Non-perturbative decoupling of massive fermions

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Outline

1. Motivation

> flavor-number-dependency of running coupling (massless fermions)

> fermion mass effects

2. Lattice methods

> used data

> gradient flow running coupling

> finite volume effects

3. Results I

> lattice gradient flow coupling data fitted

4. Renormalization scheme matching

> massive GF scheme 1-loop beta function

> relating the massive GF and massive BF-MOM schemes

5. Results II

> Comparing lattice GF data with 2-loop BF-MOM

> Potential effects from 3-loop non-universality

Beta function and running coupling in $\mathrm{SU}(N)$ gauge theories

- Gauge coupling beta function (massless fermions):

$$\mu \frac{dg^2}{d\mu} = \beta(g^2)$$

  > running gauge coupling g^2 at energy scale μ.

Beta function and running coupling in SU(N) gauge theories

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- perturbative definition:

$$\beta^{(n_l)}(g^2) = -2\,g^2 \sum_{n=0}^{n_l-1} \beta_n \left(\frac{g^2}{(4\pi)^2}\right)^{n+1}$$

> β_n in general scheme-dependent (exception(s): β_0 (, β_1)).

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[F. Herzog et al. JHEP02(2017)090]

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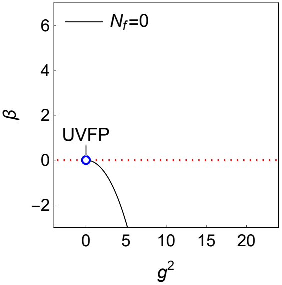

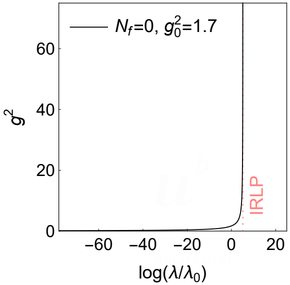

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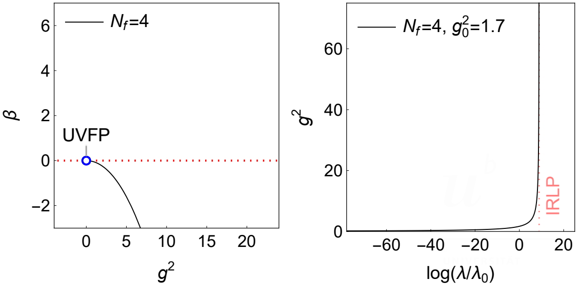

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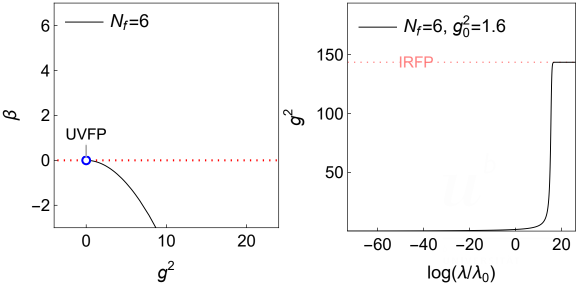

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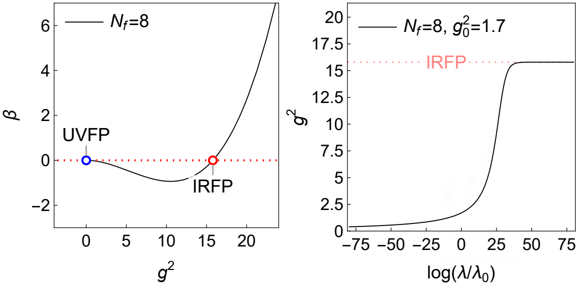

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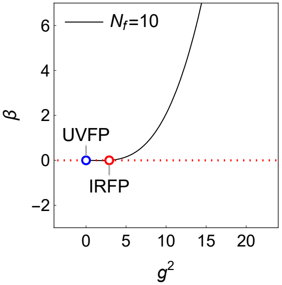

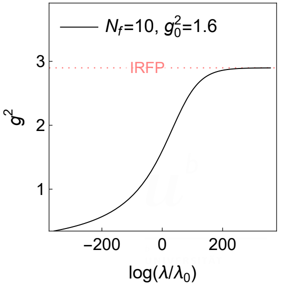

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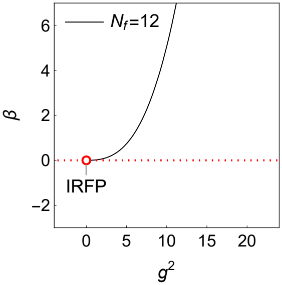

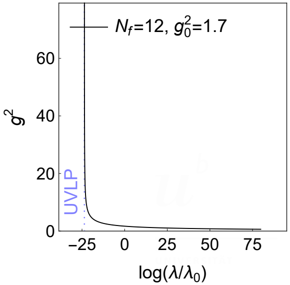

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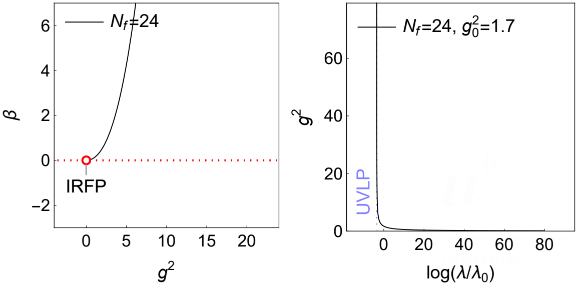

Effect of non-zero fermion mass

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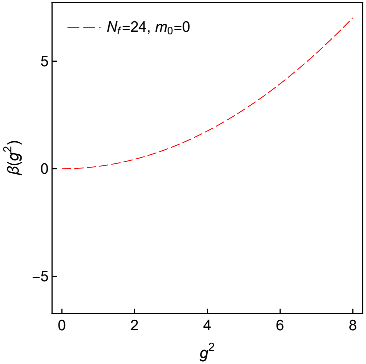

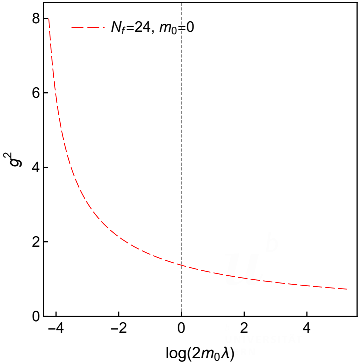

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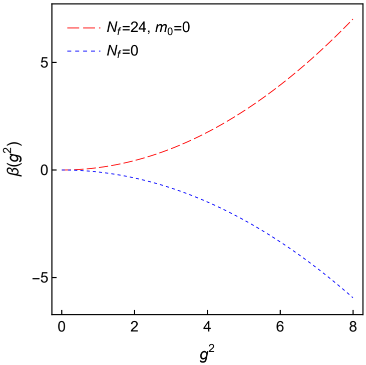

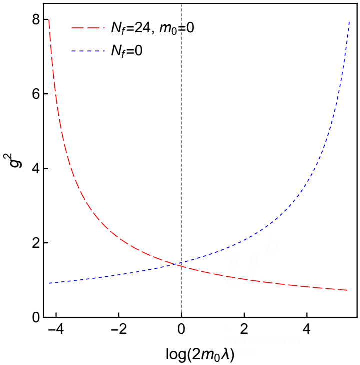

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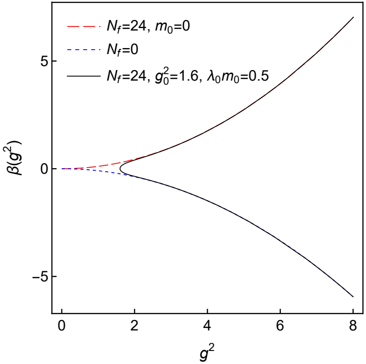

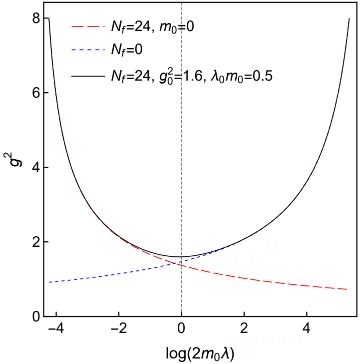

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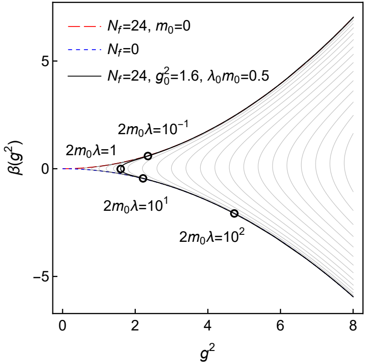

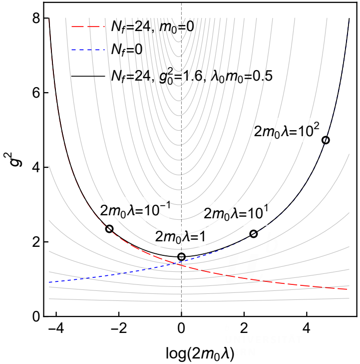

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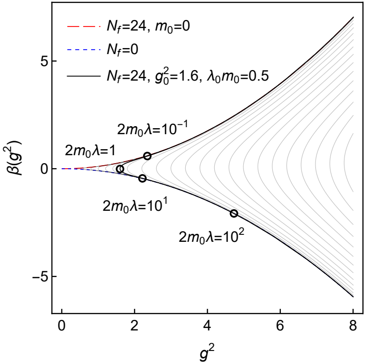

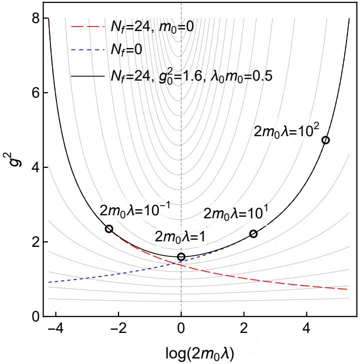

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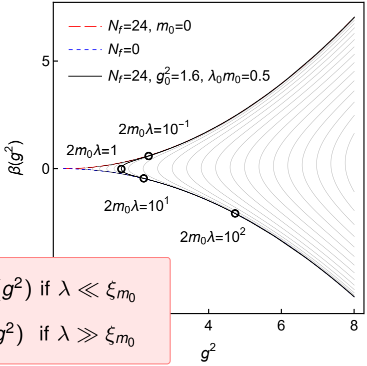
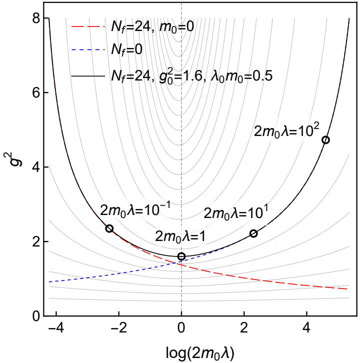
- 2-loop BF-MOM scheme result:

  [Jegerlehner & Tarasov NPB549(1999)]

  > pole mass $m = m_0$ (not running)

  > $$\beta^{(2)}_{\mathrm{BFM},N_f=24}(g^2, \lambda\, m_0) = \begin{cases} \beta^{(2)}_{\overline{\mathrm{MS}},N_f=24}(g^2) & \text{if } \lambda \ll \xi_{m_0} \\ \beta^{(2)}_{\overline{\mathrm{MS}},N_f=0}(g^2) & \text{if } \lambda \gg \xi_{m_0} \end{cases}$$





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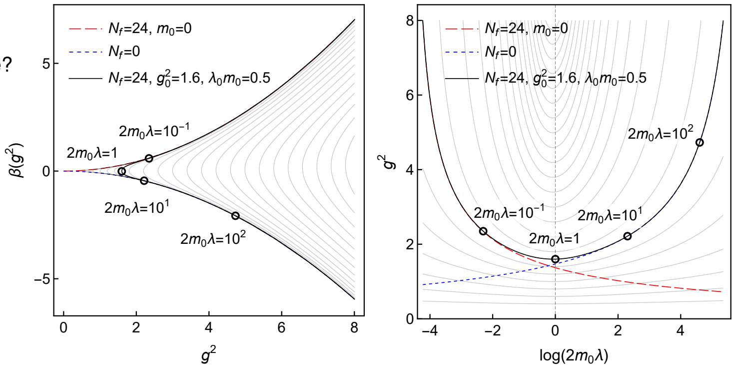

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➔ can use HMC data from previous study: [Rantaharju et al. PRD 104, 114504 (2021)]

> SU(2) lattice gauge theory with $N_f = 24$ massive Wilson fermions

> lattices of size $V = L^4$ ($L = 32a, 40a, 48a$) with periodic (gauge), resp. time-antiperiodic (fermions) BC

> lattice action: $S = S_G(U) + S_F(V) + c_{\mathrm{SW}}\, S_{\mathrm{SW}}(V)$

➔ U : fundamental, unsmeared gauge link matrices

➔ V : HEX-smeared gauge link matrices (corresponding to U) [Capitani et al. JHEP11(2006)028]

➔ S_G, S_F : Wilson gauge and Wilson fermion actions

➔ $c_{\mathrm{sw}}\, S_{\mathrm{sw}}$: clover term with Sheikholeslami-Wohlert coefficient $c_{\mathrm{SW}} = 1$

> 3 values of inverse gauge coupling, $\beta \in \{-0.25, 0.001, 0.25\}$

> for each value of β 4-6 different values of fermion hopping parameter κ

> PCAC quark mass: $a\, m_q = \left.\frac{(\partial_4^* + \partial_4) f_A(x_4)}{4\, f_P(x_4)}\right|_{x_4 = L/2}$ (axial and pseudo-scalar current correlators f_A, f_P)

> gradient flow running coupling (clover-energy, Lüscher-Weisz action)

Running coupling on the lattice

- Gradient flow (GF) scheme in the continuum: $g^2_{\mathrm{GF}}(\lambda) = \dfrac{2\,\pi^2\,\lambda^4\,\langle E(\lambda)\rangle}{3(N^2-1)}$ [Lüscher JHEP08(2010)071]
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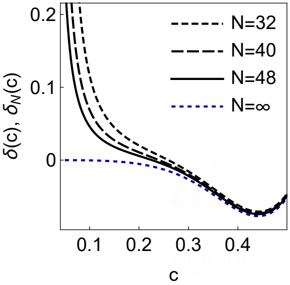

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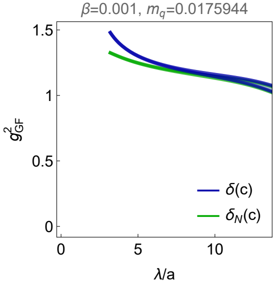
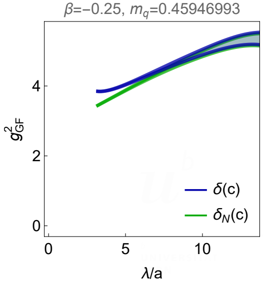
    $\Rightarrow$ improved short dist. behavior $\quad N = L/a = 32$

[Fritzsch





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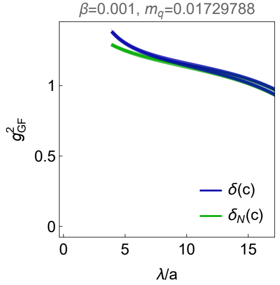
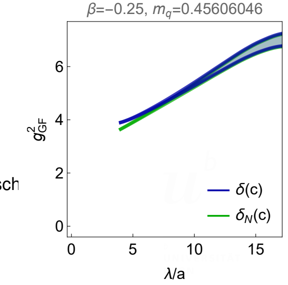
$\Rightarrow$ improved short dist. behavior $\quad N = L/a = 40$

[Fritzsch





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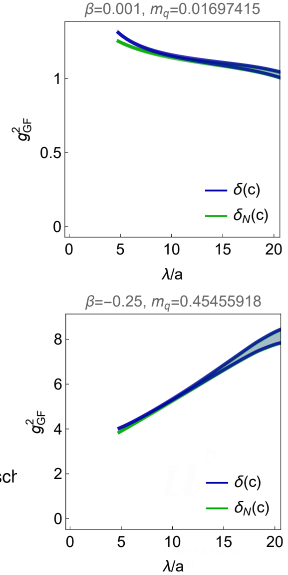

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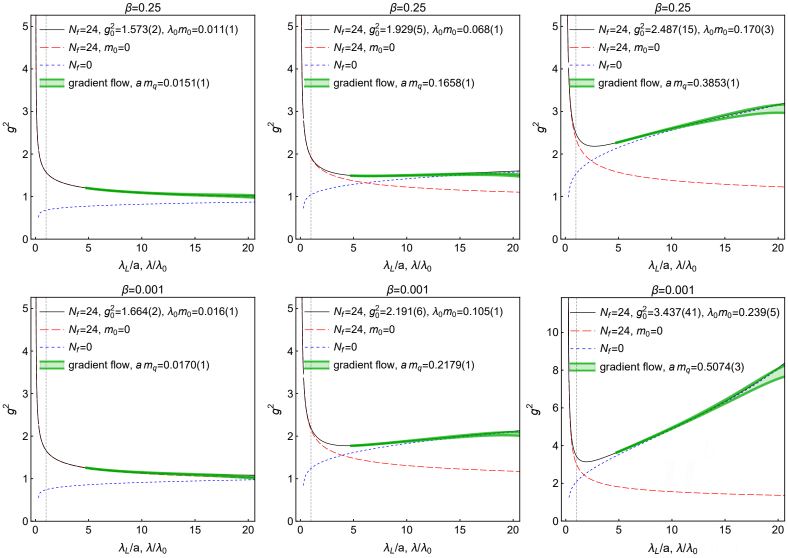
- We will consider $g^2_{\mathrm{GF}}(\lambda_L, L)$ as function of λ_L at fixed L (no step scaling).

Lattice GF-data

- $V = (48a)^4$ lattice:



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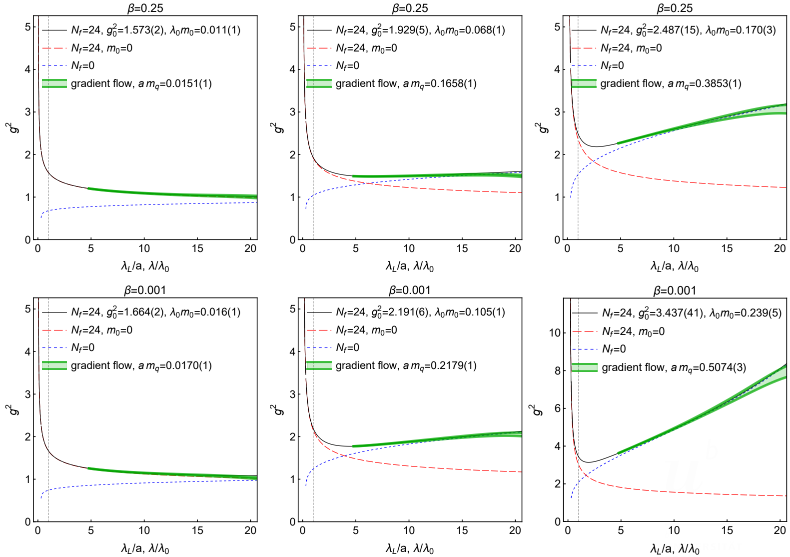
→ Observe expected change of slope as $a\,m_q$ increases



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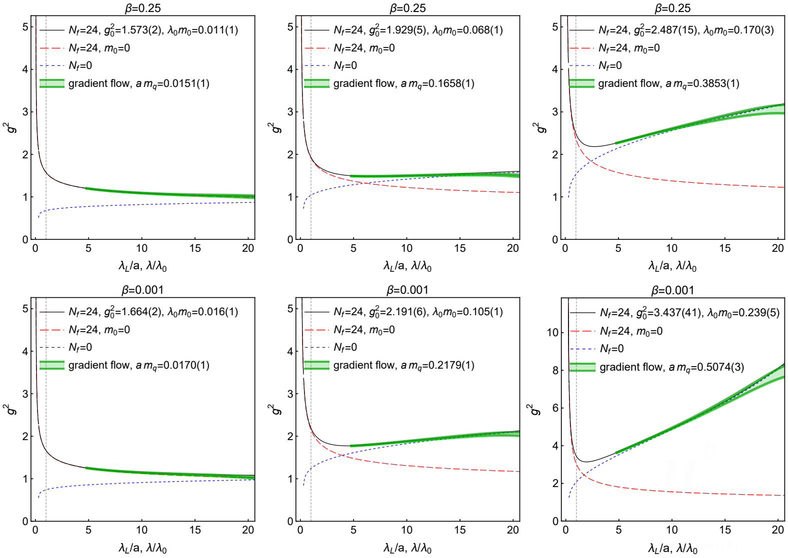
  set: $\lambda_0 = a$, $\lambda = \lambda_L$, fit: g_0^2, m_0

  $\Rightarrow r_s = \frac{\lambda_0 m_0}{a\,m_q} \approx 0.4 - 0.5$



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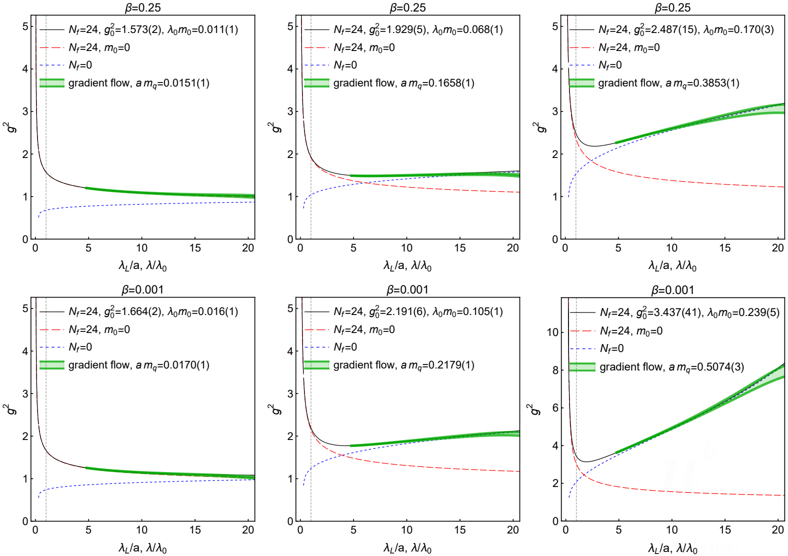

GF scheme beta function

- Continuum GF running coupling: $u_{\mathrm{GF}}(\lambda) = g^2_{\mathrm{GF}}(\lambda) = \dfrac{2\,\pi^2\,\lambda^4\,\langle E(\lambda)\rangle}{3\,(N^2-1)}$ with $\lambda = \sqrt{8\,t}$

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- use 1-loop $\overline{\mathrm{MS}}$ result for leading m_q-contribution to $\langle E(t)\rangle$ [Harlander & Neumann JHEP06(2016)161]

➔ $u_{\mathrm{GF}}(\lambda) = u_{\overline{\mathrm{MS}}}(\lambda_0)\left(1 + k_1(\lambda/\lambda_0)\dfrac{u_{\overline{\mathrm{MS}}}(\lambda_0)}{(4\pi)^2} + l_1(\lambda\, m(\lambda_0))\dfrac{u_{\overline{\mathrm{MS}}}(\lambda_0)}{(4\pi)^2} + \mathcal{O}\big(u^2_{\overline{\mathrm{MS}}}(\lambda_0)\big)\right)$ (*)

> old: $k_1(\lambda/\lambda_0) = (2\,\log(\lambda/\lambda_0) + \gamma_E)\,\beta_0 + N\left(\dfrac{52}{9} - 3\,\log(3)\right) - N_f\left(\dfrac{4}{9} - \dfrac{4}{3}\log(2)\right)$ [Lüscher JHEP08(2010)071]

> new: $l_1(y) = -\dfrac{4}{3}\,T_R\,N_f\,\Omega_{1q}\big(-1/(2y)^2\big)$

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- define GF beta function in terms of λ-derivative of (*): [Harlander PoS(LATTICE2021)489]

$$\rightarrow \quad \beta_{\mathrm{GF}} = -\lambda\frac{\mathrm{d}u_{\mathrm{GF}}}{\mathrm{d}\lambda} = -\left(\lambda\frac{\mathrm{d}k_1(\lambda/\lambda_0)}{\mathrm{d}\lambda} + \lambda\frac{\mathrm{d}l_1(\lambda\, m(\lambda_0))}{\mathrm{d}\lambda}\right)\frac{u^2_{\overline{\mathrm{MS}}}(\lambda_0)}{(4\pi)^2} + \mathcal{O}\big(u^3_{\overline{\mathrm{MS}}}\big) \;(**)$$

> invert (*): $u_{\overline{\mathrm{MS}}}(\lambda_0) = u_{\mathrm{GF}}(\lambda)\left(1 - k_1(\lambda/\lambda_0)\frac{u_{\mathrm{GF}}(\lambda)}{(4\pi)^2} - l_1(\lambda\, m(\lambda_0))\frac{u_{\mathrm{GF}}(\lambda)}{(4\pi)^2} + \mathcal{O}\big(u^2_{\mathrm{GF}}(\lambda)\big)\right)$

> and use it in (**) $\Rightarrow$ $\beta_{\mathrm{GF}}(u_{\mathrm{GF}}, \lambda\, m) = -\underbrace{\left(\lambda\frac{\mathrm{d}k_1(\lambda/\lambda_0)}{\mathrm{d}\lambda} + \lambda\frac{\mathrm{d}l_1(\lambda\, m(\lambda_0))}{\mathrm{d}\lambda}\right)}_{2\,\beta_{0,\mathrm{GF}}}\frac{u^2_{\mathrm{GF}}}{(4\pi)^2} + \mathcal{O}\big(u^3_{\mathrm{GF}}\big)$

GF scheme beta function

- $\beta_{\rm GF}(u_{\rm GF}, \lambda\, m) = -2\,\beta_{0,\rm GF}(x)\,\dfrac{u_{\rm GF}^2}{(4\pi)^2} + \mathcal{O}\big(u_{\rm GF}^3\big)$

  with $\quad \beta_{0,\rm GF}(x) = \beta_0 + \dfrac{4}{3}\, T_R\, N_f\, x\, \dfrac{\mathrm{d}\Omega_{1q}(x)}{\mathrm{d}x} \quad$ and $\quad x = -1/(2\,\lambda\, m)^2$

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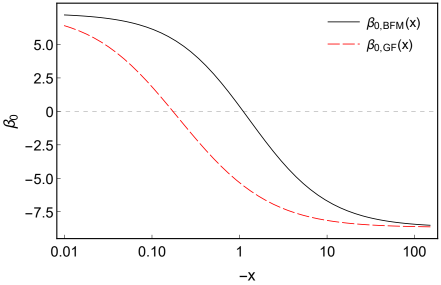

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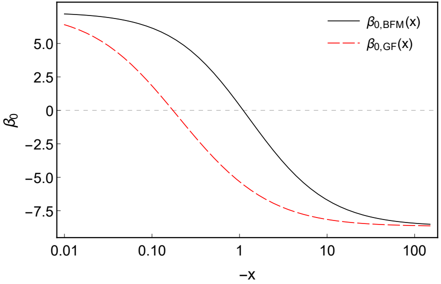
  > decoupling not near $2\,\lambda\, m_0 \approx 1 \Rightarrow$ mismatch between λ and m_0



Relating GF and BF-MOM (or BFM) scheme

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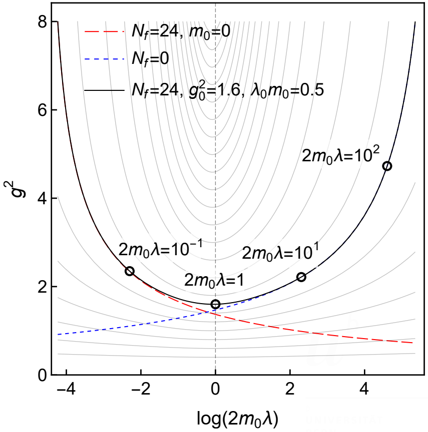

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➔ set:

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➔ Expand both sides $u_{\mathrm{R}}\big(\lambda^{(\mathrm{R})}\big) = u_0 + \sum_{n=1}^{\infty} c_n^{(\mathrm{R})}\big(u_0, x_0^{(\mathrm{R})}\big) \dfrac{\log^n\big(\lambda^{(\mathrm{R})}/\lambda_0^{(\mathrm{R})}\big)}{n!}$

Relating GF and BF-MOM (or BFM) scheme

- can match GF and BFM by rescaling $\lambda^{(\mathrm{GF})} = \rho_s\, \lambda^{(\mathrm{BFM})}$ with some $\rho_s > 0$
- scheme matching usually done at trivial fixed point $u = g^2 = 0$ (universality) ➔ not available in our case!
- if decoupling of fermions physical $\Rightarrow$ should be described equivalently in all schemes

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> $\lambda^{(\mathrm{BFM})} = \lambda \;\; , \;\lambda^{(\mathrm{GF})} = \rho_s \lambda$
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4. Renormalization scheme matching

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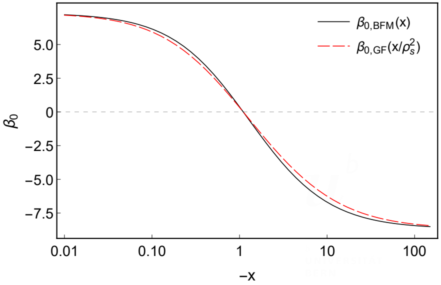
at 1-loop: $\beta_{0,\mathrm{GF}}(x_0/\rho_s^2) = \beta_{0,\mathrm{BFM}}(x_0) \;\; \Rightarrow \;\;$ $\rho_s = 2.535945\ldots$



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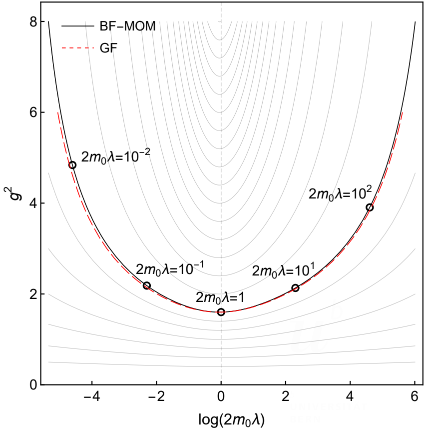

Lattice GF-data compared to 2-loop BF-MOM (no fit)

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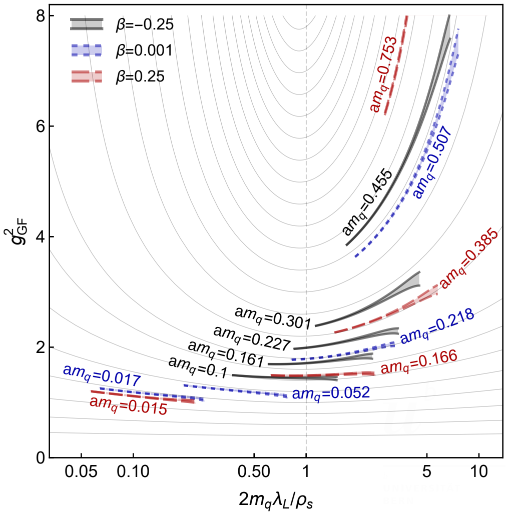

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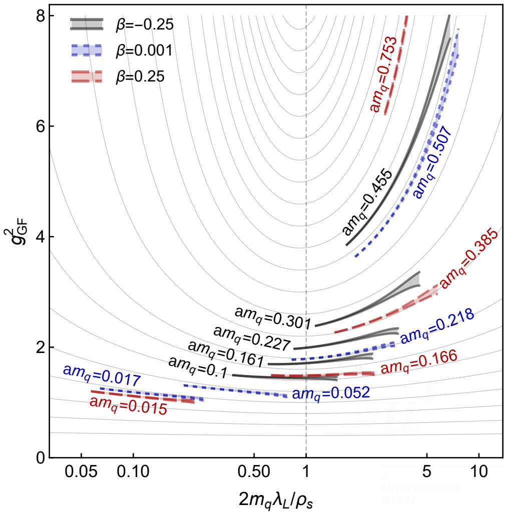
  [Dalla Brida & Ramos Eur.Phys.J.C 79,720(2019)]



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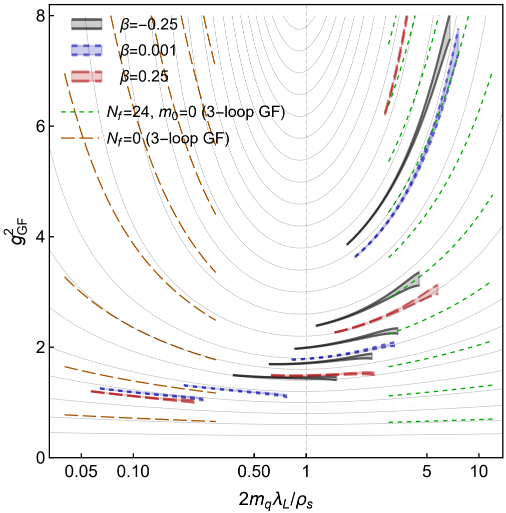

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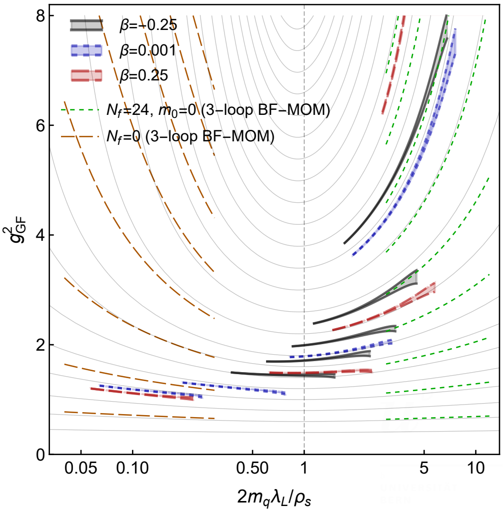

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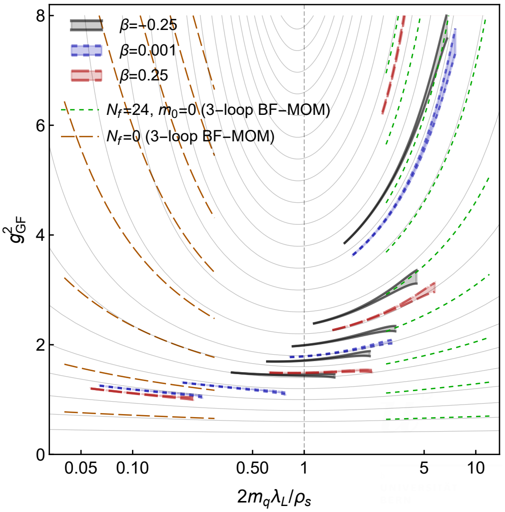

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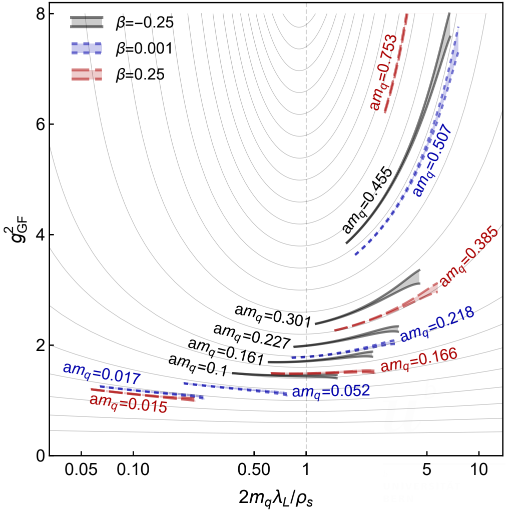
Summary

- Non-perturbative demonstration of decoupling of massive fermions at scale $\lambda \sim 1/m$



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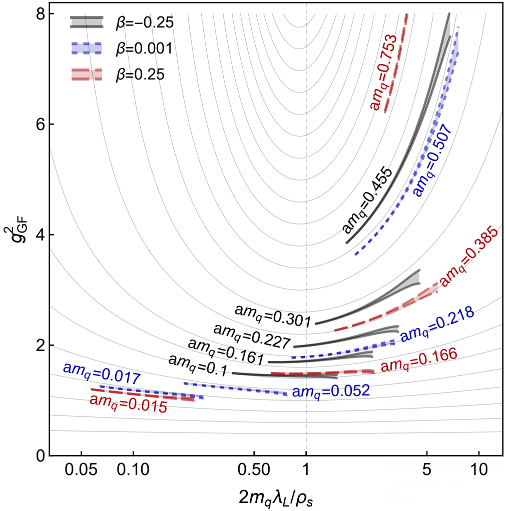

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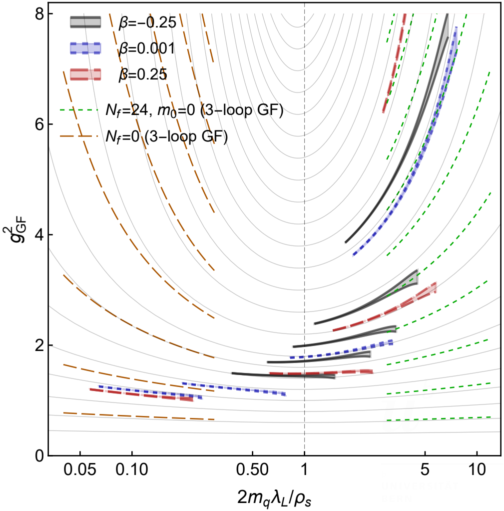

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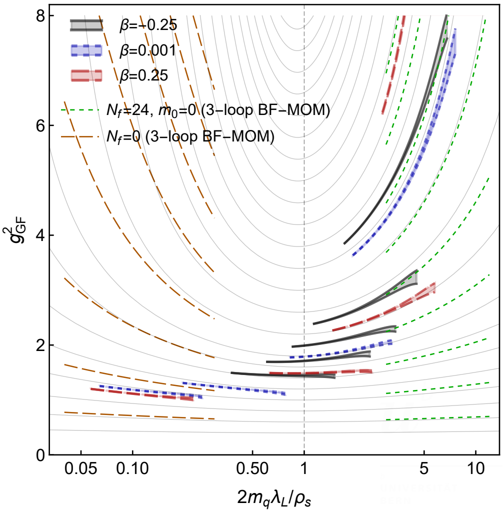

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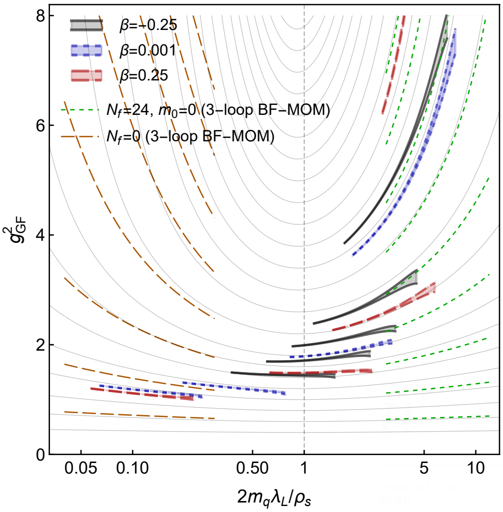
Thank you for your attention!

[PRL 129, 131601 (2022)], [arXiv:2110.13882 [hep-lat]]



7. Backup

3-loop non-universal effects

- Comparisons of the 3-loop beta function for SU(2) gauge theory with $N_f = 24$ massless flavors (left) and $N_f = 0$ flavors (right) in the $\overline{\mathrm{MS}}$ (solid, black), BF-MOM (long dashes, red) and GF scheme (short dashes, blue). For comparison also the corresponding 2-loop beta function is shown (dot-dashed, black).