Quasi-degenerate baryon energy states, the Feynman–Hellmann theorem and transition matrix elements

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This talk:

Based on two talks at the Lattice 2022 Conference

[R. Horsley + M. Batelaan]

[Write-ups + paper in preparation]

Feynman–Hellmann (FH) papers:

- ‘A Lattice Study of the Glue in the Nucleon’
  arXiv:1205.6410 (PLB)
- ‘A Feynman-Hellmann approach to the spin structure of hadrons’
  arXiv:1405.3019 (PRD)
- ‘A novel approach to nonperturbative renormalization of singlet and nonsinglet lattice operators’
  arXiv:1410.3078 (PLB)
- ‘Disconnected contributions to the spin of the nucleon’
  arXiv:1508.06856 (PRD)
- ‘Electromagnetic form factors at large momenta from lattice QCD’
  arXiv:1702.01513 (PRD)
- ‘Nucleon structure functions from lattice operator product expansion’
  arXiv:1703.01153 (PRL)
- ‘Lattice QCD evaluation of the Compton amplitude employing the Feynman-Hellmann theorem’
  arXiv:2007.01523 (PRD)
- ‘Generalized parton distributions from the off-forward Compton amplitude in lattice QCD’
  arXiv:2110.11532 (PRD)
- ‘Moments and power corrections of longitudinal and transverse proton structure functions from lattice QCD’
  arXiv:2209.04141

+ Various (Lattice) conferences

Motivation:

Need computation of non-perturbative matrix elements (MEs):

$$\langle H' | \hat{O} | H \rangle$$

General structure

- $H \sim \overline{\psi}\psi$ (meson) or $H \sim \psi\psi\psi$ (baryon)
- $\hat{O} \sim \overline{\psi}\gamma\psi \sim J$ or $\hat{O} \sim FF$ or even more complicated $\hat{O} \sim JJ$

Usual approach: determine ME via 3-point correlation functions

This talk revolves around an alternative approach using 2-point correlation functions in particular:

Generalisation of Feynman–Hellmann approach to determination of (nucleon) MEs from strictly degenerate energy states to near-degenerate or 'quasi-degenerate' energy states

- This talk: explanation of the above statement / theory / numerical tests (mainly) for transition matrix elements (eg $\Sigma \to N$) $+ \ldots$

Contents

- Feynman–Hellmann approach via transfer matrix to computation of 2-pt correlation functions
  - Quasi-degenerate states
  - Dyson expansion
  - Reduction to a Generalised EigenVector Problem (GEVP)
- Inclusion of spin
- Examples
  - Scattering and decay (or transition) matrix elements, eg $N \to N$ and $\Sigma \to N$
  - Sketches of avoided energy levels
- Numerical tests/results for transition matrix elements and scattering
- Valence versus sea components
  - Disconnected contributions
- Conclusions

HEALTH WARNING: This is a rather technical Lattice talk

Feynman–Hellmann (FH) — some Mathematical Details

Hamiltonian formalism: regard Euclidean time (at least) as continuous

Consider the 2-point nucleon correlation function

$$C_{\lambda\, B'B}(t) = {}_{\lambda}\langle 0| \; \underbrace{\hat{\tilde{B}}'(0;\vec{p}\,')}_{\text{Sink: mom op}} \; \hat{S}(\vec{q})^t \quad \underbrace{\hat{\bar{B}}(0,\vec{0})}_{\text{Source: spatial}} \quad |0\rangle_{\lambda}$$

where $\hat{S}$ is the $\vec{q}$-dependent transfer matrix

$$\hat{S}(\vec{q}) = e^{-\hat{H}(\vec{q})}$$

and in the presence of a perturbation $\quad$ [$\lambda_\alpha = |\lambda_\alpha|\zeta_\alpha$ with phase $\zeta_\alpha = \pm 1, \pm i$]

$$\hat{H}(\vec{q}) = \hat{H}_0 - \sum_\alpha \lambda_\alpha \hat{\tilde{\mathcal{O}}}_\alpha(\vec{q})$$

where $\quad$ [At leading order can drop α index]

$$\hat{\tilde{\mathcal{O}}}(\vec{q}) = \int_{\vec{x}} \left(\hat{O}(\vec{x}) e^{i\vec{q}\cdot\vec{x}} + \hat{O}^\dagger(\vec{x}) e^{-i\vec{q}\cdot\vec{x}} \right)$$

Physical situation (quasi-degenerate energies):

- Quasi-degenerate states:

  $$\hat{H}_0|B_r(\vec{p}_r)\rangle = E_{B_r}(\vec{p}_r)|B_r(\vec{p}_r)\rangle \quad r = 1, \ldots, d_S$$

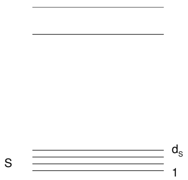
  where

  $$E_{B_r}(\vec{p}_r) = \bar{E} + \epsilon_r \qquad r = 1, \ldots, d_S$$

- Well separated from higher energy states:

  $$\hat{H}_0|X(\vec{p}_X)\rangle = E_X(\vec{p}_X)|X(\vec{p}_X)\rangle \quad E_X \gg \bar{E}$$

- Quasi-degenerate states taken as lowest energy states
- We have already applied this method to degenerate states, now generalise approach



Now insert two complete sets of unperturbed states $|X\rangle \to \frac{|X\rangle}{\sqrt{\langle X|X\rangle}}\,,\ |0\rangle \to |0\rangle$

$$\sum\!\!\!\!\!\!\!\int_{X(\vec{p}_X)} |X(\vec{p}_X))\rangle\,\langle X(\vec{p}_X)|$$

$$\equiv \quad \sum_r \underbrace{|B_r(\vec{p}_r)\rangle\langle B_r(\vec{p}_r)|}_{\text{of interest}} + \sum\!\!\!\!\!\!\!\int_{E_X \gg \bar{E}} \underbrace{|X(\vec{p}_X)\rangle\,\langle X(\vec{p}_X)|}_{\text{higher states}} = \hat{1}$$

before and after $\hat{S}^t$ to give

$$C_{\lambda\, B'B}(t) =$$

$$\sum\!\!\!\!\!\!\!\int_{X(\vec{p}_X)} \sum\!\!\!\!\!\!\!\int_{Y(\vec{p}_Y)} {}_\lambda\langle 0|\hat{\bar{B}}'(\vec{p}')|X(\vec{p}_X)\rangle\, \underbrace{\langle X(\vec{p}_X)|\hat{S}_\lambda(\vec{q})^t|Y(\vec{p}_Y)\rangle}_{\text{need}} \langle Y(\vec{p}_Y)|\hat{\bar{B}}(\vec{0})|0\rangle_\lambda$$

Time dependent perturbation theory via the Dyson Series

Dyson expansion – iterate identity

$$e^{-(\hat{H}_0-\lambda_\alpha \hat{\tilde{\mathcal{O}}}_\alpha)t} = e^{-\hat{H}_0 t} + \lambda_\alpha \int_0^t dt'\, e^{-\hat{H}_0(t-t')}\, \hat{\tilde{\mathcal{O}}}_\alpha\, e^{-(\hat{H}_0-\cancel{\lambda_\beta \hat{\tilde{\mathcal{O}}}_\beta})t'}$$

- $O(\lambda^2)$ gives Compton like amplitudes $\sim \langle \ldots | O_\alpha O_\beta | \ldots \rangle$ – not considered here
- Consider 4 possible pieces separately:

$$\begin{aligned}
\langle B_r | e^{-(\hat{H}_0-\lambda\hat{\tilde{\mathcal{O}}})t} | B_s \rangle &= e^{-\bar{E}t}\left(\delta_{rs} + tD_{rs} + O(2)\right) \\
\langle B_r | e^{-(\hat{H}_0-\lambda\hat{\tilde{\mathcal{O}}})t} | Y \rangle &= e^{-\bar{E}t}\left(\lambda \frac{\langle B_r | \hat{\tilde{\mathcal{O}}} | Y \rangle}{E_Y - E_{B_r}} + O(2)\right) + \text{more damped} \\
\ldots &= \ldots
\end{aligned}$$

This gives

$$C_{\lambda\, B'B}(t) = \sum_{rs} {}_\lambda\langle 0 | \hat{\tilde{B}}'(\vec{p}\,') | B_r(\vec{p}_r)\rangle_\lambda \; \langle B_r | e^{-(\hat{H}_0-\lambda\hat{\tilde{\mathcal{O}}})t} | B_s \rangle \; {}_\lambda\langle B_s(\vec{p}_s) | \hat{\bar{B}}(\vec{0}) | 0 \rangle_\lambda$$

with

$$\begin{aligned}
|B_s(\vec{p}_s)\rangle_\lambda &= |B_s(\vec{p}_s)\rangle + \lambda \sum\!\!\!\!\!\!\int_{E_Y \gg \bar{E}} |Y(\vec{p}_Y)\rangle \frac{\langle Y(\vec{p}_Y) | \hat{\tilde{\mathcal{O}}}(\vec{q}) | B_s(\vec{p}_s)\rangle}{E_Y - E_{B_s}} \\
D_{rs} &= -\epsilon_r \delta_{rs} + \lambda \langle B_r(\vec{p}_r) | \hat{\tilde{\mathcal{O}}}(\vec{q}) | B_s(\vec{p}_s)\rangle
\end{aligned}$$

[So a factorisation where any unwanted $|Y\rangle$ states have been absorbed into time indept renormalisation of wavefunction.]

- D_{rs} :

$$D_{rs} = -\epsilon_r \delta_{rs} + \lambda \underbrace{\langle B_r(\vec{p}_r) | \hat{\tilde{\mathcal{O}}}(\vec{q}) | B_s(\vec{p}_s) \rangle}_{a_{rs}}$$

As $d_S \times d_S$ is a dimensional Hermitian matrix:

$$D_{rs} = \sum_{i=1}^{d_S} \mu^{(i)} e_r^{(i)} e_s^{(i)\,*} \qquad \mu, e_r \text{ eigenvalues/eigenvectors}$$

with (completeness)

$$\sum_{i=1}^{d_S} e_r^{(i)} e_s^{(i)\,*} = \delta_{rs} \qquad \sum_{r=1}^{d_S} e_r^{(i)\,*} e_r^{(j)} = \delta^{ij}$$

- Re-write

$$\delta_{rs} + tD_{rs} = \sum_{i=1}^{d_S} e_r^{(i)} [1 + t\mu^{(i)}] e_s^{(i)\,*}$$

- Re-exponentiate

This gives finally:

$$C_{\lambda\, B'B}(t) = \sum_{i=1}^{d_S} A^{(i)}_{\lambda\, B'B}\, e^{-E^{(i)}_\lambda t}$$

Perturbed energies:

$$E^{(i)}_\lambda = \bar{E} - \mu^{(i)}$$

Amplitude

$$A^{(i)}_{\lambda\, B'B} = w^{(i)}_{B'}\, \bar{w}^{(i)}_B$$

with

$$w^{(i)}_{B'} = \sum_{r=1}^{d_s} Z^{B'}_r\, e^{(i)}_r \qquad \bar{w}^{(i)}_B = \sum_{s=1}^{d_s} \bar{Z}^B_s\, e^{(i)*}_s$$

where the wavefunctions or overlaps are [$\vec{p}\,' \to \vec{p}_r$]

$$Z^{B'}_r = {}_\lambda\langle 0|\hat{\bar{B}}'(\vec{p}\,')|B_r(\vec{p}_r)\rangle_\lambda \qquad \bar{Z}^B_s = {}_\lambda\langle B_s(\vec{p}_s)|\hat{\bar{B}}(\vec{0})|0\rangle_\lambda$$

- So problem is now reduced to a GEVP to determine eigenvalues $E^{(i)}_\lambda$
- GEVP eigenvectors should follow pattern of $\vec{e}^{(i)}$

Relation between momenta

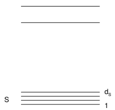
- For the matrix elements have

$$[\hat{O}(\vec{x}) = e^{-i\hat{\vec{P}}\cdot\vec{x}}\, \hat{O}(\vec{0})\, e^{i\hat{\vec{P}}\cdot\vec{x}}]$$

$$\begin{aligned}&\langle B(\vec{p}_r)|\hat{\tilde{\mathcal{O}}}(\vec{q})|B(\vec{p}_s)\rangle \\ &\quad = \quad \langle B_r(\vec{p}_r)|\hat{O}(\vec{0})|B_s(\vec{p}_s)\rangle\, \delta_{\vec{p}_r,\vec{p}_s+\vec{q}} + \langle B(\vec{p}_r)|\hat{O}^\dagger(\vec{0})|B(\vec{p}_s)\rangle\, \delta_{\vec{p}_r,\vec{p}_s-\vec{q}}\end{aligned}$$

- So matrix elements step up or down in $\vec{q} \neq \vec{0}$

$$\vec{p}_r = \vec{p}_s + \vec{q} \quad \text{or} \quad \vec{p}_r = \vec{p}_s - \vec{q}$$



[Momentum conservation]

- Diagonal matrix elements vanish
  So quasi-degenerate states have to mix
  [ie must consider degenerate perturbation theory]
- Each step up or down corresponds to another order in λ
  (Dyson expansion)
  So (eg) $O(\lambda^2)$ gives Compton like amplitudes $\sim \langle \ldots | O_\alpha O_\beta | \ldots \rangle$
  Step up step down now possible: $\vec{p} \to \vec{p} \pm \vec{q} \to \vec{p}$ relevant for DIS

Incorporation of spin – postpone

Quasi–degenerate baryon energy states I

- Flavour diagonal matrix elements – N scattering

$$O(\vec{x}) \sim (\bar{u}\gamma u)(\vec{x}) - (\bar{d}\gamma d)(\vec{x})$$

- d_S = 2-dimensional space: $r,\, s = 1,\, 2$

$$\underbrace{|B_1(\vec{p}_1)\rangle = |N(\vec{p})\rangle}_{E_{B_1}(\vec{p}_1)\equiv E_N(\vec{p})=\bar{E}+\epsilon_1} \qquad \underbrace{|B_2(\vec{p}_2)\rangle = |N(\vec{p}+\vec{q})\rangle}_{E_{B_2}(\vec{p}_2)\equiv E_N(\vec{p}+\vec{q})=\bar{E}+\epsilon_2}$$

$$\langle B_r(\vec{p}_r)|\hat{\tilde{\mathcal{O}}}(\vec{q})|B_s(\vec{p}_s)\rangle = \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix}_{rs}$$

where

$$a = \langle B_2(\vec{p}_2)|\hat{O}(\vec{0})|B_1(\vec{p}_1)\rangle \equiv \langle N(\vec{p}+\vec{q})|\hat{O}(\vec{0})|N(\vec{p})\rangle$$

Quasi–degenerate baryon energy states II

- Flavour transition matrix elements – (eg) $\Sigma(sdd) \to N(udd)$ decay

$$O(\vec{x}) \sim (\bar{u}\gamma s)(\vec{x})$$

- d_S = 2-dimensional space: $r,\, s = 1,\, 2$

$$\underbrace{|B_1(\vec{p}_1)\rangle = |\Sigma(\vec{p})\rangle}_{E_{B_1}(\vec{p}_1)\equiv E_\Sigma(\vec{p})=\bar{E}+\epsilon_1} \qquad \underbrace{|B_2(\vec{p}_2)\rangle = |N(\vec{p}+\vec{q})\rangle}_{E_{B_2}(\vec{p}_2)\equiv E_N(\vec{p}+\vec{q})=\bar{E}+\epsilon_2}$$

$$\langle B_r(\vec{p}_r)|\hat{\tilde{\mathcal{O}}}(\vec{q})|B_s(\vec{p}_s)\rangle = \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix}_{rs}$$

where

$$a = \langle B_2(\vec{p}_2)|\hat{O}(\vec{0})|B_1(\vec{p}_1)\rangle \equiv \langle N(\vec{p}+\vec{q})|\hat{O}(\vec{0})|\Sigma(\vec{p})\rangle$$

- ie similar structure to N scattering case

Diagonalising D_{rs} : $[\vec{p}_1 \equiv \vec{p},\ \vec{p}_2 \equiv \vec{p} + \vec{q}; \quad E_r = \bar{E} + \epsilon_r]$

$$D_{rs} = -\epsilon_r \delta_{rs} + \lambda \langle B_r(\vec{p}_r) | \hat{\tilde{\mathcal{O}}}(\vec{q}) | B_s(\vec{p}_s) \rangle = \begin{pmatrix} -\epsilon_1 & a^* \\ a & -\epsilon_2 \end{pmatrix}_{rs}$$

1) Eigenvalues $\mu_\pm$: [quadratic equation]

Giving energies

$$\begin{aligned} E_\lambda^{(\pm)} &= \bar{E} - \mu_\pm \\ &= \frac{1}{2}\left(E_N(\vec{p}+\vec{q}) + E_{N/\Sigma}(\vec{p})\right) \mp \frac{1}{2}\Delta E_\lambda(\vec{p},\vec{q}) \end{aligned}$$

with

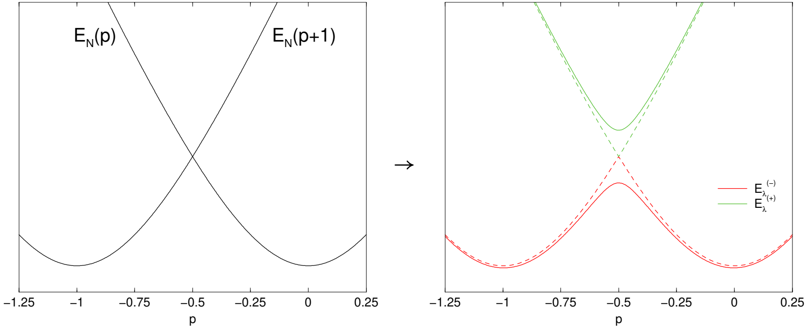
$$\Delta E_\lambda = E_\lambda^{(-)} - E_\lambda^{(+)}$$

and

$$\Delta E_\lambda = \sqrt{\left(E_N(\vec{p}+\vec{q}) - E_{N/\Sigma}(\vec{p})\right)^2 + 4\lambda^2 \underbrace{\left|\langle N(\vec{p}+\vec{q}) | \hat{O}(\vec{0}) | N/\Sigma(\vec{p}) \rangle\right|^2}_{|a|^2}}$$

Degenerate energy states – N scattering

eg 1-dimensional (exaggerated) sketch: $[\lambda^2|a|^2 = \text{const.},\ q = 1]$



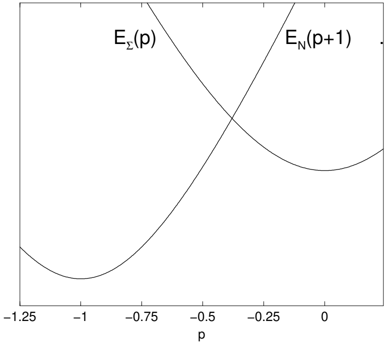
- Focus on degeneracy at: $E_N(p) = E_N(p+q)$ at $p = -q/2$

[Similarly when $E_N(p) = E_N(p-q)$ at $p = q/2$]

- Free case → Interacting case: avoided energy levels
- Sketch curves based on previously derived formulae: $E^{(+)}$, $E^{(-)}$

Quasi–degenerate energy states – $\Sigma \to N$ decay

eg 1-dimensional (exaggerated) sketch: $[\lambda^2|a|^2 = \text{const.},\ q = 1]$



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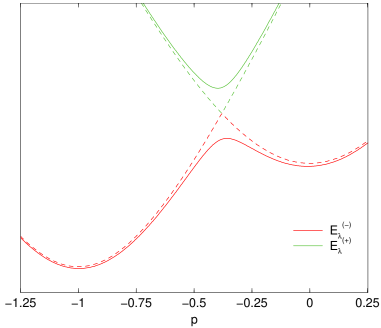


- Free case $\to$ Interacting case: avoided energy levels
- Sketch based on previous formulae

Diagonalising D_{rs} :

$$D_{rs} = -\epsilon_r \delta_{rs} + \lambda \langle B_r(\vec{p}_r) | \hat{\tilde{\mathcal{O}}}(\vec{q}) | B_s(\vec{p}_s) \rangle = \begin{pmatrix} -\epsilon_1 & a^* \\ a & -\epsilon_2 \end{pmatrix}_{rs}$$

2) Eigenvectors $e_r^{(\pm)}$:

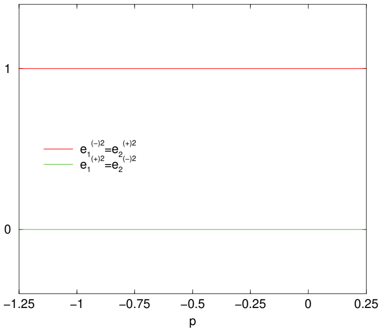
$$e_r^{(\pm)} = N^{(\pm)} \begin{pmatrix} \lambda |a| \\ \kappa_\pm \frac{a}{|a|} \end{pmatrix}_r$$

- $\kappa_\pm = \frac{1}{2}(E_{N/\Sigma} - E_N) \pm \frac{1}{2}\Delta E$
- $N^{(\pm)}$ normalisation factor
- $a/|a| = \zeta_a$ possible phase of a: ± 1, $\pm i$
  ie phase of matrix element contained in eigenvectors.
- Components related: $e_2^{(-)} = -e_1^{(+)} a/|a|$ and $e_2^{(+)} = e_1^{(-)} a/|a|$

Quasi–degenerate eigenvectors – $\Sigma \to N$ decay

$$\vec{e}^{(\pm)} = \begin{pmatrix} e_1^{(\pm)} \\ e_2^{(\pm)} \end{pmatrix}$$

eg 1-dimensional sketch: $[\lambda^2|a|^2 = \text{const.},\ \zeta_a = 1,\ q = 1]$



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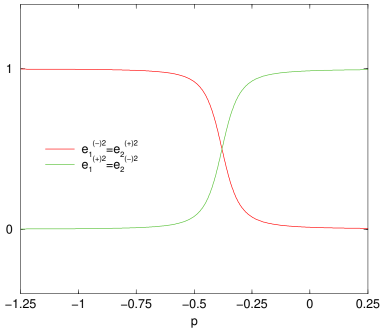


- Free case $\to$ Interacting case: change of state
- Sketch based on previous formulae

Incorporating the spin index

- $|B_r(\vec{p}_r)\rangle \to |B_r(\vec{p}_r, \sigma_r)\rangle$, $\sigma_r = \pm 1$ spin index
- D matrix doubled in size: $\sigma_r r = +1, -1, \ldots + d_S, -d_S$ ie $2d_S \times 2d_S$
- Energy states corresponding to $|B_r(\vec{p}_r, \sigma_r)\rangle$, $\sigma = \pm$ are degenerate [Kramers degeneracy] so still have d_S eigenvalues: $E_\lambda^{(i)}$

$$C_{\lambda\, rs}(t)$$

$$\equiv \sum_{\alpha\beta} \sum_{\sigma_r \sigma_s} \Gamma_{\beta\alpha}\, \underbrace{{}_\lambda\langle 0|\hat{\bar{B}}_{r\alpha}(\vec{p}_r)|B_r(\vec{p}_r, \sigma_r)\rangle_\lambda}_{Z_r\, u_\alpha^{(r)}(\vec{p}_r, \sigma_r)+\ldots} \times$$

$$\underbrace{\langle B_r(\vec{p}_r, \sigma_r)|e^{-(\hat{H}_0 - \lambda \hat{\tilde{\mathcal{O}}})t}|B_s(\vec{p}_s, \sigma_s)\rangle}_{\delta_{\sigma_r\sigma_s}\delta_{rs} + t\, \underbrace{D_{\sigma_r r, \sigma_s s}}_{-\epsilon_r \delta_{r\sigma_r, \sigma_s s} + \lambda a_{\sigma_r r, \sigma_s s}}} \times \underbrace{{}_\lambda\langle B_s(\vec{p}_s, \sigma_s)|\hat{\bar{B}}_{s\beta}(\vec{0})|0\rangle_\lambda}_{\bar{Z}_s\, \bar{u}_\beta^{(s)}(\vec{p}_s, \sigma_s)+\ldots}$$

$$a_{\sigma_r r, \sigma_s s} = \langle B_r(\vec{p}_r, \sigma_r)|\hat{\tilde{\mathcal{O}}}(\vec{q})|B_s(\vec{p}_s, \sigma_s)\rangle$$

Choice of Γ

With (eg) $\Gamma^{\mathrm{unpol}} = (1+\gamma_4)/2$ and $\bar{u}^{(r)}(\vec{p}_r,\sigma_r)\Gamma^{\mathrm{unpol}}u^{(s)}(\vec{p}_s,\sigma_s) \propto \delta_{\sigma_r,\sigma_s}$ spin sums reduce D to previous $d_S \times d_S$ matrix

$$D_{r,s} = -\epsilon_r \delta_{r,s} + \lambda a_{rs} \qquad a_{rs} = \underbrace{\frac{1}{2}\left(a_{+r,+s} + a_{-r,-s}\right)}_{\text{average}}$$

Similarly:

Γ	D_{rs}
$\Gamma^{\mathrm{unpol}} = (1+\gamma_4)/2$	$-\epsilon_r\delta_{rs} + \lambda\frac{1}{2}\left(a_{+r,+s} + a_{-r,-s}\right)$
$\Gamma_3^{\mathrm{pol}} = (1+\gamma_4)/2 \times (1 \pm i\gamma_5\gamma_3)$	$-\epsilon_r\delta_{rs} \pm \lambda\frac{1}{2}\left(a_{+r,+s} - a_{-r,-s}\right)$
$\Gamma_\pm^{\mathrm{pol}} = (1+\gamma_4)/2 \times i\gamma_5(\gamma_1 \pm i\gamma_2)$	$-\epsilon_r\delta_{rs} + \lambda a_{\pm r,\mp s}$

Explicit form factor decomposition of matrix element shows that different spin components of matrix elements related to each other:

$$a_{-r,-s} = \eta a^*_{+r,+s} \qquad a_{-r,+s} = -\eta a^*_{+r,-s} \qquad [\eta = \pm]$$

So practically pick out either $a_{+r,+s}$ or $a_{+r,-s}$ (spin-flip)

Explicit check for previous $d_S = 2$ case:

$$D_{\sigma_r r,\sigma_s s} = -\epsilon_r \delta_{r\sigma_r,\sigma_s s} + \lambda a_{\sigma_r r,\sigma_s s}$$

- $D_{\sigma_r r,\sigma_s s}$: $(2\times 2)\times(2\times 2)$ dimensional matrix
- Upshot from previous examples

$$a_{\sigma_r r,\sigma_s s} = \left(\begin{array}{c|c} 0 & a^* \\ \hline a & 0 \end{array}\right)_{\sigma_r r,\sigma_s s} \qquad a \to \begin{pmatrix} a_{++} & a_{+-} \\ a_{-+} & a_{--} \end{pmatrix}$$

- Giving

$$\Delta E_\lambda = \sqrt{(E_N(\vec{p}+\vec{q}) - E_{N/\Sigma}(\vec{p}))^2 + 4\lambda^2 \left|\det a\right|^2}$$

where

$$\left|\det a\right|^2 = \underbrace{\left|\langle N(\vec{p}+\vec{q},+)|\hat{O}(\vec{0})|N/\Sigma(\vec{p},+)\rangle\right|^2}_{|a_{++}|^2} + \underbrace{\left|\langle N(\vec{p}+\vec{q},+)|\hat{O}(\vec{0})|N/\Sigma(\vec{p},-)\rangle\right|^2}_{|a_{+-}|^2}$$

Contains previous cases

Test: Transition matrix elements $\Sigma^- \to n$ decay:

- Clover-Wilson: $N_f = 2+1$-flavours, isospin limit $\Sigma(sdd) \to N(udd)$
- Presently only considered V_4, $\vec{p} = \vec{0}$

$$Q^2 = -(M_\Sigma - E_N(\vec{q}))^2 + \vec{q}^2$$

- $32^3 \times 64$ lattice size, $O(500)$ configs, $a = 0.074\,\mathrm{fm}$, $M_\pi \sim 330\,\mathrm{MeV}$
- Momentum choices: $[\vec{q}^2 \equiv q_2^2 + \text{twisting}]$

run #	$\vec{q}^{\,2}$	Q^2 [GeV^2]
1	0.0	-0.0095
2	0.019	0.0048
3	0.0096	0.062
4	0.025	0.017
5	0.041	0.29
6	0.049	0.35

- Matrix element:

$$\begin{aligned}&\langle N(\vec{q},+)|\bar{u}\gamma_4 s|\Sigma(\vec{0},+)\rangle_{\mathrm{rel}}\\ &= \sqrt{2M_\Sigma(E_N(\vec{q})+M_N)}\\ &\quad\times\left(f_1^{\Sigma N}(Q^2) + \frac{E_N(\vec{q})-M_N}{M_N+M_\Sigma} f_2^{\Sigma N}(Q^2) + \frac{E_N(\vec{q})-M_\Sigma}{M_N+M_\Sigma} f_3^{\Sigma N}(Q^2)\right)\end{aligned}$$

Transition matrix elements for $\Sigma(sdd) \to N(udd)$ [$\Sigma^- \to n$ decay]:

$$S = S_g + \int_x (\bar{u}, \bar{s}) \underbrace{\begin{pmatrix} D_u & -\lambda\mathcal{T}' \\ -\lambda\mathcal{T}' & D_s \end{pmatrix}}_{\mathcal{M}} \begin{pmatrix} u \\ s \end{pmatrix} + \int_x \bar{d}\, D_d\, d$$

- $N_f = 2 + 1$ flavours
- $\mathcal{T}(x, y; \vec{q}) = \gamma\, e^{i\vec{q}\cdot\vec{x}}\, \delta_{x,y}$ [$\mathcal{T}' = \gamma_5 \mathcal{T}^\dagger \gamma_5$]
- [Γ^{unpol}]

$$C_{\lambda\, rs}(t) = \begin{pmatrix} C_{\lambda\, \Sigma\Sigma}(t) & C_{\lambda\, \Sigma N}(t) \\ C_{\lambda\, N\Sigma}(t) & C_{\lambda\, NN}(t) \end{pmatrix}_{rs}$$

- Generalised EigenVector Problem [GEVP]

Transition matrix elements for $\Sigma(sdd) \to N(udd)$ $[\Sigma^- \to n$ decay]:

- $$C_{\lambda\, rs}(t) = \begin{pmatrix} C_{\lambda\, \Sigma\Sigma}(t) & C_{\lambda\, \Sigma N}(t) \\ C_{\lambda\, N\Sigma}(t) & C_{\lambda\, NN}(t) \end{pmatrix}_{rs}$$

- Propagators

$$\begin{pmatrix} G_{uu} & G_{us} \\ G_{su} & G_{ss} \end{pmatrix} = \begin{pmatrix} (\mathcal{M}^{-1})_{uu} & (\mathcal{M}^{-1})_{us} \\ (\mathcal{M}^{-1})_{su} & (\mathcal{M}^{-1})_{ss} \end{pmatrix}$$

- To avoid inverting full $\mathcal{M}$ matrix expand $\mathcal{M}$ in 2×2 blocks to $O(\lambda^4)$ [Need several inversions]

$$\begin{aligned} G^{(uu)} &= (1 - \lambda^2 D_u^{-1} \mathcal{T} D_s^{-1} \mathcal{T}')^{-1} D_u^{-1} \\ G^{(ss)} &= (1 - \lambda^2 D_s^{-1} \mathcal{T}' D_u^{-1} \mathcal{T})^{-1} D_s^{-1} \end{aligned}$$

and

$$\begin{aligned} G^{(us)} &= \lambda D_u^{-1} \mathcal{T} G^{(ss)} \\ G^{(su)} &= \lambda D_s^{-1} \mathcal{T}' G^{(uu)} \end{aligned}$$

- Potential advantage: range of λ values available

Effective plots

- Diagonalise the correlation matrix (GEVP)

$$C_{\lambda\, rs}(t) = \begin{pmatrix} C_{\lambda\, \Sigma\Sigma}(t) & C_{\lambda\, \Sigma N}(t) \\ C_{\lambda\, N\Sigma}(t) & C_{\lambda\, NN}(t) \end{pmatrix}_{rs}$$

Gives two eigenvectors and eigenvalues

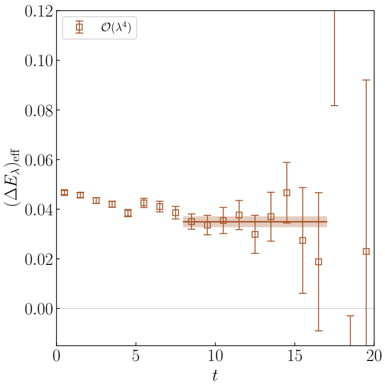
- Use the eigenvectors to project out two correlation functions

$$C^{(i)}_{\lambda}(t) = v^{(i)\,\dagger} C_{\lambda}(t) u^{(i)} \qquad i = \pm$$

- Take the ratio of the two correlators

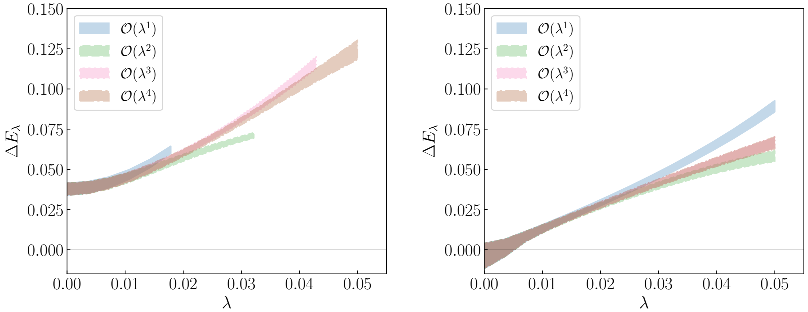
$$R_{\lambda}(t) = \frac{C^{(-)}_{\lambda}(t)}{C^{(+)}_{\lambda}(t)} \overset{t \gg 0}{\propto} e^{-\Delta E_{\lambda} t}$$

- Run #5, $O(\lambda^4)$, $\lambda = 0.025$:



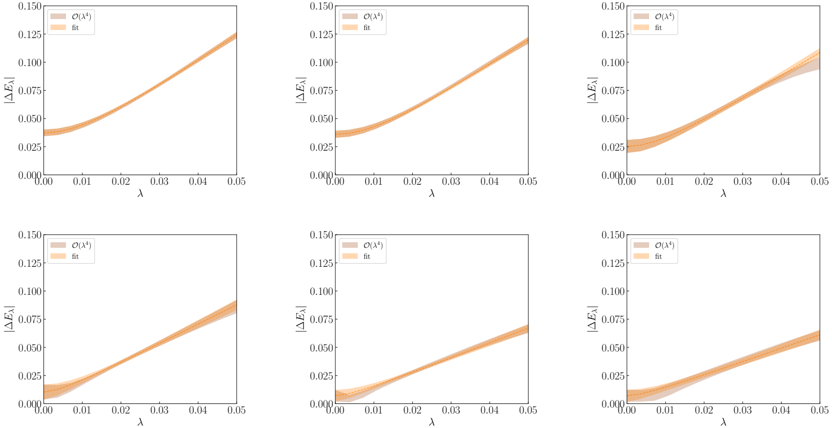
- $(\Delta E_{\lambda})_{\rm eff} = -\ln \frac{R(t+1)}{R(t)}$

λ Convergence



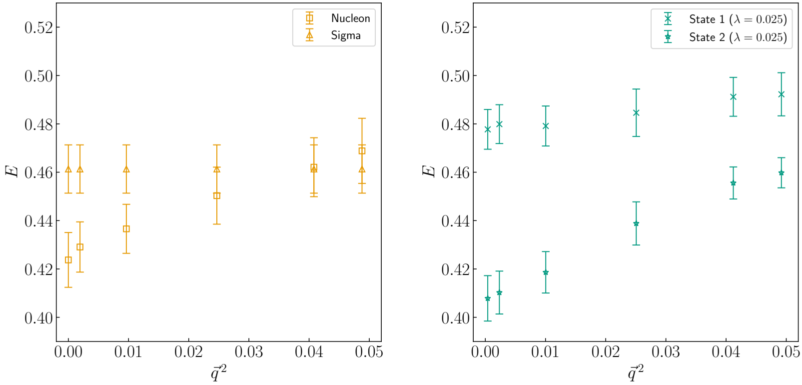
- Using $O(\lambda) - O(\lambda^4)$ terms
- LH plot: Run #1; RH plot: Run #5
- $0 \lesssim \lambda \lesssim 0.04$

Fits: Run#1 – #6 $[O(\lambda^4)$ results$]$



- Fit: $\Delta E_\lambda = \sqrt{(E_N(\vec{q}) - M_\Sigma)^2 + 4\lambda^2 \left|\langle N(\vec{q},+)|\bar{u}\gamma_4 s|\Sigma(\vec{0},+)\rangle\right|^2}$
- Pre-determine $E_N(\vec{q}) - M_\Sigma$, so one parameter fit

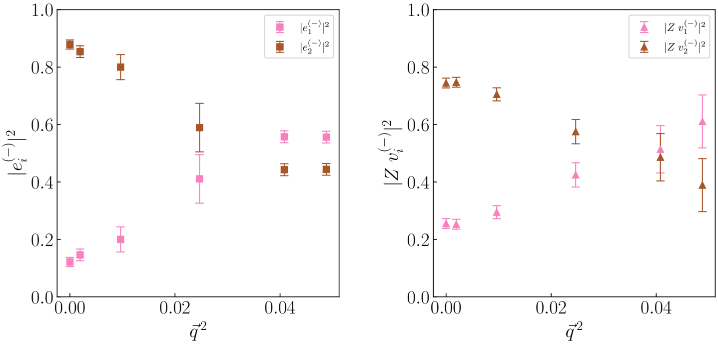
Avoided energy level crossing



- LH plot: Free case Σ (triangle) and N (squares) energy states as a function of $\vec{q}^2$
- RH plot: The mixed states $E_\lambda^{(+)}$ (crosses) and $E_\lambda^{(-)}$ (stars)
- Avoided energy level crossing

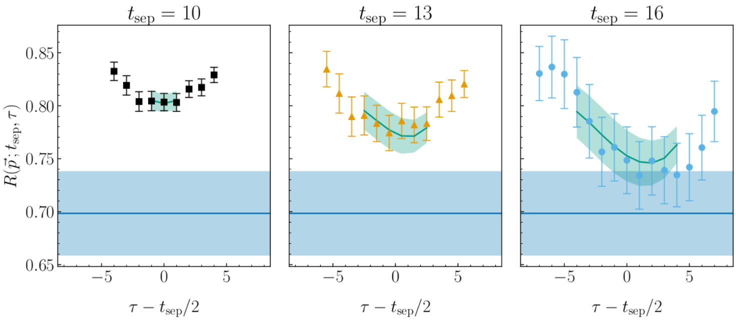
State mixing

$$\vec{e}^{(\pm)} = \begin{pmatrix} e_1^{(\pm)} \\ e_2^{(\pm)} \end{pmatrix}$$



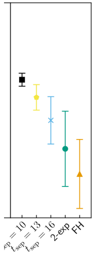
- LH plot: Eigenvectors known (from E_N, M_Σ, a_{++})
- RH plot: cf GEVP: $v_r^{(i)} \propto e_r^{(i)}$ so also track $e_r^{(i)}$
- Components flip between states

Conventional three-point function results – run #5 comparison [similiar Q^2]

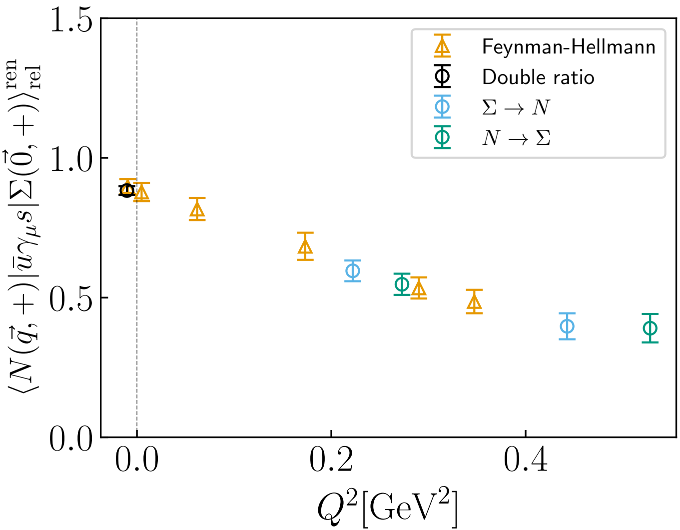


- Same number of gauge configs (~ 500)
- Both $\Sigma \to N$ and opposite 3-point functions used
- τ operator point insertion

  3 source-sink $t_{\mathrm{sep}} = 10, 13, 16$ used [$\sim 0.74, 0.96, 1.18\,\mathrm{fm}$]
- Fit ansatz includes an excited state
- Need to extrapolate



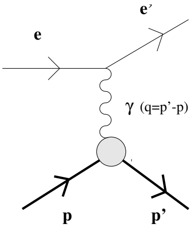
Comparison of results



Elastic nucleon scattering I

- $p' = p + q;\ Q^2 = -q^2$

$$\langle N(\vec{p}\,')|J_\mu(\vec{q})|N(\vec{p})\rangle =$$
$$\overline{u}_N(\vec{p}\,')\left[\gamma_\mu F_1(Q^2) + \sigma_{\mu\nu}\frac{q_\nu}{2M_N}F_2(Q^2)\right]u_N(\vec{p})$$



- $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d$
- Sachs form factors

$$\begin{aligned} G_E(Q^2) &= F_1(Q^2) - \frac{Q^2}{(2M_N)^2}F_2(Q^2) \\ G_M(Q^2) &= F_1(Q^2) + F_2(Q^2) \end{aligned}$$

Elastic nucleon scattering II

- Choose Breit frame geometry (electron bounces from nucleon):
- ie $\vec{p}' \equiv \vec{p} + \vec{q} = -\vec{p}$ as a trivial solution of $E_N(\vec{p}+\vec{q}) = E_N(\vec{p})$
- Degenerate (not quasi-degenerate):

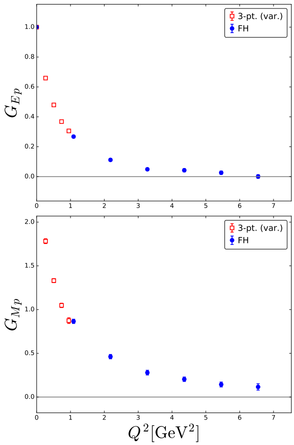
$$\Delta E_\lambda = \sqrt{\cancel{\left(E_N(\vec{p}+\vec{q}) - E_N(\vec{p})\right)^2} + 4\lambda^2 \left|\langle N(\vec{p}+\vec{q})|\hat{J}_\mu(\vec{0})|N(\vec{p})\rangle\right|^2}$$

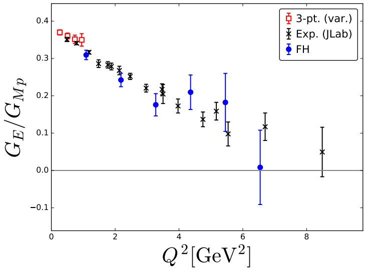
- This gives

$$\Delta E_\lambda = \begin{cases} \lambda \frac{M_N}{E_N}\, G_E & \mu = 4 \\ \lambda \frac{(\vec{e}_z \times \vec{q})_i}{E_N}\, G_M & \mu = i \end{cases}$$

- Choose $\vec{q}^{\,2}$ so large range $Q^2 \lesssim 7\,\mathrm{GeV}^2$

Results





- LH: G_E, G_M also compared to variational 3-point (on same configs)
- RH: As for LH together with JLAB experimental results

A potential problem

- Presently all results for the valence sector, as just considered correlation functions
- Including quark-line-disconnected matrix elements
  - Expensive: Need purpose generated configurations with determinant also containing the λ term
  - (H)MC problem: for probability definition need real determinant so fermion matrix must also be γ_5-Hermitian (as well as Hermitian)

$$\Longrightarrow \lambda^V, \lambda^A \text{ imaginary} \qquad\qquad [\lambda^S, \lambda^P, \lambda^T \text{ real}]$$

so E_λ develops an imaginary part for $O \sim V, A$

  - Have investigated this for axial current (and spin) and seen that this occurs (and can be measured) but is noisy (usual problem)
- Possible solution: expand Greens function (in λ) as before and take λ as imaginary
- For valence sector doesn't matter

Conclusions

- FH approach is a viable alternative to conventional method of 3-pt correlation functions for computing matrix elements
- FH approach only requires 2-pt correlation functions
- FH approach now generalised to decays
- With quasi-degenerate theory, don't need to tune for degenerate energies as before – in principle can re-use propagators for other decay/transition processes