Swiss National Science Foundation

CSCS
Centro Svizzero di Calcolo Scientifico
Swiss National Supercomputing Centre

Pseudoscalar pole contributions to HLbL at the physical point

Gurtej Kanwar (University of Bern)
In collaboration with Sebastian Burri, Marcus Petschlies, Urs Wenger + ETMC

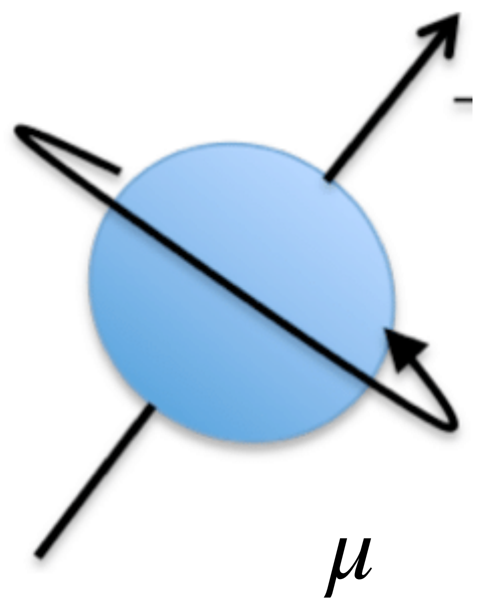
Extended Twisted Mass Collaboration

UNIVERSITÄT BERN

HU Berlin / DESY Zeuthen
Lattice Seminar | Nov 14, 2022

Anomalous magnetic moment of the muon

Precision physics, sensitive to BSM effects.



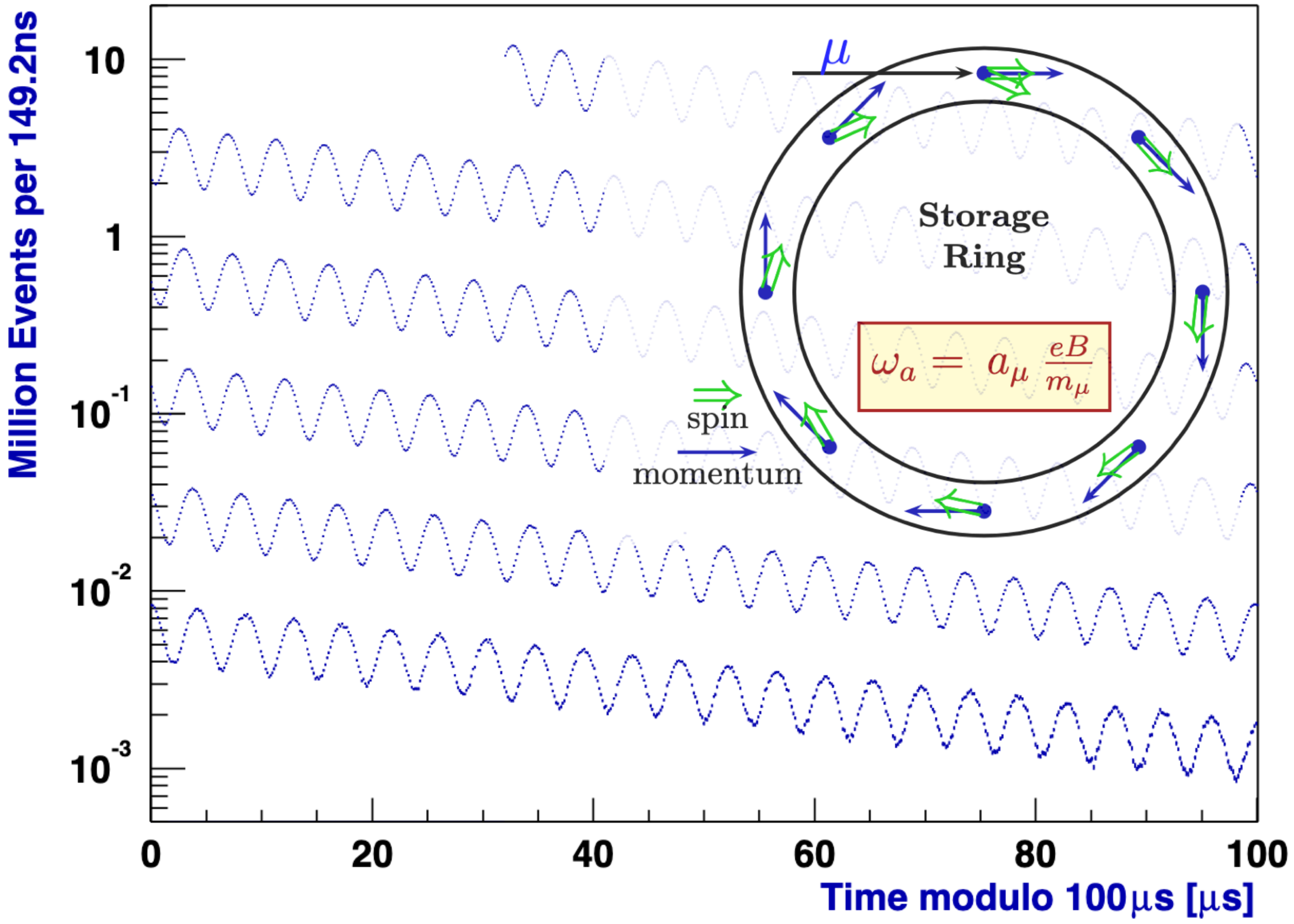
$$a_\mu = \frac{g_\mu - 2}{2}$$

- Leading order QED Schwinger term in 1948…

$$a_\ell = \frac{\alpha}{2\pi}$$

- Heavier flavors more sensitive to new physics effects
- Tau too short-lived, muon is the next-best option

Larmor precession



[Jegerlehner & Nyffeler 0902.3360]
[Bennett+ Phys. Rev. D 73 (2006) 072003]

Tensions in a_μ

Combined 4.2σ tension

$$\Delta a_\mu = 279(76) \times 10^{-11}$$

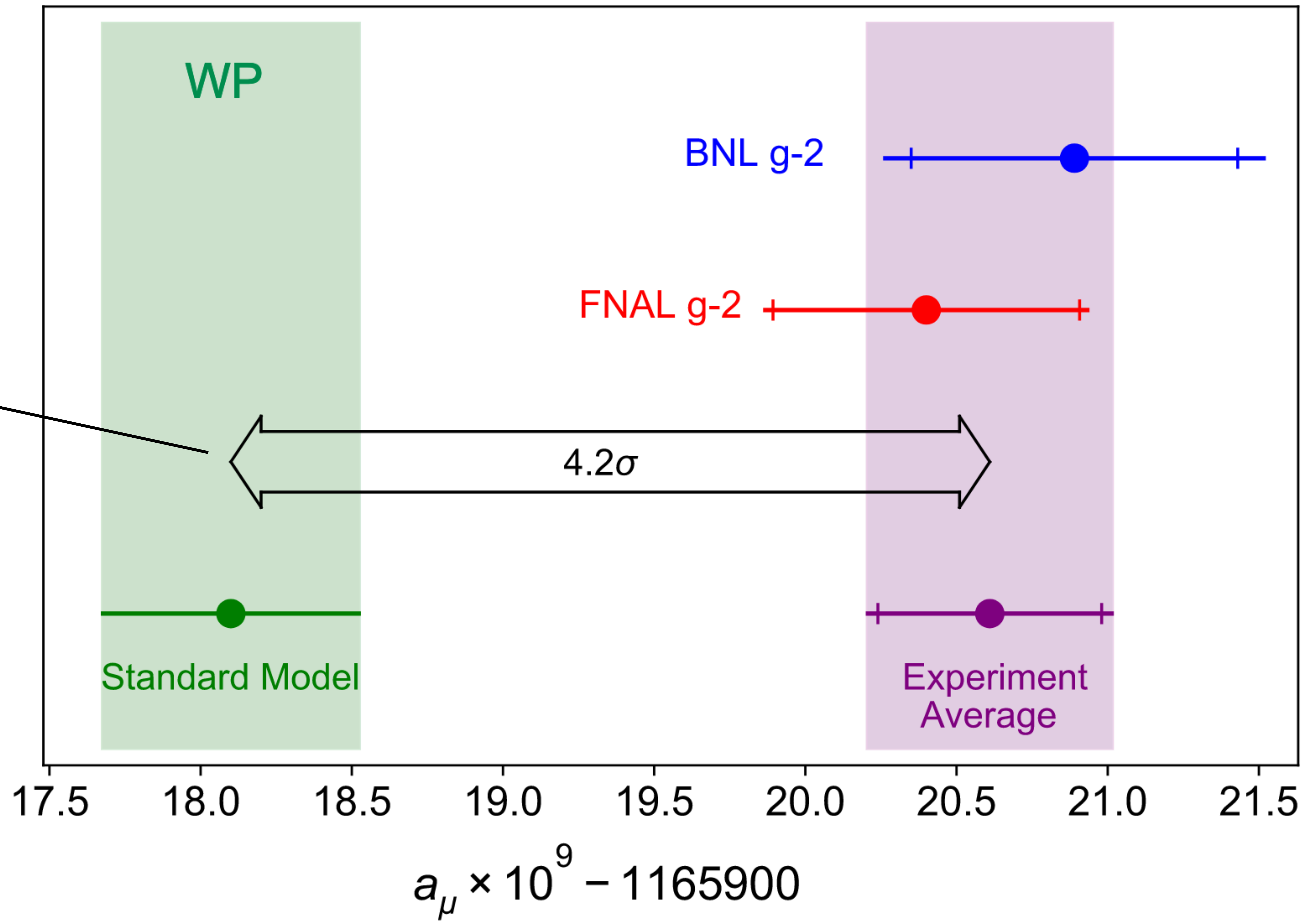
Expt: BNL E821 + Fermilab E989

- Current err ~ 60×10^{-11}
- Expected err ~ 15×10^{-11}

Theory: Perturbative + pheno + latt

- HVP err ~ 40×10^{-11}
  HLbL err ~ 20×10^{-11}

At the order of
E989 projected err



Contribution	Value $\times 10^{11}$
Experiment (E821)	116 592 089(63)
...	...
QED	116 584 718.931(104)
Electroweak	153.6(1.0)
HVP (e^+e^-, LO + NLO + NNLO)	6845(40)
HLbL (phenomenology + lattice + NLO)	92(18)
Total SM Value	116 591 810(43)
Difference: $\Delta a_\mu := a_\mu^{\text{exp}} - a_\mu^{\text{SM}}$	279(76)

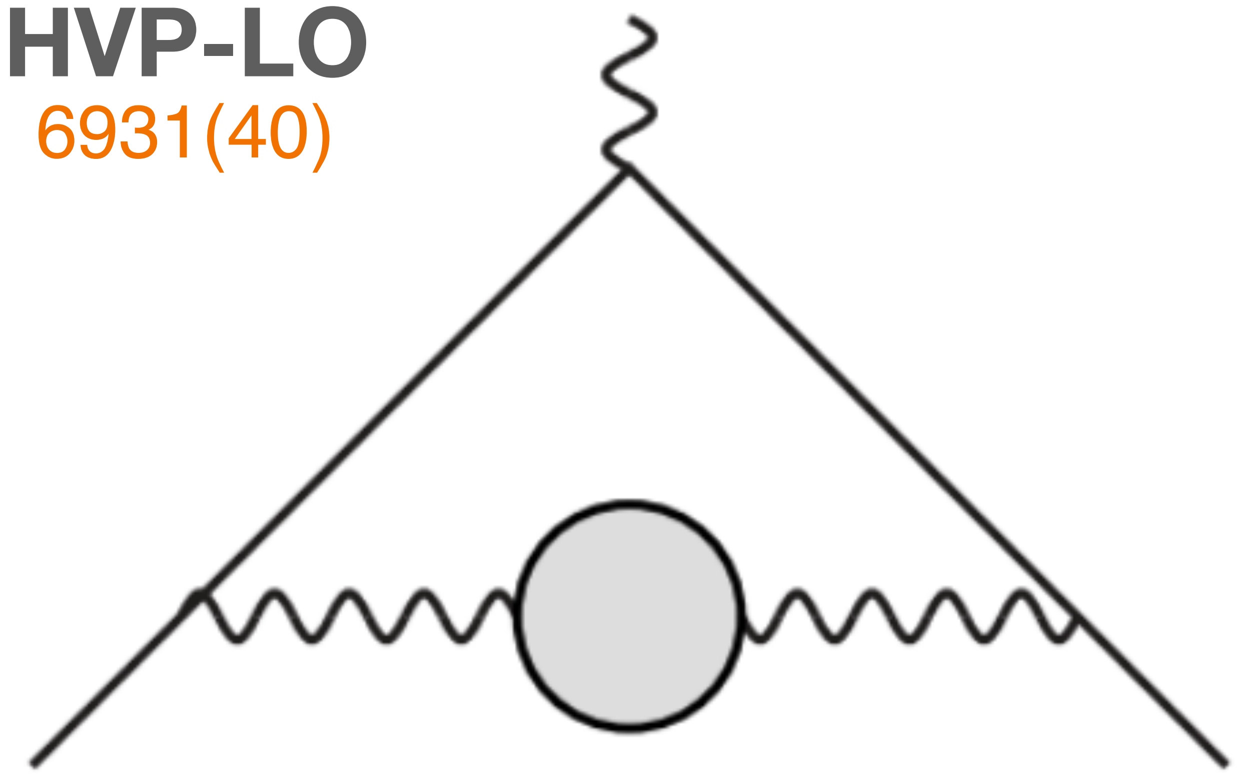
[g-2 WP, Phys. Rep. 887 (2020)]

All a_μ values $\times 10^{-11}$

Hadronic contributions to a_μ

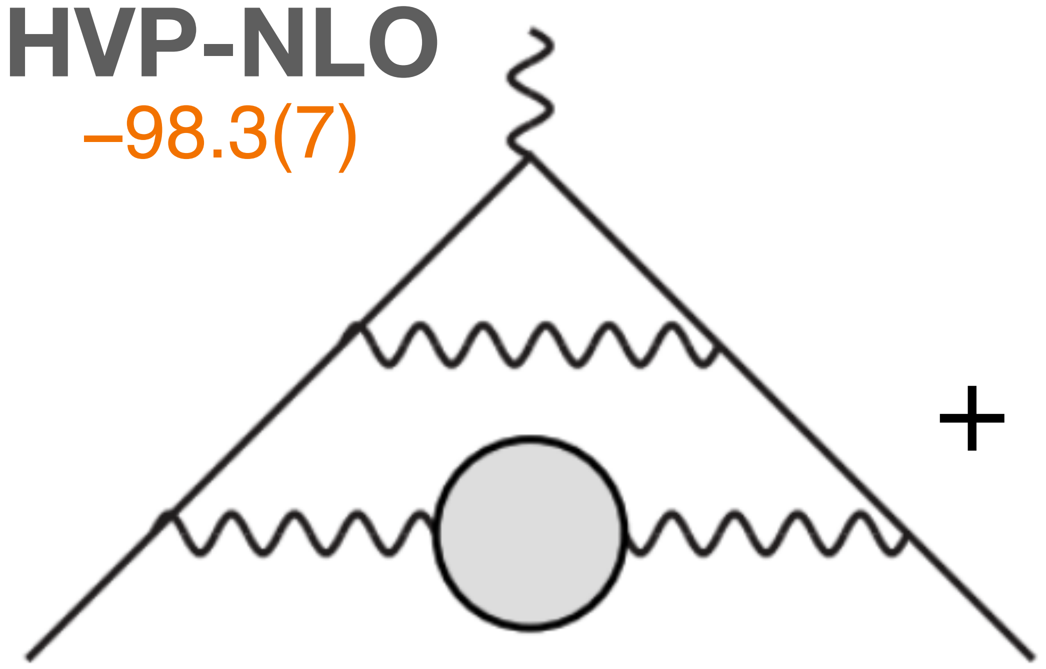
$\mathcal{O}(\alpha^2)$

HVP-LO
6931(40)



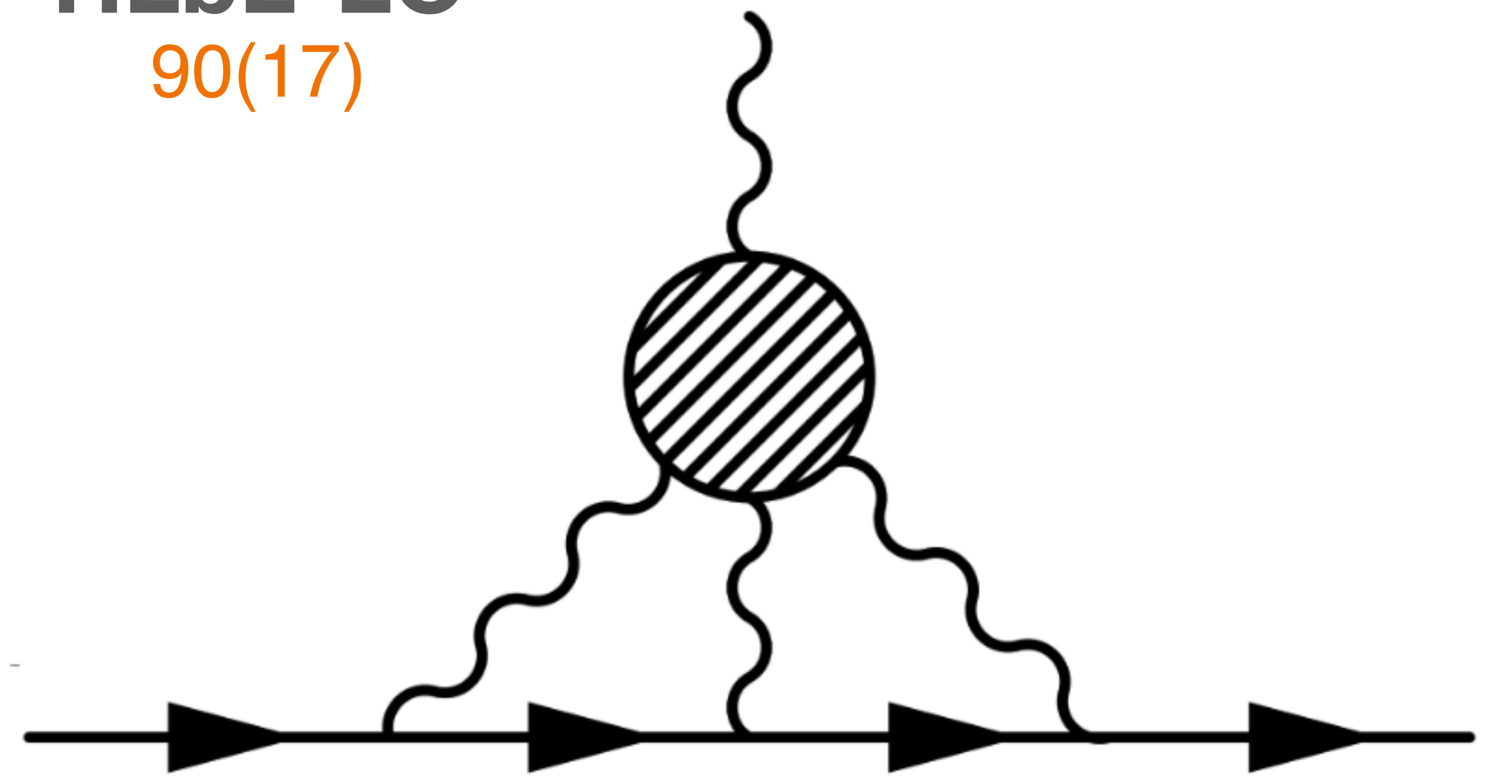
$\mathcal{O}(\alpha^3)$

HVP-NLO
–98.3(7)



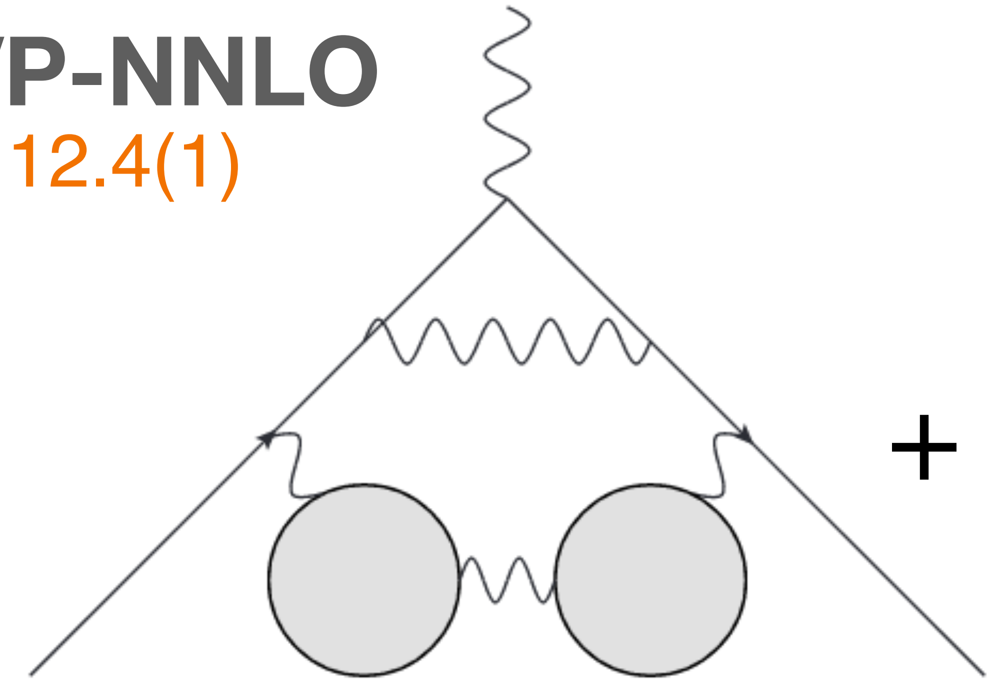
+ …

HLbL-LO
90(17)



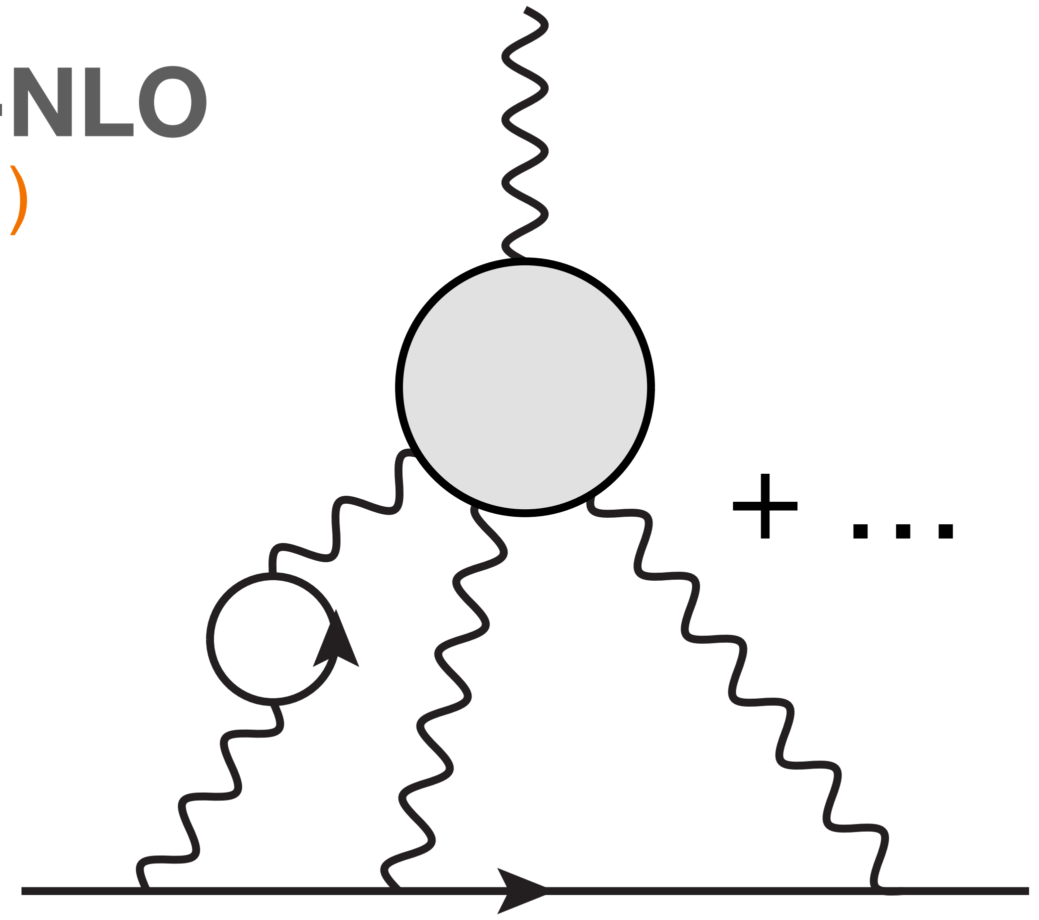
$\mathcal{O}(\alpha^4)$

HVP-NNLO
12.4(1)



+ …

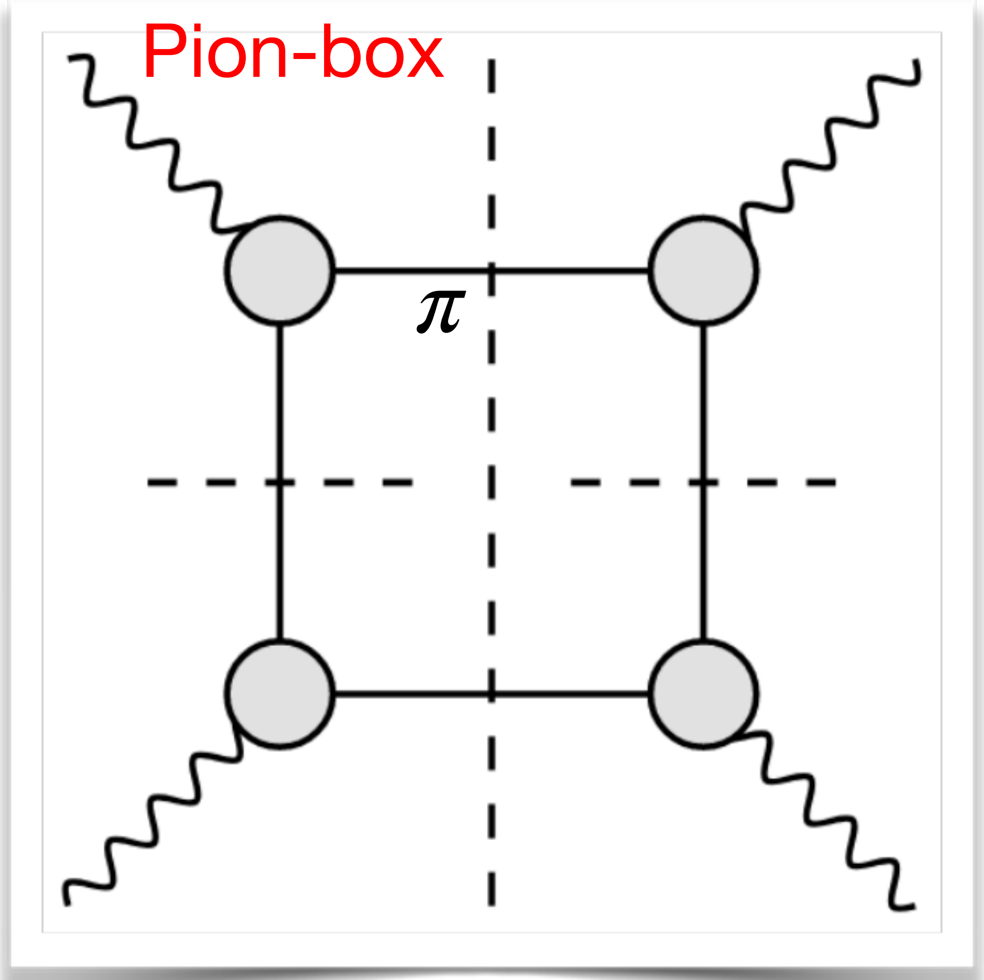
HLbL-NLO
2(1)



+ …

[Colangelo+ JHEP 1409, 091 (2014)]
[Colangelo+ PLB 738, 6 (2014)]
[Colangelo+ JHEP 1509, 074 (2015)]

HLbL decomposition



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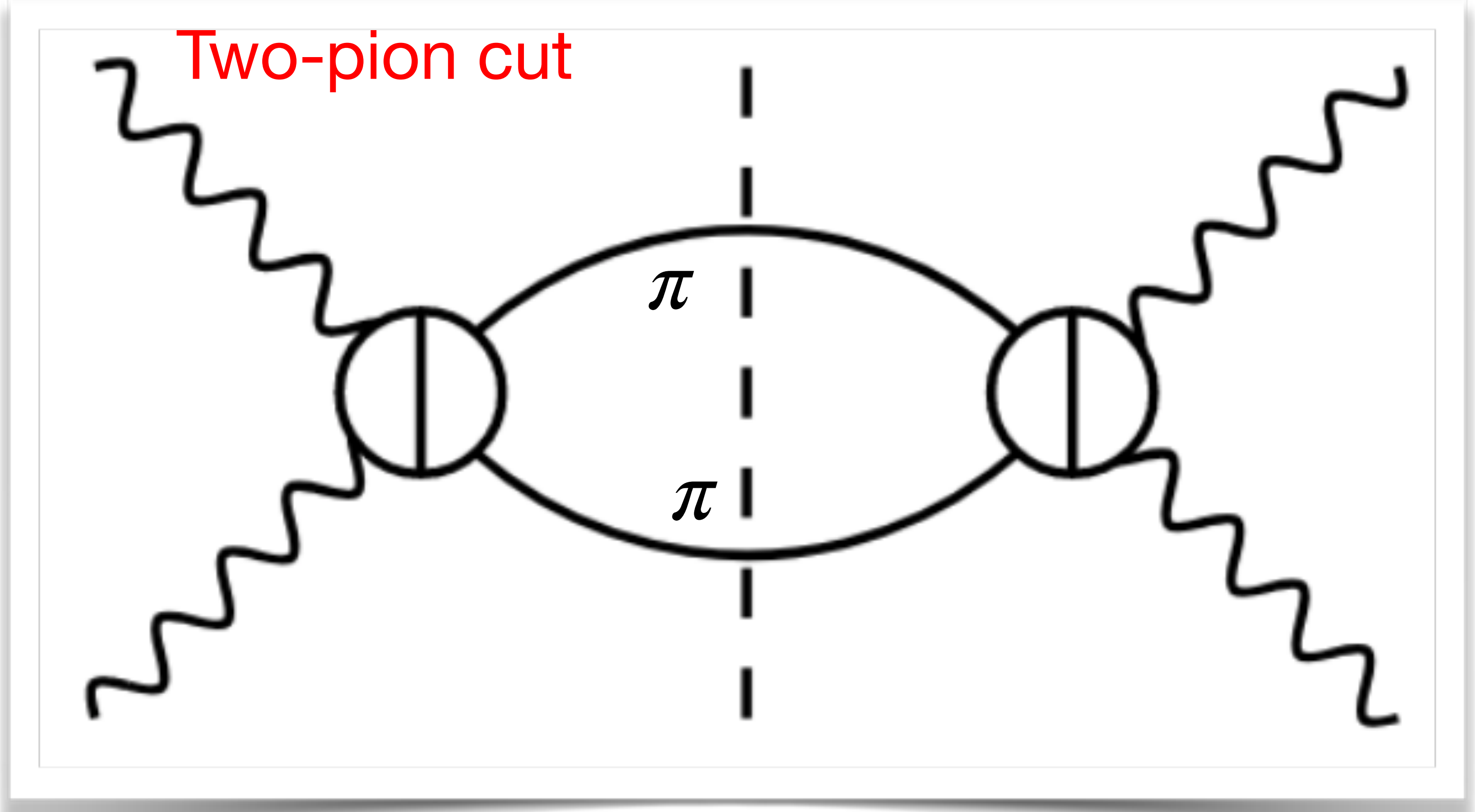

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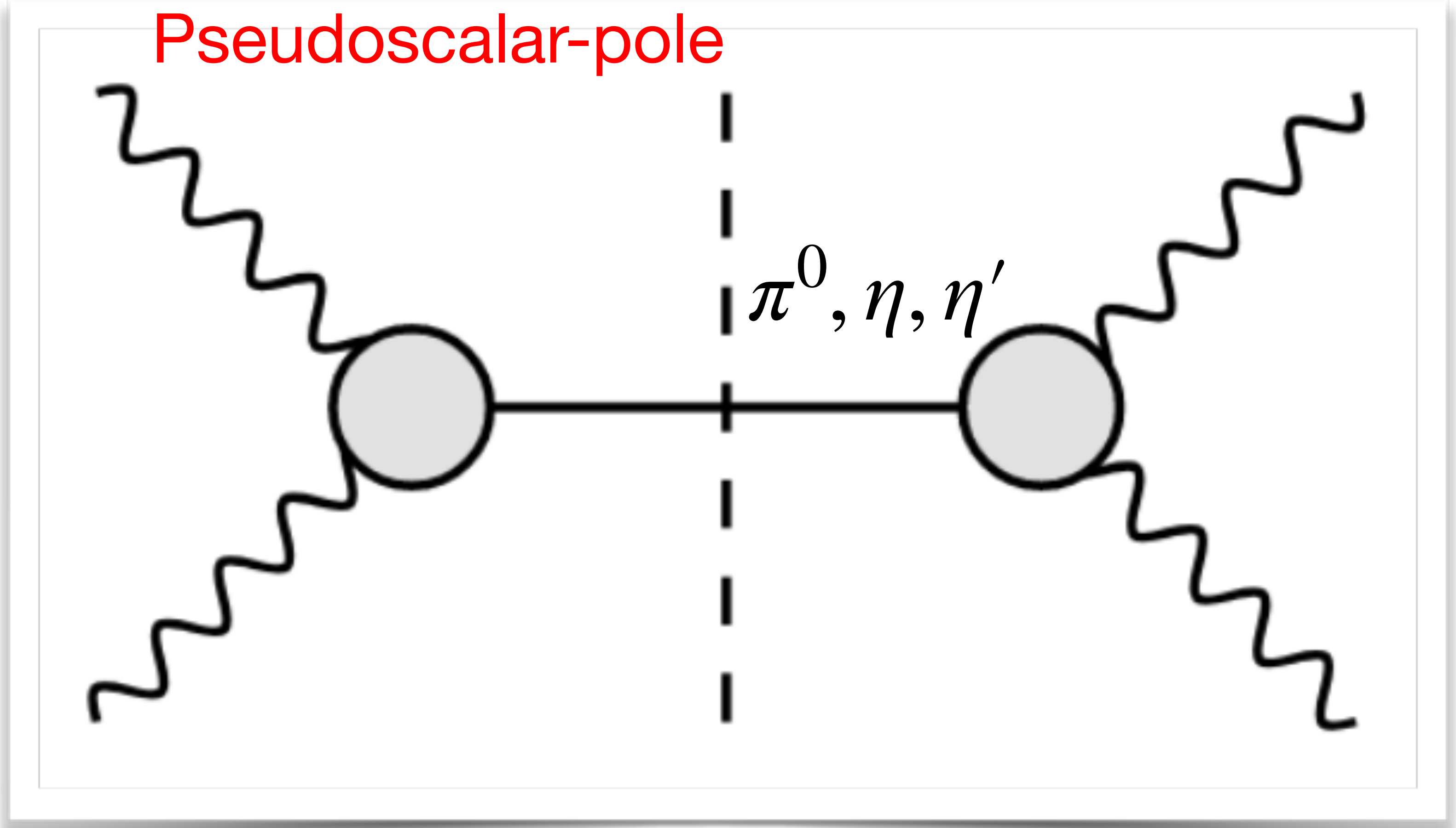
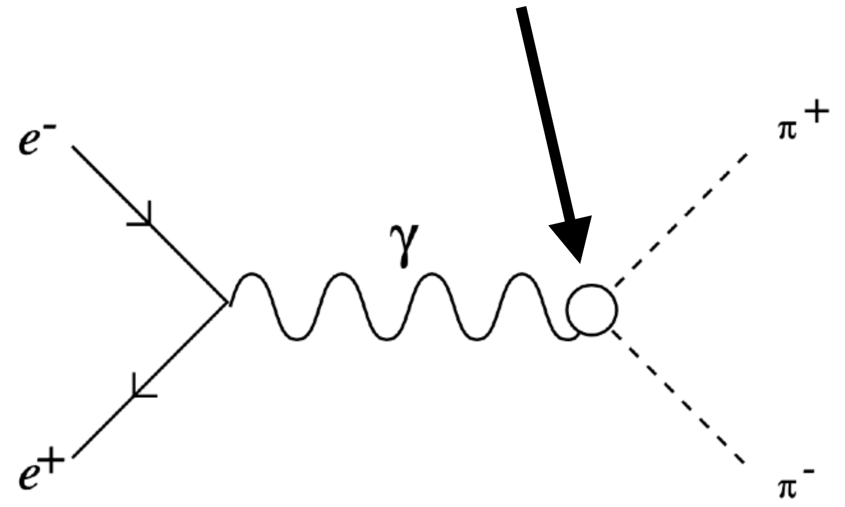


+ ...

$F_{\gamma^* \to \pi\pi}$ "VFF" well-determined experimentally ✅



Partial waves for $\gamma^*\gamma^* \to \pi\pi$ required

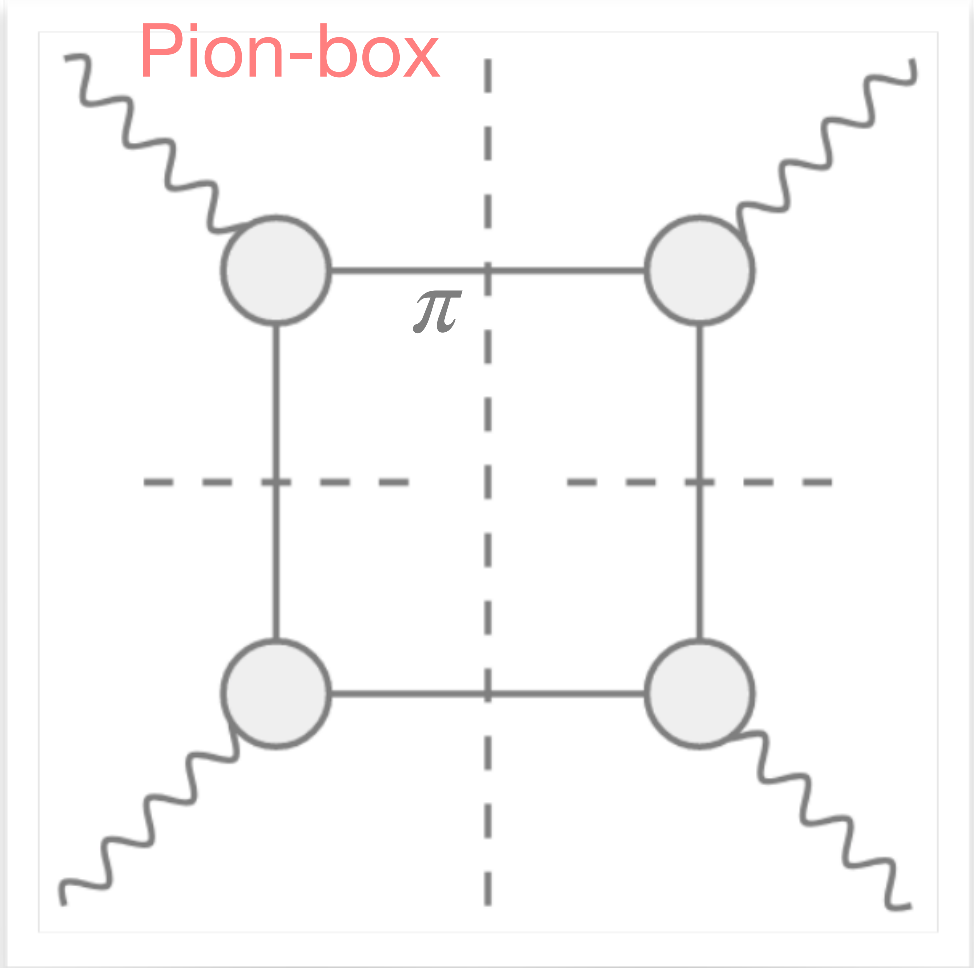
Result is small and well-constrained ✅

$F_{P \to \gamma^*\gamma}$ singly-virtual "TFF" experimentally accessible

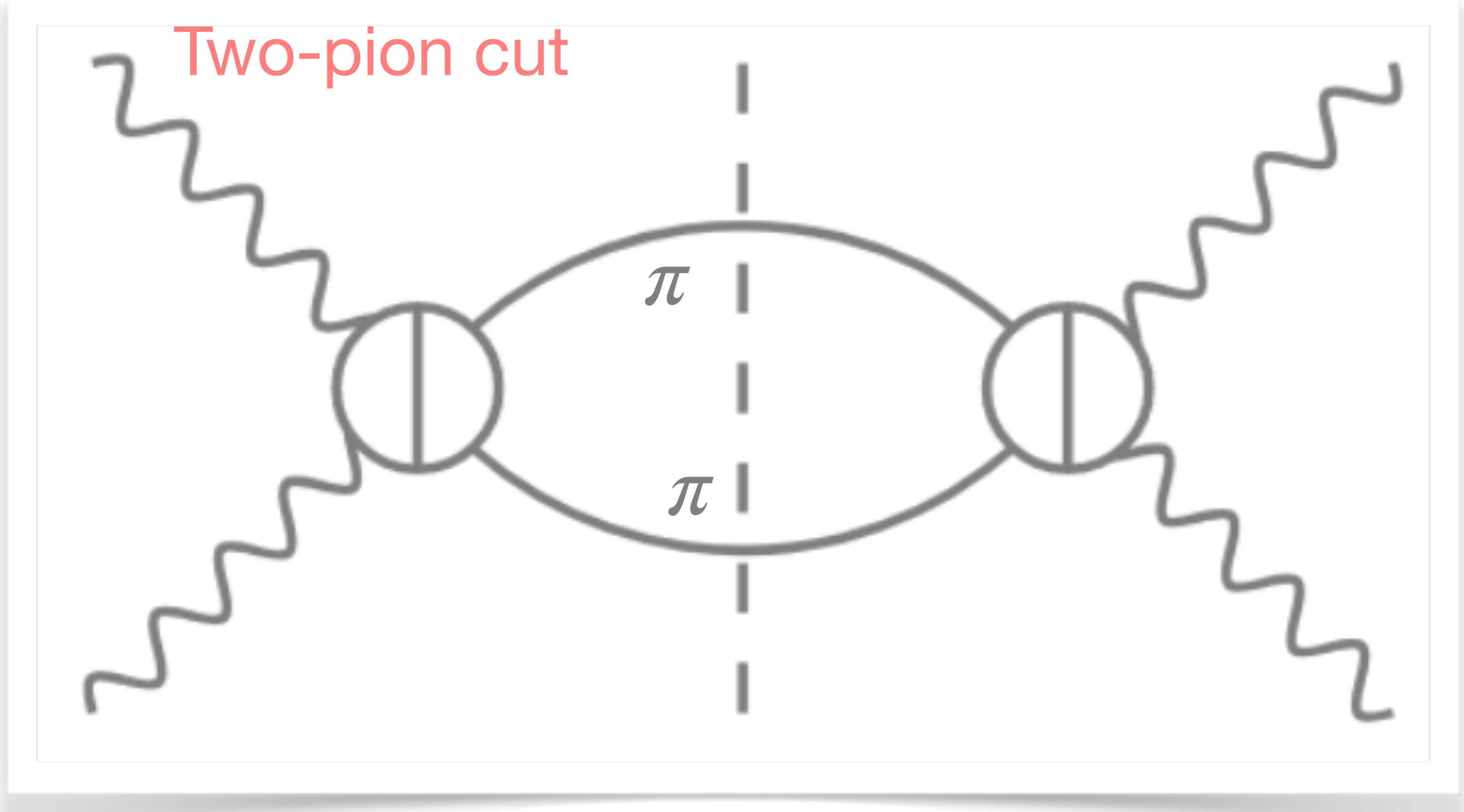
$F_{P \to \gamma^*\gamma^*}$ doubly-virtual "TFF" ~**unconstrained**

[Colangelo+ JHEP 1409, 091 (2014)]
[Colangelo+ PLB 738, 6 (2014)]
[Colangelo+ JHEP 1509, 074 (2015)]

HLbL decomposition



+



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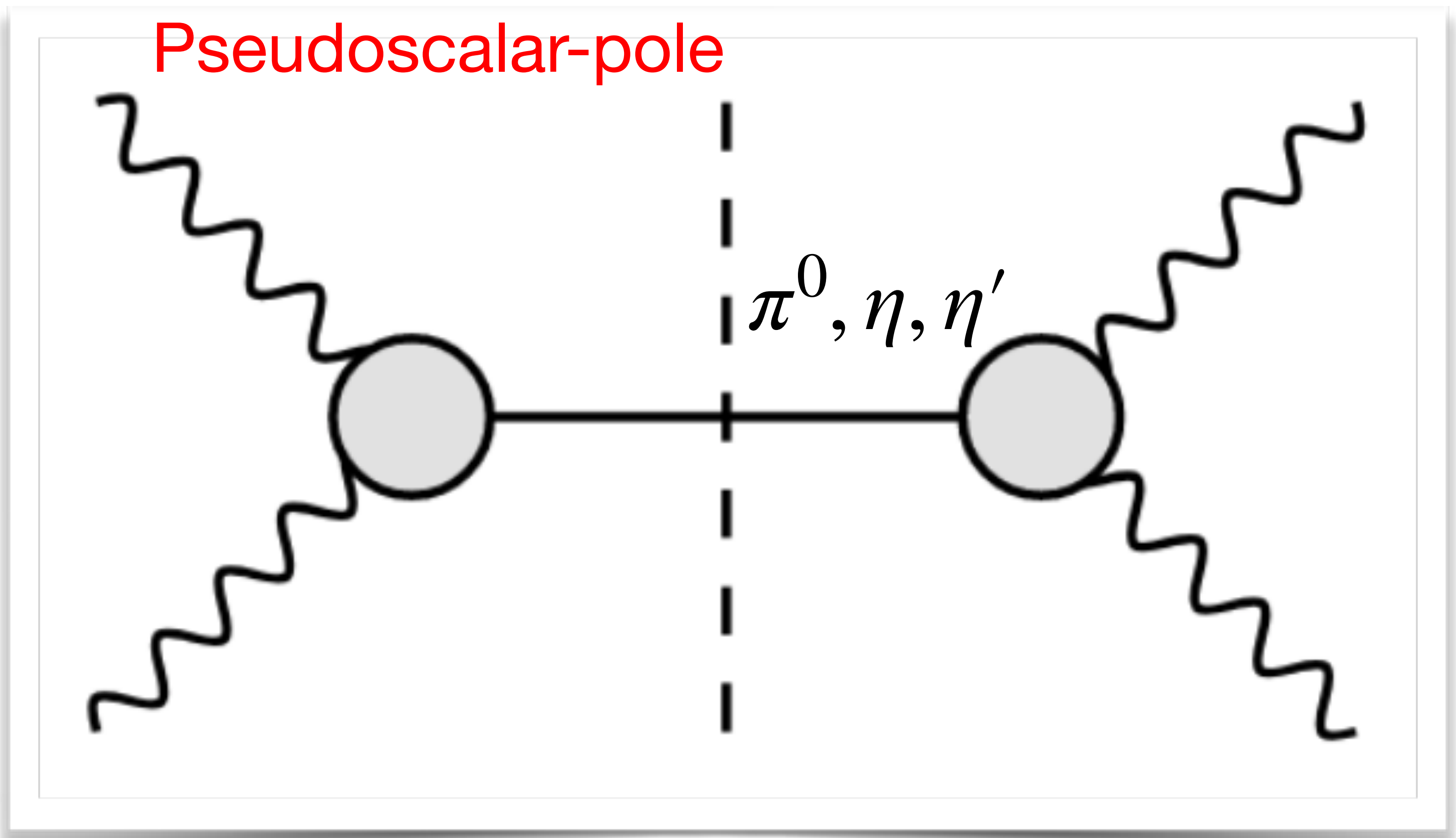
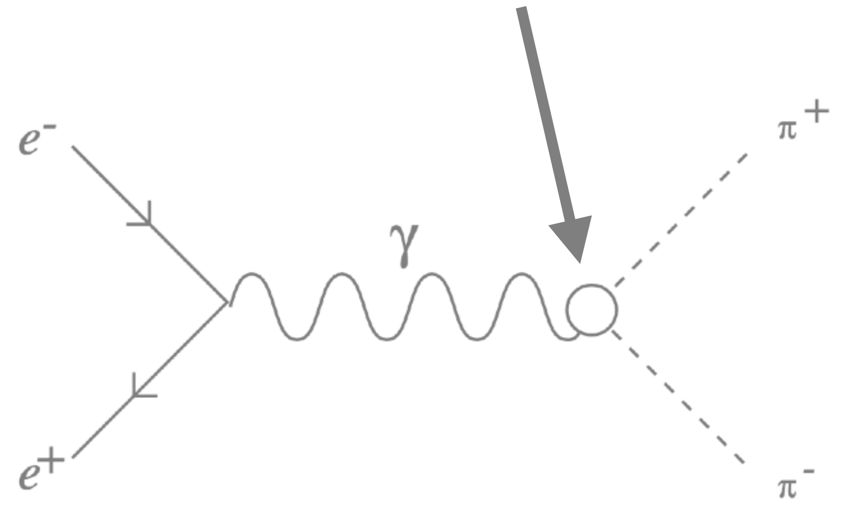


+ ...

$F_{\gamma^* \to \pi\pi}$ "VFF" well-determined experimentally ✅



Partial waves for $\gamma^*\gamma^* \to \pi\pi$ required

Result is small and well-constrained ✅

$F_{P \to \gamma^*\gamma}$ singly-virtual "TFF" experimentally accessible

$F_{P \to \gamma^*\gamma^*}$ doubly-virtual "TFF" ~**unconstrained**

THIS TALK

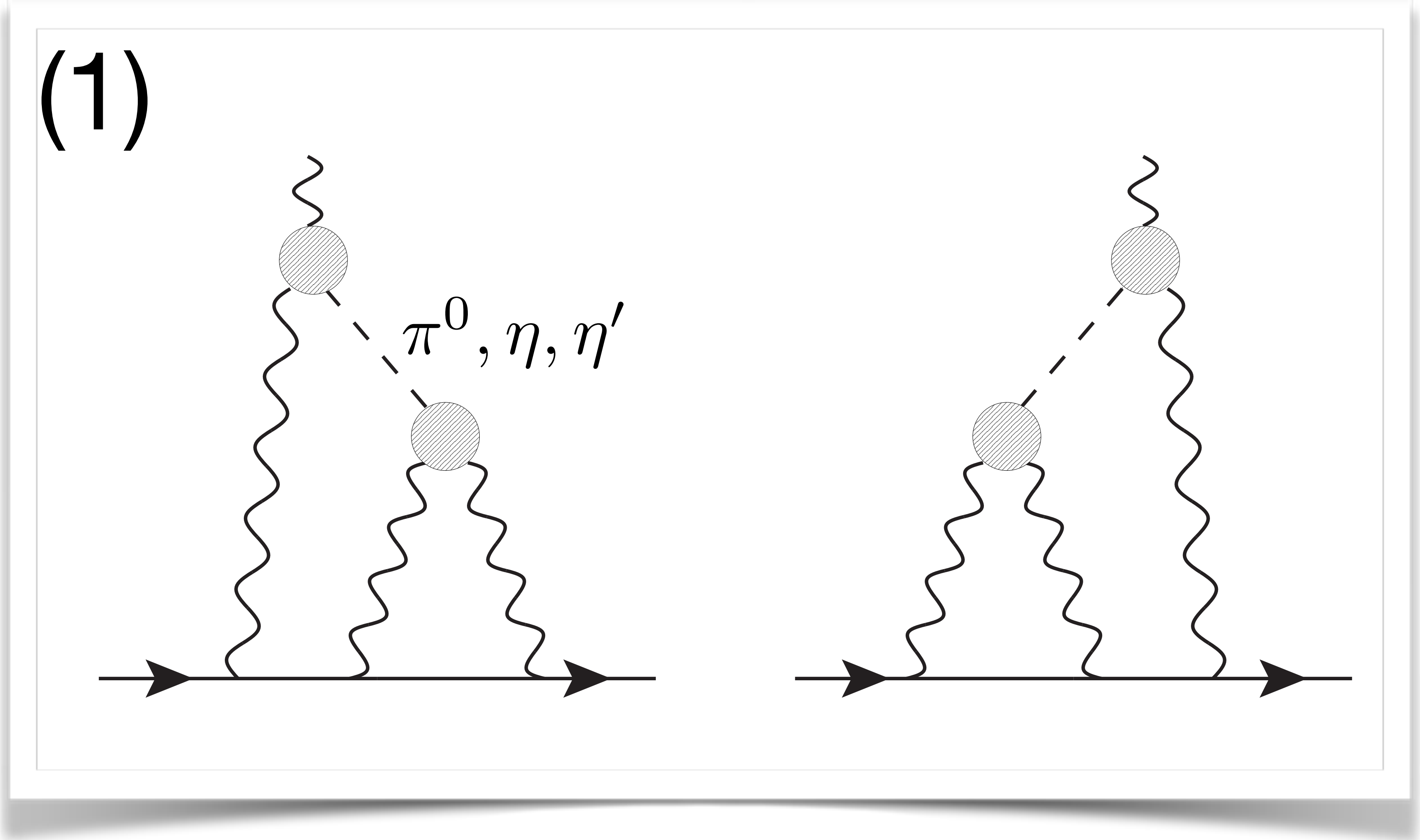
Pole contributions to HLbL

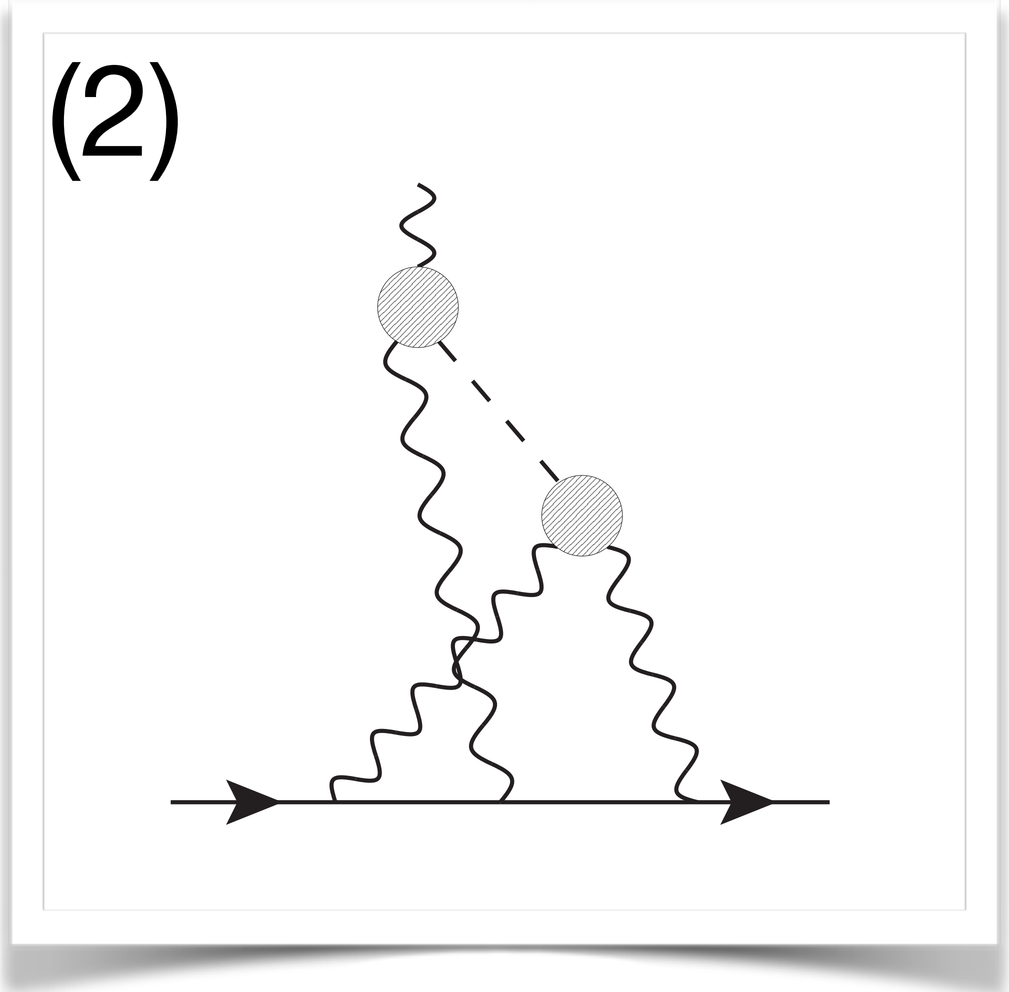
Muon g-2 contribution $a_\mu^{P-\mathrm{pole}}$

$$a_\mu^{P-\mathrm{pole},(1)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\cos\theta \; w_1(Q_1, Q_2, \cos\theta)$$
$$\times F_{P\gamma\gamma}(-Q_1^2, -(Q_1+Q_2)^2)\; F_{P\gamma\gamma}(-Q_2^2, 0)$$

$$a_\mu^{P-\mathrm{pole},(2)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\cos\theta \; w_2(Q_1, Q_2, \cos\theta)$$
$$\times F_{P\gamma\gamma}(-Q_1^2, -Q_2^2)\; F_{P\gamma\gamma}(-(Q_1+Q_2)^2, 0)$$

$$a_\mu^{P-\mathrm{pole}} = \left(\frac{\alpha}{\pi}\right)^3 \left[a_\mu^{P-\mathrm{pole},(1)} + a_\mu^{P-\mathrm{pole},(2)}\right]$$





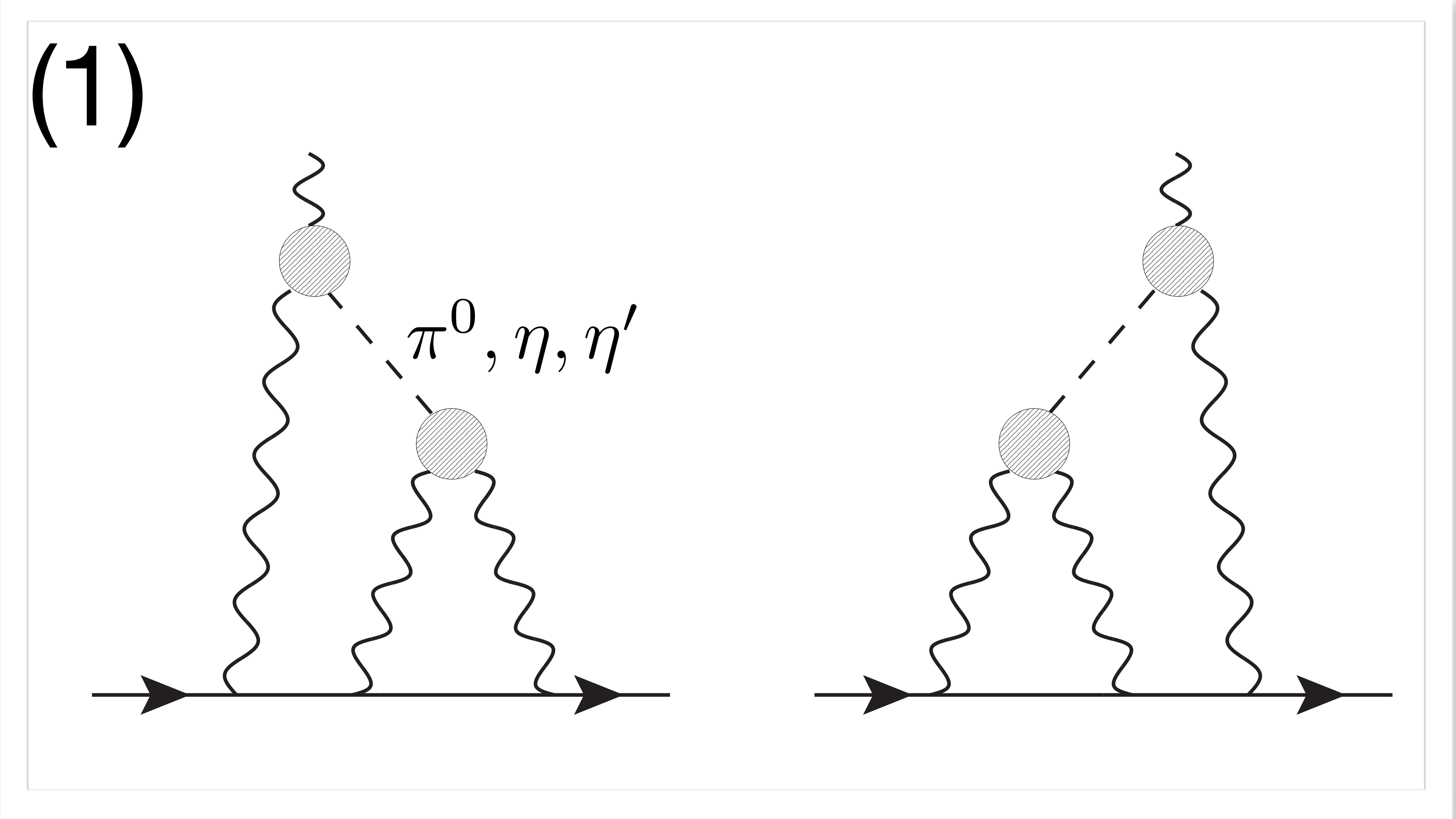
Muon g-2 contribution $a_\mu^{P-\mathrm{pole}}$

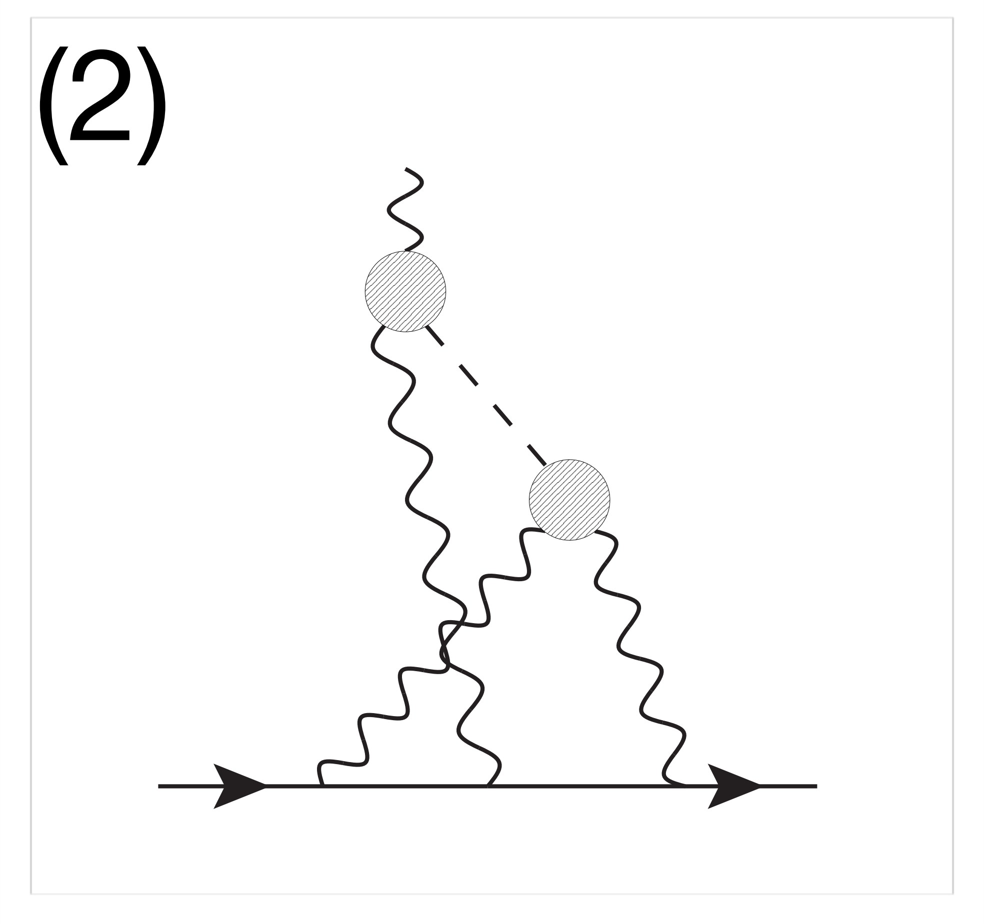
Known kinematic weight functions

$$a_\mu^{P-\mathrm{pole},(1)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\cos\theta \; w_1(Q_1, Q_2, \cos\theta)$$
$$\times F_{P\gamma\gamma}(-Q_1^2, -(Q_1+Q_2)^2)\, F_{P\gamma\gamma}(-Q_2^2, 0)$$

$$a_\mu^{P-\mathrm{pole},(2)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\cos\theta \; w_2(Q_1, Q_2, \cos\theta)$$
$$\times F_{P\gamma\gamma}(-Q_1^2, -Q_2^2)\, F_{P\gamma\gamma}(-(Q_1+Q_2)^2, 0)$$

$$a_\mu^{P-\mathrm{pole}} = \left(\frac{\alpha}{\pi}\right)^3 \left[a_\mu^{P-\mathrm{pole},(1)} + a_\mu^{P-\mathrm{pole},(2)}\right]$$





Muon g-2 contribution $a_\mu^{P-\mathrm{pole}}$

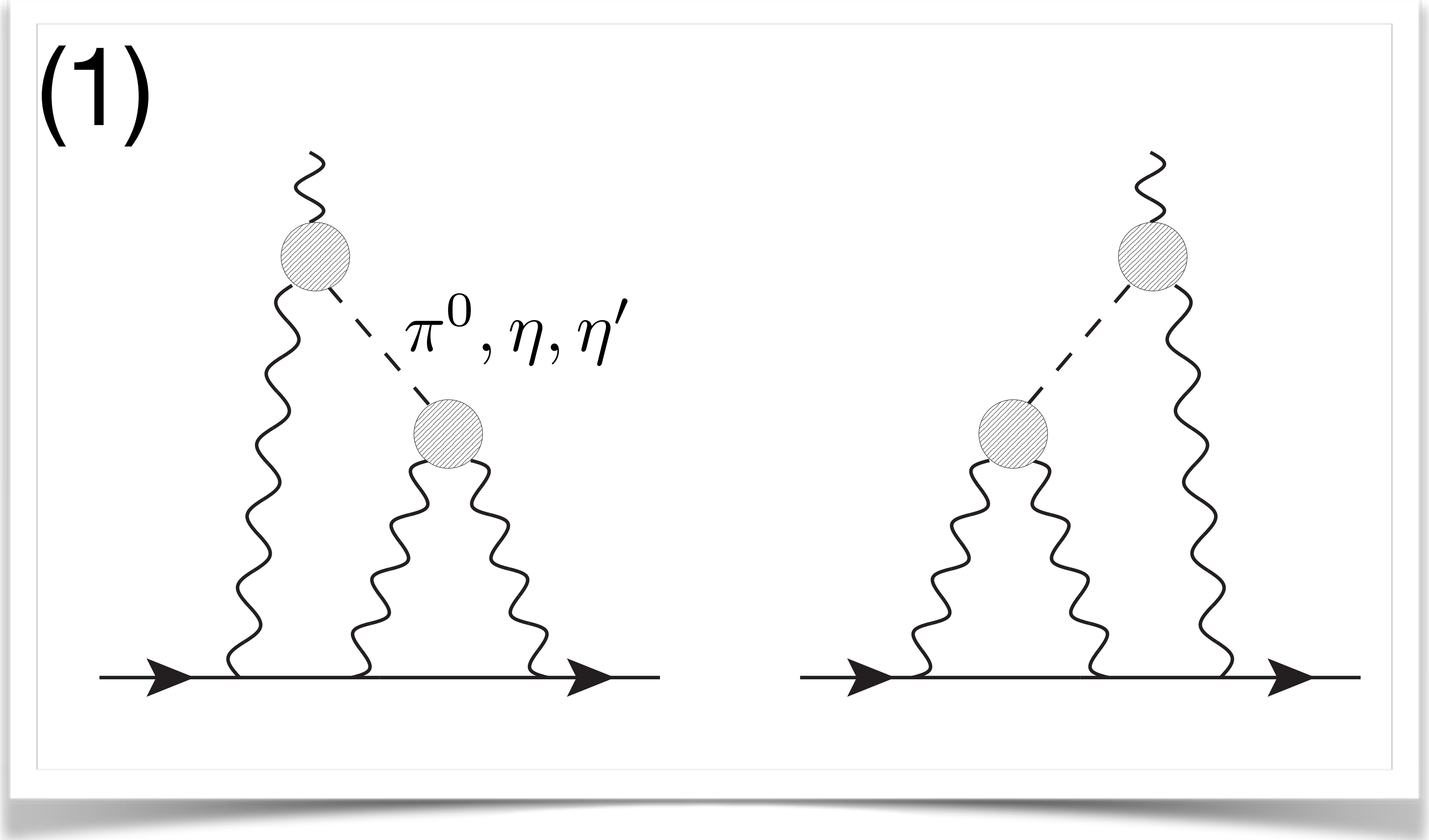
Known kinematic weight functions

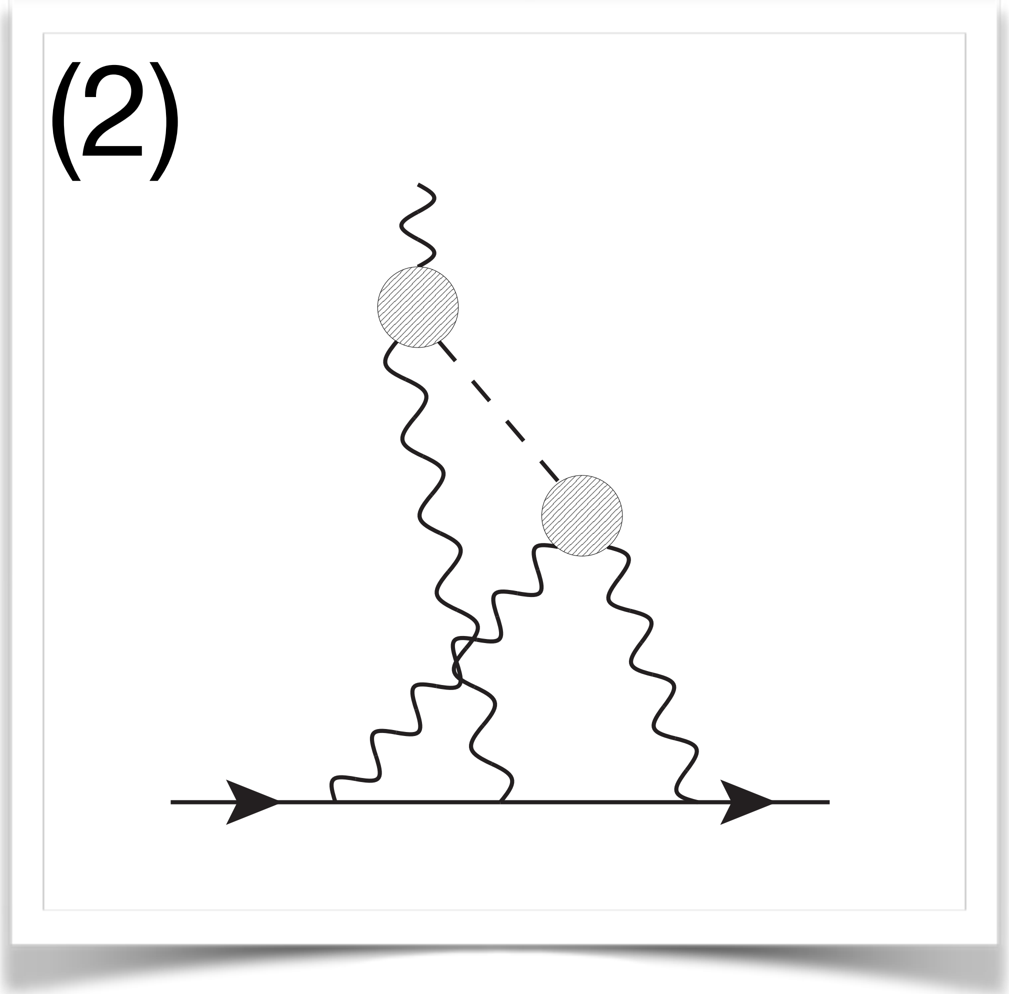
$$a_\mu^{P-\mathrm{pole},(1)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\cos\theta \; w_1(Q_1, Q_2, \cos\theta) \times F_{P\gamma\gamma}(-Q_1^2, -(Q_1+Q_2)^2)\, F_{P\gamma\gamma}(-Q_2^2, 0)$$

$$a_\mu^{P-\mathrm{pole},(2)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\cos\theta \; w_2(Q_1, Q_2, \cos\theta) \times F_{P\gamma\gamma}(-Q_1^2, -Q_2^2)\, F_{P\gamma\gamma}(-(Q_1+Q_2)^2, 0)$$

TFFs with DV x SV, spacelike structure

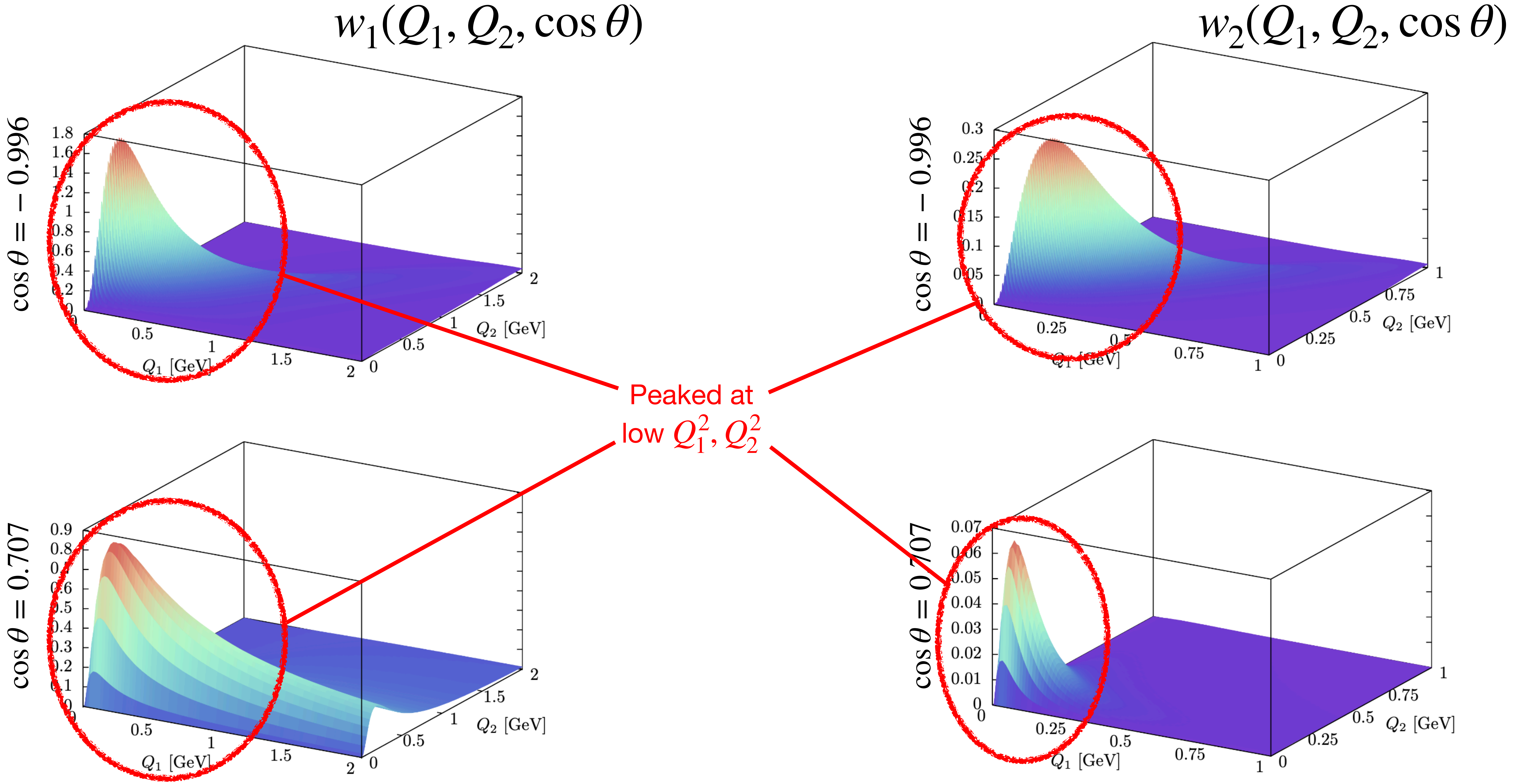
$$a_\mu^{P-\mathrm{pole}} = \left(\frac{\alpha}{\pi}\right)^3 \left[a_\mu^{P-\mathrm{pole},(1)} + a_\mu^{P-\mathrm{pole},(2)}\right]$$



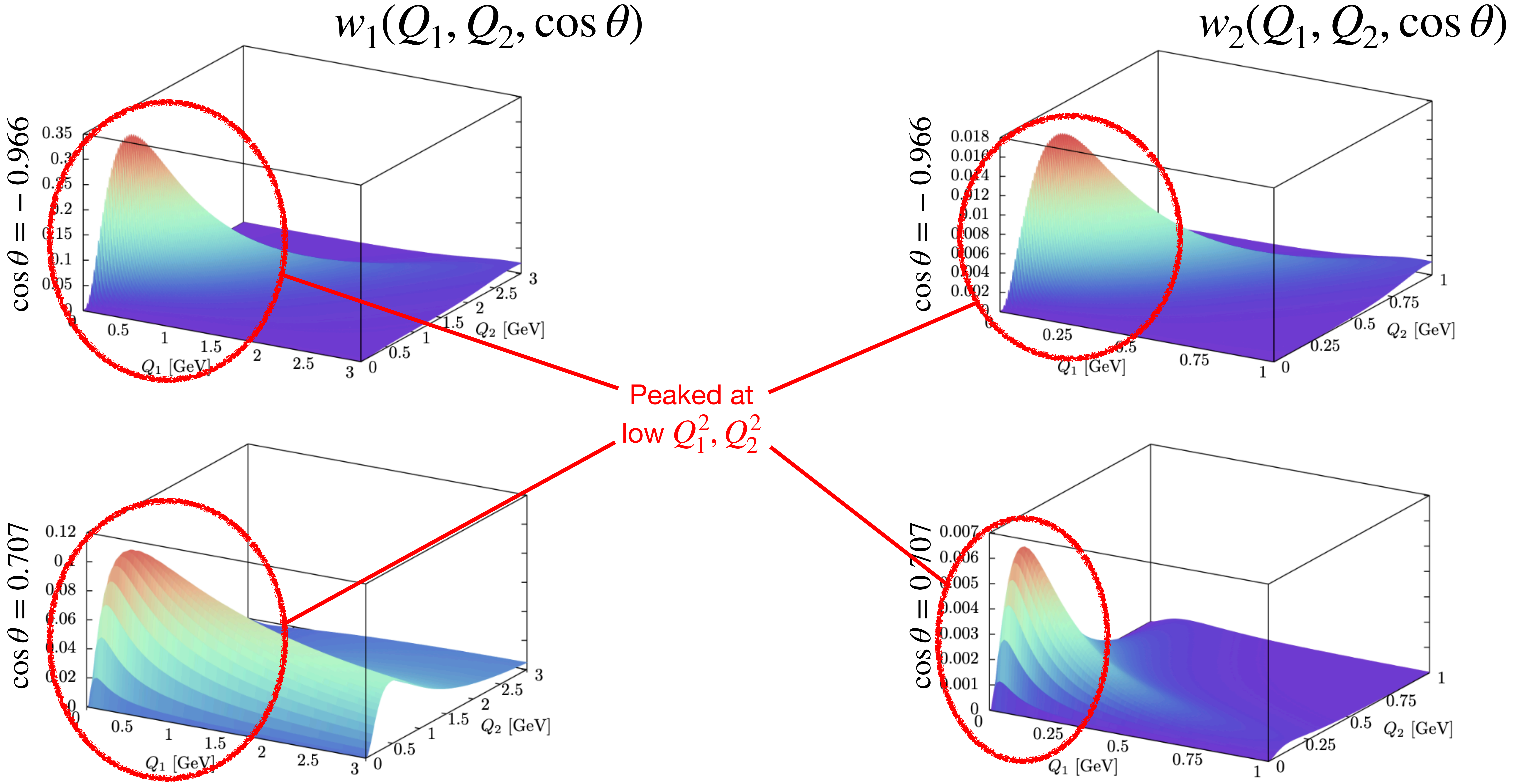


[Nyffeler PRD 94, 053006 (2016)]

Weight functions for π^0

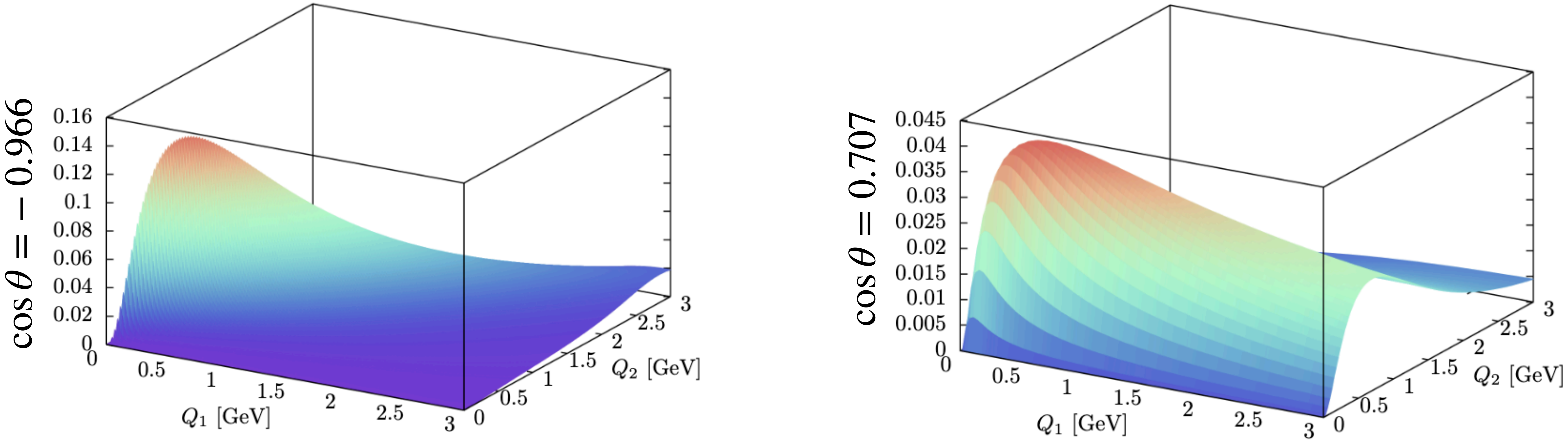


Weight functions for η



Weight functions for η'

$$w_1(Q_1, Q_2, \cos\theta)$$



Peaked at low Q_1^2, Q_2^2, but somewhat broader spread

[Nyffeler PRD 94, 053006 (2016)]

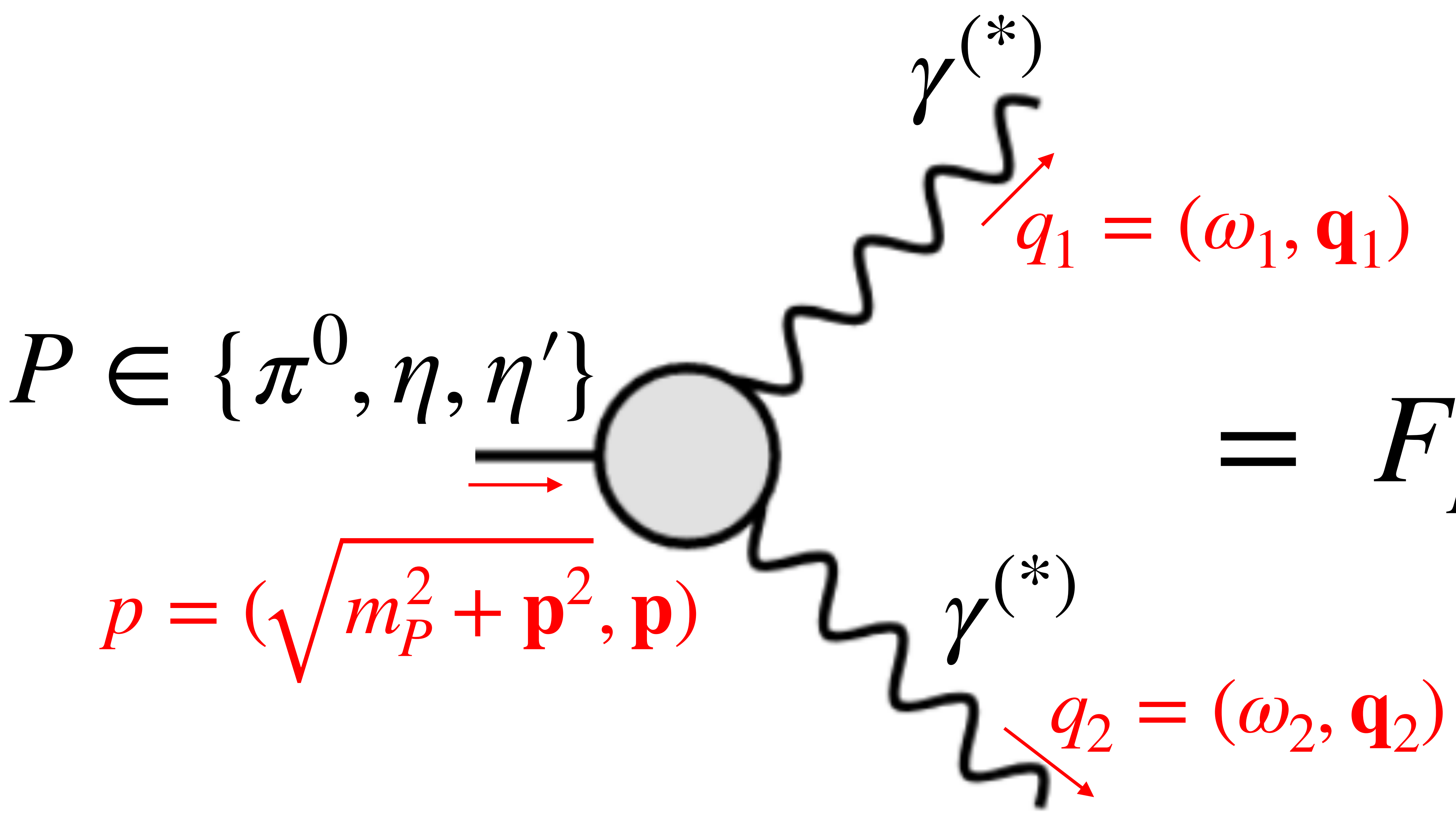
Relevant momenta and relative contribs

Λ [GeV]	π^0 [LMD+V]	π^0 [VMD]	η [VMD]	η' [VMD]
0.25	14.4 (22.9%)	14.4 (25.2%)	1.8 (12.1%)	1.0 (7.9%)
0.5	36.8 (58.5%)	36.6 (64.2%)	6.9 (47.5%)	4.5 (36.1%)
0.75	48.5 (77.1%)	47.7 (83.8%)	10.7 (73.4%)	7.8 (62.5%)
1.0	54.1 (86.0%)	52.6 (92.3%)	12.6 (86.6%)	9.9 (79.1%)
1.5	58.8 (93.4%)	55.8 (97.8%)	14.0 (96.1%)	11.7 (93.1%)
2.0	60.5 (96.2%)	56.5 (99.2%)	14.3 (98.6%)	12.2 (97.4%)
5.0	62.5 (99.4%)	56.9 (99.9%)	14.5 (100%)	12.5 (99.9%)
20.0	62.9 (100%)	57.0 (100%)	14.5 (100%)	12.5 (100%)

Integral saturates around here

$a_\mu^{P-\text{pole}}$ (models) integrating up to $Q_1, Q_2 \leq \Lambda$

Pseudoscalar TFFs



$$= F_{P\gamma\gamma}(q_1^2, q_2^2)$$

Singly-virtual, spacelike: $F_{P\to\gamma^*\gamma} = F_{P\gamma\gamma}(-Q^2, 0)$

Doubly-virtual, spacelike: $F_{P\to\gamma^*\gamma^*} = F_{P\gamma\gamma}(-Q_1^2, -Q_2^2)$

TFF kinematics

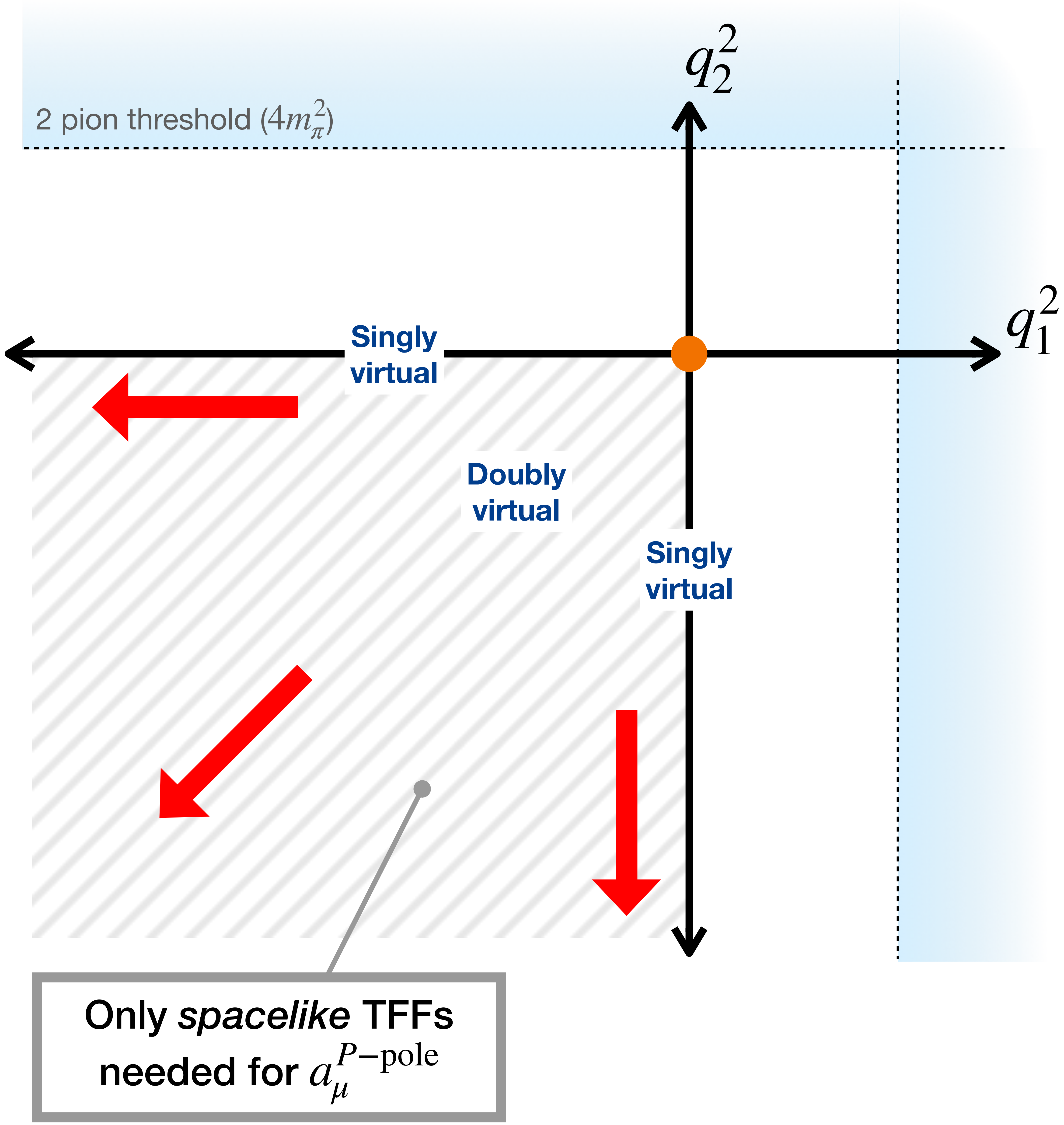
Related to decay to 2γ

$$\Gamma(P \to \gamma\gamma) = \left(\frac{\pi\alpha^2 M_P^3}{4}\right) |F_{P\gamma\gamma}(0,0)|^2$$

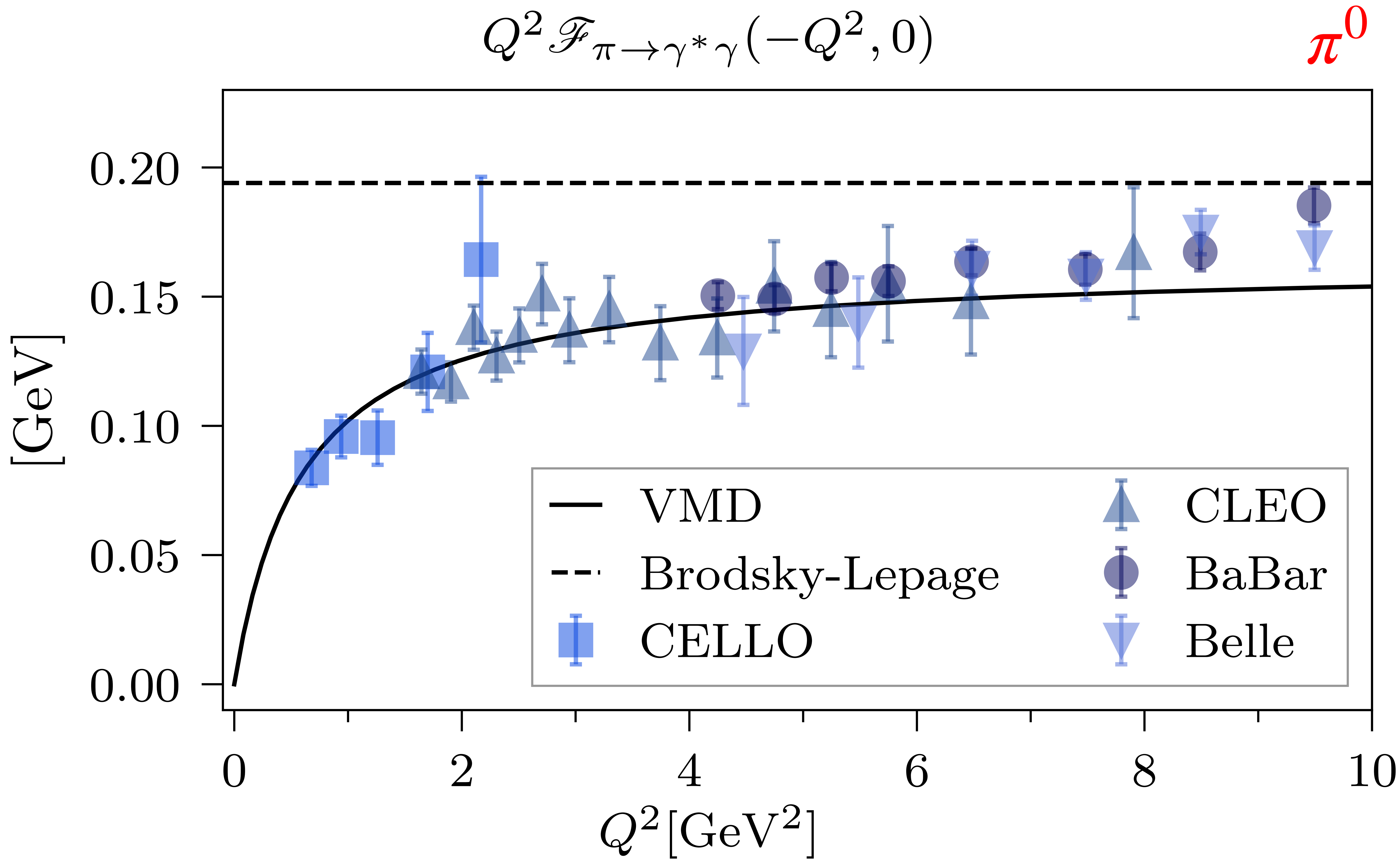
Large-Q^2 limits from pQCD

$$F_{P\gamma\gamma}(-Q^2, 0) \to \frac{2F_P}{Q^2} \text{ (Brodsky-Lepage)}$$

$$F_{P\gamma\gamma}(-Q^2, -Q^2) \to \frac{2F_P}{3Q^2} \text{ (OPE)}$$

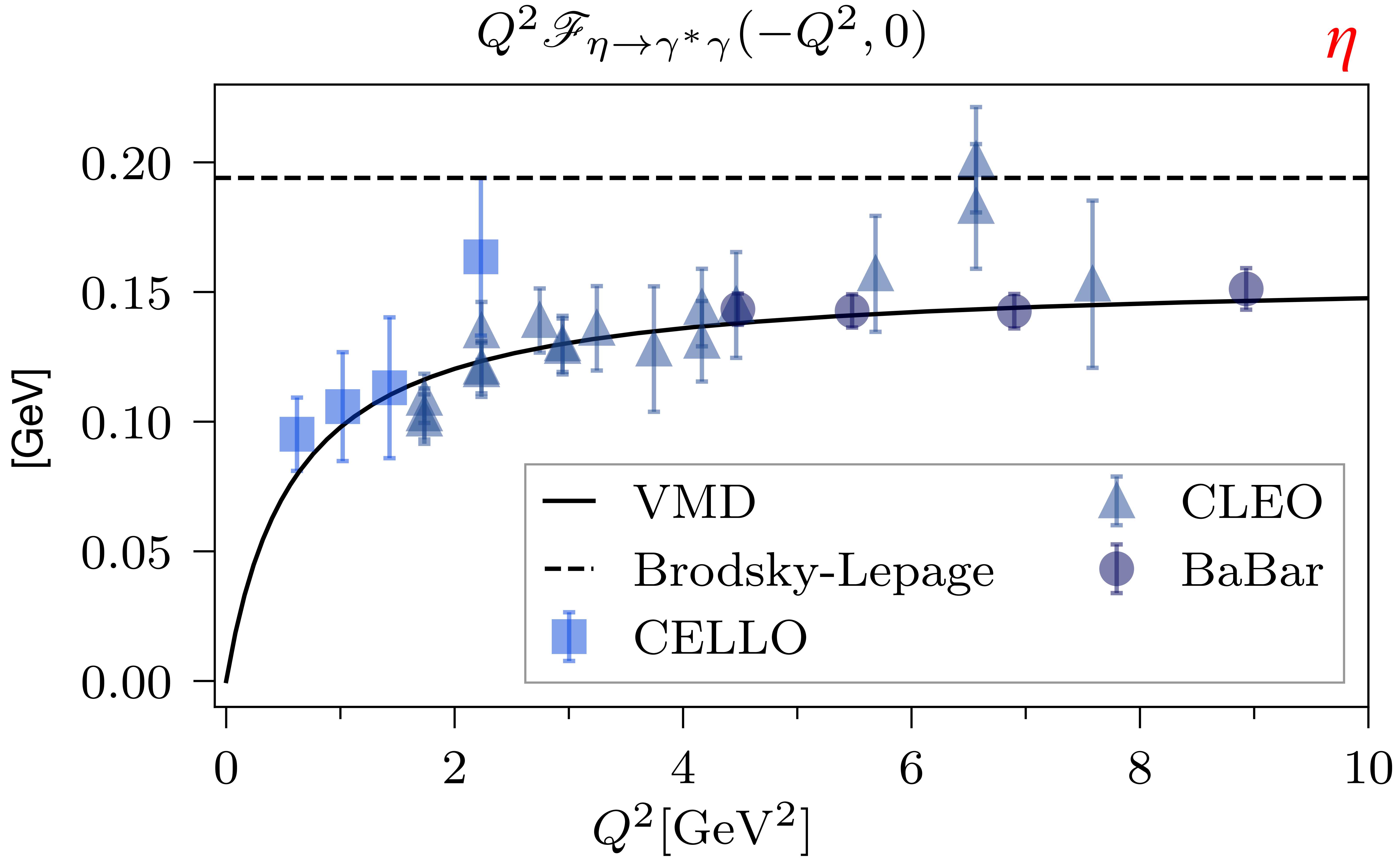


TFF data: singly virtual π^0



[H. J. Behrend+ (CELLO), Z. Phys. C49, 401 (1991)] [B. Aubert+ (BABAR), PRD 80, 052002 (2009)]
[J. Gronberg+ (CLEO), PRD 57, 33 (1998)] [P. del Amo Sanchez+ (BABAR), PRD 84, 052001 (2011)] [S. Uehara+ (Belle), PRD 86, 092007 (2012)]

TFF data: singly virtual η



Note: Singly virtual for η' is very similar.

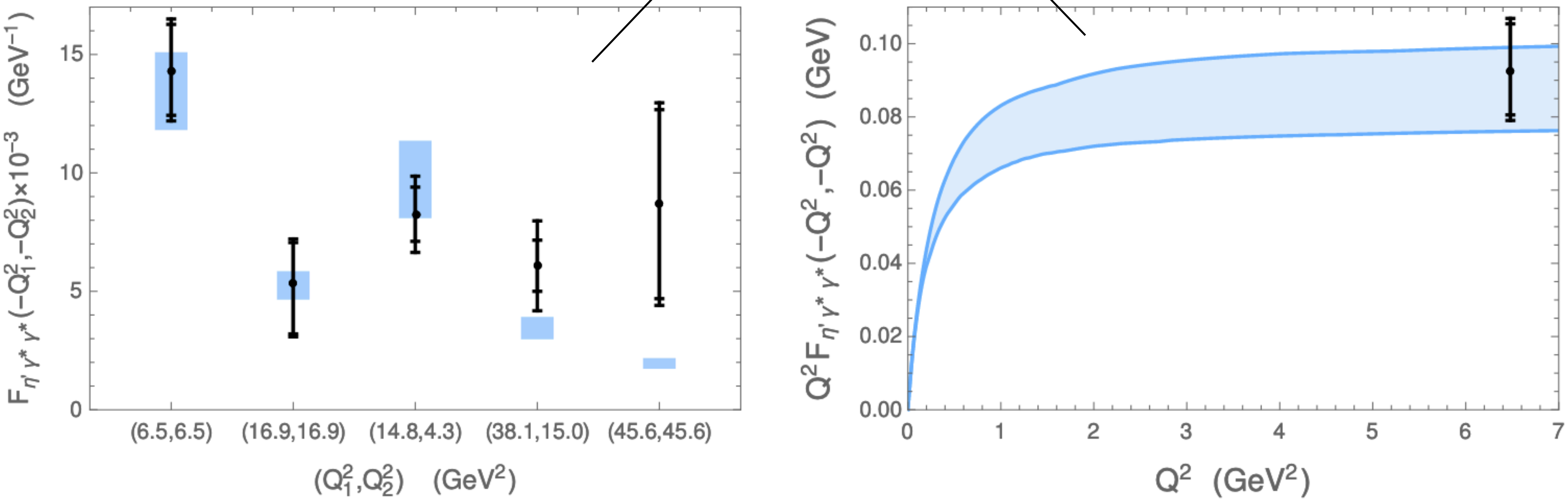
[H. J. Behrend+ (CELLO), Z. Phys. C49, 401 (1991)] [B. Aubert+ (BABAR), PRD 80, 052002 (2009)]

[J. Gronberg+ (CLEO), PRD 57, 33 (1998)] [P. del Amo Sanchez+ (BABAR), PRD 84, 052001 (2011)]

TFF data: doubly virtual

Based on ~50 events at BaBar ⚠️

η'



[J. P. Lees+ (BABAR), PRD 98, 112002 (2018)]
Figure from [g-2 WP, Phys. Rep. 887 (2020)]

Lattice can fill in the gaps

Experimentally, $F_{P\to\gamma^*\gamma}$ is easy, $F_{P\to\gamma^*\gamma^*}$ is hard (small cross sections).

On the lattice, $F_{P\to\gamma^*\gamma}$ is hard, $F_{P\to\gamma^*\gamma^*}$ is easier (coming up).

From the lattice, we can provide complementary results.

Related approaches

Other PS-pole lattice calculations

- Alternate discretization
- Chiral extrapolation

[Gérardin+ PoS(LATTICE2021)592, 2112.08101]
[Gérardin+ Phys.Rev.D 94 (2016) 7]
[Gérardin+ Phys.Rev.D 100 (2019) 3]
[Gérardin+ 2211.04159]

Direct HLbL lattice calculations

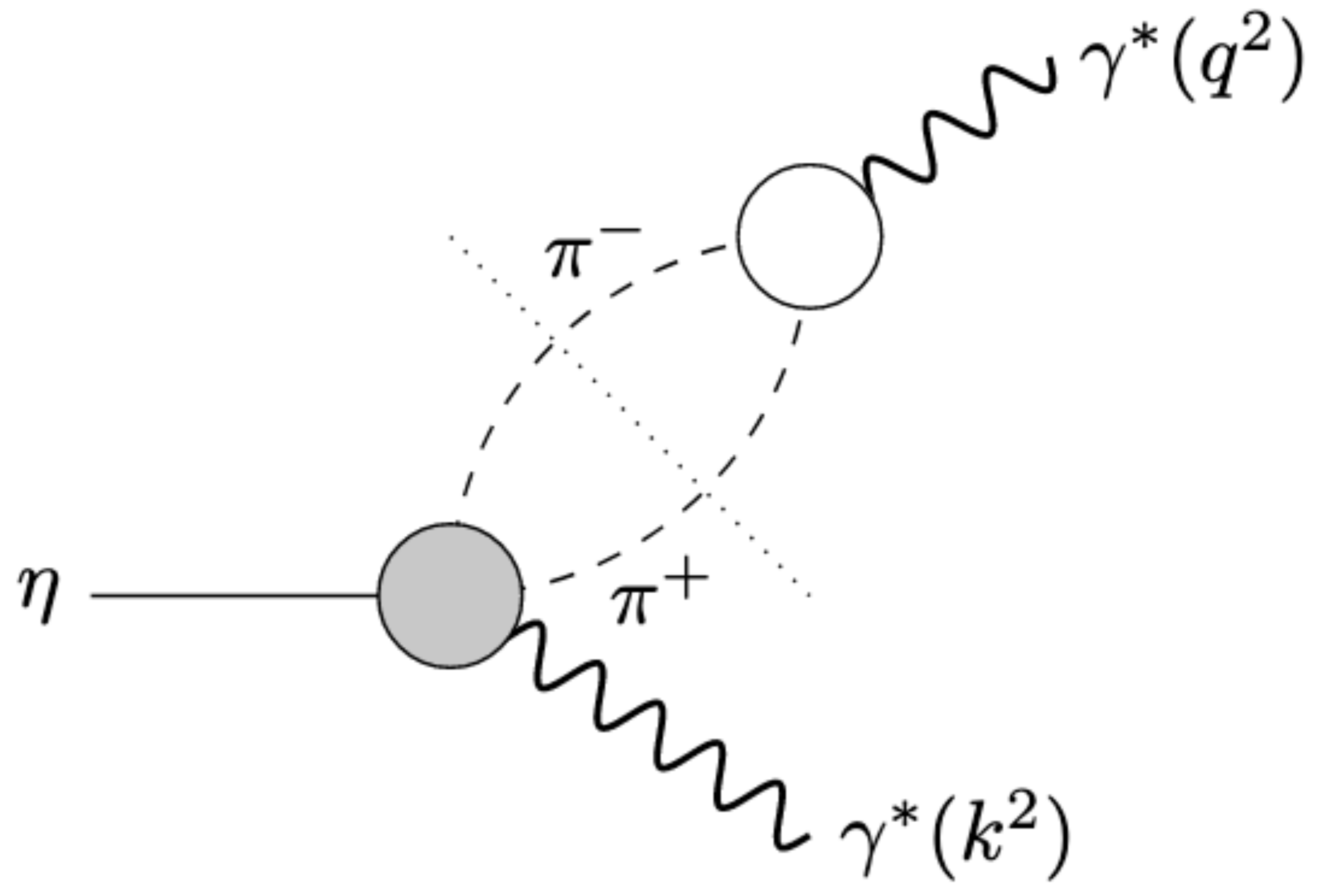
- Precision currently lower than data-driven results

[Blum+ Phys.Rev.Lett. 118, 022005 (2017)]
[Blum+ Phys.Rev.Lett. 124, 132002 (2020)]
[Chao+ Eur.Phys.J.C 81 (2021) 7]
[Asmussen+ 1801.04238]

Dispersive η, η'-pole

[Holz+ Eur. Phys. J. C 81,1002 (2021)]
[Holz+ Eur. Phys. J. C 82, 434 (2022)]

- Low-energy contribs accessible in IV channel ($q^2 \lesssim 1.0\,\mathrm{GeV}^2$)



Computing the TFFs

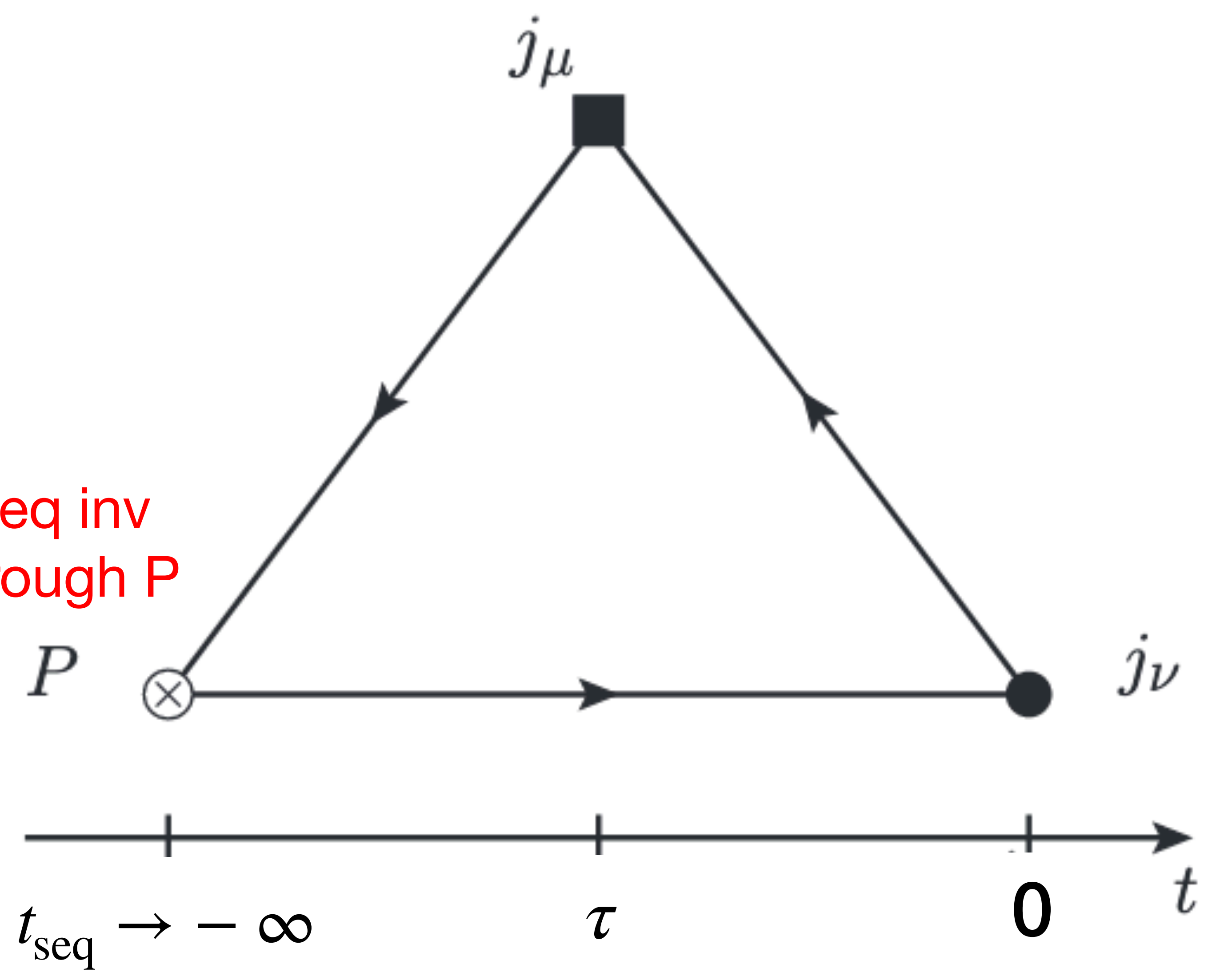
Lattice calculation of $F_{P\gamma\gamma}$

1. Euclidean time current-current matrix element

$$\tilde{A}_{\mu\nu}(\tau) = \sum_{\mathbf{x}} e^{-i\mathbf{q}_1\cdot\mathbf{x}} \left\langle 0 \middle| j_\mu(\tau;\mathbf{x})\, j_\nu(0;\mathbf{0}) \middle| P(\mathbf{p}) \right\rangle$$

Note: renormalized currents
using precisely known Z_A, Z_V

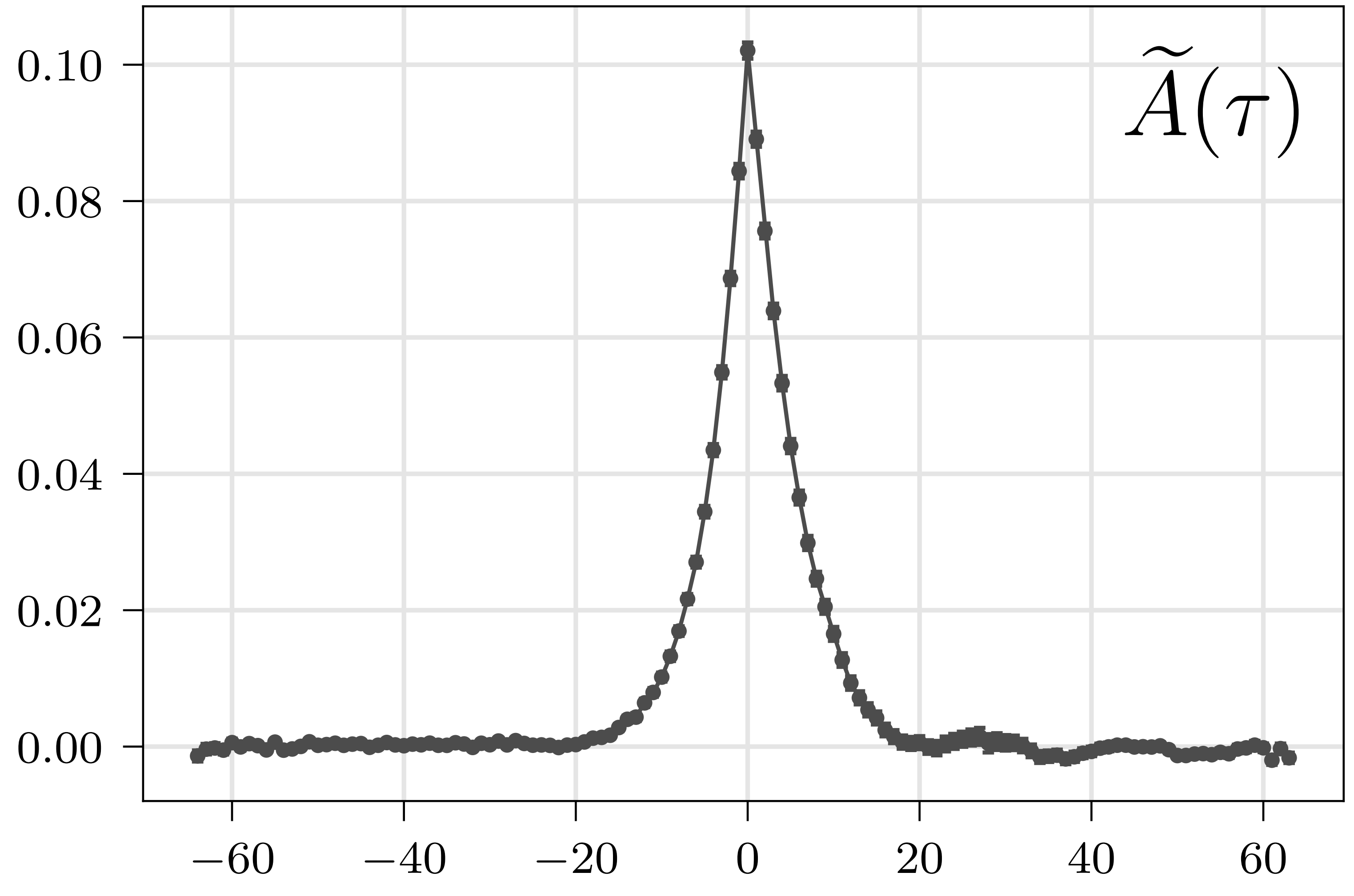
Seq inv
through P



2. Laplace transform

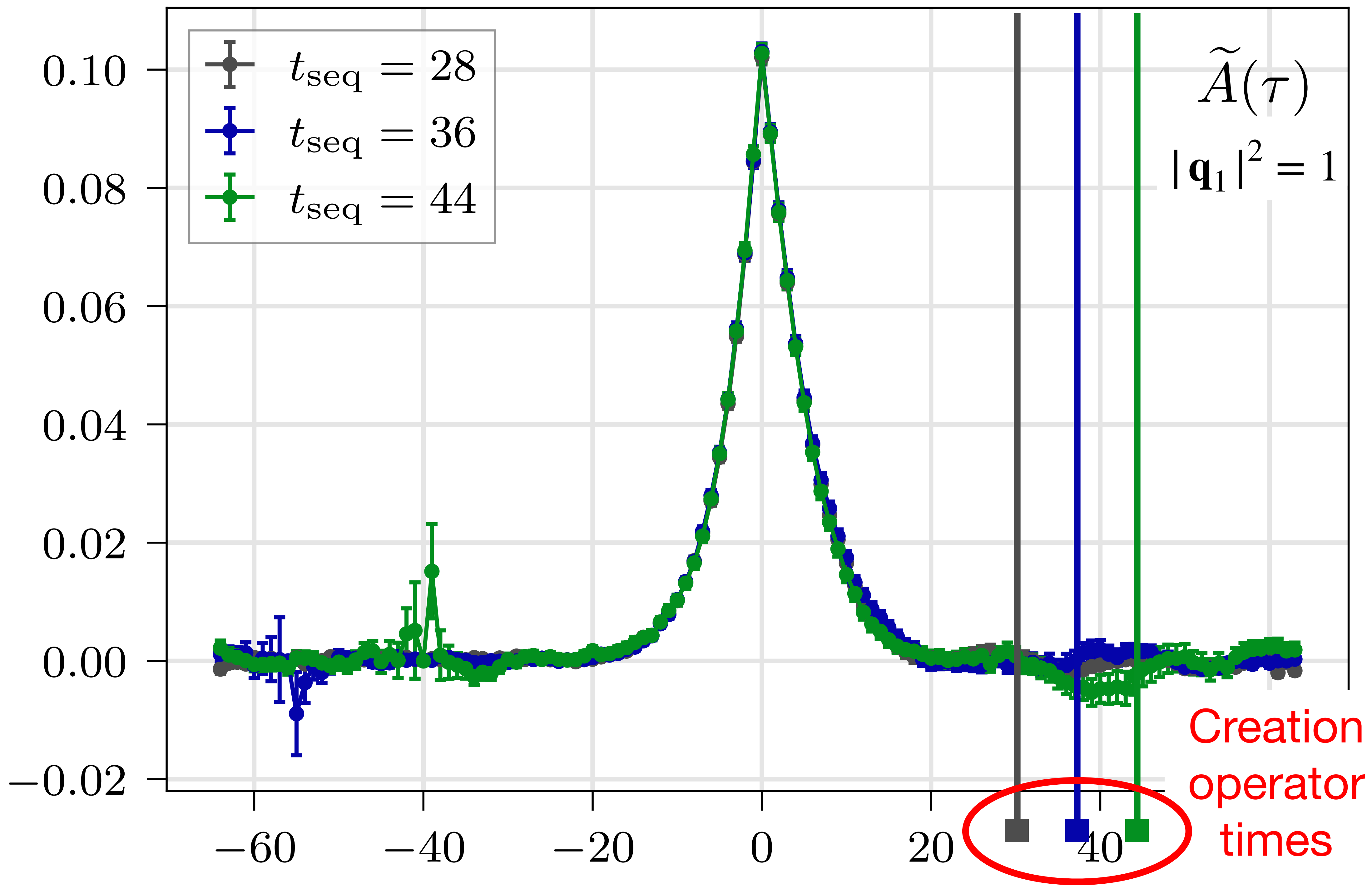
$$\epsilon^{\mu\nu\rho\sigma} q_{1\rho} q_{2\sigma} F_{P\gamma\gamma}(q_1^2, q_2^2) = -\, i^{n_0} \int_{-\infty}^{\infty} d\tau\, e^{\omega_1 \tau}\, \tilde{A}_{\mu\nu}(\tau)$$

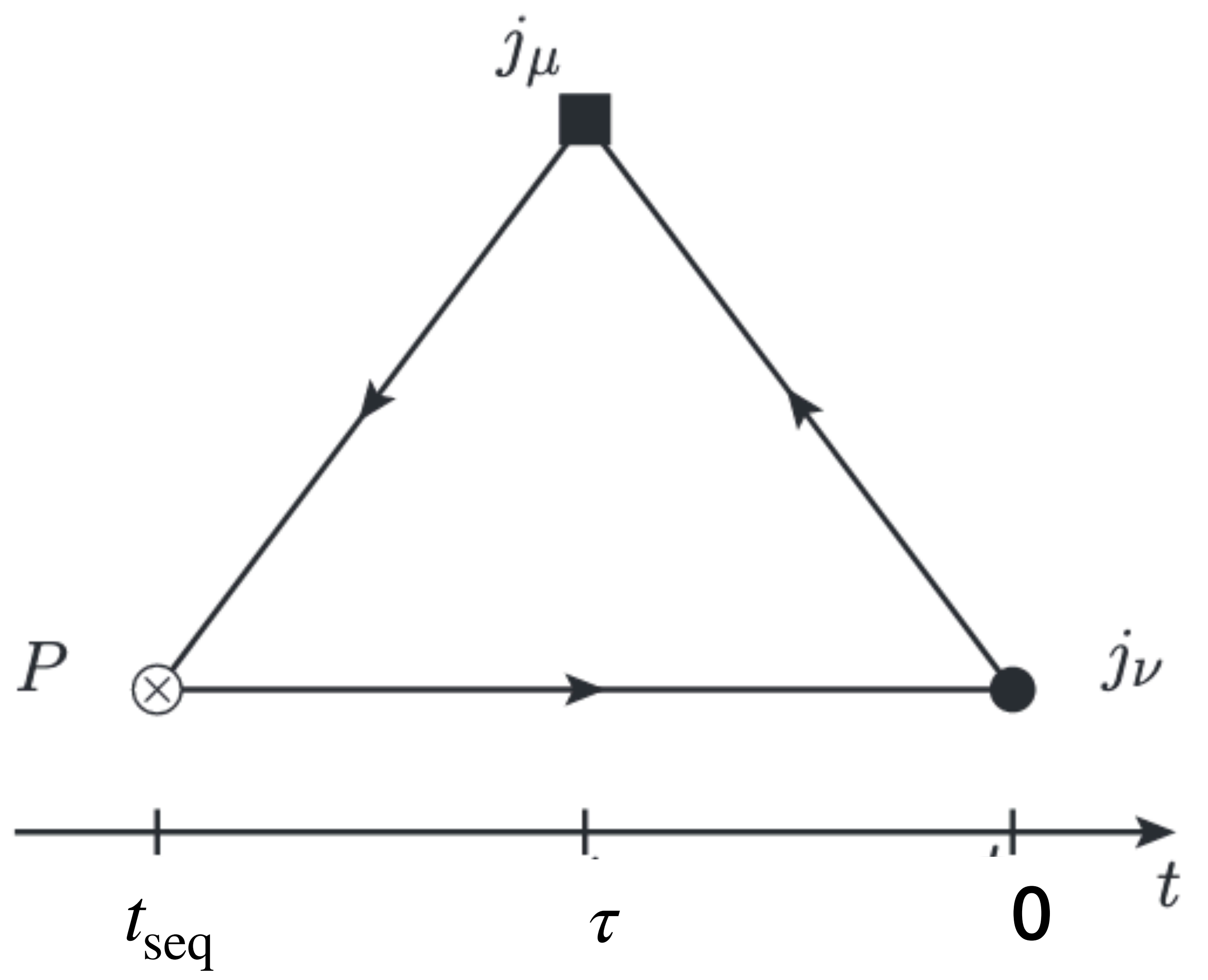
where $\mathbf{q}_2 = \mathbf{p} - \mathbf{q}_1$ by momentum conservation



Matrix element from 3-pt function

$$\tilde{A}_{\mu\nu}(\tau) = \boxed{\lim_{t_{\mathrm{seq}}\to\infty}} \frac{2E_P}{Z_P} e^{E_P t_{\mathrm{seq}}} \sum_{\mathbf{x},\mathbf{y}} e^{-i\mathbf{q}_1\cdot\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{y}} \left\langle j_\mu(\tau,\mathbf{x}) j_\nu(0,\mathbf{0}) P^\dagger(-t_{\mathrm{seq}},\mathbf{y}) \right\rangle$$





Interpolating operators

Note: λ^i are $SU(3)_f$ generators (Gellmann basis)

$$\lambda^3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

$$\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

For π^0 ordinary local interpolator is sufficient:

$$\mathcal{O}_{\pi^0} = i\bar{\psi}\lambda^3\gamma_5\psi$$

For η, η', mixing in principle requires diagonalizing between

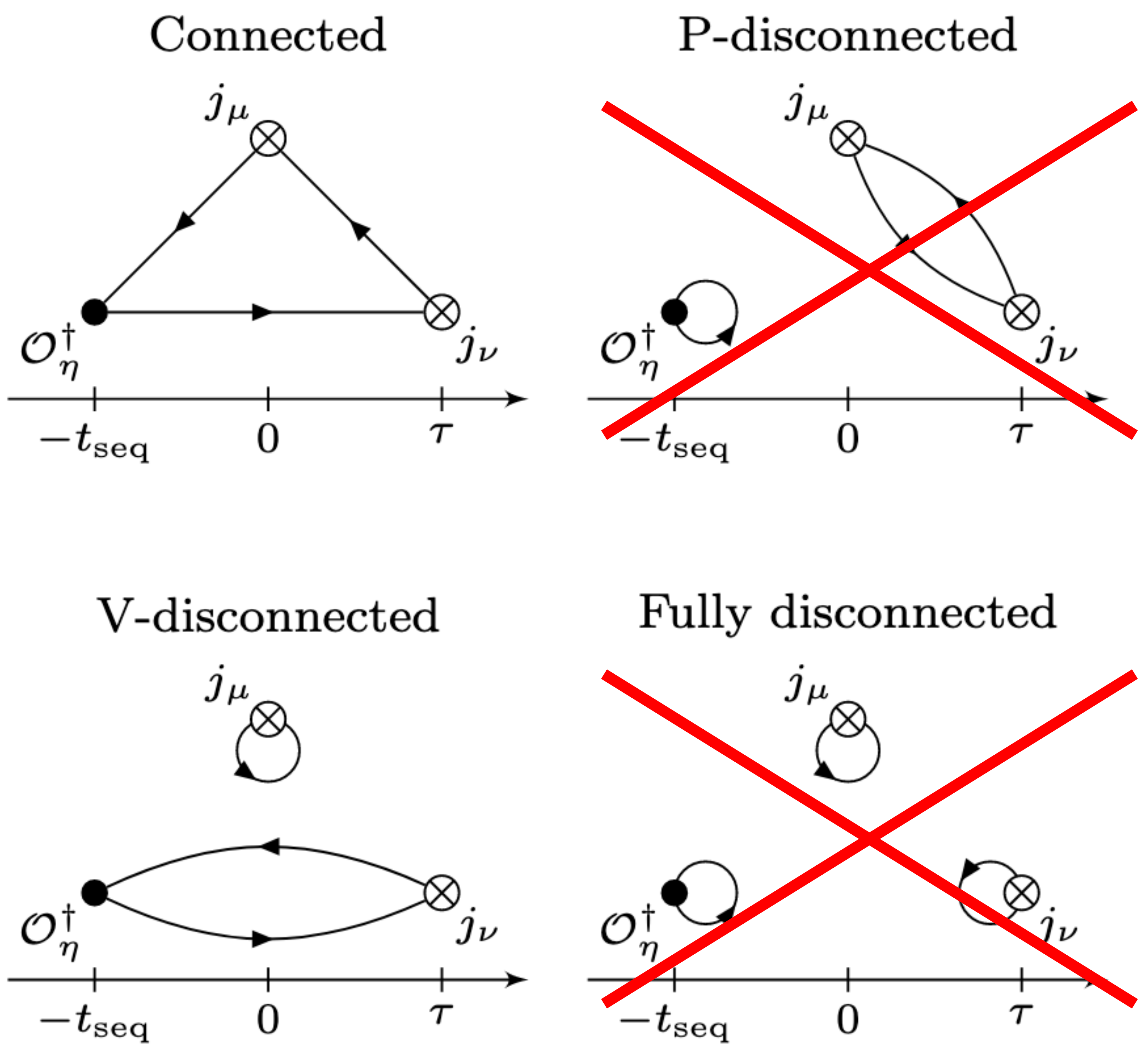
$$\mathcal{O}_{\eta^8} = i\bar{\psi}\lambda^8\gamma_5\psi \quad \text{and} \quad \mathcal{O}_{\eta^0} = i\bar{\psi}\gamma_5\psi$$

to separate η' from the correlator ground state (the η)

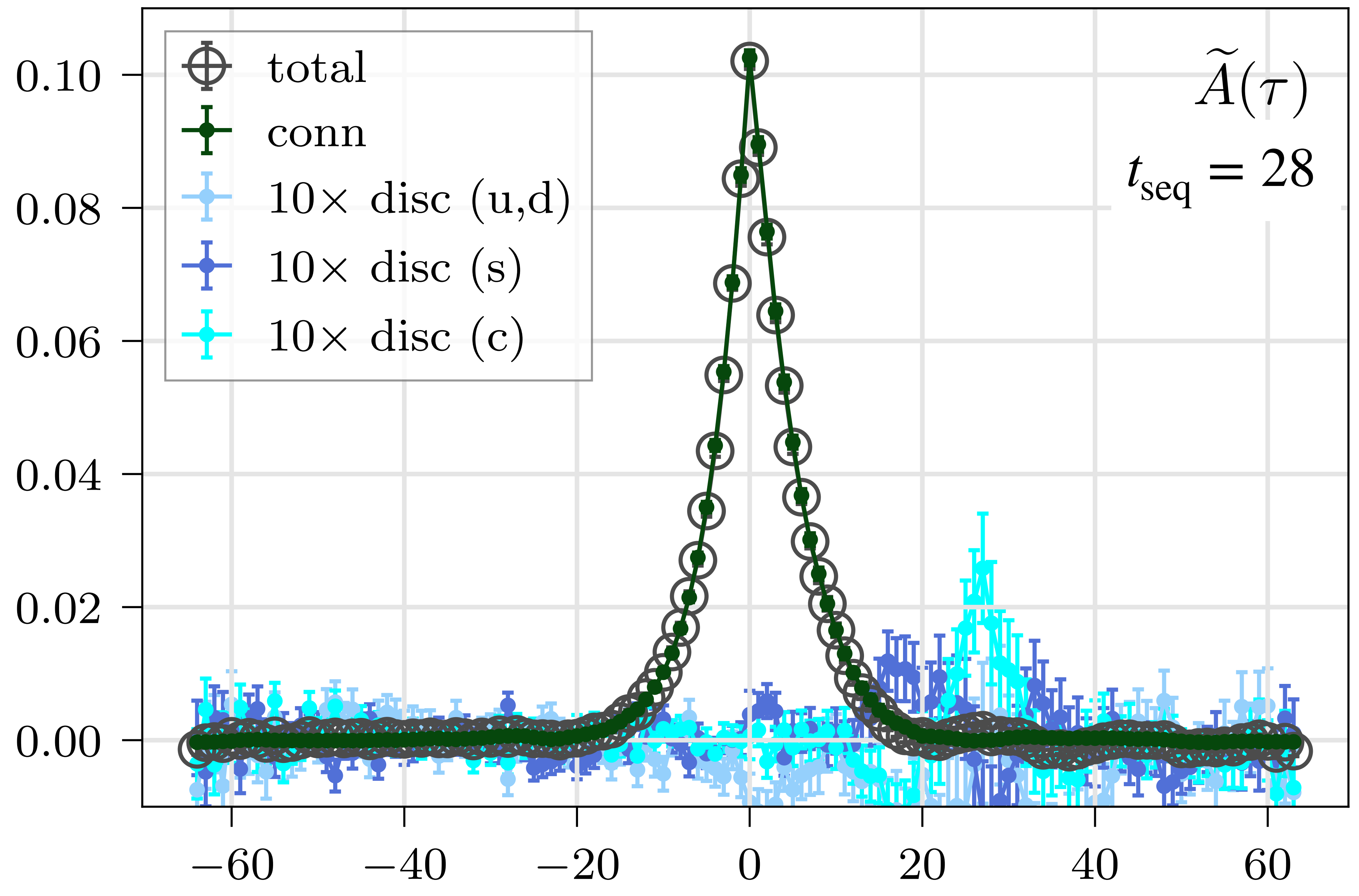
- Analysis for η proceeds with $\mathcal{O}_{\eta^8}$
- Noise reduction & GEVP required for future exploration of η'

Wick contractions: π^0 isospin rotation

Isospin symmetry* allows removing P-disconnected diagrams:



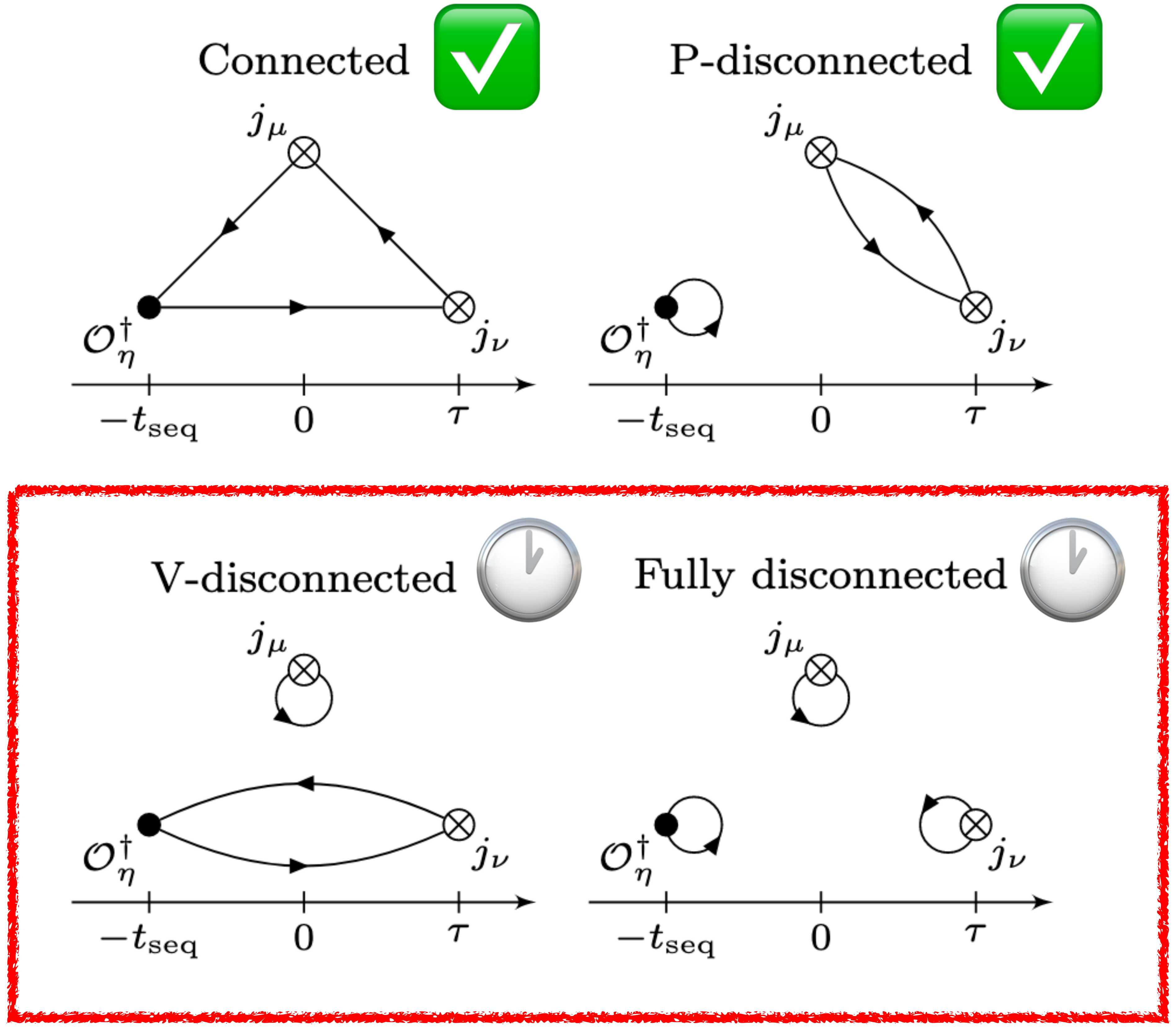
Remaining V-disconnected diagrams are included, but numerically small:



* $O(a^2)$ isospin breaking with twisted mass is part of continuum extrapolation

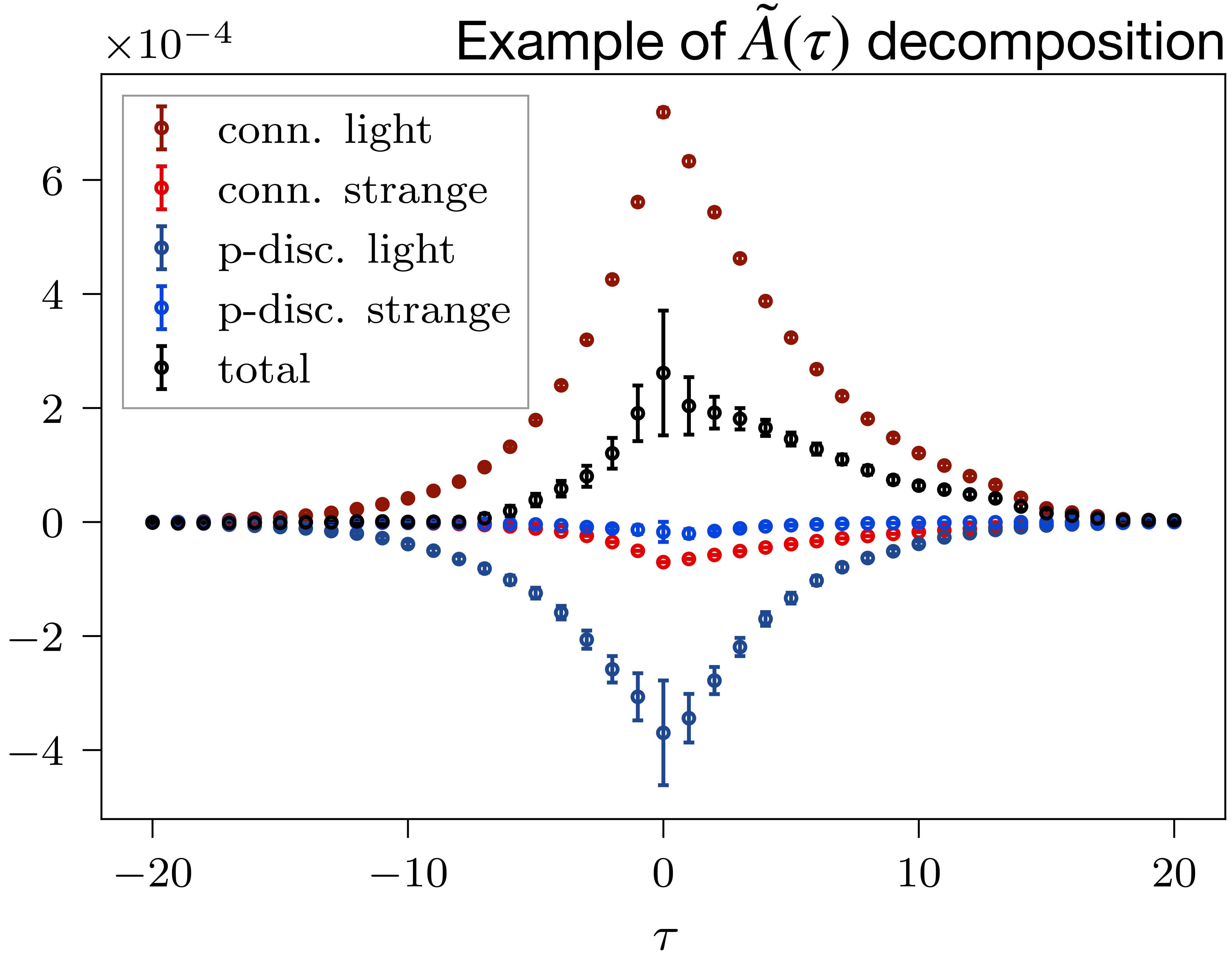
Wick contractions: η diagrams

All diagrams required even in isospin limit.



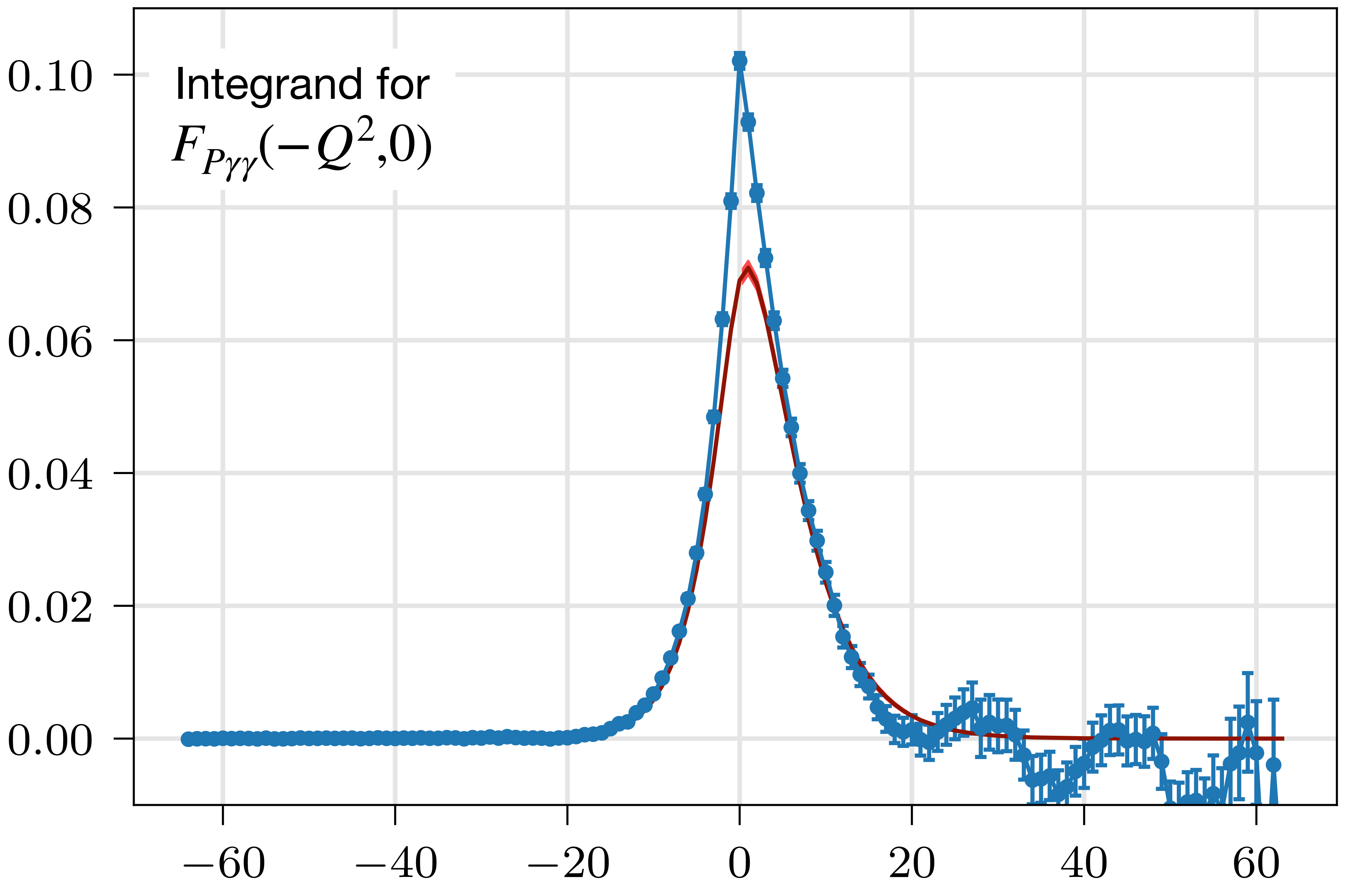
Excluded at the present stage of analysis,
but expected to be subleading.

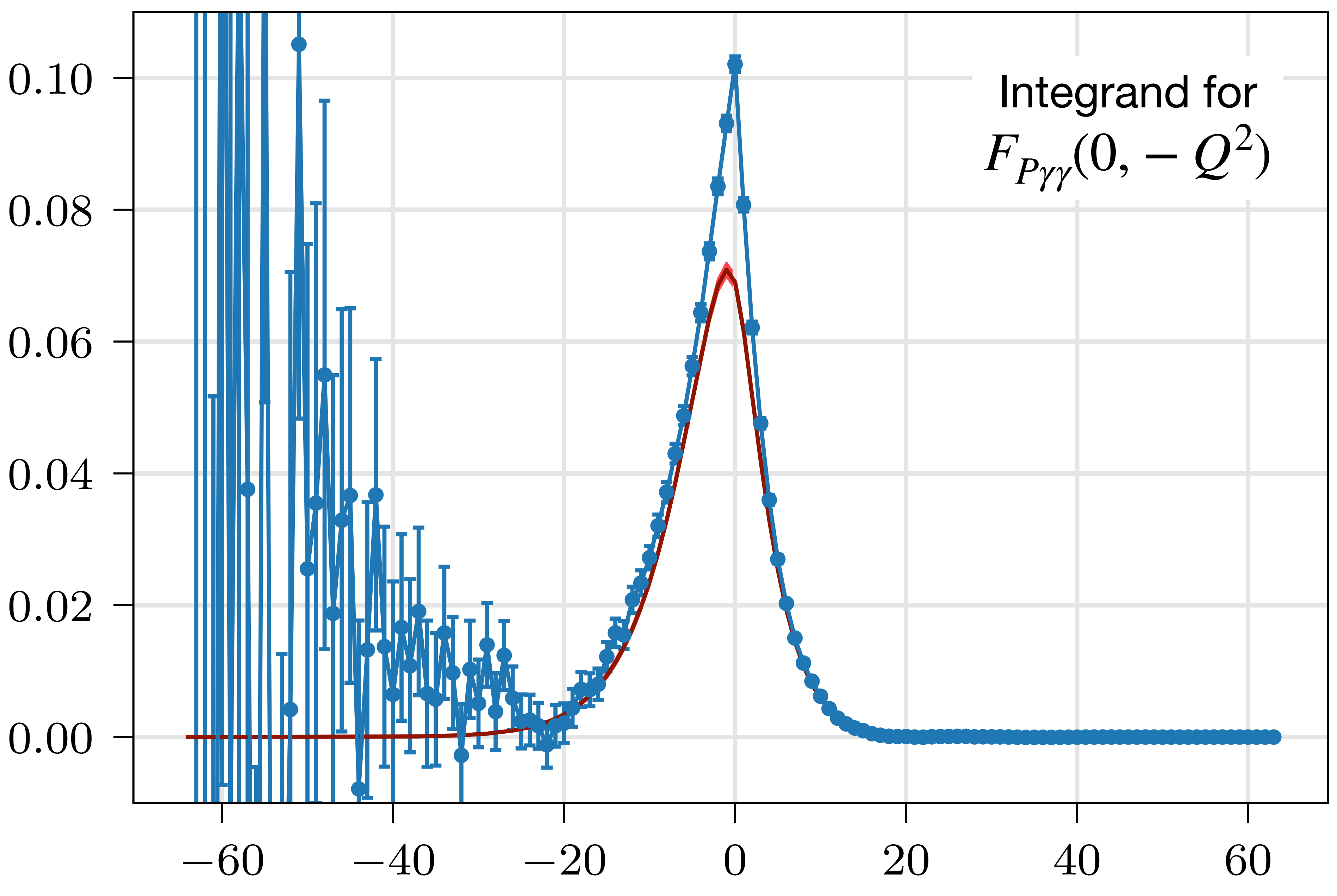
[Gérardin+ 2211.04159]



Tail fitting and integration

- Exponentially growing noise in the tails of $\tilde{A}_{\mu\nu}(\tau)$
  - Probed in the Laplace transform $\int_{-\infty}^{\infty} d\tau\, \boxed{e^{\omega_1 \tau}}\, \tilde{A}_{\mu\nu}(\tau)$





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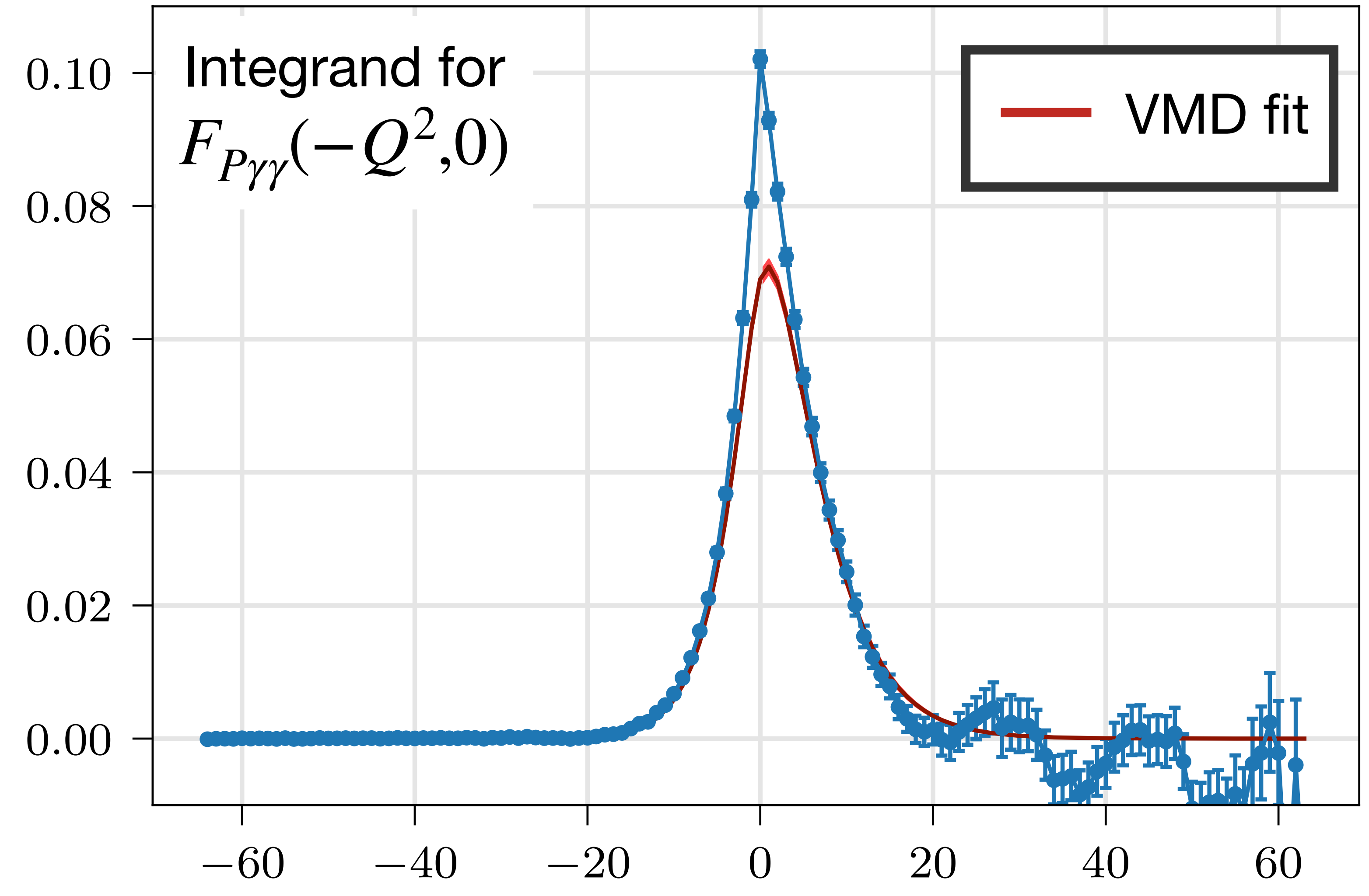
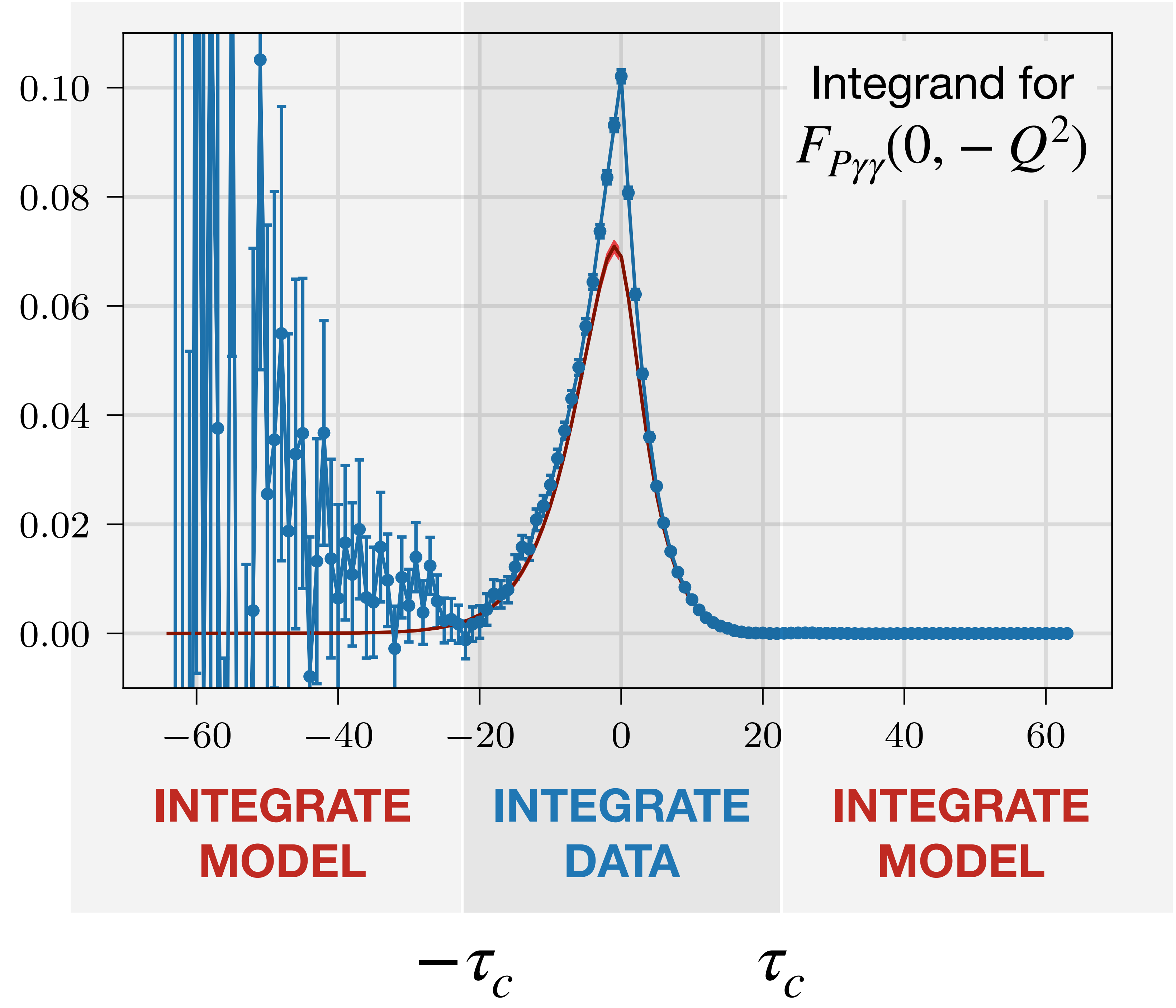
- To handle this…
  - Fit Vector Meson Dominance (VMD) or Lowest Meson Dominance (LMD) models to the tails
  - Integrate **data** in peak ($|\tau| < \tau_c$)
    Integrate **model** in tails ($|\tau| > \tau_c$)





VMD & LMD

$$F_{P\gamma\gamma}^{\mathrm{LMD}}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(q_1^2 - M_V^2)(q_2^2 - M_V^2)}$$

and

$$F_{P\gamma\gamma}^{\mathrm{VMD}}(q_1^2, q_2^2) = F_{P\gamma\gamma}^{\mathrm{LMD}}(q_1^2, q_2^2)\,|_{\beta=0}$$

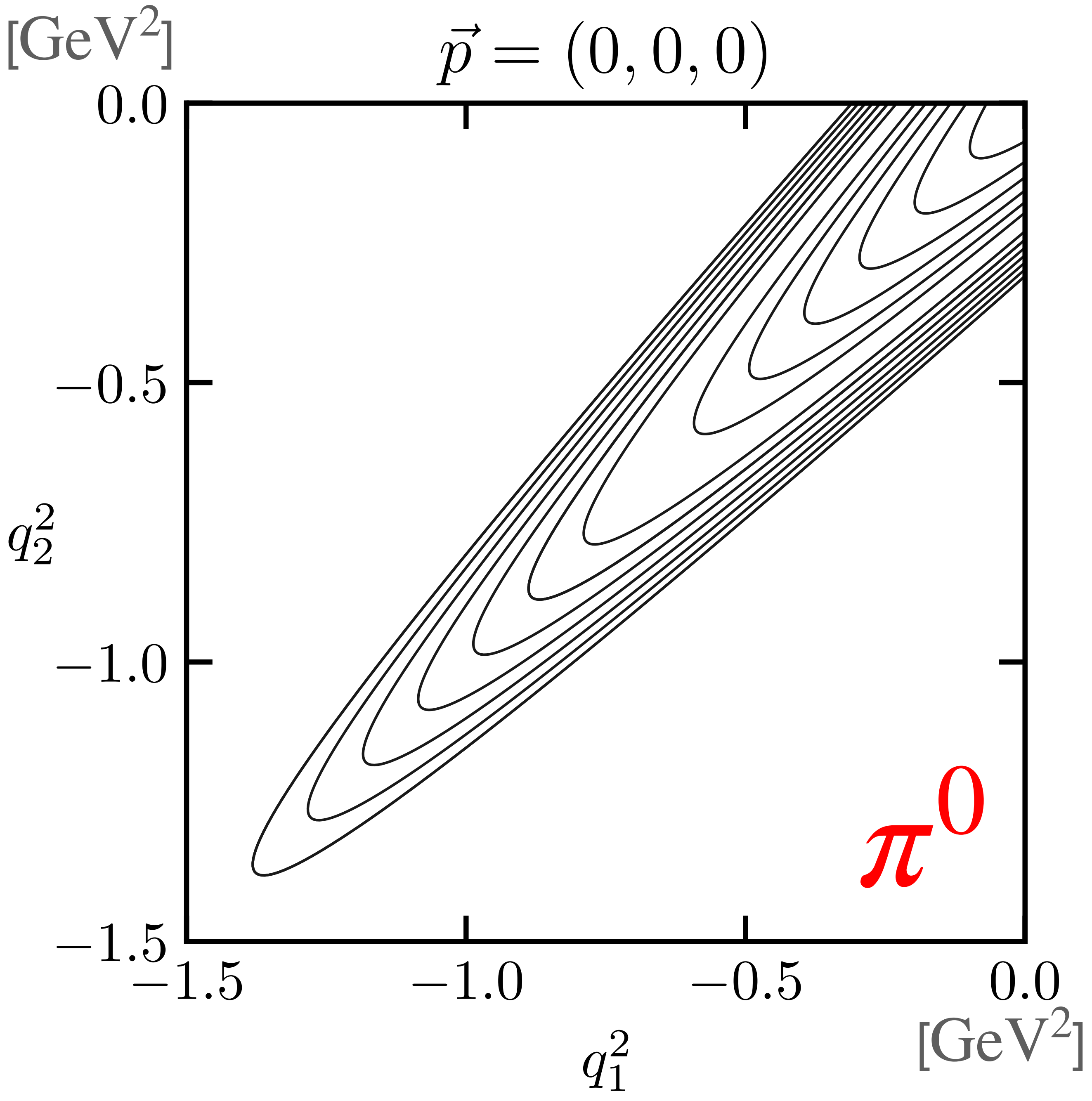
$\Rightarrow$ $\tilde{A}^{\mathrm{LMD}}(\tau)$ and $\tilde{A}^{\mathrm{VMD}}(\tau)$ by inverse Laplace transform

According to VMD, $M_V = M_\rho$, but left as a free fit parameter in this work

Extrapolating in (q_1^2, q_2^2) plane

Finite volume momenta $\mathbf{q}_1$, $\mathbf{p}$, arbitrary ω_1

- Parabolic "orbits" of (q_1^2, q_2^2) accessible
- Singly virtual coverage ($\mathbf{p} \neq 0$ & larger m_P help!)



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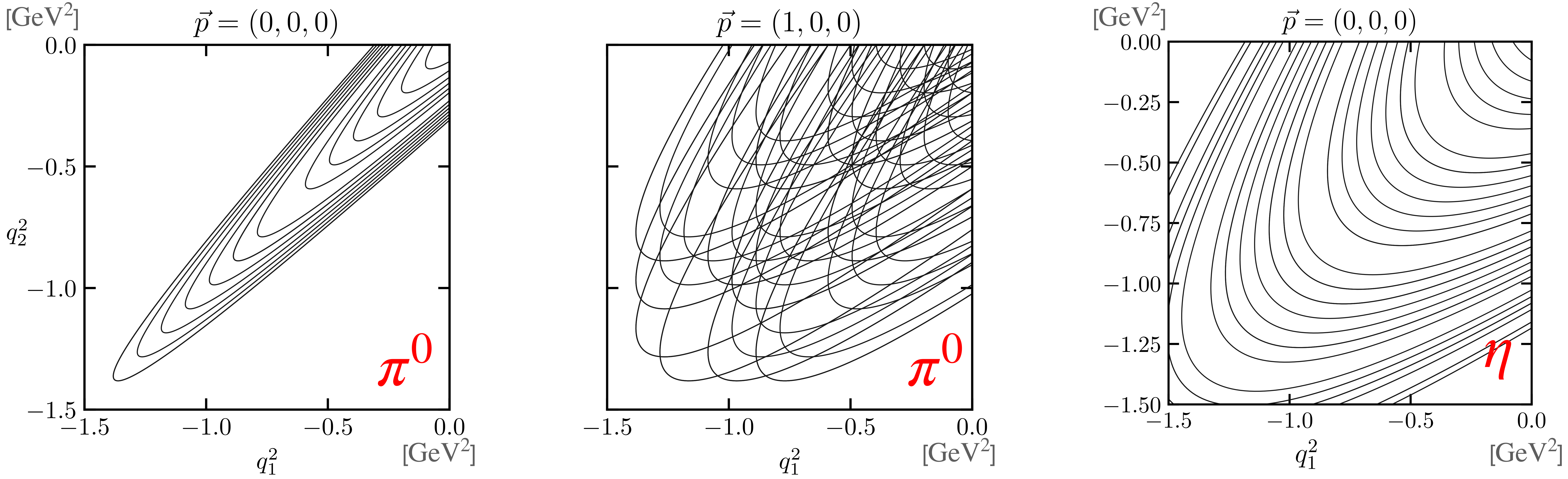
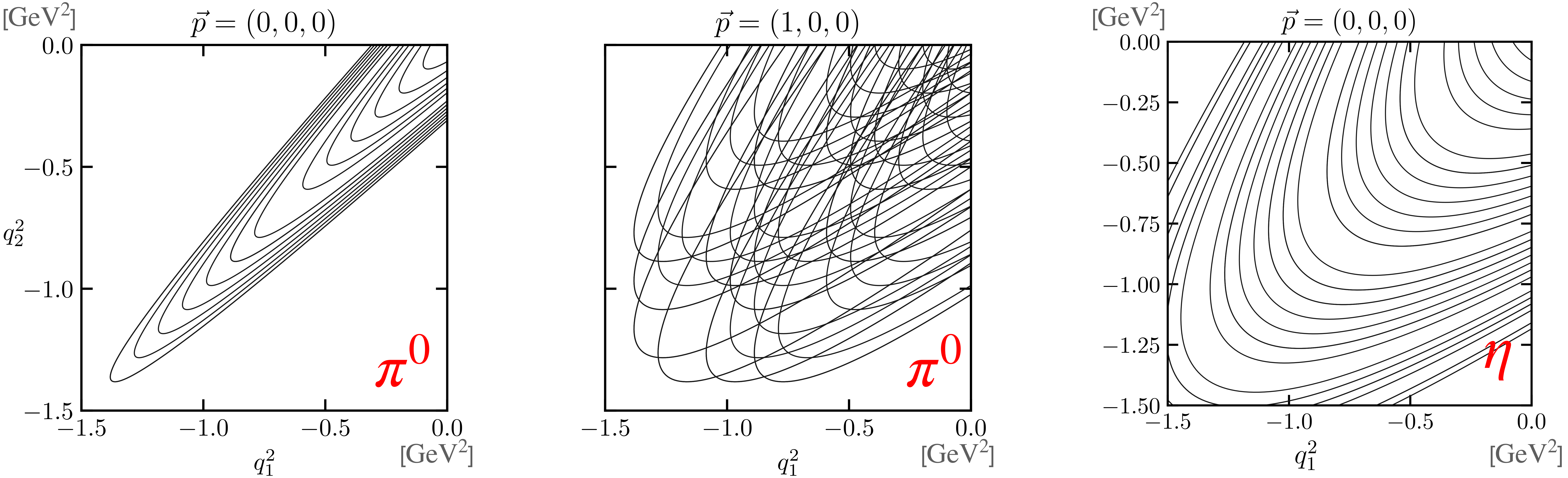


Extrapolating in (q_1^2, q_2^2) plane

Finite volume momenta $\mathbf{q}_1$, $\mathbf{p}$, arbitrary ω_1

- Parabolic "orbits" of (q_1^2, q_2^2) accessible
- Singly virtual coverage ($\mathbf{p} \neq 0$ & larger m_P help!)

→ z-expansion for full (q_1^2, q_2^2) dependence



Results

Lattice details

Action: Nf = 2+1+1 twisted clover, Iwasaki gauge action, *physical point*

Pion analysis:

- Three lattice spacings ✅
- Continuum extrapolation ✅ (preliminary)

ensemble	$L^3 \cdot T/a^4$	m_π [MeV]	a [fm]	$a \cdot L_x$ [fm]	$m_\pi \cdot L_x$
cB072.64	$64^3 \cdot 128$	140.2	0.080	5.09	3.62
cC060.80	$80^3 \cdot 160$	136.7	0.068	5.46	3.78
cD054.96	$96^3 \cdot 192$	140.8	0.057	5.46	3.90

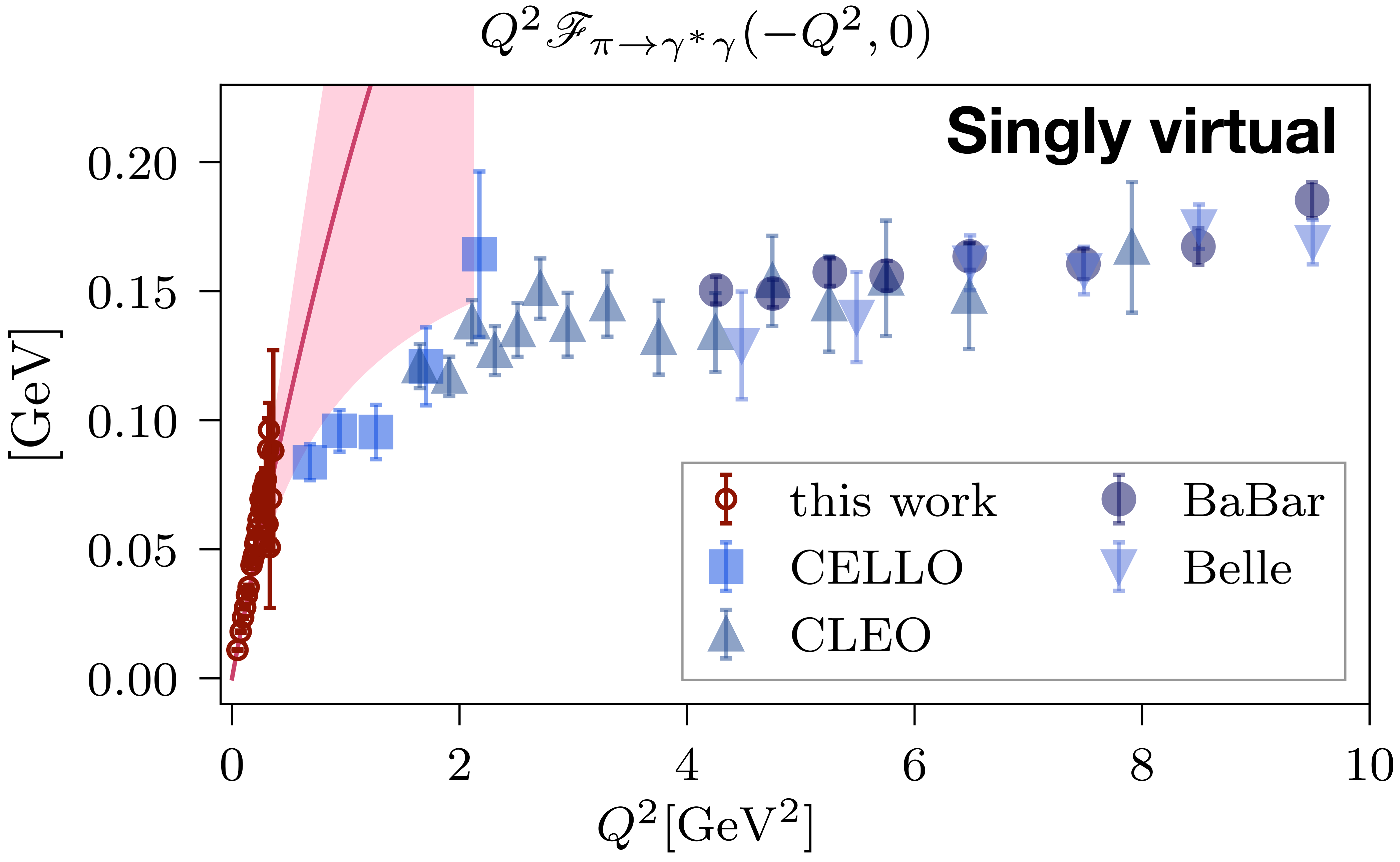
Eta analysis:

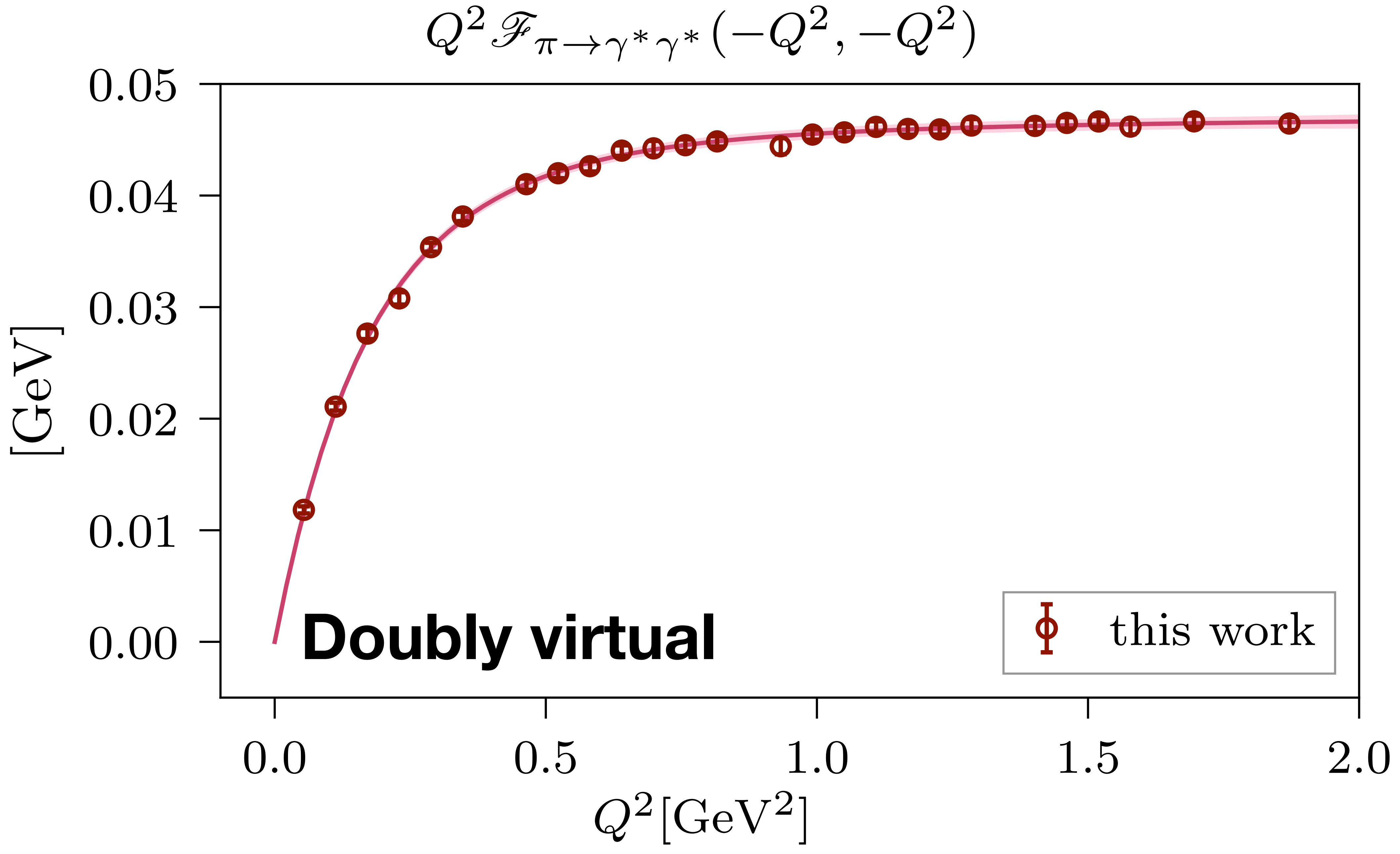
- One lattice spacing (cB072.64, 0.08fm) ✅
- Runs for cC, cD in next allocation cycle 🕐

Eta' analysis:

- Found to be too noisy so far ⚠️

π^0 TFF results





Shown for one choice of ensemble and analysis procedure.

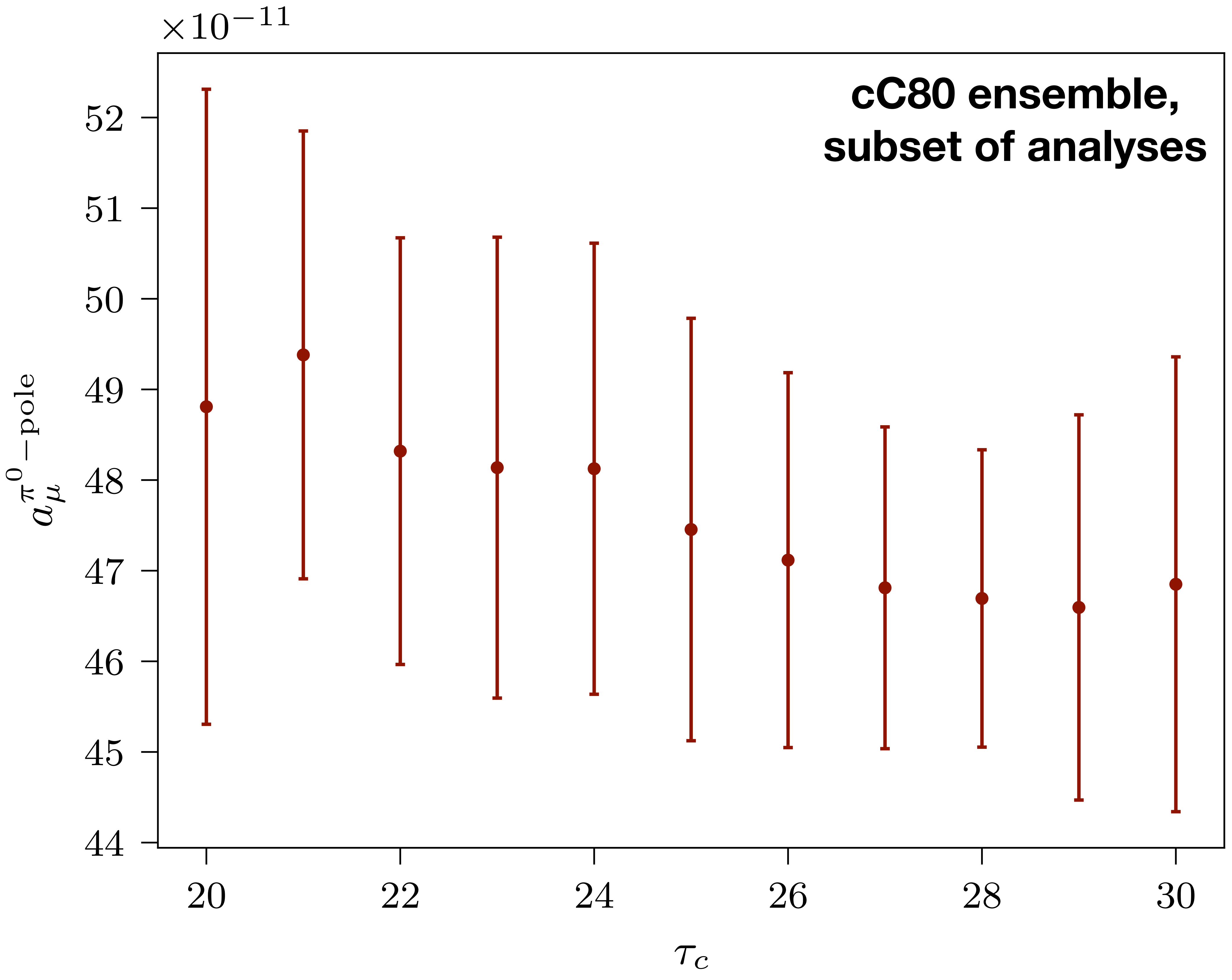
Doubly virtual much more precise than singly virtual due to kinematics.

Muon g-2: π^0 pole

1. z-expansion fits to FF data per choice of tail model, tail fit window, tail cut τ_c

2. Integrated $a_\mu^{\pi^0\mathrm{-pole}}$ per FF

3. Systematic error accounting:
   - AIC weighted model averaging
     [Borsanyi+ Science 347, 1452 (2015)]
   - Stability against other analysis choices (z-exp fit ranges, t_{seq})



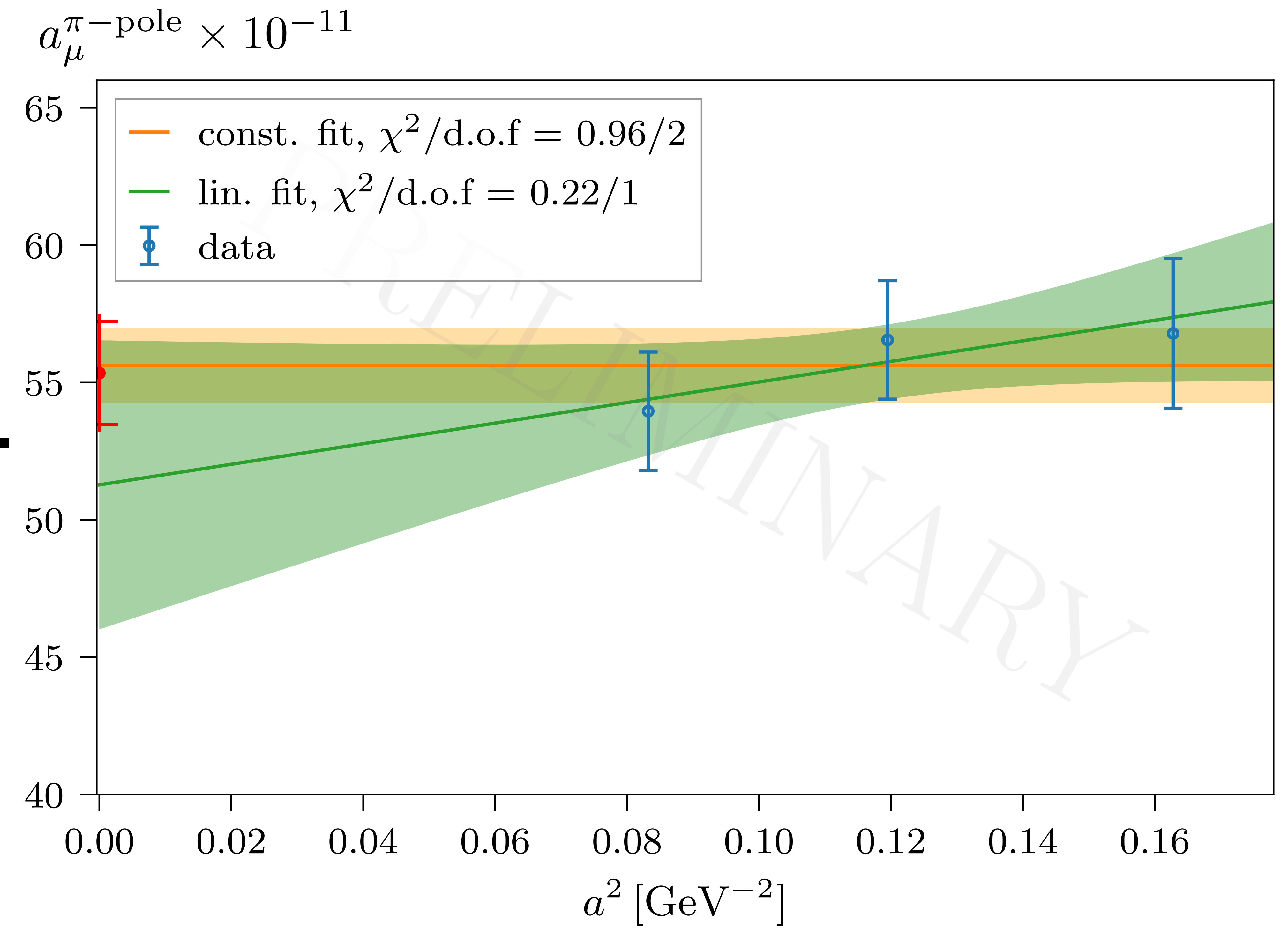
Muon g-2: π^0 pole

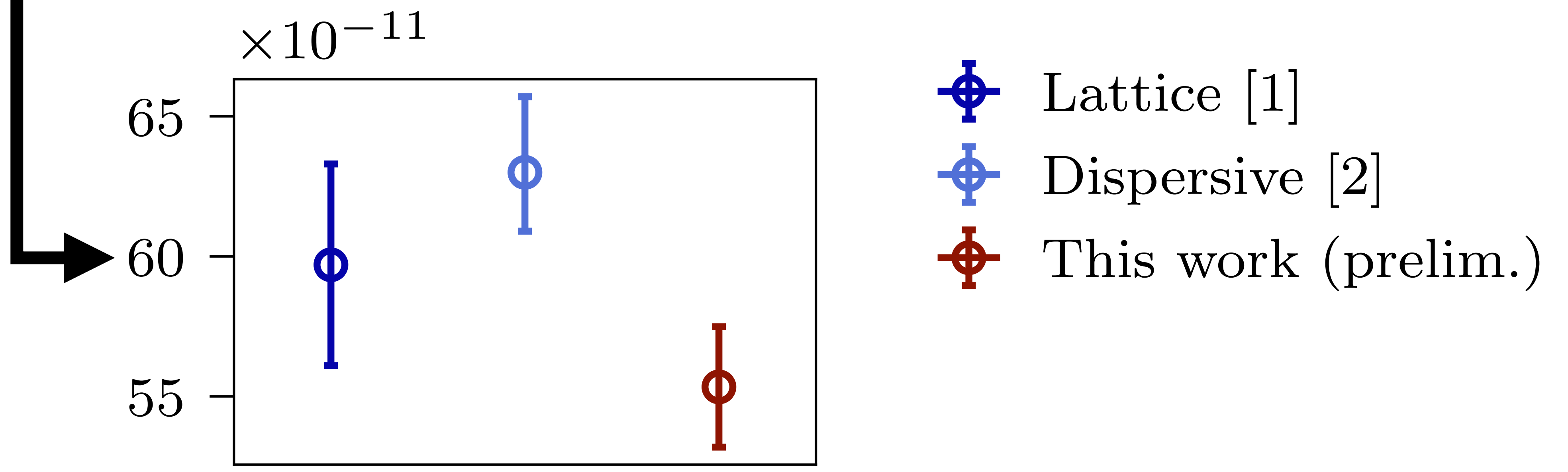
Preliminary continuum limit:

$$a_\mu^{\pi^0-\mathrm{pole}} = 55.3(1.9)_{\mathrm{ctm}}(1.0)_{\mathrm{ctm-syst}} \times 10^{-11}$$

Constant vs linear in a^2 fits.

- Constant: underestimated error
- Linear: overfitting
- Preliminary result: weighted average, variation between fits as syst. err [Alexandrou+ (ETMC) 2104.13408]
- Likely too aggressive, will be refined

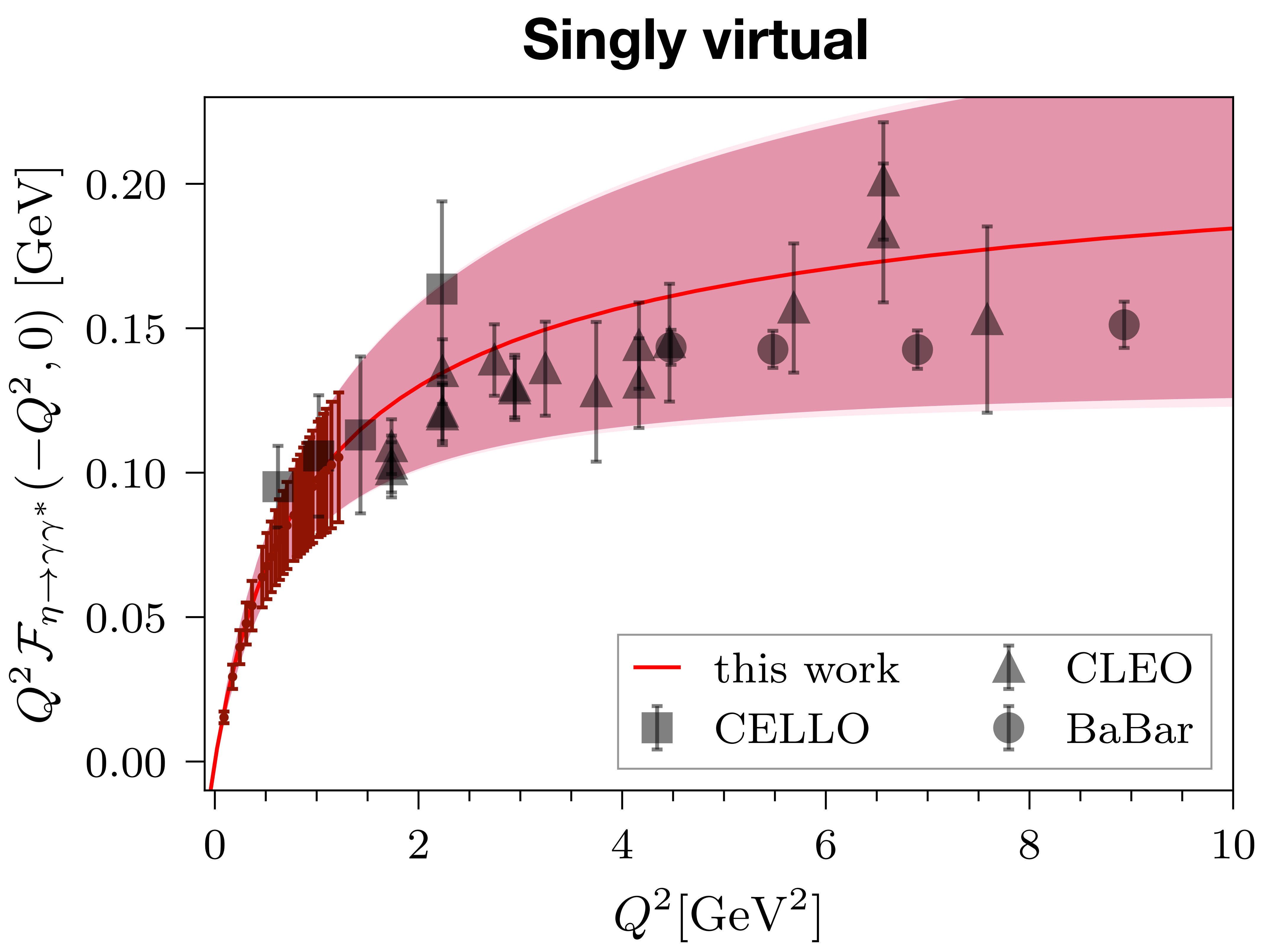


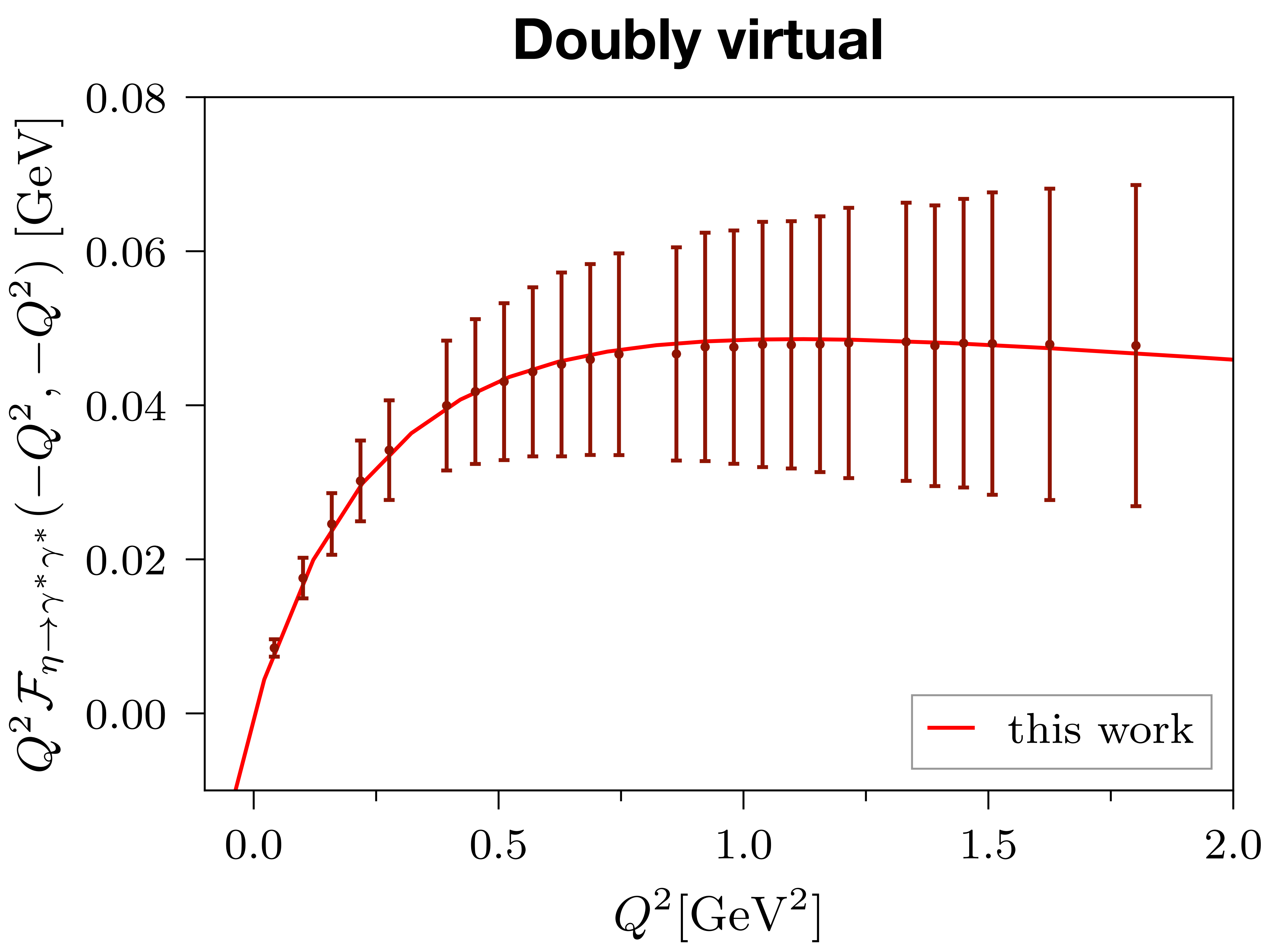


[1] Gérardin+ PRD 100 (2019) 3
[2] g-2 WP, Phys. Rep. 887 (2020)

η TFF results

Statistical errors (dark band) dominate systematic errors (light outer band).





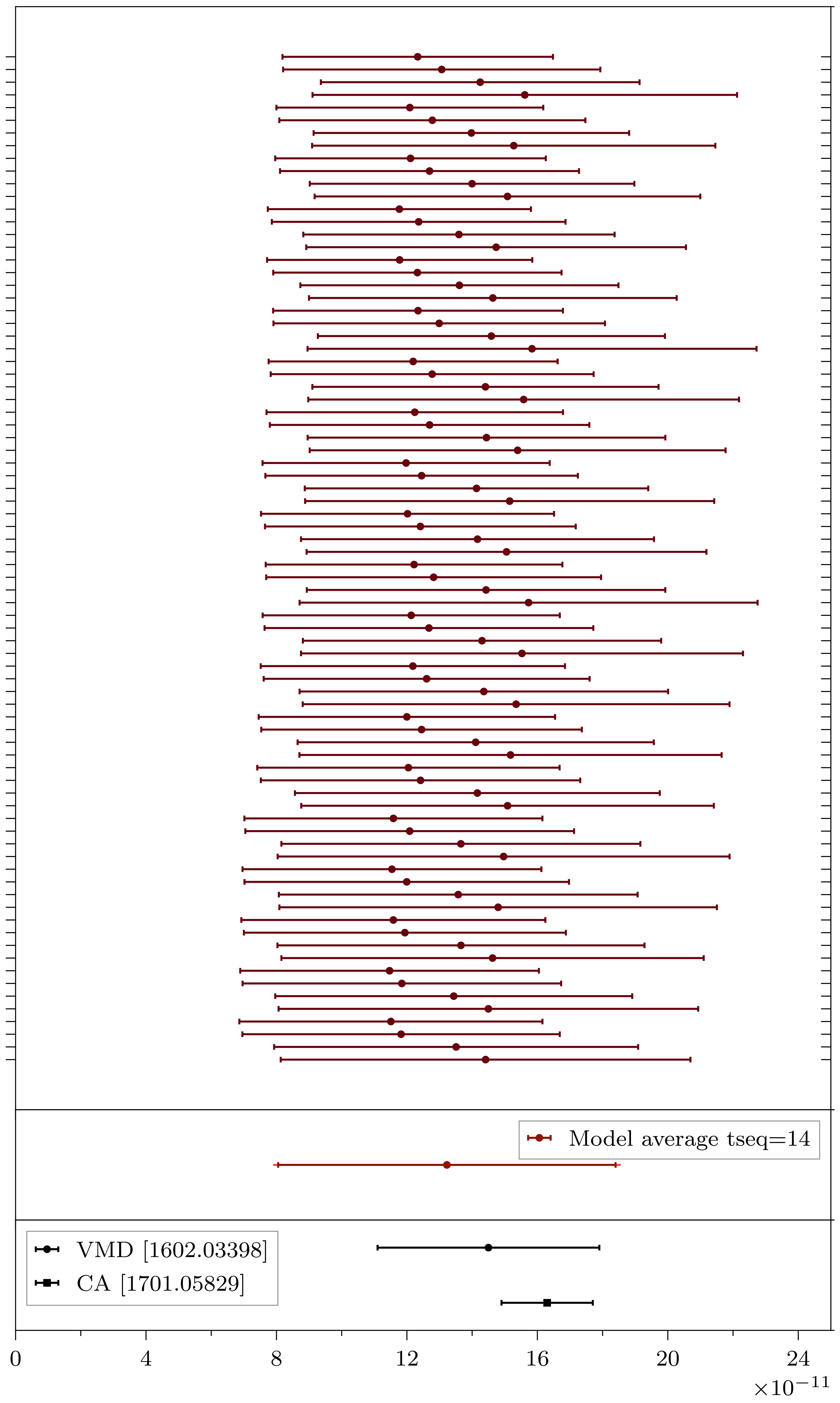
Muon g-2: η pole (0.08 fm)

- Same model averaging procedure as for the pion

- Results found to be quite stable

- Statistical error also dominates g-2

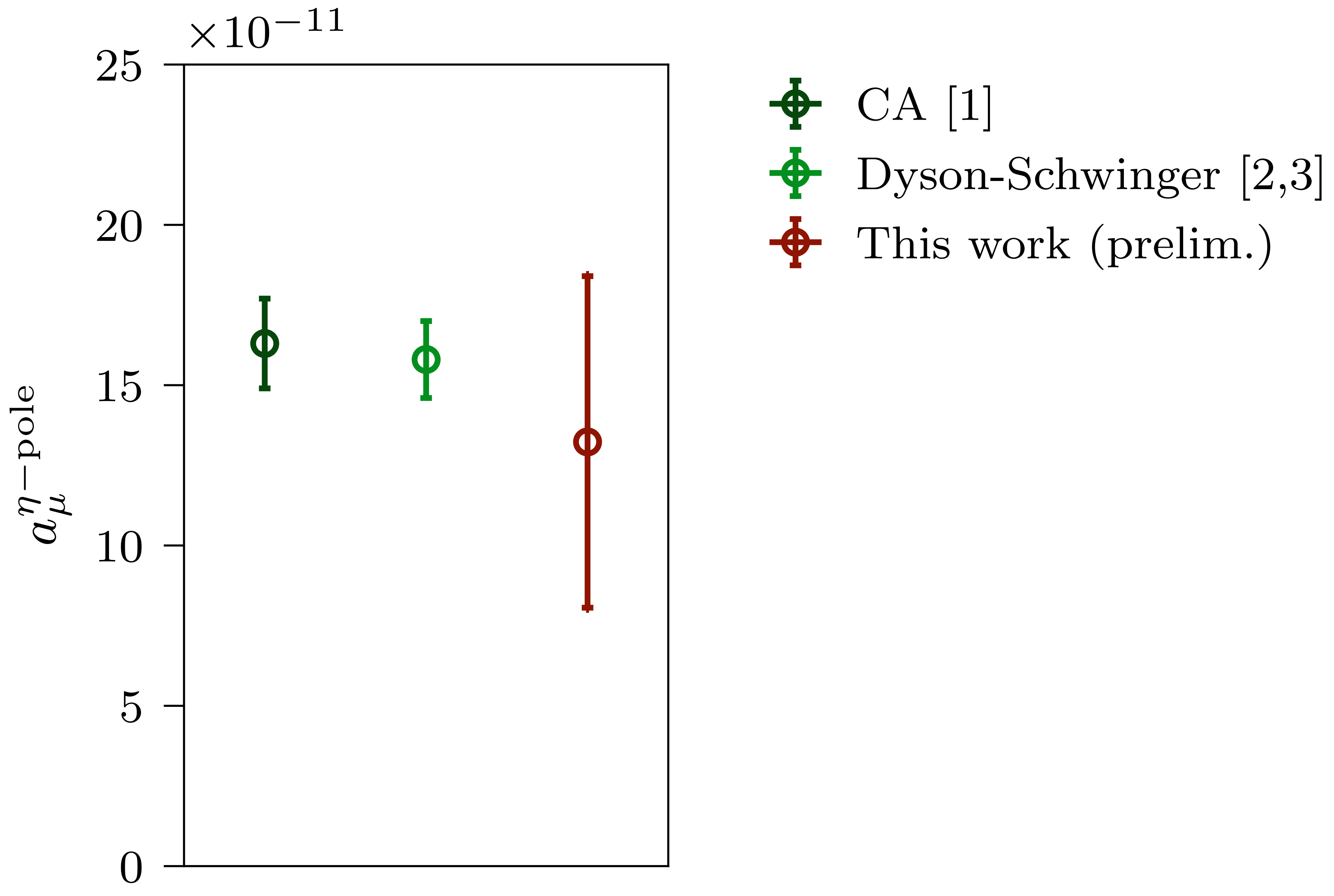
Various fit windows, integration cuts, z-exp orders



Muon g-2: η pole (0.08 fm)

Preliminary 0.08fm (cB ens.):

$$a_{\mu,0.08\mathrm{fm}}^{\eta-\mathrm{pole}} = 13.2(5.2)_{\mathrm{stat}}(1.3)_{\mathrm{syst}} \times 10^{-11}$$

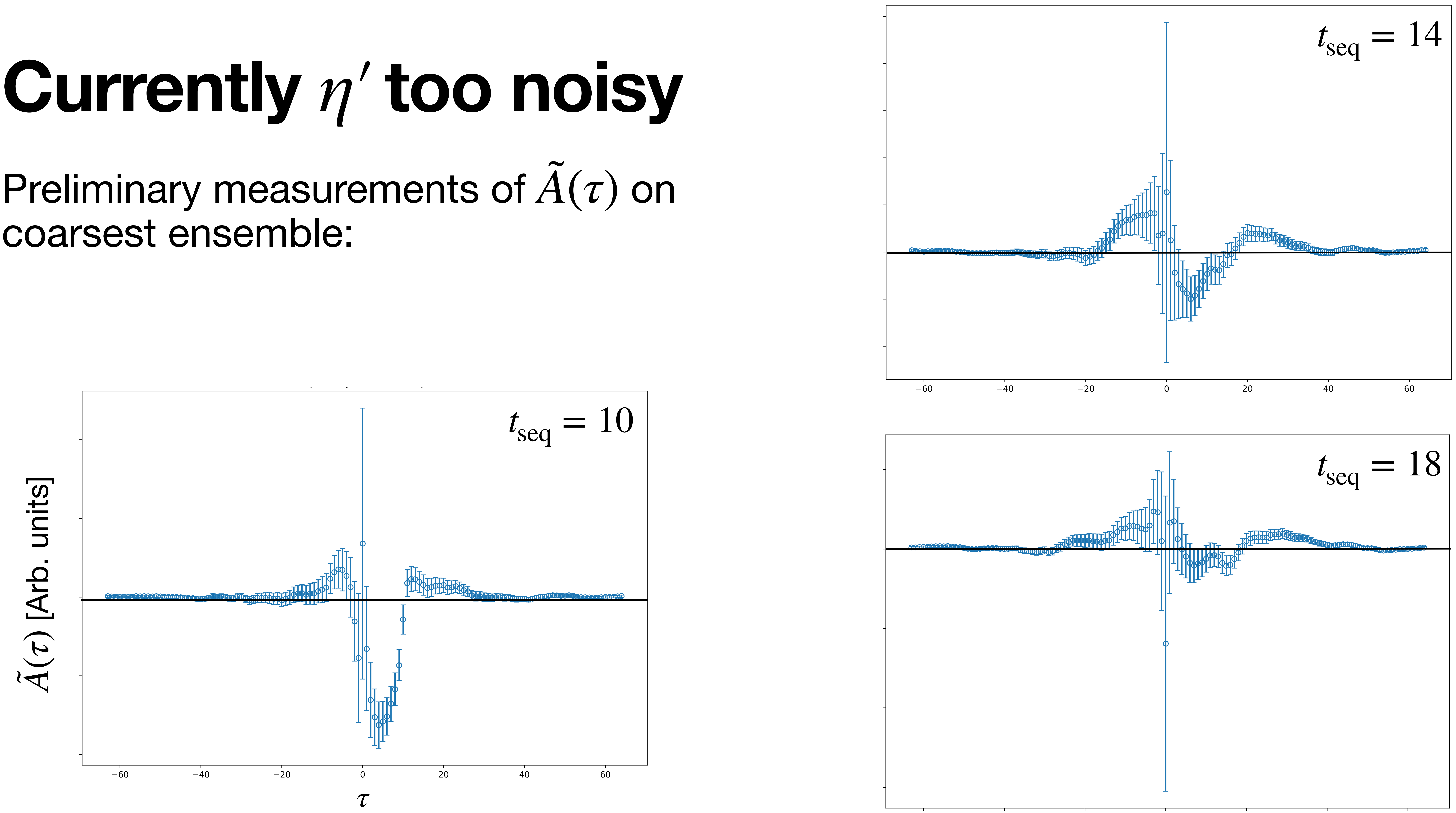


- Largest $t_{\mathrm{seq}} = 14$ used (conservative choice)
- Results consistent with previous work

[1] Masjuan, Sanchez-Puertas PRD95 (2017) 054026.
[2] Eichmann+ PLB797 (2019) 134855.
[3] Khépani+ PRD101 (2020) 074021.

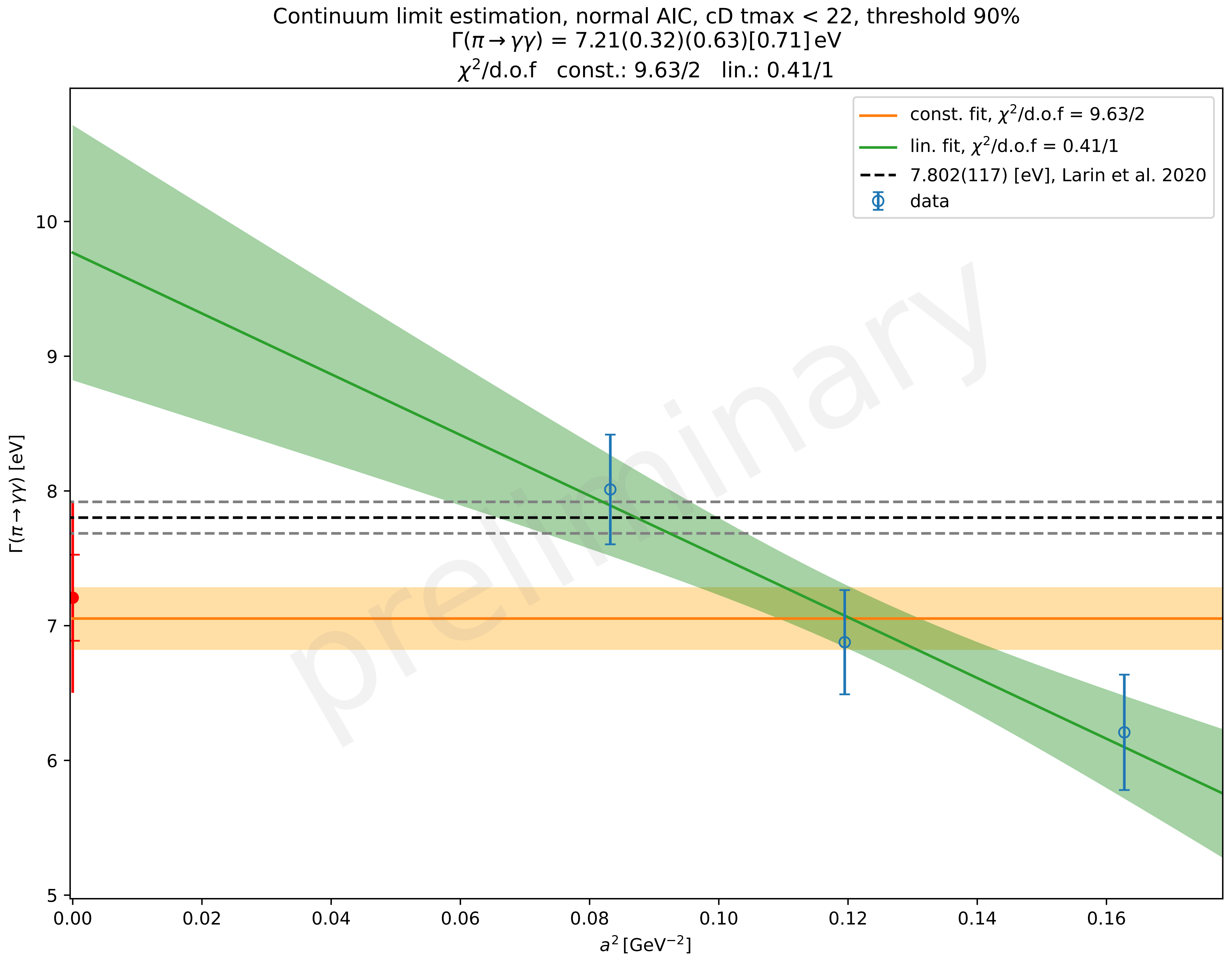
Currently η' too noisy

Preliminary measurements of $\tilde{A}(\tau)$ on coarsest ensemble:

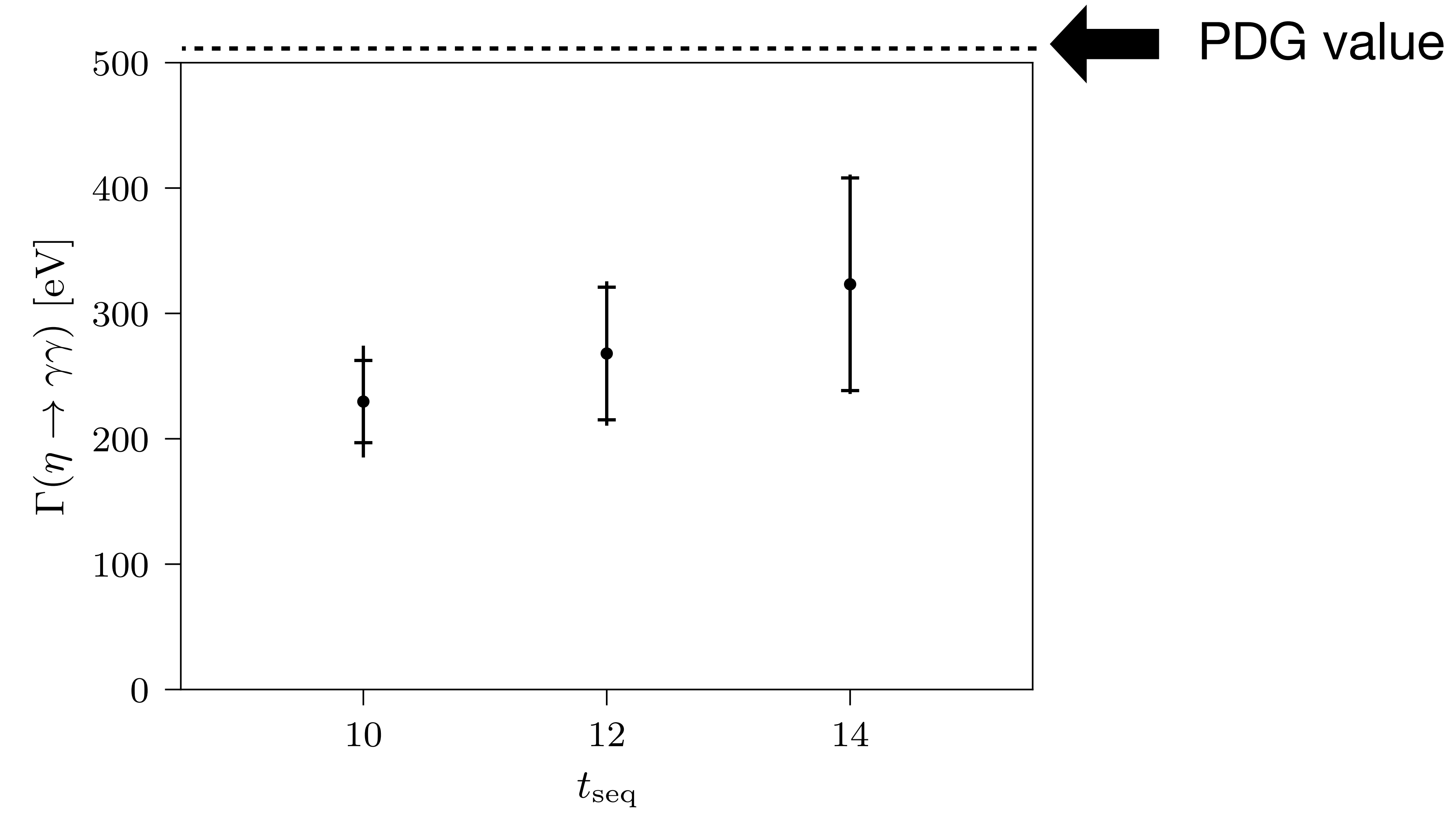


Other observables: $\Gamma(P \to \gamma\gamma)$

π^0 results (incl. continuum limit):



η results (cB 0.08fm ens. only):



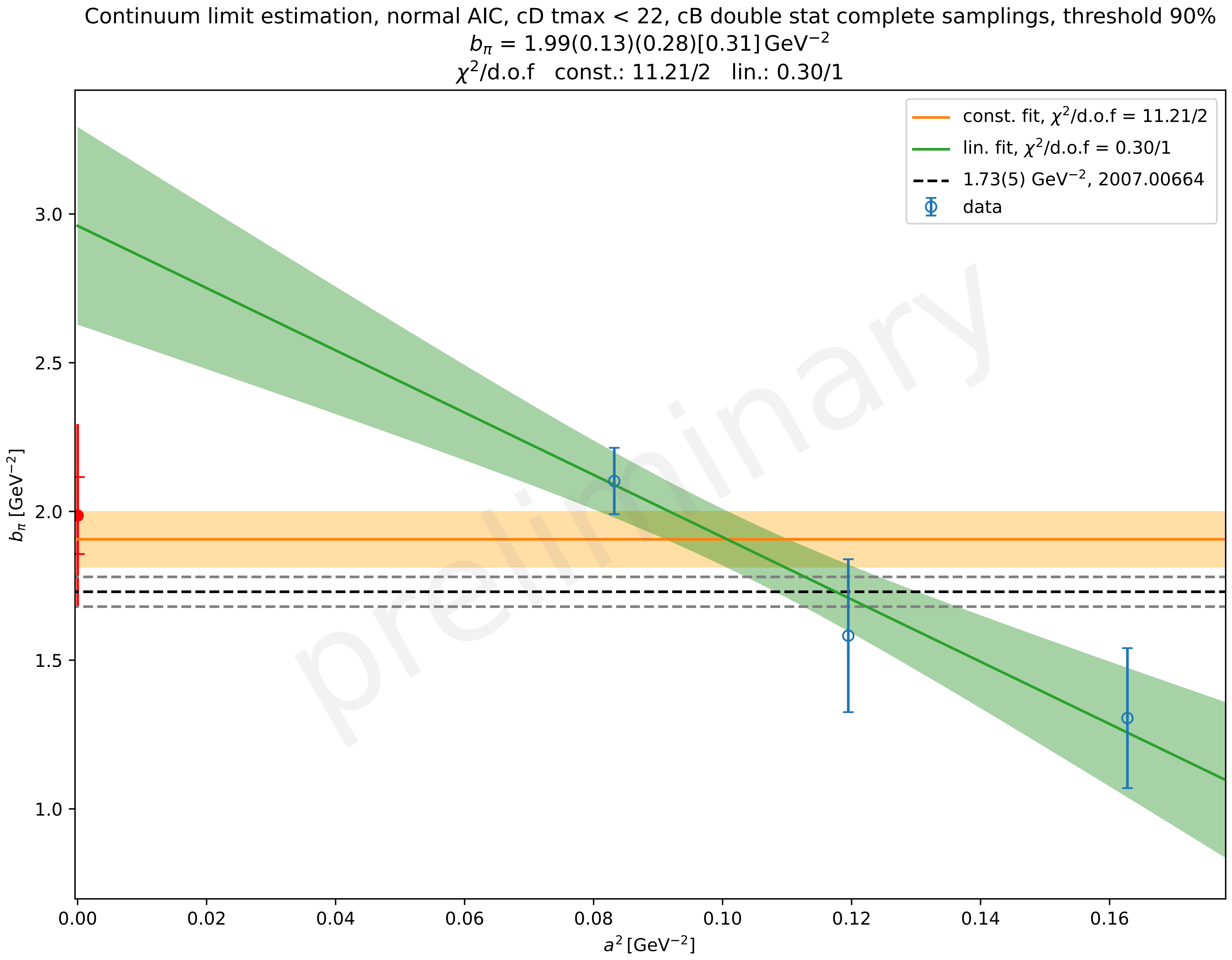
$$\Gamma(\eta \to \gamma\gamma)^{0.08\text{fm}} = 323.2(84.8)_{\text{stat}}(21.5)_{\text{syst}}\,\text{eV}$$

vs.

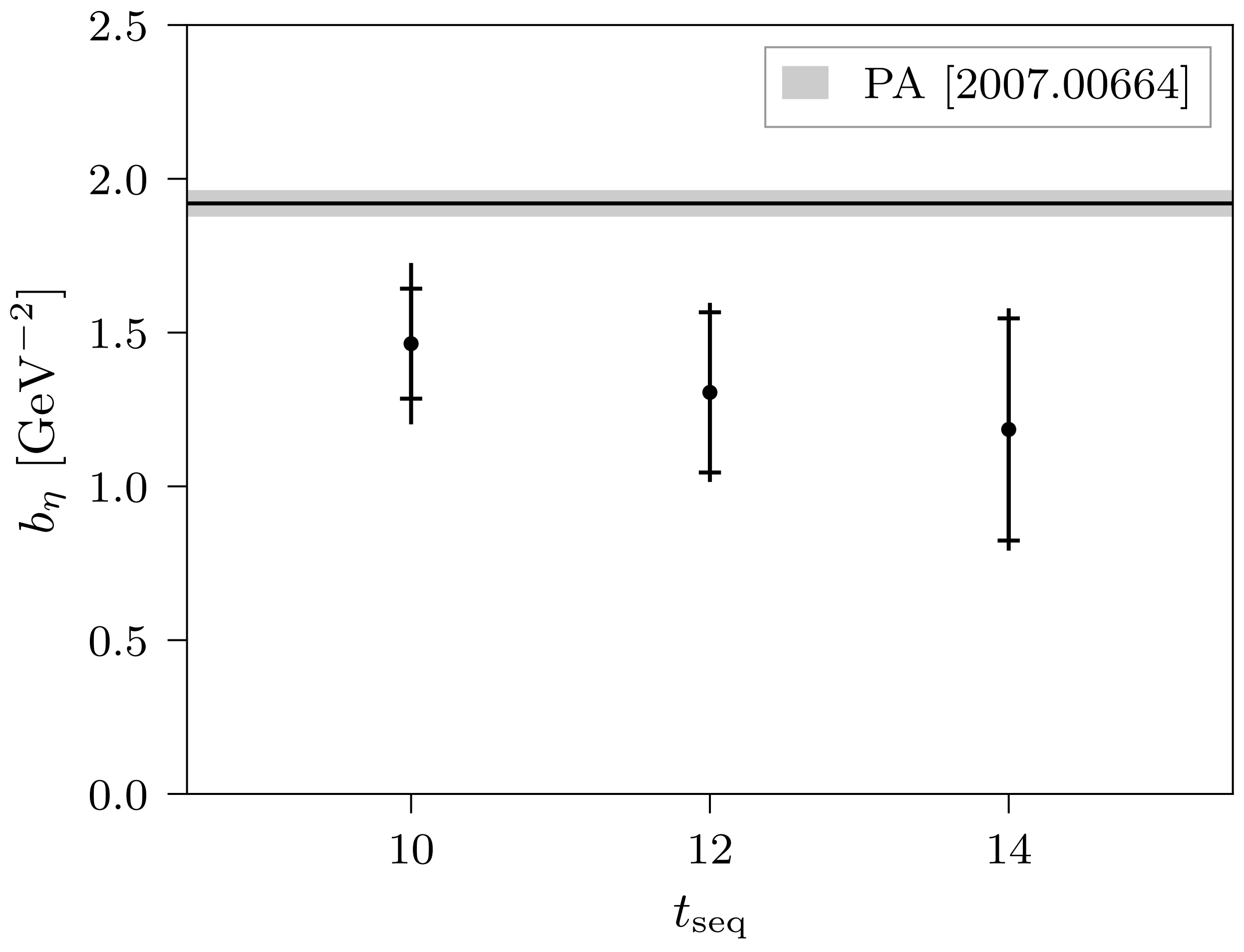
$$\Gamma(\eta \to \gamma\gamma) = 516(18)\,\text{eV}$$ [PDG]

Other observables: $b_P = -\frac{1}{\mathcal{F}(0,0)}\frac{\mathrm{d}}{\mathrm{d}Q^2}\mathcal{F}_{P\gamma^*\gamma}(0,-Q^2)\Big|_{Q^2=0}$

π^0 results (incl. continuum limit):



η results (cB 0.08fm ens. only):



$$b_\eta^{0.08\mathrm{fm}} = 1.185(0.361)_{\mathrm{stat}}(0.157)_{\mathrm{syst}}\mathrm{GeV}^{-2}$$

vs.

$$b_\eta = 1.92(0.04)\,\mathrm{GeV}^{-2}$$ [1]

[1] Gan+ Phys. Rept. 945 (2022), 2007.00664

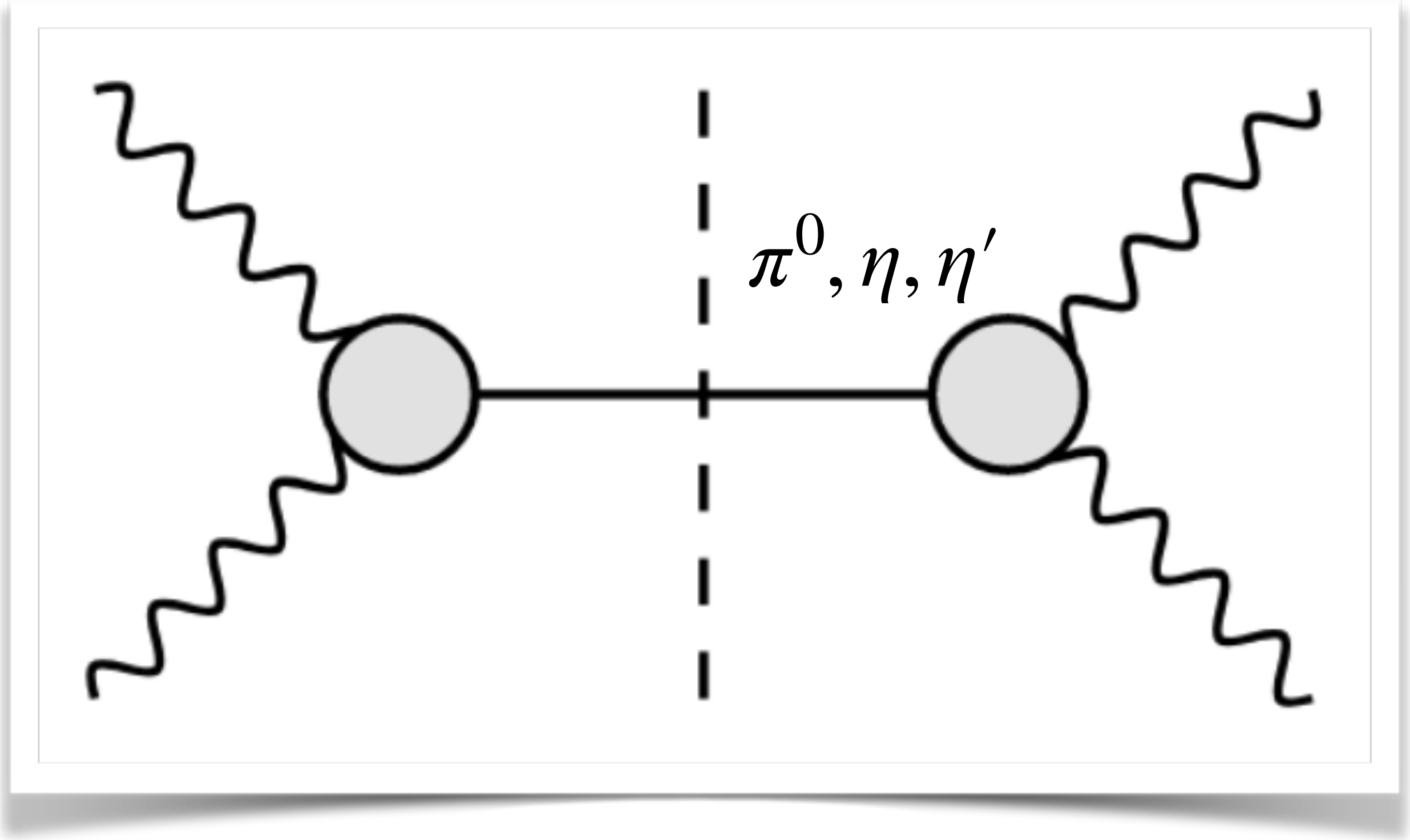
Summary

1. Unambiguous decomposition of terms of HLbL
    - Pseudoscalar-pole contribution significant uncertainty

2. Form factors are the key input
    - π^0 dispersively from expt data, η, η' not yet
    - Lattice especially important for **doubly virtual**

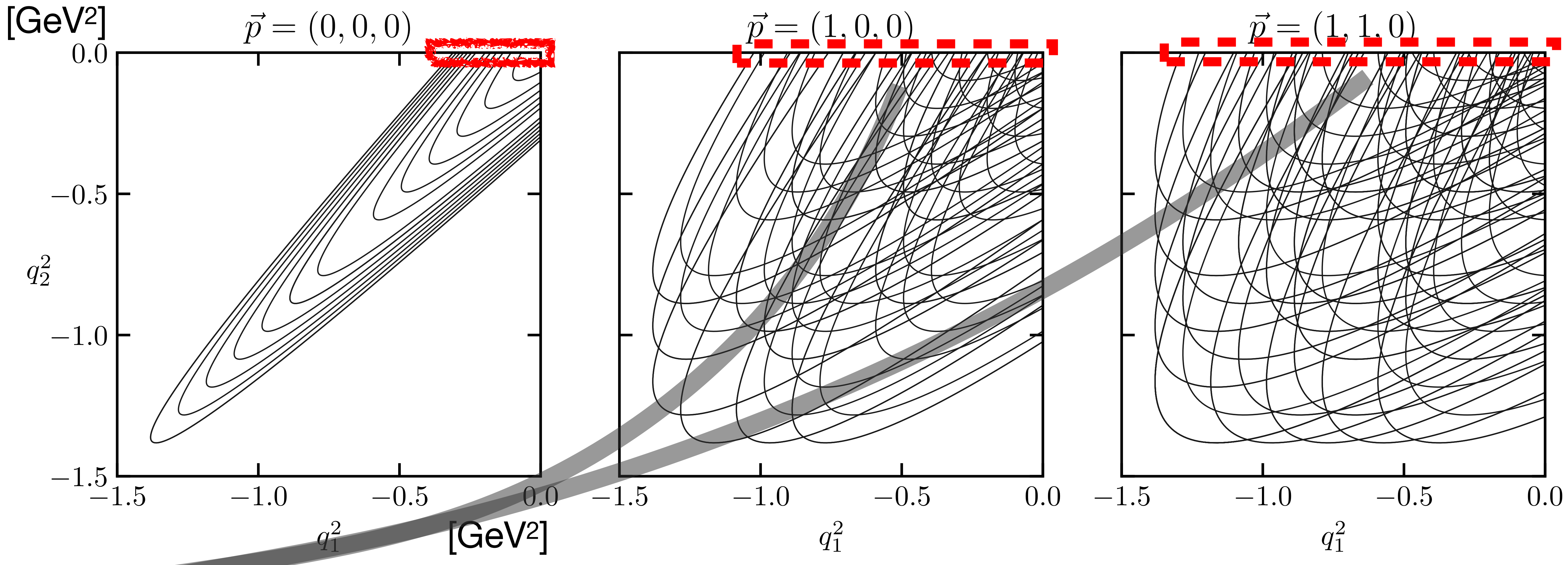
3. Comparable determination of $a_\mu^{\pi^0-\text{pole}}$ at physical point

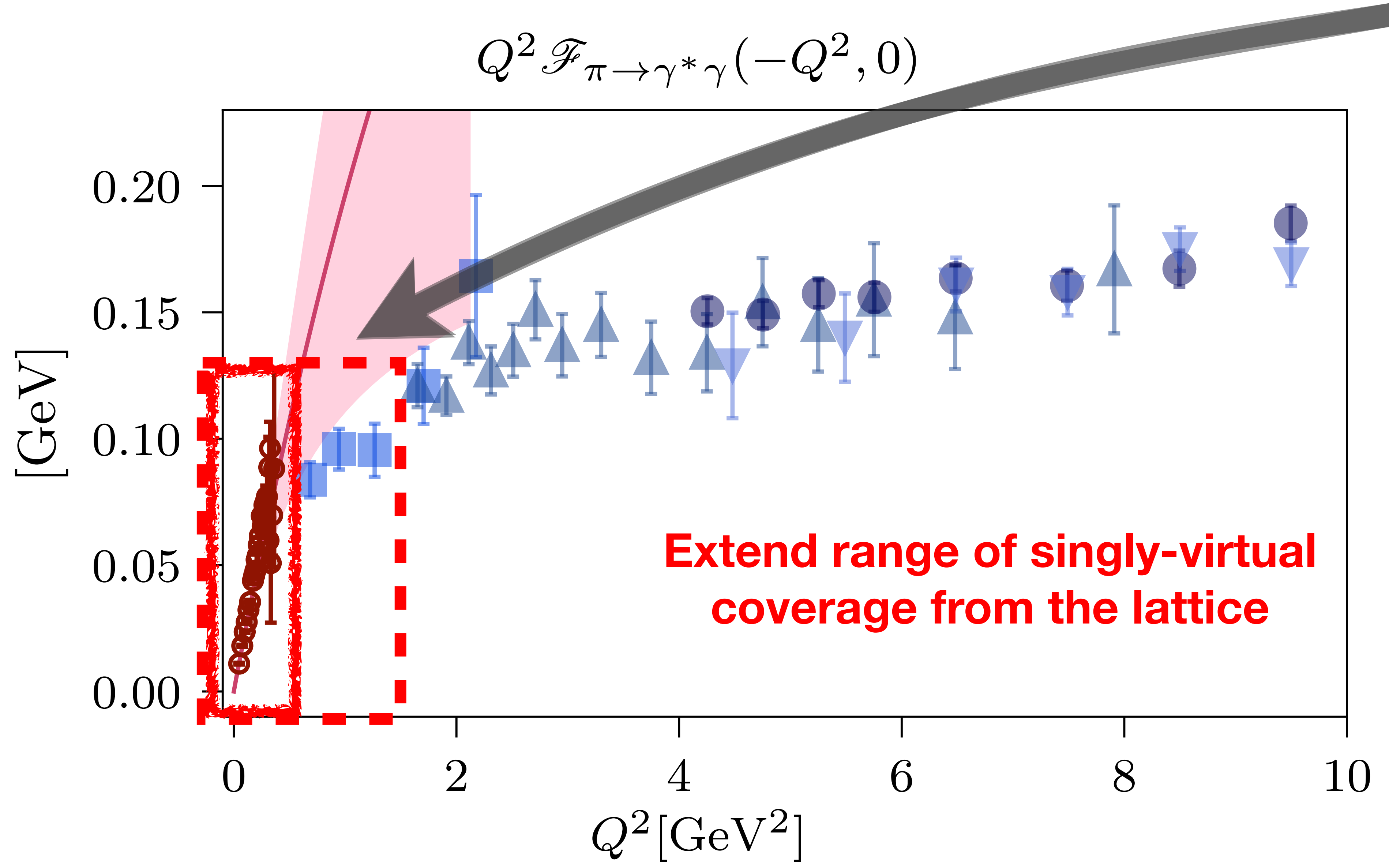
4. Progress towards η TFF and $a_\mu^{\eta-\text{pole}}$
    - Continuum limit required



Outlook

1. Boosted frame for π^0

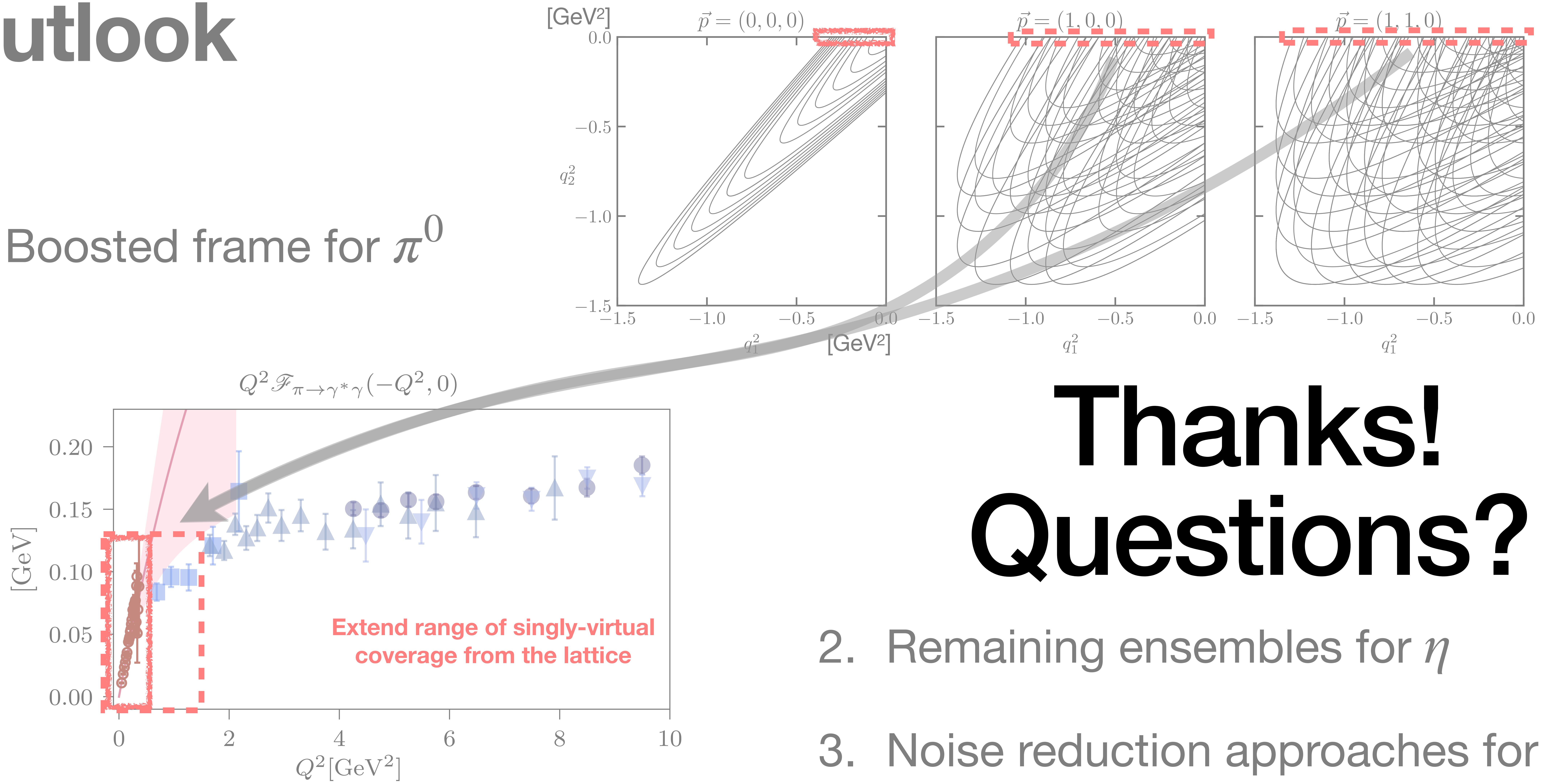




2. Remaining ensembles for η
3. Noise reduction approaches for η' ?
4. Other terms in HLbL decomposition?

Outlook

1. Boosted frame for π^0



Thanks! Questions?

2. Remaining ensembles for η
3. Noise reduction approaches for η' ?
4. Other terms in HLbL decomposition?

Backup slides

z-expansion

Conformal transformation:

$$z_k = \frac{\sqrt{t_c - Q_k^2} - \sqrt{t_c - t_0}}{\sqrt{t_c + Q_k^2} + \sqrt{t_c - t_0}}$$

$k \in \{1,2\}$

$t_c = 4m_\pi^2$ given by 2π threshold

t_0 chosen to minimize $|z_k|$ over range of interest

Polynomial fit to cutoff order $N \in \{1,2\}$ used in this work.

Multiplicative factor to match B-L and OPE asymptotics:

$$P(Q_1^2, Q_2^2) = 1 + \frac{Q_1^2 + Q_2^2}{M_V^2}$$

Other terms in HLbL decomposition

Contribution	Ref. [6]
π^0, η, η'-poles	93.8(4.0)
π, K-loops/boxes	−16.4(2)
S-wave $\pi\pi$ rescattering	−8(1)
subtotal	69.4(4.1)
scalars tensors	− 1(3)
axial vectors	6(6)
u, d, s-loops / short-distance	15(10)
c-loop	3(1)
total	92(19)

Addressed by this work

AV form factors accessible on the lattice… future work?

Theoretical ambiguities actively being addressed by data-driven community

[Colangelo+ Snowmass White Paper, 2203.15810]

Renormalization factors Z_V and Z_A

Determined extremely precisely using a hadronic method based on the lattice Ward identities in [Alexandrou+ 2206.15084].

ensemble	Z_V	Z_A
cB211.072.64	0.706378 (16)	0.74284 (23)
cB211.072.96	0.706402 (15)	0.74274 (20)
cC211.060.80	0.725405 (13)	0.75841 (16)
cD211.054.96	0.744105 (11)	0.77394 (10)

AIC averaging

- Average analysis results, weighted e.g. with reduced χ^2 or Akaike Information Criterion (AIC).
- Procedure with modified AIC from [Sz. Borsanyi et al., Nature, 593(7857) 51–55, (2021) and refs. therein] can be used to estimate the systematic errors.

$$\mathrm{AIC} \sim \exp\left[-\frac{1}{2}\left(\chi^2 + 2n_{\mathrm{par}} - n_{\mathrm{data}}\right)\right]$$

- Normalize weights, i.e. $\sum_i w_i = 1$.
- Build cumulative distribution function (CDF)

$$P(y;\lambda) = \int_{-\infty}^{y} dy' \sum_i w_i N(y'; m_i, \sigma_i\sqrt{\lambda}).$$

[Slide credit: Sebastian Burri]

- Median of CDF defines central value, total error given by 16% and 84% percentiles, i.e.

$$\sigma_{\text{total}}^2 \equiv \left[\frac{1}{2}\left(y_{84} - y_{16}\right)\right]^2 \equiv \sigma_{\text{stat}}^2 + \sigma_{\text{sys}}^2,$$

  with $P(y_{16}; 1) = 0.16$ and $P(y_{84}; 1) = 0.84$.
- Rescaling each σ_i^2 by λ increases σ_{stat}^2 by the same factor, i.e.

$$\lambda\sigma_{\text{stat}}^2 + \sigma_{\text{sys}}^2 \equiv \left[\frac{1}{2}\left(\tilde{y}_{84} - \tilde{y}_{16}\right)\right]^2 \equiv \tilde{\sigma}_{\text{total}}^2,$$

  with $P(\tilde{y}_{16}; \lambda) = 0.16$ and $P(\tilde{y}_{84}; \lambda) = 0.84$.
- Maximal effective sample size can be estimated by the magnitude of the correlation matrix R,

$$|R| = \sup\left\{\frac{1}{a^* R a} : a \in \mathbb{R}^n, \sum a_i = 1\right\},$$

  with $R_{ij} = \text{cov}(X_i, X_j)/\sigma_{X_i}\sigma_{X_j} \in [-1, 1],\ R_{ii} = 1$.