Correlators, topological field theory and categorification

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Based on years of work with many people, including G. Felder, J. Fröhlich, J. Fuchs, I. Runkel, J. Fjelstad,
most recent developments: Jürgen Fuchs and Yang Yang

January 13, 2023

Overview

Chapter 1

Solving a quantum field theory

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How can the idea of “solving a QFT” be translated into a precise mathematical problem?

What we need

1. A class of “space times”.

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- World sheet theory for strings
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Field insertions and invariance under mapping class groups
$\rightsquigarrow$ promote to a bicategory of extended cobordisms.
It is actually convenient to include embeddings as additional 1-morphisms and to work with a double category.

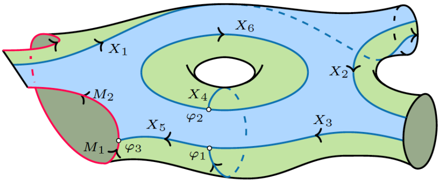
Boundaries and defects

For structural investigations, boundaries and defects give additional insights.

Motivation:

1. D-branes in string theory
2. Defects in systems of condensed matter physics
3. Percolation probabilities
4. (Topological) defects describe symmetries and dualities

Quantum field theory on stratified manifolds



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2. Boundary conditions, defect types.
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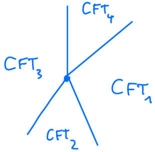
3. Specify the field content.
   Most general field: multipronged field, with several adjacent CFTs, separated by topological defects:



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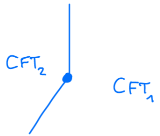

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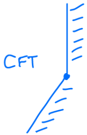
   Other special case: boundary fields:



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1. A class of “space times”.
2. Boundary conditions, defect types.
   Here: conformal boundary conditions, topological defects
3. Specify the field content.
4. Task: ideally, compute all correlation functions as a function of the insertion points and the “shape of the manifold”.

What we need to determine

1. A class of “space times”.
2. Boundary conditions, defect types.
   Here: conformal boundary conditions, topological defects
3. Specify the field content.

Plan

- Restrict to theories with “enough symmetries”
- Define the notion of a “consistent set of correlators” and prove its existence
- Extract computable quantities
- Control relations between different correlators

What are these symmetries?

Answer: symmetries of a two-dimensional conformal field theory are encoded in a vertex algebra $\mathcal{V}$

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What are vertex algebras?
Brief answer: generalizations of chiral current symmetries.
details are not relevant for this talk.
Vertex algebras are a very active field. They are the topic of the Emmy-Noether group of Sven Möller that is affiliated with QU.

What are these symmetries?

How do you think about a quantum mechanical system with $\mathrm{SU}(2)$-symmetry?

First way:

Endow the three-sphere $\mathbb{S}^3$ with a group structure by identifying it with complex square matrices

$$\begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix} \quad \text{with } |z_1|^2 + |z_2|^2 = 1$$

and define a unitary action on a Hilbert space.

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Second way:

Know the irreducible unitarizable representations of $\mathrm{SU}(2)$ - they are labelled by spin $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

In quantum mechanics, we highlight the importance of:

- Coupling of spins, e.g. $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$.
- Clebsch-Gordan coefficients

$$|J, M, j_1, j_2\rangle = \sum_{m_1, m_2} |j_1, m_1; j_2, m_2\rangle \langle j_1, m_1; j_2, m_2 | J, M, j_1, j_2\rangle$$

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Recall Wigner-Eckhart theorem for a tensor operator $T^{(k)}_q$

$$\langle \alpha', j' m' | T^{(k)}_q | \alpha, jm\rangle = \langle jm; kq | j' m'\rangle \langle \alpha' j' \| T^{(k)} \| \alpha j\rangle$$

One factor is determined by the symmetries of the system, the other contains information on the dynamics of the specific system.

What are these symmetries?

We will work in a similar way:

- Organize field content in representations of the vertex algebra $\mathcal{V}$
- Get constraints on correlation functions which lead to building blocks for correlators (conformal blocks).
  They are invariant tensors as the Clebsch-Gordan coefficients are.
- Find a geometric description for the conformal blocks
  and then a geometric prescription for the Wigner-Eckhart coefficients.

Conformal blocks

Correlators are solutions of differential equations expressing the $\mathcal{V}$-symmetry. Famous example: Knizhnik–Zamolodchikov equations for current symmetries based on Lie algebra $\mathfrak{g}$ with basis t^a:

$$\left((k+h)\partial_{z_i} + \sum_{j \neq i} \frac{\sum_{a,b} \eta_{ab} t_i^a \otimes t_j^b}{z_i - z_j} \right) \langle \Phi(v_N, z_N) \dots \Phi(v_1, z_1) \rangle = 0$$

Conformal blocks

Correlators are solutions of differential equations expressing the $\mathcal{V}$-symmetry.
Fact: solutions of differential equations on complex domains are multivalued.

Example

Hypergeometric equation:

$$z(1-z)w'' - (1 - \frac{5}{2}z)w' - \frac{1}{2}w = 0$$

has the multivalued solution $(1-z)^{-\frac{1}{2}}$.

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Conformal blocks

Multivalued functions over the moduli space $\mathcal{M}_{g,n}$ of n-punctured Riemann surfaces of genus g.

$\longleftrightarrow$ Representations of $\pi_1(\mathcal{M}_{g,n}) = \mathrm{Map}(\Sigma_{g,n})$ Mapping class group.

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We have traded vertex algebras and conformal blocks for a set of representations of $\mathrm{Map}(\Sigma_{g,n})$, consistent with gluing ('modular functor")

Definition 2.1. An *open-closed modular functor* is a symmetric monoidal 2-functor

$$\mathrm{Bl} : \quad \mathcal{B}ord^{\mathrm{or}}_{2,\mathrm{o/c}} \longrightarrow \mathcal{P}rof_{\Bbbk} \tag{2.1}$$

from the bicategory $\mathcal{B}ord^{\mathrm{or}}_{2,\mathrm{o/c}}$ of two-dimensional open-closed bordisms to the bicategory of $\Bbbk$-linear profunctors.

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Constraints for correlators:

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- compatible under gluing (factorization)

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Constraints for correlators:

Definition 2.20. A *consistent system of correlators*, for a given spherical fusion category $\mathcal{C}$, is a collection of vectors $\mathrm{Cor}(\mathcal{S}) \in \mathrm{Bl}_{\mathcal{C}}(\mathcal{S})$, one for each world sheet $\mathcal{S}$, that satisfy:
(i) Invariance under the mapping class group of $\mathcal{S}$:

$$\gamma(\mathrm{Cor}(\mathcal{S})) = \mathrm{Cor}(\mathcal{S}) \tag{2.49}$$

for every mapping class $\gamma \in \mathrm{Map}(\mathcal{S})$.
(ii) Compatibility with sewing:

$$\mathrm{Cor}(\cup_{b,b'}\mathcal{S}) = \mathrm{s}_{b,b'}(\mathrm{Cor}(\mathcal{S})) \tag{2.50}$$

for every sewing of a world sheet along gluing circles, and

$$\mathrm{Cor}(\cup_{r,r'}\mathcal{S}) = \mathrm{s}_{r,r'}(\mathrm{Cor}(\mathcal{S})) \tag{2.51}$$

for every sewing of a world sheet along gluing intervals.

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Reference (January 2023)

Chapter 2

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Task: Find a concise description of conformal blocks based on data extracted from $\mathcal{C} = \mathcal{V}\mathrm{-rep}$.

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Avoid Reshetikhin-Turaev (=CS type) TFT in 3 dimensions:

- 3-manifolds involve "mental gymnastics"
- existence on arbitrary 3-manifolds conceptually restrictive

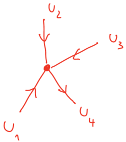
Rather, using progress in the last decade:

- Two-dimensional approach
- Geometric implementation of the mapping class group

Stringnets (Lewin-Wen 2005)

Find a concise description of conformal blocks. Keep the following data from $\mathcal{C} = \mathcal{V}\text{−rep}$.
Keep the following data:

- The representations ("spins") and their intertwiners
- Couplings=invariant tensors



$$\mathrm{Hom}_{\mathcal{C}}(1, U_1 \otimes U_2 \otimes U_3 \otimes U_4^*)$$

cyclic invariance

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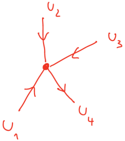

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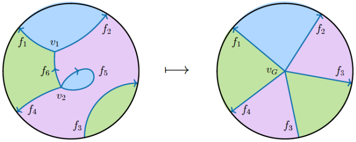
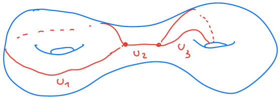
cyclic invariance

Graphical calculus on discs



Comments on string nets



String-net space for Σ closed oriented:

$$\mathrm{SN}_{\mathcal{C}}(\Sigma) = \mathcal{C}\text{-Graph}(\Sigma)/\text{local relations}$$

- Graphical calculus on discs gives local relations
- With suitable input data, this construction can be generalized to higher dimensions.
  Topological field theory in higher dimensions is a very active field. It is the topic of the Emmy-Noether group of David Reutter who is also affiliated with QU.

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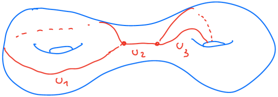

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- Canonically isomorphic to the ones obtained by state-sum construction

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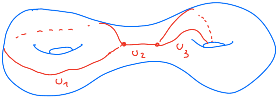

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- $\mathcal{C} \boxtimes \mathcal{C}^{rev}$ describes the combination of chiral and antichiral symmetries.
  Bulk fields are thus naturally objects of $\mathcal{C} \boxtimes \mathcal{C}^{rev} \cong Z(\mathcal{C})$.
- Geometric action of mapping class group

Chapter 3

Fields and correlators

We have now a geometric and tractable description of the holonomies of conformal blocks which are chiral quantities.

We now turn to fields and correlators of a full, local two-dimensional conformal field theory.

Bulk fields

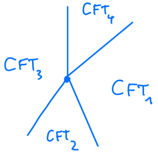
- Combination of left movers and right movers

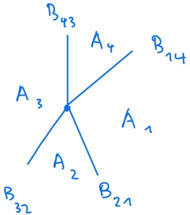
$$\mathbb{F} \in \mathcal{C} \boxtimes \mathcal{C}^{rev} \cong Z(\mathcal{C})$$

- Chiral symmetries are not enough to fix $\mathbb{F}$:
  depend on "choice of modular invariant"
- Categorification:
  $\mathcal{C}$ has a tensor product (coupling of spins), i.e. is a like a ring or an algebra.
  Look for a representation of this algebra in the world of categories:

$$c \otimes m \in \mathcal{M} \quad \text{for } c \in \mathcal{C} \text{ and } m \in \mathcal{M}$$

  Mixed tensor product: Module category.
- Realize $M = \text{mod-}A$ for an algebra $A \in \mathcal{C}$.
- Find descritpion for bulk fields $\mathbb{F}_{\mathcal{M}}$ and, more generally for multipronged defect fields:





A_i are (special symmetric) Frobenius algebras
B_{ij} are (traced) bimodules

Bulk fields in the cylinder category

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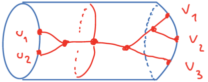

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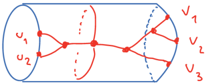

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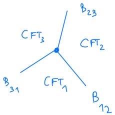
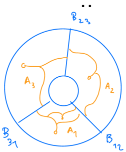


take idempotent



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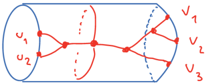

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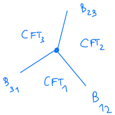

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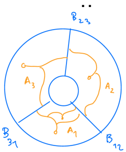

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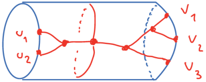

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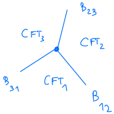

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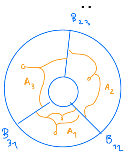


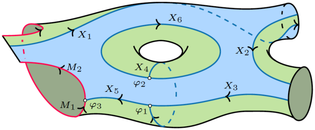
Remarks

- The bulk fields can be proven to be modular invariant.
- Generalizes all known expressions of the bulk fields. The most beautiful expression I know for them:

$$\mathbb{F}_{\mathcal{M}} = \int_{m \in \mathcal{M}} \underline{\mathrm{Hom}}(m, m) \in \mathcal{Z}(\mathcal{C})$$

Correlators

Now we construct correlators as elements of the string net space: for the worldsheet



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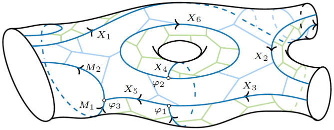

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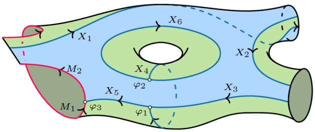

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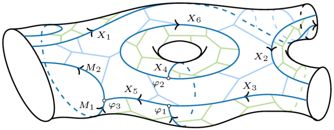
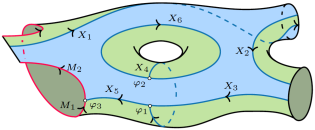


Remarks

- Invariance under the mapping class group action follows from the interplay of Frobenius algebras and Pachner moves.
- Factorization is a theorem.

Correlators

Now we construct correlators as elements of the string net space: for the worldsheet



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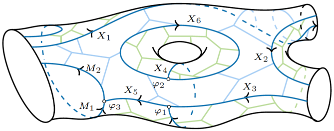

Theorem 3.28. *Assigning to any world sheet* $\mathcal{S}$ *the string-net correlator* $\mathrm{Cor}_{\mathrm{SN}}(\mathcal{S})$ *provides a consistent system of correlators in the sense of Definition* 2.20.

Chapter 4

Consequences, universal correlators

OPE coefficients and partition functions

The coefficients of partition functions can be computed.
For the torus partition function, one finds the result

$$Cor(T) = \sum_{ij} \dim \mathrm{Hom}_{A|A}(i \otimes^{-} \otimes A \otimes^{+} j, A)\ i \boxtimes j$$

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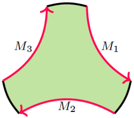
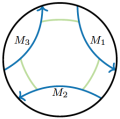
Operator product coefficients can be expressed as invariants of graphs.

For three boundary fields $\mathrm{D}_{M_1,M_2,M_3} =$  we find 

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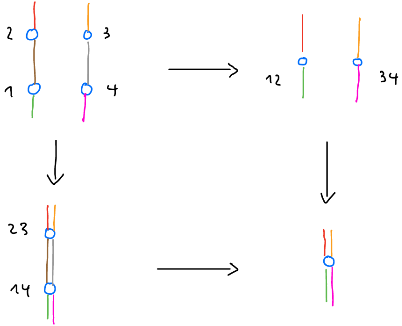
An Eckmann-Hilton theorem for OPEs can be proven:



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An Eckmann-Hilton theorem for OPEs can be proven:

Theorem 5.1. *Let $\mathcal{C}$ be a modular fusion category and A, A' and A'' be simple special symmetric Frobenius algebras in $\mathcal{C}$. Then the diagram*

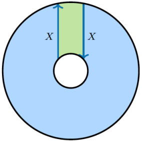
$$\begin{tikzcd}
 & \mathbb{D}^{X_4,X_6} \otimes \mathbb{D}^{X_2,X_4} \otimes \mathbb{D}^{X_3,X_5} \otimes \mathbb{D}^{X_1,X_3} \arrow[dl, "\mathrm{id} \otimes \gamma_{\mathbb{D}^{X_2,X_4};\mathbb{D}^{X_3,X_5}} \otimes \mathrm{id}"'] \arrow[dr, "\underline{\mu}_{\mathrm{ver}} \otimes \underline{\mu}_{\mathrm{ver}}"] & \\
\mathbb{D}^{X_4,X_6} \otimes \mathbb{D}^{X_3,X_5} \otimes \mathbb{D}^{X_2,X_4} \otimes \mathbb{D}^{X_1,X_3} \arrow[d, "\underline{\mu}^{\mathrm{l}}_{\mathrm{hor}} \otimes \underline{\mu}^{\mathrm{l}}_{\mathrm{hor}}"'] & & \mathbb{D}^{X_2,X_6} \otimes \mathbb{D}^{X_1,X_5} \arrow[d, "\underline{\mu}^{\mathrm{l}}_{\mathrm{hor}}"] \\
\mathbb{D}^{X_3 \otimes_{A'} X_4, X_5 \otimes_{A'} X_6} \otimes \mathbb{D}^{X_1 \otimes_{A'} X_2, X_3 \otimes_{A'} X_4} \arrow[rr, "\underline{\mu}_{\mathrm{ver}}"] & & \mathbb{D}^{X_1 \otimes_{A'} X_2, X_5 \otimes_{A'} X_6}
\end{tikzcd} \tag{5.3}$$

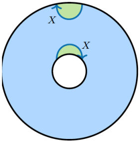
involving the horizontal and vertical compositions of internal natural transformations and the half-braiding γ of $\mathbb{D}^{X_2,X_4}$ commutes for all $X_1, X_3, X_5 \in A$-mod-A' and all $X_2, X_4, X_6 \in A'$-mod-A''.

Mapping class group in the presence of defects

Invariance under the mapping class group becomes much more subtle in the presence of defects!

Suppose that X is an invertible topological defect. Then the two worldsheets

$\mathcal{S}_1 :=$ 

and 

$=: \mathcal{S}_2$.

have the same correlators and cannot be distinguished by the CFT.

Is the Dehn twist in the mapping class group or not?

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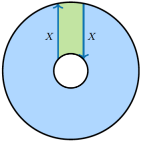
$\mathcal{S}_1 :=$ 

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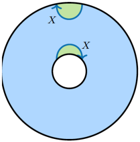

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Is the Dehn twist in the mapping class group or not?

The answer is surprisingly interesting and leads to a conceptually new role of defects.

Universal correlators and bicategorical string nets

- Defect data are bicategorical (=three-layered):

$$\left\{ \begin{array}{l} \text{algebras} \\ \text{bimodules} \\ \text{morphisms of bimodules} \end{array} \right\} =: \mathcal{F}rob(\mathcal{C})$$

- Bicategorical string net construction from bicategorical graphical calculus: vector space $\mathrm{SN}_{\mathcal{F}rob(\mathcal{C})}(\Sigma)$ space of quantum world sheets for geometric surface Σ. This space is constructed from defect data.

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- Bicategorical string net construction from bicategorical graphical calculus: vector space $\mathrm{SN}_{\mathcal{F}rob(\mathcal{C})}(\Sigma)$ space of quantum world sheets for geometric surface Σ. This space is constructed from defect data.
- Every worldsheet $\mathcal{S}$ gives (tautologically) a vector in this space

$$\mathbb{C} \xrightarrow{\mathcal{S}} \mathrm{SN}_{\mathcal{F}rob(\mathcal{C})}(\Sigma)$$

The two world sheets $\mathcal{S}_1$ and $\mathcal{S}_2$ give the same vector.
- Correlators factor through quantum world sheets:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\mathcal{S}"] \arrow[d, "Cor_{\mathcal{S}}"'] & \mathrm{SN}_{\mathcal{F}rob(\mathcal{C})}(\Sigma) \arrow[dl, dashed] \\ \mathrm{SN}_{\mathcal{C}}(\Sigma) & \end{tikzcd}$$

Universal correlators

Correlators factor through vector space of quantum world sheets:

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Here, the map of the universal correlator is induced by a ridig separable Frobenius monoidal functor

$$\mathcal{F}rob(\mathcal{C}) \to *//\mathcal{C}$$

and an isomorphism.

Universal correlators

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\end{tikzcd}$$

$\mathrm{Map}(\Sigma)$ acts on $\mathrm{SN}_{\mathcal{F}rob(\mathcal{C})}(\Sigma)$ and the appropriate mapping class group for the worldsheet $\mathcal{S}$ with quantum world sheet $v_{\mathcal{S}} \in \mathrm{SN}_{\mathcal{F}rob(\mathcal{C})}(\Sigma)$ is the stabilizer:

$$\mathrm{Map}(\mathcal{S}) := \mathrm{Stab}_{\mathrm{Map}(\Sigma)}(v_{\mathcal{S}})$$

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The conformal field theory cannot detect the complete world sheet geometry, only the image in the space of quantum world sheets. The latter is determined by defect data.

Outlook

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- Construction is set up to go beyond semisimplicity (logarithmic conformal field theory)
- Construction is set up to go beyond rigidity (percolation)
- Construction is set up to go to homotopical versions (3dSYM/VOA correspondence)

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Outlook

- Universal correlators in other classes of quantum field theories? New conceptual role of defect data?
- Consider (dynamical, even realistic) quantum field theories “in the background of a topological field theory” / modular functor
  $\rightarrow$ invertible symmetries
  $\rightarrow$ more constructions?