IBP Reduction with Gröbner bases

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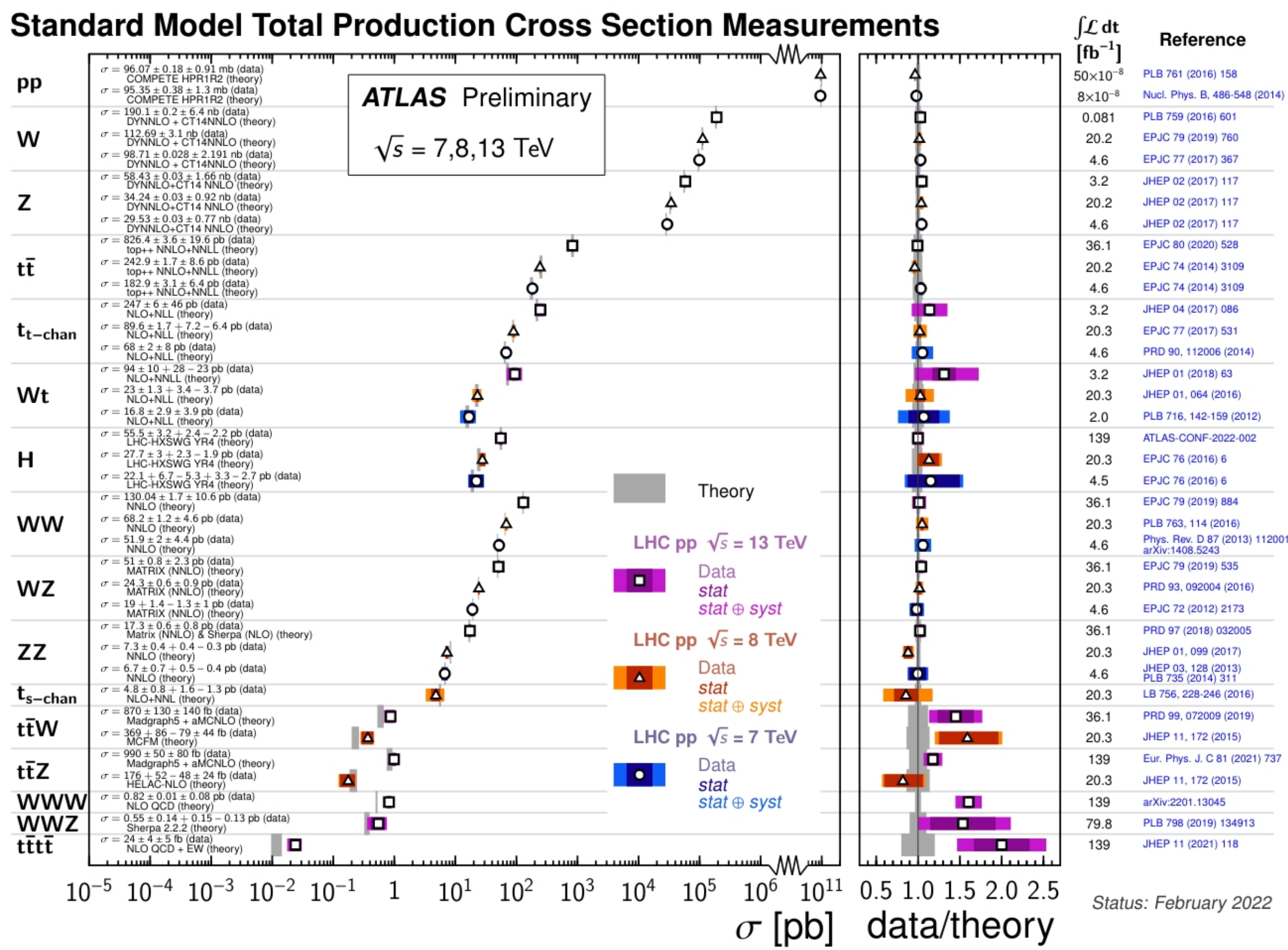
Precision physics

The standard model of particle physics is extremely successful!



Precision physics

The standard model of particle physics is extremely successful!

However: No new particle beyond the standard model discovered so far!

Discovery through precision:

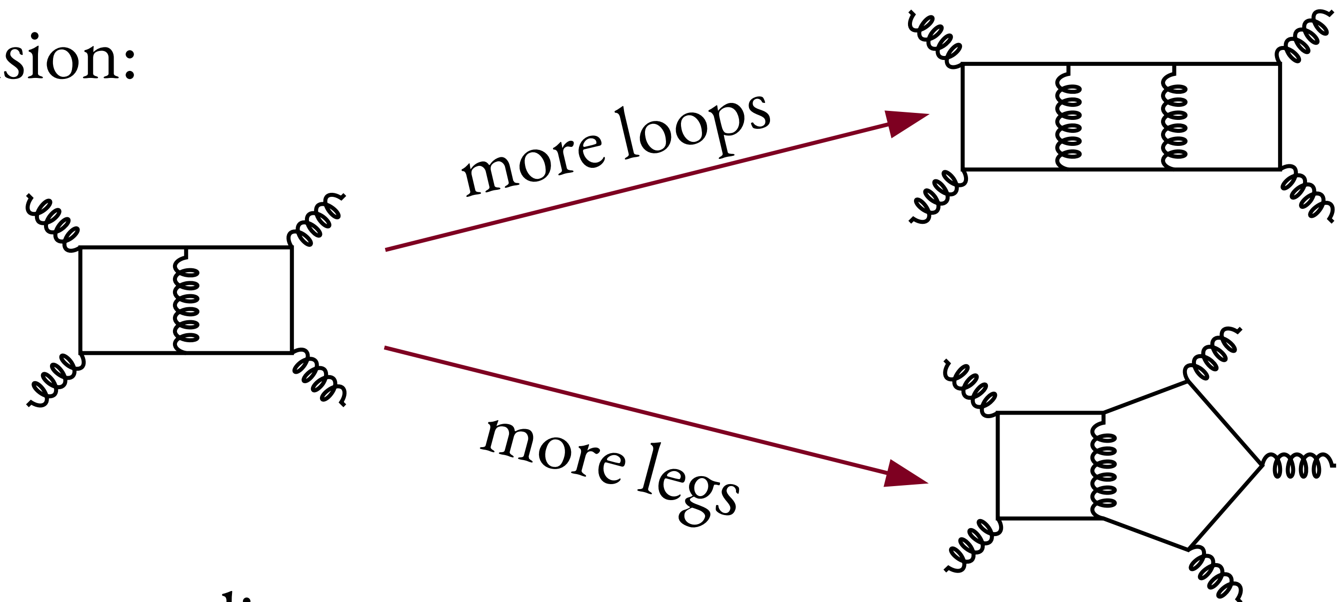
experimental measurements
+
theoretical predictions

→

search for deviations
from the standard model

Loop calculations

Increasing precision:



Challenges:

- number of Feynman diagrams
- size of intermediate expressions
- complexity of loop integrals
- ...

integration-by parts
(IBP) reduction

This talk:

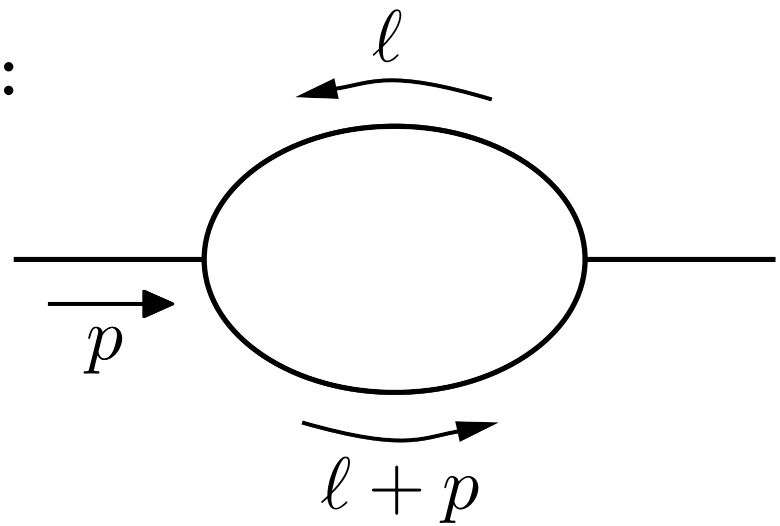
- new conceptually insights into IBP reduction

Overview

1. IBP reduction
2. Gröbner bases
3. IBP reduction with Gröbner bases
4. Linear algebra ansatz

IBP Reduction

Bubble integrals:

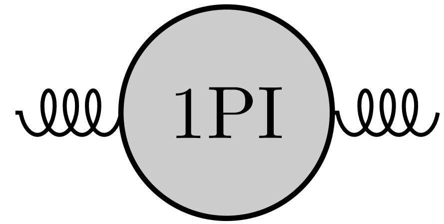


$(p^2 \neq 0)$

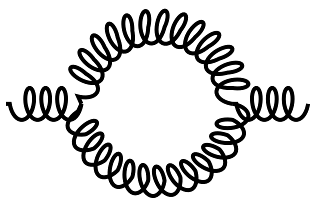
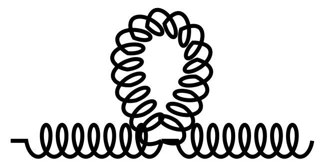
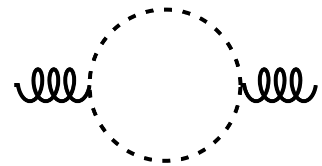
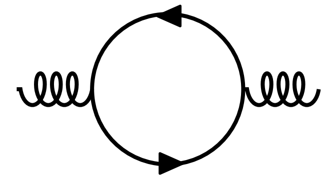
$$F(a_1, a_2) = \int \frac{\mathrm{d}^d \ell}{i\pi^{d/2}} \frac{1}{D_1^{a_1} \, D_2^{a_2}} \,, \qquad a_i \in \mathbb{Z}$$

$$\begin{aligned} D_1 &= -\ell^2 \\ D_2 &= -(\ell + p)^2 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \ell^2 &= -D_1 \\ \ell \cdot p &= \frac{1}{2}(D_1 - D_2 - p^2) \end{aligned}$$

Vacuum polarization

$$i\Pi^{\mu\nu} = i(p^2 g^{\mu\nu} - p^\mu p^\nu)\Pi(p^2) =$$ 

One-loop (R_ξ gauge):

$\Pi_{\text{1-loop}} =$  $+$  $+$  $+$ 

$$\propto C_A\Big[c_{1,1}F(1,1) + c_{1,2}F(1,2) + c_{2,1}F(2,1) + c_{2,2}F(2,2)\Big]$$

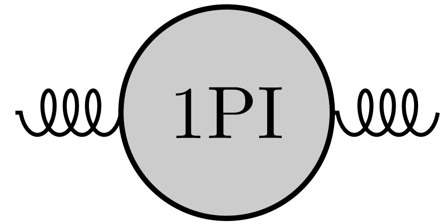
$$+ \frac{2(d-2)}{(d-1)} T_F n_f F(1,1)$$

IBP reduction

$$\propto \widetilde{c}_{1,1}F(1,1)$$

$\in \mathbb{Q}(d, p^2)$
(rational in d and p^2)
+ polynomial in ξ

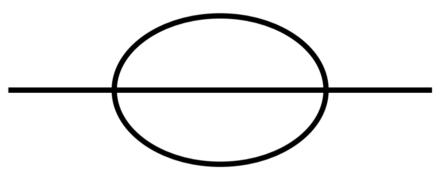
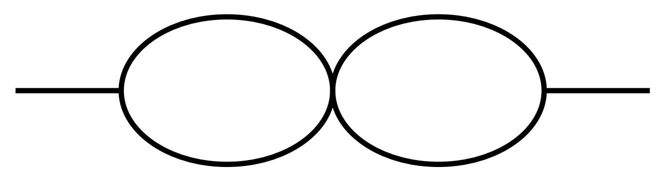
Vacuum polarization

$$i\Pi^{\mu\nu} = i(p^2 g^{\mu\nu} - p^\mu p^\nu)\Pi(p^2) =$$ 

Two-loop (R_ξ gauge):

$$\Pi_{\text{2-loop}} = \sum_{i=1}^{213} c_i \times (\text{scalar integral})_i$$

IBP reduction

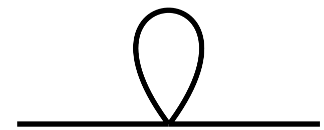
$$= \widetilde{c}_1 \times$$  $$+ \widetilde{c}_2 \times$$ 

two master integrals

Scaleless Integrals

Consider:

$$F(a_1, 0) = \int \frac{\mathrm{d}^d \ell}{i\pi^{d/2}} \frac{1}{[-\ell^2]^{a_1}}$$



Rescale loop momenta: $\ell \to \lambda \ell$

$$\Rightarrow \quad F(a_1, 0) = \lambda^{d-2a_1} F(a_1, 0) \quad \Rightarrow \quad F(a_1, 0) = 0$$

In general we have:

$$F(a_1, a_2) = 0\,, \text{if } a_1 \leq 0 \text{ or } a_2 \leq 0\,.$$

(Scaleless integrals can efficiently be identified with the help of the Feynman or Lee-Pomeransky representation.)

[Pak, Smirnov, '11 | Lee, '13]

Symmetry

The bubble integral family has the symmetry

$$F(a_1, a_2) = F(a_2, a_1)$$

This symmetry is realized by the following shift of loop momentum: $\ell \to -\ell - p$

reminder: $D_1 = -\ell^2$

$$D_2 = -(\ell + p)^2$$

(Symmetries can efficiently be found with the help of the Feynman or Lee-Pomeransky representation.) [Pak, '12 | Hoff '15]

IBP equations

[Tkachov, '81 | Chetyrkin, Tkachov, '81]

Starting point:

$$0 = \int \frac{\mathrm{d}^d \ell}{i\pi^{d/2}} \frac{\partial}{\partial \ell^\mu} \frac{v^\mu}{D_1^{a_1} \, D_2^{a_2}}$$

standard IBP equations:

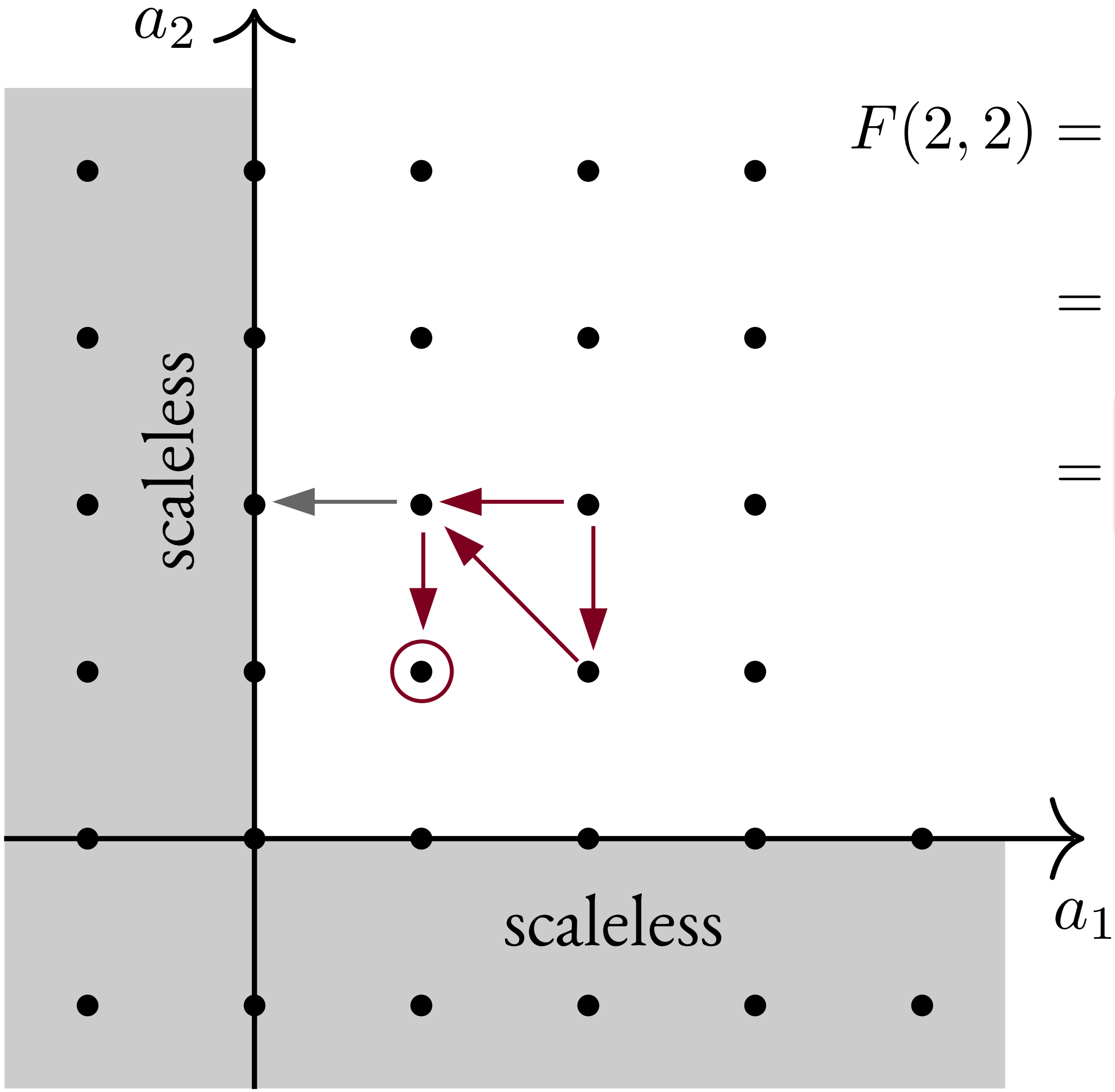
$\underline{v = \ell:}$

$$\begin{aligned} 0 = {} & (d - a_2 - 2a_1)F(a_1, a_2) - a_2 p^2 F(a_1, a_2 + 1) \\ & - a_2 F(a_1 - 1, a_2 + 1) \end{aligned}$$

$\underline{v = p:}$

$$\begin{aligned} 0 = {} & (a_1 - a_2)F(a_1, a_2) + a_2 p^2 F(a_1, a_2 + 1) - a_1 p^2 F(a_1 + 1, a_2) \\ & + a_2 F(a_1 - 1, a_2 + 1) - a_1 F(a_1 + 1, a_2 - 1) \end{aligned}$$

IBP Reduction by hand

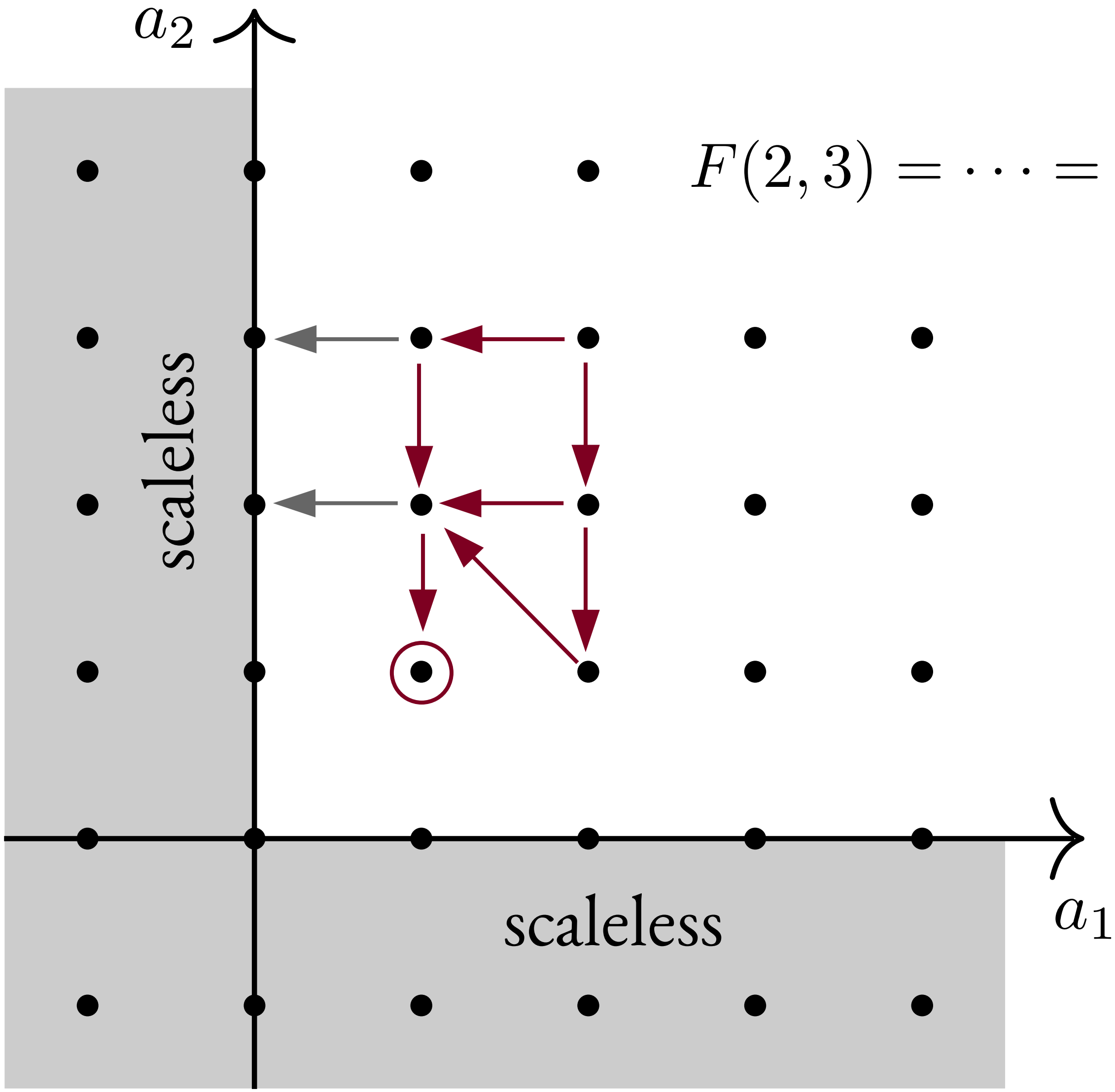


(using standard IBP eq.)

$$F(2,2) = \frac{1}{p^2}\left[(d-5)F(1,2) - F(2,1)\right]$$
$$= \frac{(d-6)}{p^2}F(1,2)$$
$$= \frac{(d-6)(d-3)}{p^4}F(1,1)$$
$$- \frac{(d-6)}{p^4}F(0,2)$$

master integral

IBP Reduction by hand



$$F(2,3) = \cdots = \frac{(d-8)(d-5)(d-3)}{2\,p^6} F(1,1)$$

master integral

(the bubble integral family has only one master integral)

(using standard IBP eq.)

Laporta's algorithm

Generate linear system of equations:

→ specialize set of IBP equations to a range of integer indices

$$\{0 = (d - a_2 - 2a_1)F(a_1, a_2) - a_2 p^2 F(a_1, a_2 + 1) \\ \qquad - a_2 F(a_1 - 1, a_2 + 1)\,, \\ \ldots\}$$

$$(a_1, a_2) \in \{(1,1),\ (1,2),\ (2,1),\ \ldots\}$$

Laporta's algorithm

Generate linear system of equations:

include symmetries

$$
\begin{aligned}
0 &= F(2,1) - F(1,2) \\
0 &= (d-3)F(1,1) - p^2 F(1,2) \\
0 &= (d-4)F(1,2) - 2p^2 F(1,3) \\
0 &= 2p^2 F(1,3) - p^2 F(2,2) - F(1,2) - F(2,1) \\
0 &= (d-5)F(2,1) - p^2 F(2,2) - F(1,2) \\
&\vdots
\end{aligned}
$$

directly discard scaleless integrals

Solve system by expressing "complicated" integrals in terms of "simpler" integrals.

[Laporta, '01]

IBP reduction

IBP reduction is a cornerstone of multi-loop calculations

Public IBP programs:

- FIRE6 [Smirnov, Chukharev, '19]
- Kira 2 [Klappert, Lange, Maierhöfer, Usovitsch, '20]
- Reduce 2 [Manteuffel, Studerus, '19]
- FiniteFlow [Peraro, '19]
- LiteRed [Lee, '13]

Gröbner bases

Warmup

Univariate polynomial division:

remainder

$$(x^3 + 2x)/(x+1) = x^2 - x + 3 + \frac{\boxed{-3}}{x+1}$$

Extend to multivariate case with several polynomials:

$$p = x_1^3 x_2^2 + x_2^2\,, \qquad q_1 = x_1^2 x_2 + 1\,, \qquad q_2 = x_1 x_2^2 + 1$$

⟶ decomposition: $p = p_1\, q_1 + p_2\, q_2 + \boxed{h}$

remainder

polynomials

⟶ Ideal membership problem

Ideal

Let R be a ring over the field $\mathbb{K}$. A subset $I \subset R$ is an ideal if:

- I is an additive subgroup of R
- $y \in R \wedge z \in I \Rightarrow yz \in I$

In our example we have a polynomial ring $R = \mathbb{Q}[x_1, x_2]$ with $p, q_1, q_2 \in \mathbb{Q}[x_1, x_2]$ and the ideal is given by:

$$I = \langle q_1, q_2 \rangle = \{f_1 q_1 + f_2 q_2 | f_1, f_2 \in \mathbb{Q}[x_1, x_2]\}$$

Question: $p \in I$?

Some Definitions

Multi-index notation: $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

A monomial order is a total order: $x^\alpha > x^\beta \Rightarrow x^\alpha x^\gamma > x^\beta x^\gamma$

Lexicographic order:

$$x^\alpha >_{\text{lex}} x^\beta \Leftrightarrow \text{ first nonzero entry of } \alpha - \beta > 0$$

Degree lexicographic order:

$$\mathrm{x}^\alpha >_{\text{dlex}} x^\beta \Leftrightarrow \begin{array}{c} (\deg x^\alpha > \deg x^\beta) \text{ or } (\deg x^\alpha = \deg x^\beta \text{ and} \\ \text{the first nonzero entry of } \alpha - \beta > 0) \end{array}$$

Example: $x_1^2 x_2 >_{\text{lex}} x_1 x_2^5 >_{\text{lex}} x_1 x_2^2$

$$x_1 x_2^5 >_{\text{dlex}} x_1^2 x_2 >_{\text{dlex}} x_1 x_2^2$$

Some Definitions

Multi-index notation: $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

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For $p \in R$, the lead term $\mathrm{LT}(p)$ is the largest term of p w.r.t. $>$

Division

Back to our example:

$$p = x_1^3 x_2^2 + x_2^2\,, \qquad q_1 = x_1^2 x_2 + 1\,, \qquad q_2 = x_1 x_2^2 + 1$$

First possibility:

$$p = (x_1 x_2) \cdot q_1 \; - x_1 x_2 + x_2^2$$

$\mathrm{LT}(p)/\mathrm{LT}(q_1)$ not divisible by $\mathrm{LT}(q_2)$

(using $>_{\text{dlex}}$)

Division

Back to our example:

$$p = x_1^3 x_2^2 + x_2^2\,, \qquad q_1 = x_1^2 x_2 + 1\,, \qquad q_2 = x_1 x_2^2 + 1$$

First possibility:

$$p = (x_1 x_2) \cdot q_1 + 0 \cdot q_2 \boxed{- x_1 x_2 + x_2^2}$$

(using $>_{\mathrm{dlex}}$)

Division

Back to our example:

$$p = x_1^3 x_2^2 + x_2^2\,, \qquad q_1 = x_1^2 x_2 + 1\,, \qquad q_2 = x_1 x_2^2 + 1$$

First possibility:

$$p = (x_1 x_2) \cdot q_1 + 0 \cdot q_2 \boxed{- x_1 x_2 + x_2^2}$$

Second possibility:

different non-vanishing remainders!

$$p = 0 \cdot q_1 + x_1^2 \cdot q_2 \boxed{- x_1^2 + x_2^2}$$

$\longrightarrow$ Naively $p \notin I$ (wrong!)

(using $>_{\text{dlex}}$)

Gröbner bases

Let I be an Ideal. A finite subset $G = \{g_1, \ldots, g_n\} \subset I$ is a Gröbner basis for I if $\mathrm{LT}(G) = \mathrm{LT}(I)$.

$\langle \mathrm{LT}(q) | q \in I \rangle$

Properties:

- G generates I
- unique remainder $p = \sum_k p_k g_k + \boxed{h}$

$$\longrightarrow h = 0 \Leftrightarrow p \in I$$

Remainder is called normal form $\boxed{\mathrm{NF}_G(p) = h}$

Gröbner basis can be computed with the Buchberger algorithm.

Back to our example:

$$p = x_1^3 x_2^2 + x_2^2\,, \qquad q_1 = x_1^2 x_2 + 1\,, \qquad q_2 = x_1 x_2^2 + 1$$

Ideal: $I = \langle q_1, q_2 \rangle$

Gröbner basis: $G = \{g_1, g_2\} = \{x_1 - x_2,\ x_2^3 + 1\}$

Decomposition: $p = (x_1^2 x_2^2 + x_1 x_2^3 + x_2^4) \cdot g_1 + x_2^2 \cdot g_2 + 0$

$\longrightarrow \boxed{p \in I}$

(using $>_{\text{dlex}}$)

Gröbner bases and IBP reduction

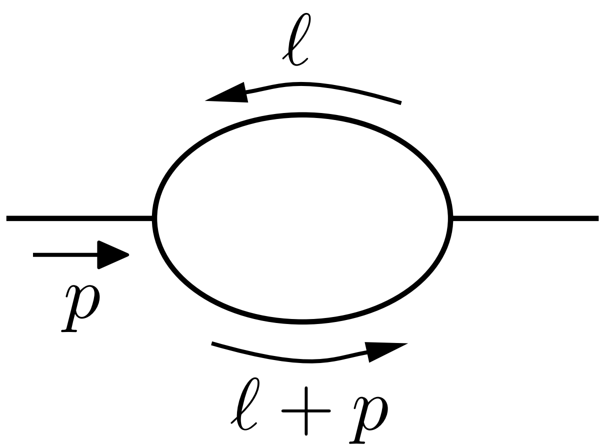
Related work

We are not the first to consider Gröbner bases for IBP reductions:

- Gröbner bases in combination with partial differential equation and dimension shift [Tarasov, '98, '04]
- Gröbner bases with shift algebras [Gerdt '05 (but chosen example is scaleless)]
- Maple implementation of IBP reduction with shift algebras [Gerdt, Robertz, '06]
- Sector bases [Smirnov, Smirnov, '05 - '08]
- IBPs and scaleless integrals as left and right ideal [Lee, '08]

Setup

Consider again the bubble integrals



$$F(a_1, a_2) = \int \frac{\mathrm{d}^d \ell}{i\pi^{d/2}} \frac{1}{D_1^{a_1}\, D_2^{a_2}}$$

Introduce operators with partial right action

$$F(a_1, a_2) \bullet \widehat{a}_1 = a_1 F(a_1, a_2) \qquad F(a_1, a_2) \bullet \widehat{D}_1 = F(a_1 - 1, a_2)$$

$$F(a_1, a_2) \bullet \frac{1}{\widehat{a}_1} = \frac{1}{a_1} F(a_1, a_2) \qquad F(a_1, a_2) \bullet \widehat{D}_1^{-} = F(a_1 + 1, a_2)$$

$a_1 \neq 0$

not scaleless

Algebra

We have a **non-commutative rational double-shift algebra**

$$Y := \mathbb{Q}(d, p^2)(\widehat{a}_1, \widehat{a}_2)\langle \widehat{D}_1, \widehat{D}_1^-, \widehat{D}_2, \widehat{D}_2^- \rangle$$

with relations:

$$[\widehat{a}_i, \widehat{D}_j] = \delta_{ij}\widehat{D}_i\,, \quad [\widehat{a}_i, \widehat{D}_j^-] = -\delta_{ij}\widehat{D}_i^-\,, \quad \widehat{D}_i\widehat{D}_i^- = \widehat{D}_i^-\widehat{D}_i = 1\,,$$

$$[\widehat{a}_i, \widehat{a}_j] = [\widehat{D}_i, \widehat{D}_j] = [\widehat{D}_i^-, \widehat{D}_j^-] = [\widehat{D}_i, \widehat{D}_j^-] = 0$$

(no summation over repeated indices)

IBP operators

First IBP operator $(v = \ell)$:

$$\begin{aligned} 0 &= (d - a_2 - 2a_1)F(a_1, a_2) - a_2 p^2 F(a_1, a_2 + 1) \\ &\quad - a_2 F(a_1 - 1, a_2 + 1) \\ &= F(a_1, a_2) \bullet \underbrace{\left[(d - \widehat{a}_2 - 2\widehat{a}_1) - p^2 \widehat{a}_2 \widehat{D}_2^- - \widehat{a}_2 \widehat{D}_1 \widehat{D}_2^-\right]}_{= \widehat{r}_1} \end{aligned}$$

Second IBP operator $(v = p)$:

$$\widehat{r}_2 = -\widehat{a}_1 \widehat{D}_1^- \widehat{D}_2 + \widehat{a}_2 \widehat{D}_1 \widehat{D}_2^- - p^2 \widehat{a}_1 \widehat{D}_2^- + (\widehat{a}_1 - \widehat{a}_2)$$

IBPs as a left ideal

IBP operators form a **left** ideal:

$$I_{\mathrm{IBP}} = \langle \widehat{r}_1, \widehat{r}_2 \rangle_Y = \{\widehat{f}_1 \widehat{r}_1 + \widehat{f}_2 \widehat{r}_2 | \widehat{f}_1, \widehat{f}_2 \in Y\}$$

$$f \in Y\,, r \in I_{\mathrm{IBP}} \quad \Rightarrow \quad f r \in I_{\mathrm{IBP}} \quad \text{but} \quad r f \notin I_{\mathrm{IBP}}$$

IBPs as a left ideal

IBP operators form a **left** ideal:

$$I_{\mathrm{IBP}} = \langle \widehat{r}_1, \widehat{r}_2 \rangle = \{ \widehat{f}_1 \widehat{r}_1 + \widehat{f}_2 \widehat{r}_2 | \widehat{f}_1, \widehat{f}_2 \in Y \}$$

By construction we have

$$\boxed{F(a_1, a_2)\widehat{r} = 0 \quad \text{for all} \quad \widehat{r} \in I_{\mathrm{IBP}}}$$

Reduced rational Gröbner basis (4 elements)

$$\begin{aligned} G = \{ &(d - \widehat{a}_1 - \widehat{a}_2)(d - 2\widehat{a}_1 - 2\widehat{a}_2 + 2)\widehat{D}_2 - p^2(\widehat{a}_2 - 1)(d - 2\widehat{a}_2), \\ &(d - \widehat{a}_1 - \widehat{a}_2)(d - 2\widehat{a}_1 - 2\widehat{a}_2 + 2)\widehat{D}_1 - p^2(\widehat{a}_1 - 1)(d - 2\widehat{a}_1), \\ &p^2 \widehat{a}_2 (d - 2\widehat{a}_2 - 2)\widehat{D}_2^- - (d - \widehat{a}_1 - \widehat{a}_2 - 1)(d - 2\widehat{a}_1 - 2\widehat{a}_2), \\ &p^2 \widehat{a}_1 (d - 2\widehat{a}_1 - 2)\widehat{D}_1^- - (d - \widehat{a}_1 - \widehat{a}_2 - 1)(d - 2\widehat{a}_1 - 2\widehat{a}_2) \} \end{aligned}$$

IBP reduction

$$F(2,2) = F(1,1) \bullet \left[\widehat{D}_1^- \widehat{D}_2^-\right]$$

reference integral

IBP reduction

$$
\begin{aligned}
F(2,2) &= F(1,1) \bullet \left[\widehat{D}_1^- \widehat{D}_2^-\right] \\
&= F(1,1) \bullet \left[\sum_{i=1}^{4} \underset{\uparrow}{f_i g_i} + \mathrm{NF}_G\left(\widehat{D}_1^- \widehat{D}_2^-\right)\right]
\end{aligned}
$$

annihilates any integral

IBP reduction

$$
\begin{aligned}
F(2,2) &= F(1,1) \bullet \left[\widehat{D}_1^- \widehat{D}_2^-\right] \\
&= F(1,1) \bullet \left[\sum_{i=1}^{4} f_i g_i + \mathrm{NF}_G\left(\widehat{D}_1^- \widehat{D}_2^-\right)\right] \\
&= F(1,1) \bullet \mathrm{NF}_G\left(\widehat{D}_1^- \widehat{D}_2^-\right)
\end{aligned}
$$

only need to compute normal form

IBP reduction

$$
\begin{aligned}
F(2,2) &= F(1,1) \bullet \left[\widehat{D}_1^- \widehat{D}_2^-\right] \\
&= F(1,1) \bullet \left[\sum_{i=1}^{4} f_i g_i + \mathrm{NF}_G\left(\widehat{D}_1^- \widehat{D}_2^-\right)\right] \\
&= F(1,1) \bullet \mathrm{NF}_G\left(\widehat{D}_1^- \widehat{D}_2^-\right) \\
&= F(1,1) \bullet \tfrac{(d-\widehat{a}_1-\widehat{a}_2-1)(d-\widehat{a}_1-\widehat{a}_2-2)(d-2\widehat{a}_1-2\widehat{a}_2)(d-2\widehat{a}_1-2\widehat{a}_2-2)}{p^4\widehat{a}_1\widehat{a}_2(d-2\widehat{a}_1-2)(d-2\widehat{a}_2-2)} \\
&= \frac{(d-6)(d-3)}{p^4} F(1,1)
\end{aligned}
$$

Note: Can compute IBP reduction w.r.t. any reference integral (e.g. $F(11,42)$) $\rightarrow$ different master integral

IBP reduction

Scaleless integrals are correctly identified:

$$F(1,1) \bullet \widehat{D}_1 = F(1,1) \bullet \mathrm{NF}_G(\widehat{D}_1) = F(1,1) \bullet (\widehat{a}_1 - 1)(\dots) = 0$$

$$F(0,1) = 0$$

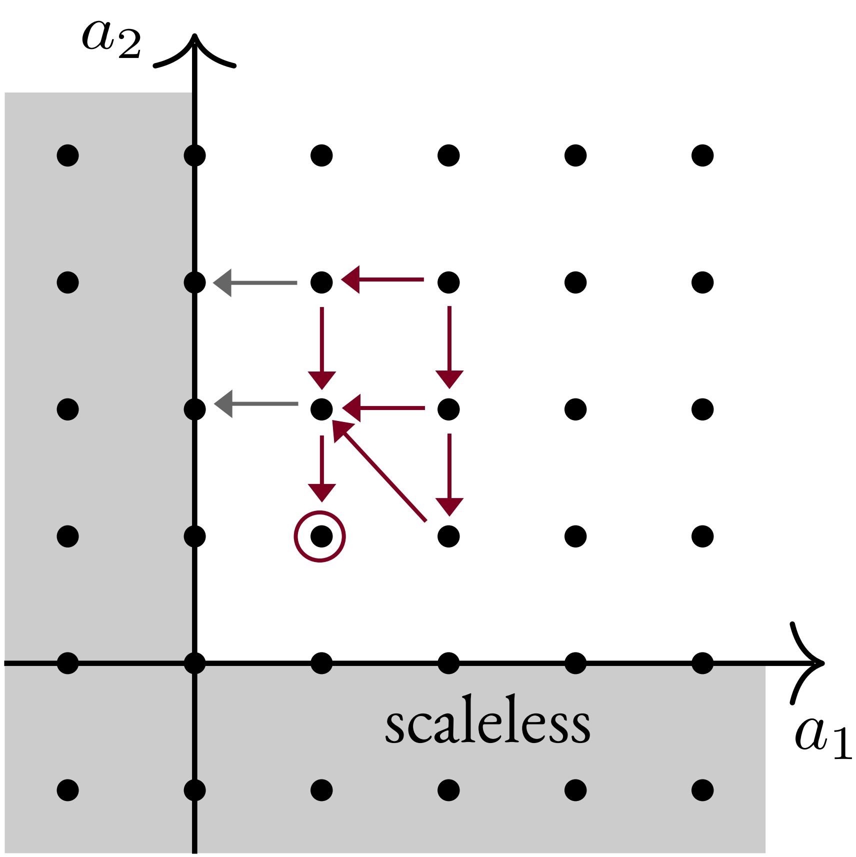
Particular interesting are IBP operators of the form

$$\widehat{R}_i = \widehat{a}_i \widehat{D}_i^- - \mathrm{NF}_G(\widehat{a}_i \widehat{D}_i^-) \in I_{\mathrm{IBP}}$$

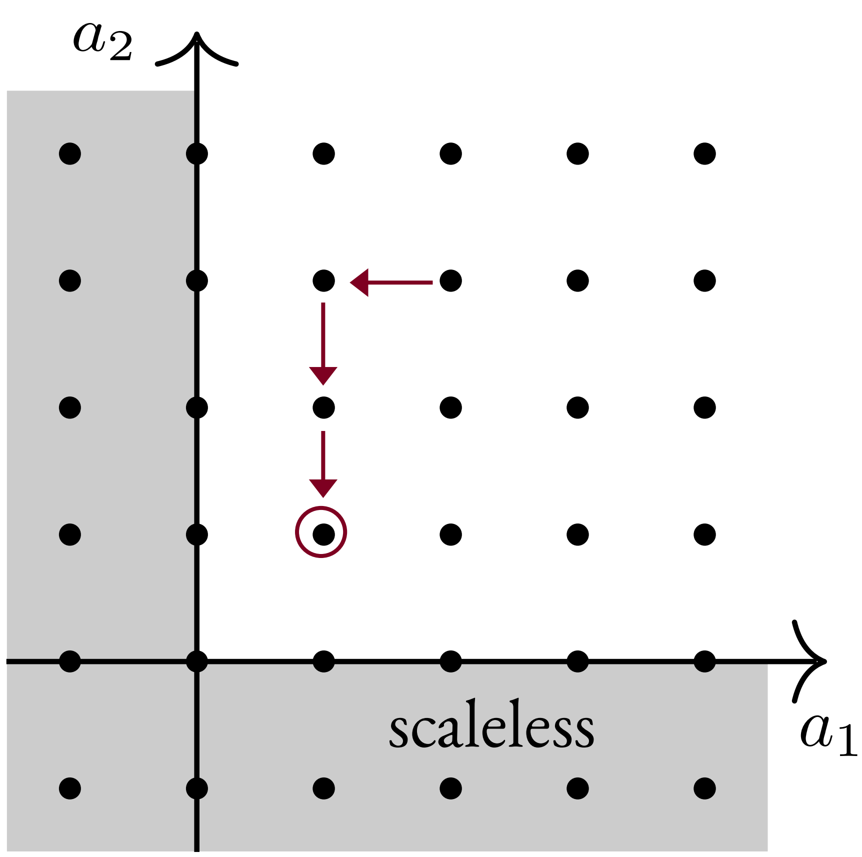
with: $\mathrm{NF}_G(\widehat{a}_1 \widehat{D}_1^-) = \dfrac{(d - \widehat{a}_1 - \widehat{a}_2 - 1)(d - 2\widehat{a}_1 - 2\widehat{a}_2)}{(d - 2\widehat{a}_1 - 2)p^2}$

Reduction of $F(3,2)$:

using standard IBPs

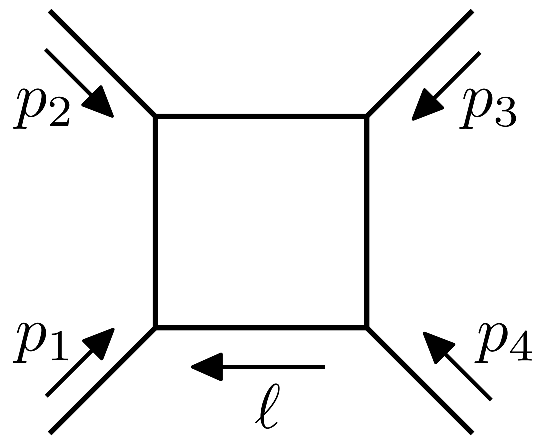


using $\widehat{R}_i = \widehat{a}_i \widehat{D}_i^- - \mathrm{NF}_G(\widehat{a}_i \widehat{D}_i^-)$



Box integrals

$$F(a_1, a_2, a_3, a_4) =$$

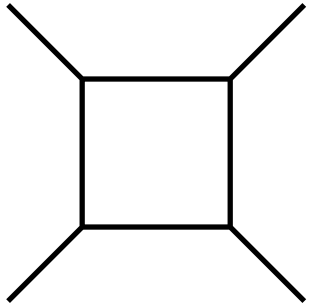


$$D_1 = -\ell^2$$
$$D_2 = -(\ell - p_1)^2$$
$$D_3 = -(\ell - p_1 - p_2)^2$$
$$D_4 = -(\ell + p_4)^2$$

- kinematics: $p_i^2 = 0\,, \quad p_1 \cdot p_2 = s_{12}\,, \quad p_1 \cdot p_4 = s_{14}$
- 4 standard IBP equations
- algebra: $Y := \mathbb{Q}(d, s_{12}, s_{14})(\widehat{a}_1, \ldots, \widehat{a}_4)\langle \widehat{D}_i, \widehat{D}_i^- \,|\, i = 1, \ldots, 4\rangle$

Box integrals

The Gröbner basis has 9 elements:



$$G = \left\{ \widehat{D}_4 - \widehat{D}_2 + \frac{(\widehat{a}_2 - \widehat{a}_4)s_{12}}{d - \widehat{a}_{1234}}, \ldots \right\}$$

IBP reduction with Gröbner basis easy to implement in Mathematica or FORM (+ fast and parallelization trivial)

$F(10,10,10,10)$, ~ 5 sec.

```
FORM 4.2.1 (Jul  7 2022, v4.2.1-40-g982111a) 64-bits
    #-
    Local Expr = Int(10,10,10,10);
    #call ReductionBox

    .sort
    On Statistics;
    .end

Time =       4.51 sec    Generated terms =         3
             Expr        Terms in output =         3
                         Bytes used      =    202756
```

$F(10,10,10,-10)$, ~ 3 sec.

```
FORM 4.2.1 (Jul  7 2022, v4.2.1-40-g982111a) 64-bits
    #-
    Local Expr = Int(10,10,10,-10);
    #call ReductionBox

    .sort
    On Statistics;
    .end

Time =       2.86 sec    Generated terms =         1
             Expr        Terms in output =         1
                         Bytes used      =     18044
```

Standard monomials

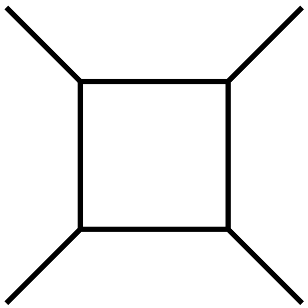
A standard monomial f is a monomial in $\widehat{D}_i, \widehat{D}_i^-$, such that:

$$\mathrm{NF}_G(f) = f$$

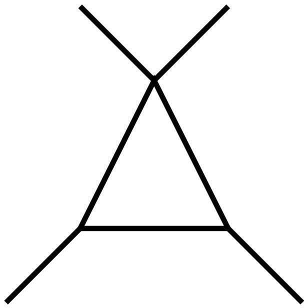
→ corresponds to master integral

Example box integrals:

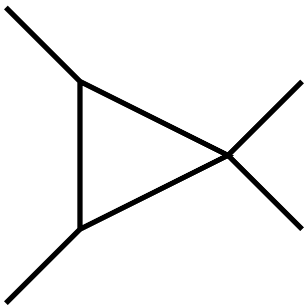
$\mathrm{NF}_G(1) = 1$



$\mathrm{NF}_G(\widehat{D}_3) = \widehat{D}_3$



$\mathrm{NF}_G(\widehat{D}_4) = \widehat{D}_4$



(reference integral: $F(1,1,1,1)$)

Computing a Gröbner bases

The computations were done with the GAP package `LoopIntegrals`

https://homalg-project.github.io/pkg/LoopIntegrals

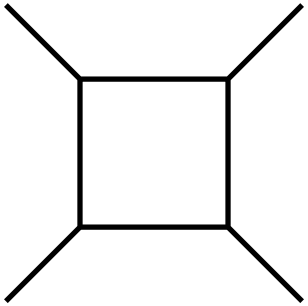
Dependencies:

[Decker, Greuel, Pfister, Schönemann, '19]

- SINGULAR for commutative Gröbner bases in polynomial rings
- PLURAL for non-commutative Gröbner bases in the double-shift algebra with polynomial coefficients $(\to \widehat{a}_i)$ [Levandovskyy, Schönemann, '03]
- Chyzak's Maple package `Ore_algebra` for non-commutative Gröbner bases in the double-shift algebra with rational coefficients $(\to \widehat{a}_i)$ [Chyzak, '98]
- The Julia package HECKE for simulating the Gröbner bases computation with linear algebra to obtain IBPs of the form $\widehat{R}_i = \widehat{a}_i \widehat{D}_i^- - \mathrm{NF}_G(\widehat{a}_i \widehat{D}_i^-)$ (last part of this talk)

LoopIntegrals

Box integrals



```
In [17]: LoadPackage( "LoopIntegrals" )

In [18]: R = RingOfLoopDiagram( LD )

Out[18]: GAP: Q[D,s12,s14][D1,D2,D3,D4]

In [19]: Ypol = DoubleShiftAlgebra( R )

Out[19]: GAP: Q[D,s12,s14][a1,a2,a3,a4]<D1,D1_,D2,D2_,D3,D3_,D4,D4_>/( D1*D1_-1, D2*D2_-1, D3*D3_-1, D4
*D4_-1 )
```

polynomial $\widehat{a}_i$

```
In [20]: ibps = MatrixOfIBPRelations( LD )

Out[20]: GAP: <A 4 x 1 matrix over a residue class ring>

In [21]: r1 = ibps[1,1]

Out[21]: GAP: |[ -a2*D1*D2_-s12*a3*D3_-a3*D1*D3_-a4*D1*D4_+D-2*a1-a2-a3-a4 ]|

In [22]: bas_pol = BasisOfRows( ibps )

Out[22]: GAP: <A non-zero 28 x 1 matrix over a residue class ring>

In [23]: NormalForm( "a1*D1_" / Ypol, bas_pol )

Out[23]: GAP: |[ a1*D1_ ]|
```

reduction does not work!

The following command needs Chyzak's Maple package `Ore_algebra` for the noncommutative Gröbner bases of the rational double-shift algebra:

```
In [24]: Y = RationalDoubleShiftAlgebra( R )

Out[24]: GAP: Q(D,s12,s14)(a1,a2,a3,a4)<D1,D1_,D2,D2_,D3,D3_,D4,D4_>/( D1*D1_-1, D2*D2_-1, D3*D3_-1, D4
         *D4_-1 )
```

rational $\widehat{a}_i$

```
In [25]: ribps = Y * ibps

Out[25]: GAP: <A 4 x 1 matrix over a residue class ring>

In [26]: bas = BasisOfRows( ribps )

Out[26]: GAP: <A non-zero 9 x 1 matrix over a residue class ring>

In [27]: NormalForm( "a1*D1_" /  Y, bas )

Out[27]: GAP: |[ -2*(D-2*a1-2*a2-2*a4)*(D-a1-a2-a3-a4)*(-a4-1+D-a1-a2-a3)*D3/(D-2*a1-2*a4-2)/(D-2*a1-2*
         a2-2)/s12/s14+4*(a3-1)*(D-a1-a2-a3-a4)*(-a4-1+D-a1-a2-a3)*D4/(D-2*a1-2*a4-2)/(D-2*a1-2*a2-2)/s
         12/s14+(-a4-1+D-a1-a2-a3)*(D-2*a1-2*a3-2*a4)/(D-2*a1-2*a4-2)/s12 ]|
```

reduction works!

```
In [28]: NormalFormWrtInitialIntegral( "D1_" / Y, bas )

Out[28]: GAP: |[ -2/(D-6)*(D-4)*(-5+D)*D3/s12/s14+(-5+D)/s12 ]|

In [29]: NormalFormWrtInitialIntegral( "D3_" / Y, bas )

Out[29]: GAP: |[ -2/(D-6)*(D-4)*(-5+D)*D3/s12/s14+(-5+D)/s12 ]|

In [30]: NormalFormWrtInitialIntegral( "D1*D2" / Y, bas )

Out[30]: GAP: |[ 0 ]|
```

reduction w.r.t. $F(1,1,1,1)$

scaleless integral

The Jupyter notebook can be found online:

https://homalg-project.github.io/nb/1LoopBox/

Linear algebra ansatz

Linear algebra ansatz

Especially interested in the IBPs:

$$\widehat{R}_i = \widehat{a}_i \widehat{D}_i^- - \mathrm{NF}_G(\widehat{a}_i \widehat{D}_i^-)$$

- allow for easy reduction (at least of the top-level sector)
- in all considered problems these generate the left ideal:

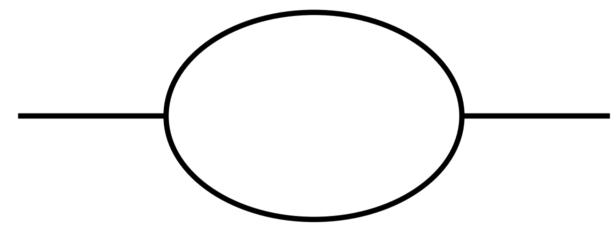
$$\langle \widehat{R}_i | i = 1, \ldots, n \rangle_Y = I_{\mathrm{IBP}}$$

number of propagators

do NOT expect to be true for all integral families!

Idea: Use linear algebra to compute $\widehat{R}_i$ when Gröbner basis is not available.

Consider again the bubble integrals



Start with a generating set of I_{IBP}, e.g. the standard IBPs

$$
\begin{aligned}
\widehat{r}_1 &= (d - \widehat{a}_2 - 2\widehat{a}_1) - p^2 \widehat{a}_2 \widehat{D}_2^- - \widehat{a}_2 \widehat{D}_1 \widehat{D}_2^- && \text{“} = 0\text{”} \\
\widehat{r}_2 &= -\widehat{a}_1 \widehat{D}_1^- \widehat{D}_2 + \widehat{a}_2 \widehat{D}_1 \widehat{D}_2^- - p^2 \widehat{a}_1 \widehat{D}_2^- + (\widehat{a}_1 - \widehat{a}_2) && \text{“} = 0\text{”} \\
\widehat{D}_1^- \widehat{r}_1 &= (d - \widehat{a}_2 - 2\widehat{a}_1 - 2) \widehat{D}_1^- - p^2 \widehat{a}_2 \widehat{D}_1^- \widehat{D}_2^- - \widehat{a}_2 \widehat{D}_2^- && \text{“} = 0\text{”} \\
&\vdots
\end{aligned}
$$

$(\widehat{D}_1^-)^{j_1} (\widehat{D}_2^-)^{j_2} \widehat{r}_i$

treat as unknowns

schematically

Solve for $\widehat{D}_1^-$ and $\widehat{D}_2^-$

$\longrightarrow$ reproduce result of Gröbner basis calculation

Special IBPs

Choice on generating set for I_{IBP} has impact on required max. values for j_1 and j_2 in $(\widehat{D}_1^-)^{j_1}(\widehat{D}_2^-)^{j_2}\widehat{r}_i$

Observation: Special IBPs (unitarity compatible) need lower max. values than standard IBPs

$$0 = \int \frac{\mathrm{d}^d \ell}{i\pi^{d/2}} \frac{\partial}{\partial \ell^\mu} \frac{v^\mu}{D_1^{a_1}\, D_2^{a_2}}$$

$$v^\mu = C_1(D_1, D_2)\ell^\mu + C_2(D_1, D_2)p^\mu$$

polynomial dependence

consider

$$v^\mu \frac{\partial}{\partial \ell^\mu} \frac{1}{D_1^{a_1} D_2^{a_2}} = \sum_k v^\mu \frac{\partial D_k}{\partial \ell^\mu} \boxed{\frac{\partial}{\partial D_k} \frac{1}{D_1^{a_1} D_2^{a_2}}}$$

increases exponent $\to \widehat{D}_k^-$

Special IBPs

Choose $C_i(D_1, D_2)$ such that $v^\mu \frac{\partial D_k}{\partial \ell^\mu} \propto D_k$

$$C_1(D_1, D_2)\ell^\mu + C_2(D_1, D_2)p^\mu$$

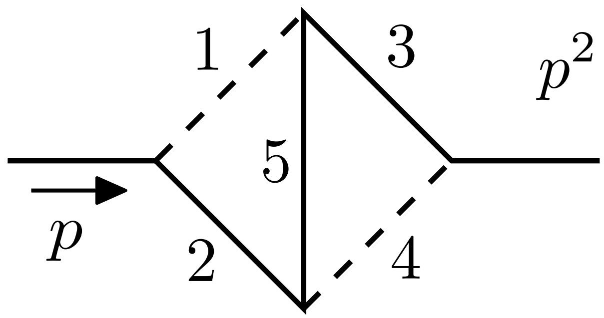
⟶ IBPs free of $\widehat{D}_i^-$

⟶ special or unitarity compatible IBPs

[Kosower et al. '10, '18 | Schabinger et al. '11, '20 | Ita '15 | Böhm et al. '17]

To find $C_i(D_1, D_2)$ we need to compute **syzygies**. However, the computation takes place in a polynomial ring.

On-shell kite



$$
\begin{aligned}
D_1 &= -\ell_1^2 \\
D_2 &= -(\ell_1 + p)^2 + m^2 \\
D_3 &= -(\ell_2 + p)^2 + m^2 \\
D_4 &= -\ell_2^2 \\
D_5 &= -(\ell_1 + \ell_2 + p)^2 + m^2
\end{aligned}
$$

Gröbner basis not available

⟶ use linear algebra ansatz to compute $\widehat{R}_i = \widehat{a}_i \widehat{D}_i^- - \mathrm{NF}(\widehat{a}_i \widehat{D}_i^-)$

$$\mathrm{NF}(\widehat{a}_1 \widehat{D}_1^-) = \frac{p_{10}}{4 d_1 d_2 d_3 d_4 d_7 d_8 s} + \frac{p_{12}\widehat{D}_2 + p_{13}\widehat{D}_3 + p_{14}\widehat{D}_4 + p_{15}\widehat{D}_5}{16 d_1 d_2 d_3 d_4 d_7 d_8 d_9 m^4}$$

- p_{ij} too large for printing (polynomial in $\widehat{a}_i$, d)
- d_i simple, e.g. $d_3 = 2\widehat{a}_1 + \widehat{a}_2 - d + 1$
- Allows for reduction of the top level sector

Conclusion

Conclusion

Gröbner basis theory can be used to perform IBP reductions of loop integrals:

- IBPs form a left ideal I_{IBP} in a rational double-shift algebra Y
- Reduction via computing normal forms w.r.t. Gröbner basis

  $\longrightarrow$ fast + parallelization trivial
- Problem: Gröbner basis is computationally expensive

Linear algebra ansatz to compute normal forms when Gröbner basis is not available.

Backup

Buchberger algorithm

Ideal: I= $\langle q_1, q_2 \rangle$ with $\mathrm{q}_1 = x_1^2 x_2 + 1$, $q_2 = x_1 x_2^2 + 1$

Idea: The lead term of any element of the ideal should be divisible by the lead term of at least one generator

Consider: $q_3 = -x_1 + x_2$

- division algorithm: $q_3 = 0 \cdot q_1 + 0 \cdot q_2 - x_1 + x_2$
- however: $q_3 = x_2 \cdot q_1 - x_1 \cdot q_2 \in I$

lin. comb. removes leading terms of q_1, q_2

(using $>_{\mathrm{dlex}}$)

Buchberger algorithm

Ideal: I$= \langle q_1, q_2 \rangle$ with q$_1 = x_1^2 x_2 + 1\,,\; q_2 = x_1 x_2^2 + 1$

Idea: The lead term of any element of the ideal should be divisible by the lead term of at least one generator

Consider: $-x_1 + x_2$

- division algorithm: $-x_1 + x_2 = 0 \cdot q_1 + 0 \cdot q_2 - x_1 + x_2$
- however: $-x_1 + x_2 = x_2 \cdot q_1 - x_1 \cdot q_2 \in I$

$$\frac{\mathrm{LT}(q_2)}{\gcd(\mathrm{LT}(q_1), \mathrm{LT}(q_2))} = \frac{x_1 x_2^2}{x_1 x_2}$$

(using $>_{\mathrm{dlex}}$)

Buchberger algorithm

S-polynomial:

$$S(q_i, q_j) = \frac{\mathrm{LT}(q_j)}{\gcd(\mathrm{LT}(q_i), \mathrm{LT}(q_j))} q_i - \frac{\mathrm{LT}(q_i)}{\gcd(\mathrm{LT}(q_i), \mathrm{LT}(q_j))} q_j = -S(q_j, q_i)$$

Buchberger's criterion:

$G = \{g_1, \ldots, g_n\} \subset I$ is a Gröbner basis iff the remainders of all S-polynomials $S(g_i, g_j)$ are zero.

⟶ use for algorithm

Buchberger algorithm

Start: $S(q_1, q_2) = 0 \cdot q_1 + 0 \cdot q_2 \underbrace{\boxed{- x_1 + x_2}}_{=:\, q_3}$ (add to basis)

Next: $S(q_1, q_3) = 0$ (nothing to do)

Next: $S(q_1, q_3) = 0 \cdot q_1 + 0 \cdot q_2 + 0 \cdot q_3 = \underbrace{\boxed{-x_2^3 - 1}}_{=:\, q_4}$ (add to basis)

Done: $S(q_i, q_j) = 0\,,\ i, j = 1, \ldots, 4$ (Buchberger's criterion)

Gröbner basis: $G' = \{q_1, q_2, q_3, q_4\}$

(simplify)

Reduced Gröbner basis: $G = \{x_1 - x_2,\ x_2^3 + 1\}$

(using $>_{\text{dlex}}$)