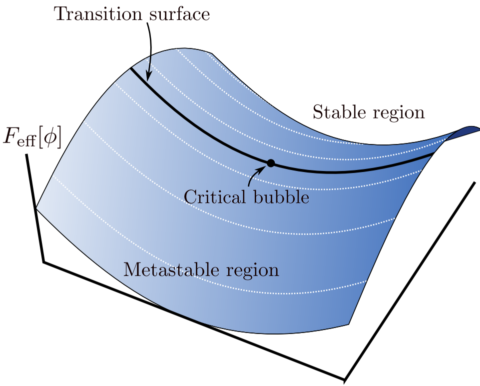
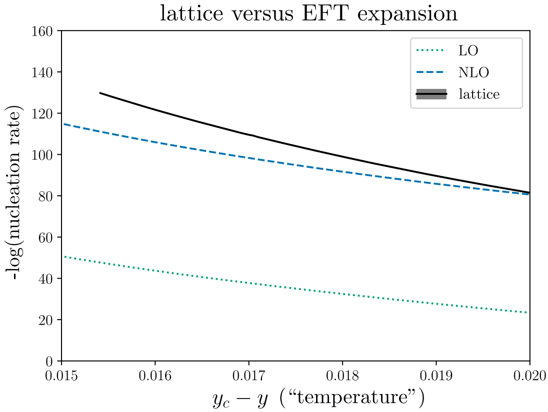



Figure: Dotted lines of $u = \text{const}$ perpendicular to transition surface.

# SU(2) Higgs model - bubble nucleation




https://doi.org/10.5281/zenodo.6548608

Moore & Rummukainen '00

OG, Güyer & Rummukainen 2205.07238

# High temperature effective field theory

# Equilibrium thermodynamics

- Can be formulated in $\mathbb{R}^3 \times S^1$.


- Fields are expanded into Fourier (Matsubara) modes:

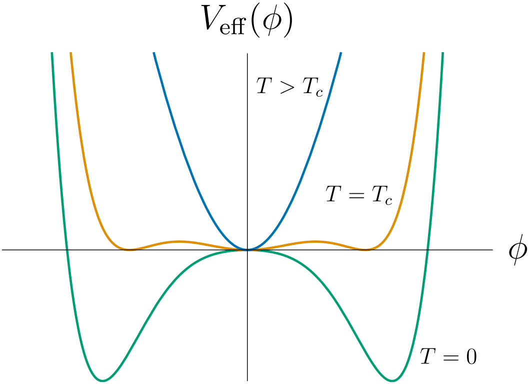
$$\Phi(\mathrm{x}, \tau) = \sum_{n \text{ even}} \phi_n(\mathrm{x}) e^{i(n\pi T)\tau} \leftarrow \text{boson}$$

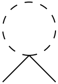
$$\Psi(\mathrm{x}, \tau) = \sum_{n \text{ odd}} \psi_n(\mathrm{x}) e^{i(n\pi T)\tau} \leftarrow \text{fermion}$$

- Masses of Matsubara modes are

$$m_n^2 = m^2 + (n\pi T)^2$$

# Thermal mass hierarchies

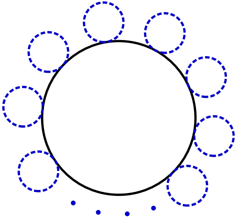



$m_{0,\text{eff}}^2 = m^2 +$  $= m^2 + \# g^2 T^2$

- At $T \gg T_c$, thermal corrections dominate, so $m_{0,\text{eff}} \sim gT$ which is much less than $\pi T$.
- Near $T = T_c$, cancellations typically give $m_{0,\text{eff}} \ll gT$.

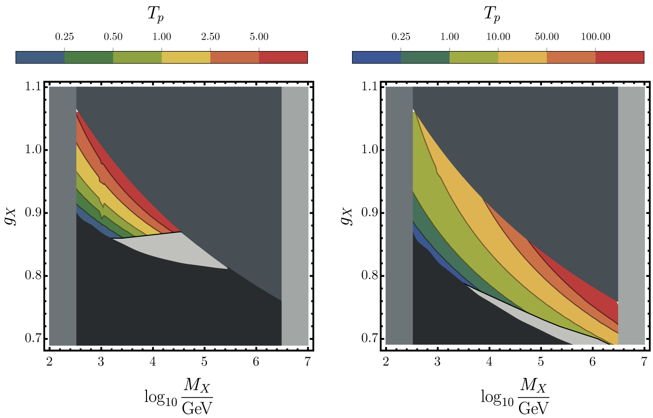
# Resumming UV problems

$$\mathscr{L}_0 = \underbrace{\frac{1}{2}(\nabla\phi_0)^2 + \frac{1}{2}m^2\phi_0^2}_{\mathscr{L}_{\text{free}}} + \underbrace{\frac{1}{4!}g^2\phi_0^4}_{\mathscr{L}_{\text{int}}}.$$



Resummation by changing split between $\mathscr{L}_{\text{free}}$ and $\mathscr{L}_{\text{int}}$,

$$\mathscr{L}_{\text{free}} \to \mathscr{L}_{\text{free}} + \frac{1}{2}(m_{0,\text{eff}}^2 - m^2)\phi_0^2,$$
$$\mathscr{L}_{\text{int}} \to \mathscr{L}_{\text{int}} + \frac{1}{2}(m^2 - m_{0,\text{eff}}^2)\phi_0^2.$$




Percolation temperatures for two different RG scales in

$$\mathscr{L} = \mathscr{L}_{\mathsf{SM}} + \frac{1}{4} U^a_{\mu\nu} U^a_{\mu\nu} + \lambda_2 (\Phi^\dagger \Phi) \Psi \dagger \Psi$$
$$+ \frac{1}{2} (D_U \Psi)^2 + m^2_\Psi (\Psi^\dagger \Psi) + \lambda_3 (\Psi \dagger \Psi)^2$$

Kierkla et al. arXiv:2210.07075