

# Inclusive semi-leptonic $B_{(s)}$ mesons decay at the physical $b$ quark mass

**Alessandro Barone**

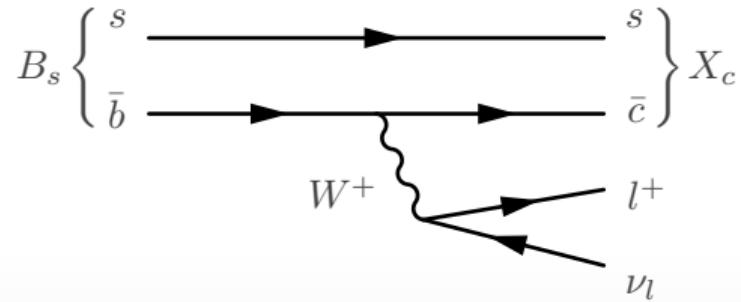
in collaboration with

Andreas Jüttner, Shoji Hashimoto,  
Takashi Kaneko, Ryan Kellermann

HU Berlin / NIC DESY Zeuthen joint lattice seminar  
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# Semileptonic decays: dictionary

Focus on **weak semi-leptonic** decays

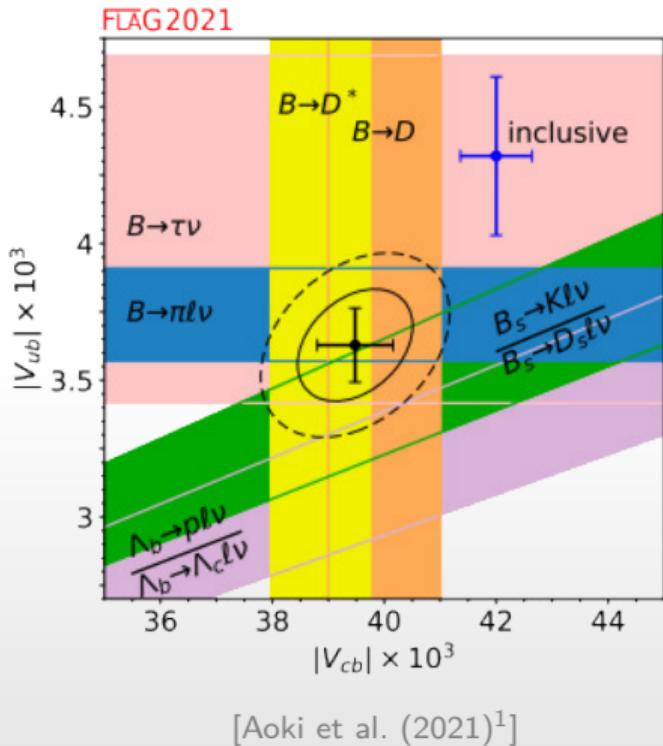


mediated by the weak hamiltonian

$$H_W = \frac{4G_F}{\sqrt{2}} V_{cb} [\bar{b}_L \gamma^\mu c_L] [\bar{\nu}_l \gamma_\mu l]$$

- ▶ **EXCLUSIVE:**  $B_s \rightarrow D_s l \nu_l$ , with just one hadron in the final state
- ▶ **INCLUSIVE:**  $B_s \rightarrow X_c l \nu_l$  , with multi-particle states

# Introduction and motivations

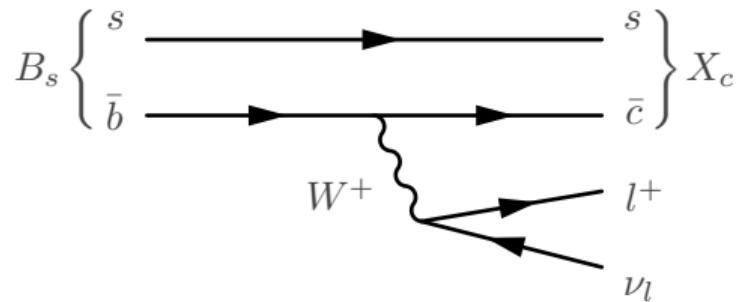


- $\sim 3\sigma$  discrepancy (in the plot) between inclusive/exclusive determination;
- lattice QFT represents a fully non-perturbative theoretical approach to QCD;
- no current predictions from lattice QCD for the inclusive decays.

This talk: Pilot study  $B_s \rightarrow X_c l \bar{\nu}_l$

- improve existing strategies for inclusive decays on the lattice;
- compare two different methods for the analysis.

# Differential decay rate



**Decay rate:** 
$$\frac{d\Gamma}{dq^2 dq_0 dE_l} = \frac{G_F^2 |V_{cb}|^2}{8\pi^3} [L_{\mu\nu}] [W^{\mu\nu}],$$

Leptonic tensor  $\leftarrow$  Hadronic tensor  $\rightarrow$

$$[W^{\mu\nu}] = \sum_{X_c} (2\pi)^3 \delta^{(4)}(p - q - r) \frac{1}{2E_B} \langle B_s(\mathbf{p}) | J^{\mu\dagger} | X_c(\mathbf{r}) \rangle \langle X_c(\mathbf{r}) | J^\nu | B_s(\mathbf{p}) \rangle.$$

→ contains all the **non-perturbative QCD**

# Hadronic tensor

The Hadronic tensor can be decomposed into 5 **Lorentz invariant structure functions**

$$W_i(q^2, v \cdot q) = W_i(q^2, \omega), \quad \omega = E_{X_c},$$

$$W^{\mu\nu} = -g^{\mu\nu} [W_1] + v^\mu v^\nu [W_2] - i\epsilon^{\mu\nu\alpha\beta} v_\alpha q_\beta [W_3] + q^\mu q^\nu [W_4] + (v^\mu q^\nu + v^\nu q^\mu) [W_5].$$

contribute to total decay rate  
disappear after integration over  $E_l$  (massless limit)  
relevant only for  $\tau$   
i.e.  $m_l \neq 0$

# Total decay rate

$$\Gamma = \frac{G_F^2 |V_{cb}|^2}{24\pi^3} \int_0^{\mathbf{q}_{\max}^2} d\mathbf{q}^2 \sqrt{\mathbf{q}^2} \bar{X} ,$$

kinematics

$$\bar{X} = \int_{\omega_{\min}}^{\omega_{\max}} d\omega \begin{matrix} \uparrow \\ k_{\mu\nu} \end{matrix} \times \begin{matrix} \downarrow \\ W^{\mu\nu} \end{matrix} = \sum_{l=0}^2 \int_{\omega_{\min}}^{\omega_{\max}} d\omega X^{(l)} .$$

portal to compute the  $\Gamma/|V_{cb}|^2$

from lattice?

$$X^{(0)} = \mathbf{q}^2 W_{00} + \sum_i (q_i^2 - \mathbf{q}^2) W_{ii} + \sum_{i \neq j} q^i W_{ij} q^j ,$$

$$X^{(1)} = -q_0 \sum_i q^i (W_{0i} + W_{i0}) ,$$

$$X^{(2)} = q_0^2 \sum_i W_{ii} .$$

# Inclusive decays on the lattice

[Hansen et al. (2017)<sup>2</sup>, Hashimoto (2017)<sup>3</sup>, Gambino and Hashimoto (2020)<sup>4</sup>]

We need the non-perturbative calculation of the hadronic tensor

$$W^{\mu\nu}(\mathbf{q}, \omega) \sim \sum_{X_c} \langle B_s | J^{\mu\dagger} | X_c \rangle \langle X_c | J^\nu | B_s \rangle.$$

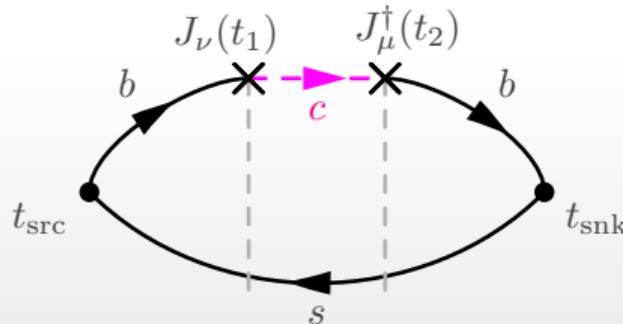
# Inclusive decays on the lattice

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On the lattice, this is achieved with a **4pt correlation function**:



- ▶  $t_{\text{src}}, t_2, t_{\text{snk}}$  fixed
- ▶  $t_{\text{src}} \leq t_1 \leq t_2$
- ▶  $t = t_2 - t_1$
- ▶  $t$  small → signal-to-noise ratio deteriorate with  $t$

$$C^{\mu\nu}(t) \simeq \frac{C_{4\text{pt}}^{\mu\nu}(t_{\text{snk}}, t_2, t_1, t_{\text{src}})}{C_{2\text{pt}}(t_{\text{snk}}, t_2)C_{2\text{pt}}(t_1, t_{\text{src}})} \leftrightarrow \langle B_s | \tilde{J}^{\mu\dagger}(\mathbf{q}, 0) e^{-t\hat{H}} \tilde{J}^\nu(\mathbf{q}, 0) | B_s \rangle.$$

# Lattice data (Euclidean)



finite/discrete number of  
time-slices  $t = -i\tau$

**lattice data**  
(correlation function)

$$C(t) = \int_0^\infty d\omega \rho(\omega) e^{-\omega t}$$

spectral function:

$$\sum_j \langle 0 | \mathcal{O} | j \rangle \langle j | \mathcal{O}^\dagger | 0 \rangle \delta(\omega - E_j)$$

$$\rho(\omega) \xrightleftharpoons[\text{ill-posed problem}]{\text{trivial}} C(t)$$

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Extracting the hadronic tensor is an ill-posed problem (**inverse problem**)

**lattice data**  
for inclusive

$$\leftarrow C_{\mu\nu}(t) = \int_0^\infty d\omega W_{\mu\nu}(\mathbf{q}, \omega) e^{-\omega t}$$

$$\sum_{X_c} \langle B_s | J_\mu^\dagger | X_c \rangle \langle X_c | J_\nu | B_s \rangle \delta(\omega - E_{X_c})$$

# Example of inverse problem

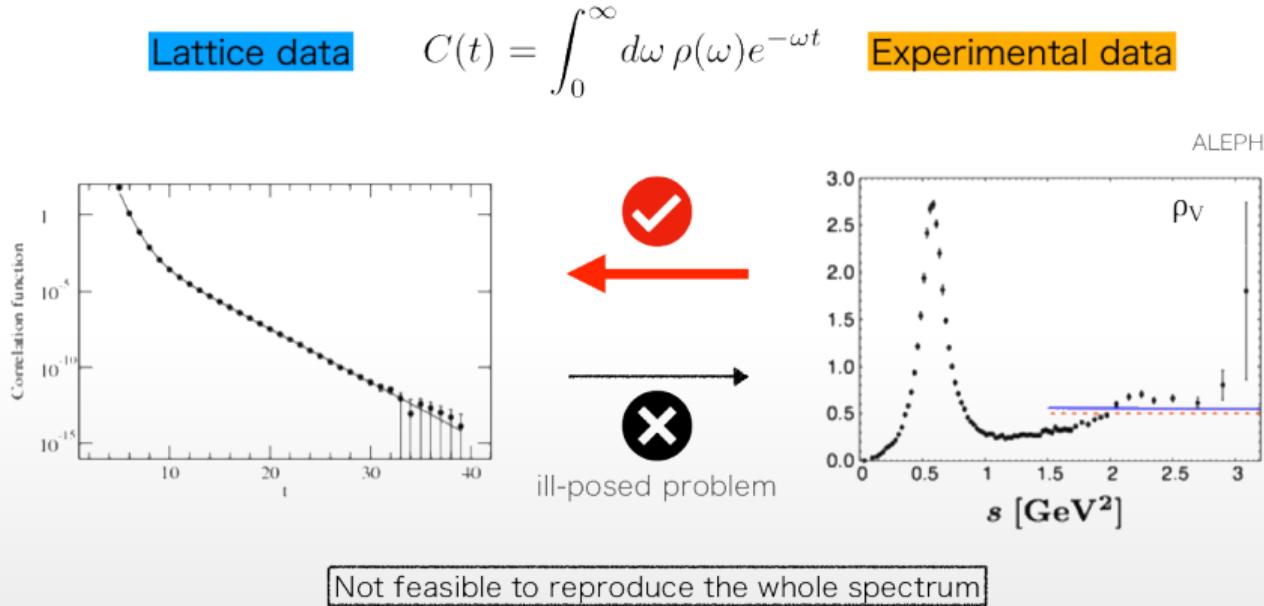


Figure: Slide from Shoji Hashimoto.

# Decay rate from lattice data

$$\begin{aligned} \bar{X} &= \int_{\omega_{\min}}^{\omega_{\max}} d\omega W^{\mu\nu} k_{\mu\nu}(\mathbf{q}, \omega) \\ &= \int_{\omega_0}^{\infty} d\omega W^{\mu\nu} k_{\mu\nu}(\mathbf{q}, \omega) \theta(\omega_{\max} - \omega) \rightarrow \text{kernel operator} \\ 0 \leq \omega_0 \leq \omega_{\min} &\leftarrow \boxed{\omega_0} \\ &= \int_{\omega_0}^{\infty} d\omega W^{\mu\nu} K_{\mu\nu}(\mathbf{q}, \omega) \end{aligned}$$

↑  
kinematics factors

Here we are NOT extracting the hadronic tensor  $W_{\mu\nu}$ !

We are addressing directly the integral  $\bar{X}$  using techniques common to a typical inverse problem.

# Decay rate from lattice data

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↑  
kinematics factors

Can we trade

$$\int_{\omega_0}^{\infty} d\omega W^{\mu\nu} K_{\mu\nu}(\mathbf{q}, \omega) \leftarrow ? \rightarrow \boxed{C^{\mu\nu}(t)} = \int_{\omega_0}^{\infty} d\omega W^{\mu\nu} e^{-\omega t}$$

↓  
lattice data

# Decay rate from lattice data

$$\begin{aligned}
 \bar{X} &= \int_{\omega_{\min}}^{\omega_{\max}} d\omega W^{\mu\nu} \boxed{k_{\mu\nu}(\mathbf{q}, \omega)} \\
 &= \int_{\omega_0}^{\infty} d\omega W^{\mu\nu} \boxed{k_{\mu\nu}(\mathbf{q}, \omega) \theta(\omega_{\max} - \omega)} \rightarrow \text{kernel operator} \\
 0 \leq \omega_0 \leq \omega_{\min} &\leftarrow \boxed{\omega_0} \\
 &= \int_{\omega_0}^{\infty} d\omega W^{\mu\nu} K_{\mu\nu}(\mathbf{q}, \omega)
 \end{aligned}$$

kinematics factors

We can expand in  $K_{\mu\nu}$  in power of  $e^{-\omega}$

$$K_{\mu\nu} \simeq c_{\mu\nu,0} + c_{\mu\nu,1} e^{-\omega} + \cdots + c_{\mu\nu,N} e^{-\omega N},$$

$$\Rightarrow \bar{X} \simeq c_{\mu\nu,0} \underbrace{\int_{\omega_0}^{\infty} d\omega W^{\mu\nu}}_{C^{\mu\nu}(0)} + c_{\mu\nu,1} \underbrace{\int_{\omega_0}^{\infty} d\omega W^{\mu\nu} e^{-\omega}}_{C^{\mu\nu}(1)} + \cdots + c_{\mu\nu,N} \underbrace{\int_{\omega_0}^{\infty} d\omega W^{\mu\nu} e^{-\omega N}}_{C^{\mu\nu}(N)}.$$

## Polynomial approximation strategies

$$K(\omega) : [\omega_0, \infty) \rightarrow \mathbb{R}, \quad K(\omega) \simeq \sum_j^N c_j p_j(\omega).$$

$\omega_0 \in [0, \omega_{\min}]$       family of polynomials in  $e^{-\omega}$

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↓  
 $\omega_0 \in [0, \omega_{\min})$ 
↓  
 family of polynomials in  $e^{-\omega}$

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## Chebyshev approach (1)

Standard Chebyshev polynomials  $T_k(\omega) : [-1, 1] \rightarrow [-1, 1]$

$$\Rightarrow \tilde{T}_k(\omega) = T_k(h(\omega))$$

Shifted Chebyshev polynomials  $\tilde{T}_k(\omega) : [\omega_0, \infty] \rightarrow [-1, 1]$

generic shifted Chebyshev

$$\tilde{T}_k(\omega) = \sum_{j=0}^k \tilde{a}_j^{(k)} e^{-j\omega}$$

$$h : [\omega_0, \infty] \rightarrow [-1, 1], \quad h(\omega) = Ae^{-\omega} + B,$$

$$\begin{cases} h(\omega_0) &= -1 \\ h(\infty) &= 1 \end{cases}$$

map between domains

$$K(\omega) \simeq \frac{\tilde{c}_0}{2} + \sum_{k=1}^N \tilde{c}_k \tilde{T}_k(\omega), \quad \tilde{c}_k = \langle K, \tilde{T}_k \rangle = \int_{\omega_0}^{\infty} d\omega K(\omega) \tilde{T}_k(\omega) \tilde{\Omega}(\omega).$$

# Polynomial approximation strategies

## Backus-Gilbert approach (2)

We minimize the functional ( $L_2$ -norm)

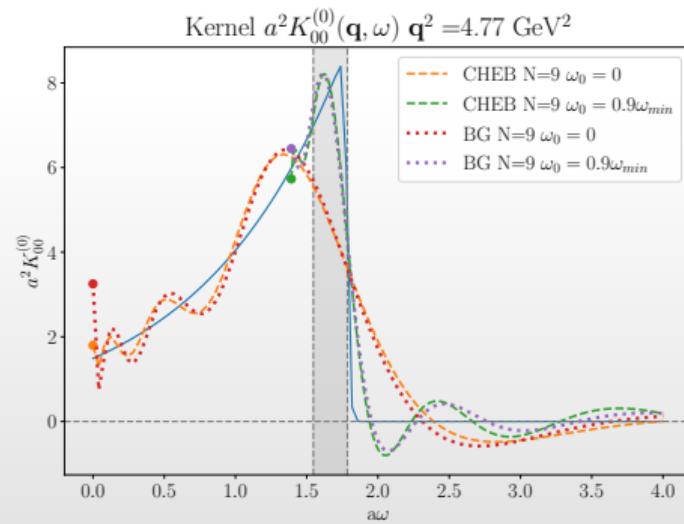
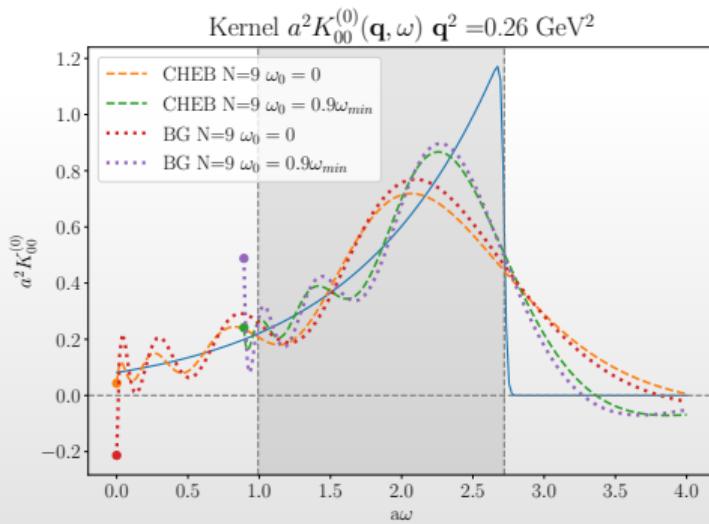
$$A[g] = \int_{\omega_0}^{\infty} d\omega \left[ K(\omega) - \sum_{j=1}^N g_j e^{-j\omega} \right]^2,$$

$$g_j \quad \leftrightarrow \quad \frac{\delta A}{\delta g_j} = 0.$$

# Kernel: polynomial approximation

$$K_{\mu\nu}(\mathbf{q}, \omega; t_0) = e^{2\omega t_0} k_{\mu\nu}(\mathbf{q}, \omega) \theta_\sigma(\omega_{\max} - \omega)$$

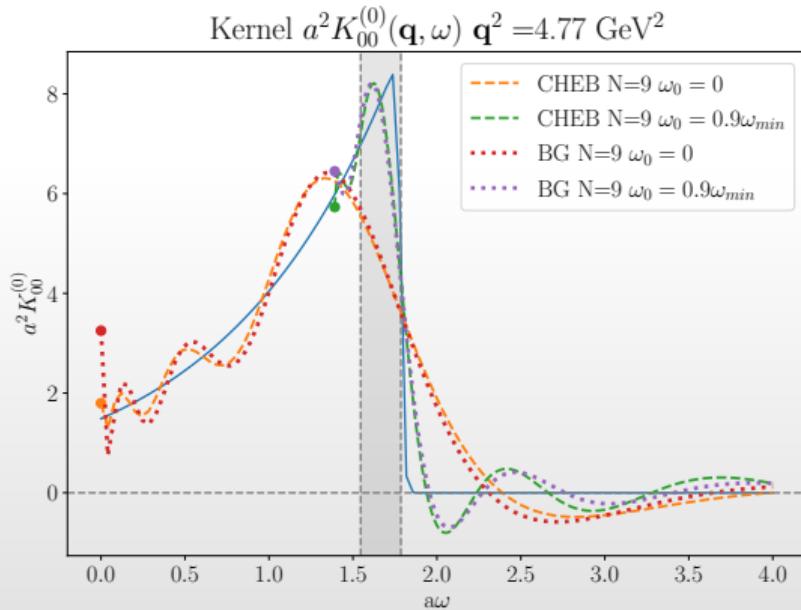
smooth step-function (sigmoid):  
cut the unphysical states above  $\omega_{\max}$



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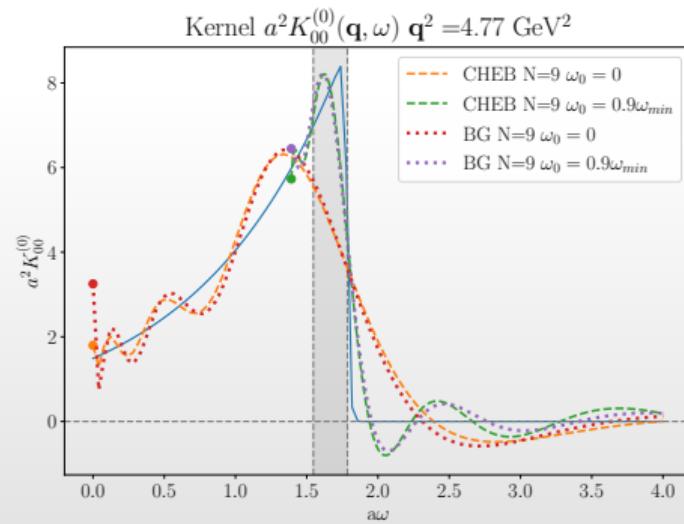
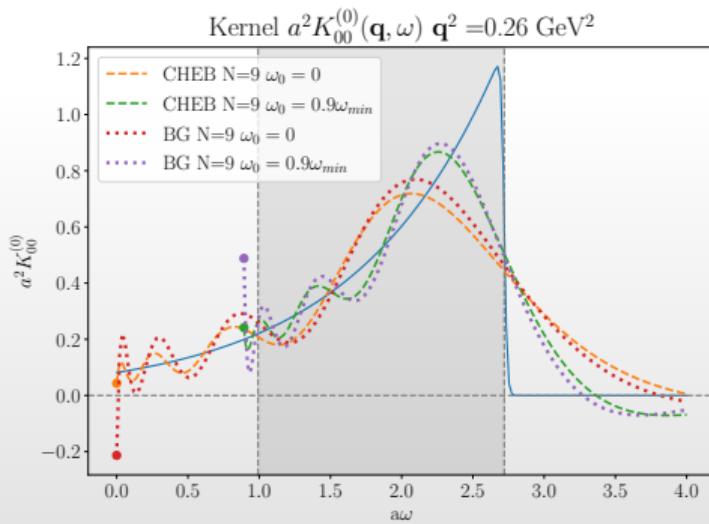
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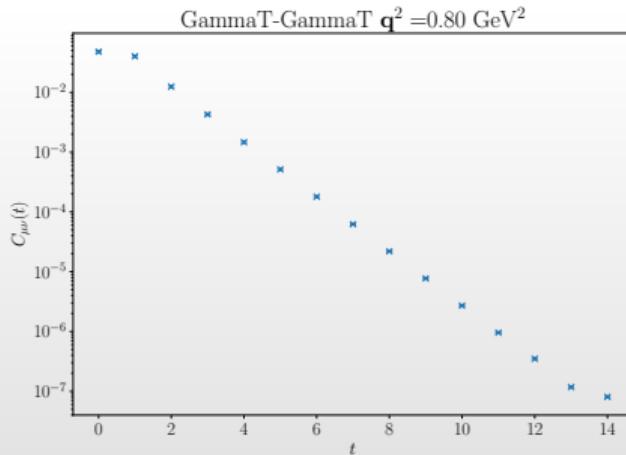


# Analysis strategy

$$\bar{X} = \int_{\omega_0}^{\infty} W^{\mu\nu} K_{\mu\nu}(\mathbf{q}, \omega) \simeq \sum_j^N c_{\mu\nu,j} C^{\mu\nu}(j)$$

## Problems

- noise from the data adds up and error on  $\bar{X}$  explodes;
- time-slice  $t = 0$  must be avoided.



## Analysis strategy (2)

$$\bar{X} = \int_{\omega_0}^{\infty} d\omega W^{\mu\nu} K_{\mu\nu}(\omega, \mathbf{q}; t_0) e^{+2t_0\omega} e^{-2t_0\omega} \Rightarrow \boxed{\bar{X} \simeq \sum_{j=0}^N c_{\mu\nu,j} C^{\mu\nu}(j + 2t_0)}$$

- $j \leftrightarrow t$ : degree corresponds to a certain time-slice, so  $N$  is limited by the available data and the noise of the signal;
- we take  $2t_0 = 1$ , i.e. as small as possible.

To control the noise we have 2 options:

- act on the data  $C^{\mu\nu}$  (Chebyshev approach);
- act on the coefficients  $c_{\mu\nu,j}$  (Backus-Gilbert).

# Analysis strategy: Chebyshev

[Bailas et al. (2020)<sup>5</sup>]

We can expand the kernel  $K_{\mu\nu}(\omega, \mathbf{q}; t_0)$  with Chebyshev polynomials as

$$\bar{X} \simeq \frac{\tilde{c}_{\mu\nu,0}}{2} \int_{\omega_0}^{\infty} d\omega \mathbf{W}^{\mu\nu} \tilde{T}_0(\omega) e^{-2\omega t_0} + \cdots + \tilde{c}_{\mu\nu,N} \int_{\omega_0}^{\infty} d\omega \mathbf{W}^{\mu\nu} \tilde{T}_N(\omega) e^{-2\omega t_0},$$

where

$$\int_{\omega_0}^{\infty} d\omega \mathbf{W}^{\mu\nu} \tilde{T}_k(\omega) e^{-2\omega t_0} = \int_{\omega_0}^{\infty} d\omega \mathbf{W}^{\mu\nu} \left[ \sum_{j=0}^k \tilde{t}_j^{(k)} e^{-j\omega} \right] e^{-2\omega t_0} = \sum_{j=0}^k \tilde{t}_j^{(k)} C^{\mu\nu}(j + 2t_0).$$

Chebyshev polynomials are **bounded**, so we normalize

$$\int_{\omega_0}^{\infty} d\omega \mathbf{W}^{\mu\nu} \tilde{T}_j(\omega) e^{-2\omega t_0} \quad \rightarrow \quad -1 \leq \frac{\int_{\omega_0}^{\infty} d\omega \mathbf{W}^{\mu\nu} \tilde{T}_j(\omega) e^{-2\omega t_0}}{\boxed{\int_{\omega_0}^{\infty} d\omega \mathbf{W}^{\mu\nu} \tilde{T}_0(\omega) e^{-2\omega t_0}}} \leq 1. \rightarrow C^{\mu\nu}(2t_0)$$

## Analysis strategy: Chebyshev (2)

[Bailas et al. (2020)<sup>5</sup>, Gambino and Hashimoto (2020)<sup>4</sup>]

The new relation is

$$\langle \tilde{T}_k \rangle_{\mu\nu} \equiv \frac{\int_{\omega_0}^{\infty} d\omega W^{\mu\nu} \tilde{T}_j(\omega) e^{-2\omega t_0}}{\int_{\omega_0}^{\infty} d\omega W^{\mu\nu} \tilde{T}_0(\omega) e^{-2\omega t_0}} = \sum_{j=0}^k \tilde{t}_j^{(k)} \left[ \frac{C_{\mu\nu}(j+2t_0)}{C_{\mu\nu}(2t_0)} \right], \quad |\langle \tilde{T}_j \rangle_{\mu\nu}| \leq 1,$$

$\rightarrow \bar{C}_{\mu\nu}(j)$

$\downarrow$  Chebyshev matrix elements

and value of  $\bar{X}$  can be obtained as

$$\bar{X} \simeq C_{\mu\nu}(2t_0) \left[ \frac{\bar{X}}{C_{\mu\nu}(2t_0)} \right] \rightarrow \equiv \bar{X}_{\bar{C}\mu\nu}, \text{ depends on } \bar{C}_{\mu\nu}$$

We can then calculate it through the Chebyshev matrix elements as

$$\bar{X}_{\bar{C}\mu\nu} \simeq \frac{\tilde{c}_{\mu\nu,0}}{2} + \sum_{j=1}^N \tilde{c}_{\mu\nu,j} \langle \tilde{T}_j \rangle_{\mu\nu}.$$

$\downarrow$

We need to determine these from the data

# Chebyshev fit

The relations between data and Chebyshev matrix elements are

$$\langle \tilde{T}_k \rangle_{\mu\nu} = \sum_{j=0}^k \tilde{t}_j^{(k)} \bar{C}_{\mu\nu}(j), \quad \bar{C}_{\mu\nu}(k) = \sum_{j=0}^k \tilde{a}_j^{(k)} \langle \tilde{T}_j \rangle_{\mu\nu}$$

So far this are related by a linear transformation

$$\begin{pmatrix} \bar{C}_{\mu\nu}(0) \\ \bar{C}_{\mu\nu}(1) \\ \vdots \\ \vdots \\ \bar{C}_{\mu\nu}(N) \end{pmatrix} = \begin{pmatrix} \tilde{a}_0^{(0)} & 0 & \cdots & \cdots & 0 \\ \tilde{a}_0^{(1)} & \tilde{a}_1^{(1)} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \tilde{a}_0^{(N)} & \tilde{a}_1^{(N)} & \cdots & \cdots & \tilde{a}_N^{(N)} \end{pmatrix} \begin{pmatrix} \langle \tilde{T}_0 \rangle_{\mu\nu} \\ \langle \tilde{T}_1 \rangle_{\mu\nu} \\ \vdots \\ \vdots \\ \langle \tilde{T}_N \rangle_{\mu\nu} \end{pmatrix}$$

This is not taking into account the bounds on the Chebyshev matrix elements  $\langle \tilde{T}_k \rangle_{\mu\nu}$ .  
⇒ We address it through a **Bayesian fit with constraints**.

## Chebyshev fit (2)

We address the extraction of the Chebyshev through a fit with the following steps

- ▶ start from a frequentist fit;
- ▶ enforce the bounds;
- ▶ stabilise the fit augmenting the  $\chi^2$  with some priors.

The frequentist  $\chi^2$  (**Maximum Likelihood**) looks like

$$\chi^2 = \sum_{i,j=1}^N \left( \bar{C}(i) - \sum_{\alpha=1}^i \tilde{a}_{\alpha}^{(i)} \langle \tilde{T}_{\alpha} \rangle \right) \text{Cov}_{ij}^{-1} \left( \bar{C}(j) - \sum_{\alpha=1}^j \tilde{a}_{\alpha}^{(j)} \langle \tilde{T}_{\alpha} \rangle \right)$$

So far this is equivalent to the linear system. We can enforce the bounds substituting  $\tilde{T}_{\alpha} = f(\tau_{\alpha})$  with  $f : (-\infty, +\infty) \rightarrow [-1, 1]$

$$\chi^2 = \sum_{i,j=1}^N \left( \bar{C}(i) - \sum_{\alpha=1}^i \tilde{a}_{\alpha}^{(i)} f(\tau_{\alpha}) \right) \text{Cov}_{ij}^{-1} \left( \bar{C}(j) - \sum_{\alpha=1}^j \tilde{a}_{\alpha}^{(j)} f(\tau_{\alpha}) \right)$$

## Chebyshev fit (3)

We need to stabilize the fit! We use **Maximum a Posteriori (MAP)** + bounds

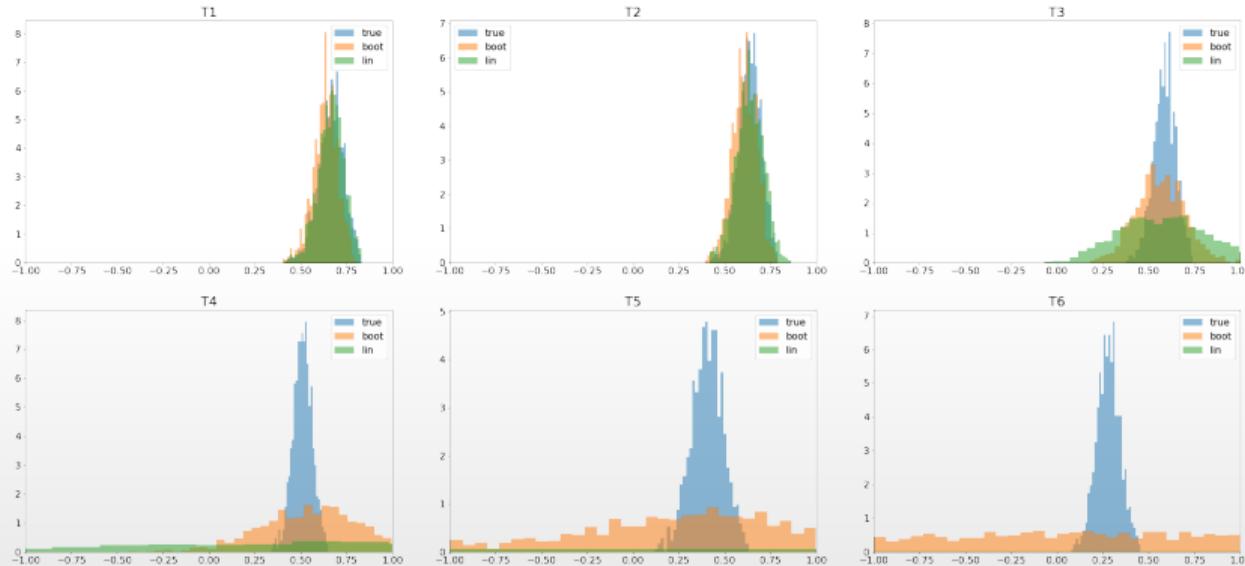
$$\chi^2_{\text{aug}} = \sum_{i,j=1}^N \left( \bar{C}(i) - \sum_{\alpha=1}^i \tilde{a}_{\alpha}^{(i)} f(\tau_{\alpha}) \right) \text{Cov}_{ij}^{-1} \left( \bar{C}(j) - \sum_{\alpha=1}^j \tilde{a}_{\alpha}^{(j)} f(\tau_{\alpha}) \right) + \chi^2_{\text{prior}}$$
$$\chi^2_{\text{prior}} = \sum_{\alpha=1}^N \frac{(\tau_{\alpha} - \bar{\tau}_{\alpha})^2}{\bar{\sigma}_{\alpha}^2}.$$

sampled from  $\mathcal{N}(0, 1)$   $\forall$  bootstrap bin  
 $\bar{\sigma}_{\alpha}^2 \rightarrow = 1$  (weak prior)

- $\tau_{\alpha}$  are assumed to be gaussian distributed
- $\tilde{T}_{\alpha} = f(\tau_{\alpha})$  and the function  $f(\tau_{\alpha}) = \text{erf}\left(\frac{\tau_{\alpha}}{\sqrt{2}}\right)$
- $\chi^2_{\text{prior}}$  has the role to stabilize the fit and correspond to a flat distribution for  $\tilde{T}_{\alpha}$

# Chebyshev fit: example

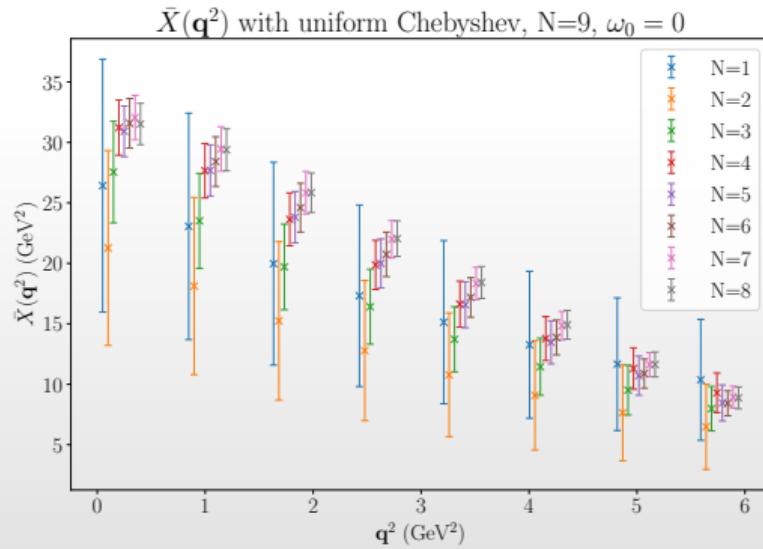
True chebyshev distribution → data → data + noise → analysis



# Chebyshev with uniform distribution

The idea is:

- ▶ we take a priori all the ( $N=9$ ) Chebyshev to be uniform (result is expected to be correct, but noisy);
- ▶ we introduce the actual Chebyshev determined from the fit step by step (starting from the lowest degrees) to see how the situation changes.



## Analysis strategy: Backus-Gilbert

[Backus and Gilbert (1968)<sup>6</sup>, Hansen et al. (2019)<sup>7</sup>, Bulava et al. (2021)<sup>8</sup>]

Aside from the functional  $A[g]$ , which approximates the target function (kernel), we include some information on the data

$$A[g] = \int_{\omega_0}^{\infty} d\omega \left[ K_{\mu\nu}(\mathbf{q}, \omega; t_0) - \sum_{j=1}^N g_j e^{-j\omega} \right]^2 \rightarrow \text{exponential basis}$$

$$B[g] = \sigma_X^2 = \sum_{i,j=1}^N g_i \text{Cov} [\bar{C}_{\mu\nu}(i), \bar{C}_{\mu\nu}(j)] g_j, \quad \bar{C}_{\mu\nu}(i) = \frac{C_{\mu\nu}(i + 2t_0)}{C_{\mu\nu}(2t_0)}$$

We minimise

$$W_\lambda[g] = (1 - \lambda) \frac{A[g]}{A[0]} + \lambda B[g].$$

The parameter  $\lambda$  control the interplay between the 2 functionals, i.e. the balance between **statistical** and **systematic** errors.

# Analysis strategy: Backus-Gilbert generalised

[Alexandrou et al. (2022)<sup>9</sup>]

We can generalise to allow the use of an arbitrary basis of polynomials  $\tilde{P}_k(\omega)$  in  $e^{-\omega}$

$$\tilde{P}_k(\omega) = \sum_{j=0}^k \tilde{p}_j^{(k)} e^{-j\omega}, \quad \omega \in [\omega_0, \infty)$$

such that the functionals read

$$A[g] = \int_{\omega_0}^{\infty} d\omega \boxed{\tilde{\Omega}(\omega)} \left[ K_{\mu\nu}(q, \omega; t_0) - \sum_{j=0}^N g_j \boxed{\tilde{P}_j(\omega)} \right]^2 \rightarrow \text{arbitrary basis}$$
$$B[g] = \sum_{k,l=1}^N g_k \text{Cov} \left[ \boxed{\bar{C}_{\mu\nu}^P(k)}, \bar{C}_{\mu\nu}^P(l) \right] g_l$$
$$\downarrow \bar{C}_{\mu\nu}^P(k) = \sum_{j=0}^k \tilde{p}_j^{(k)} \bar{C}_{\mu\nu}(j)$$

## Analysis strategy: Backus-Gilbert generalised (2)

Why generalising? The solution of the Backus-Gilbert problem with  $\lambda = 0$  is given by

$$\mathbf{A} \cdot \mathbf{g} = \mathbf{K} \quad \leftrightarrow \quad \mathbf{g} = \mathbf{A}^{-1} \cdot \mathbf{K}$$

with

$$A_{ij} = \int_{\omega_0}^{\infty} d\omega \tilde{\Omega}(\omega) \tilde{P}_i(\omega) \tilde{P}_j(\omega),$$

$$K_i = \int_{\omega_0}^{\infty} d\omega \tilde{\Omega}(\omega) \tilde{P}_i(\omega) K(\omega)$$

In general the matrix  $\mathbf{A}$  is ill-conditioned and its inverse requires arbitrary precision. If we choose the  $\tilde{\Omega}$  and  $\tilde{P}_j = \tilde{T}_j$  from the Chebyshev we can take advantage of their orthogonality property such that  $\mathbf{A}$  is diagonal! The solution for  $\lambda \neq 0$  is

$$\mathbf{g}_\lambda = \mathbf{W}_\lambda^{-1} \cdot \mathbf{K}, \quad \mathbf{W}_\lambda = (1 - \lambda)\mathbf{A} + \lambda A[0]\mathbf{B}$$

## Analysis strategy: Backus-Gilbert - a different perspective

With the previous idea we can put things in an equivalent but different perspective. We can write the coefficients as

$$g_i = \boxed{c_i} + \epsilon_i \quad \text{↑ case } \lambda = 0$$

and require that the correction  $\epsilon_i$  approximate the null function through the minimisation of  $W_\lambda[\epsilon]$

$$W_\lambda[\epsilon] = (1 - \lambda)A[\epsilon] + \lambda B[\epsilon]$$

$$A[\epsilon] = \int_{\omega_0}^{\infty} d\omega \tilde{\Omega}(\omega) \left[ \sum_{j=0}^N \epsilon_j \tilde{P}_j(\omega) \right]^2,$$

$$B[\epsilon] = \sum_{i,j=1}^N [2\epsilon_i \sigma_{ij}^P c_j + \epsilon_i \sigma_{ij}^P \epsilon_j], \quad \sigma_{ij}^P = \text{Cov} [\bar{C}_{\mu\nu}^P(i), \bar{C}_{\mu\nu}^P(j)]$$

NB: this is equivalent to the previous case! It's just a different perspective which may give more insight in particular with the comparison with the Chebyshev case.

## Analysis strategy: Backus-Gilbert constraints

[Bulava et al. (2021)<sup>8</sup>]

On top of that, we also include a constraint on the area:

$$\int_{\omega_0}^{\infty} d\omega \tilde{\Omega}(\omega) \sum_{j=0}^N g_j \tilde{P}_j(\omega) = \int_{\omega_0}^{\infty} d\omega \tilde{\Omega}(\omega) K_{\mu\nu}(\mathbf{q}, \omega).$$

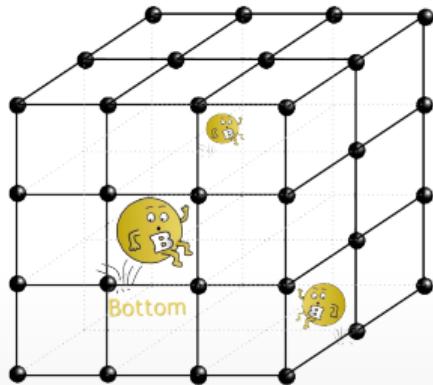
The value of  $\lambda$  can be in principle tuned arbitrarily. In practice, we choose the value of optimal balance  $\lambda^*$  between statistical and systematic errors with

$$W(\lambda) = W_\lambda[g^\lambda], \quad \left. \frac{dW(\lambda)}{d\lambda} \right|_{\lambda^*} = 0$$

$$\Rightarrow \frac{A[g^{\lambda^*}]}{A[0]} = B[g^{\lambda^*}]$$

# Inclusive decays on the lattice: setup

Simulations carried out on the DiRAC Extreme Scaling service at the University of Edinburgh using the **Grid** [Boyle et al.<sup>10</sup>] and **Hadrons** [Portelli et al.<sup>11</sup>] software packages



**Pilot study** with RBC/UKQCD ensembles

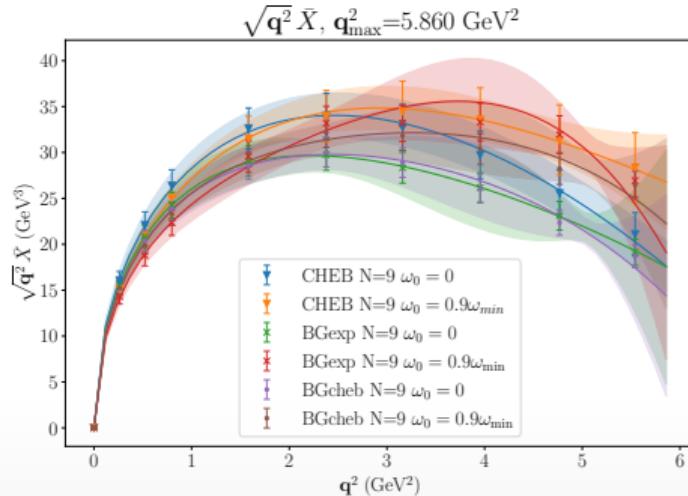
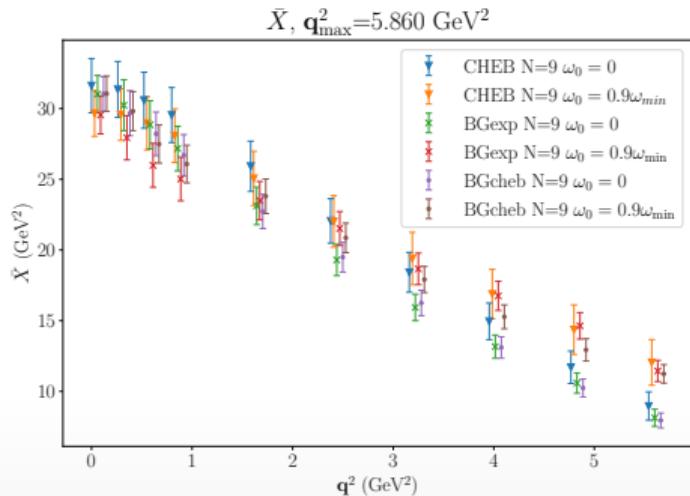
[Allton et al. (2008)<sup>12</sup>]:

- ▶ lattice  $24^3 \times 64$ ;
- ▶ lattice spacing  $a^{-1} = 1.79 \text{ GeV}$ ;
- ▶  $M_\pi \simeq 330 \text{ MeV}$ ;
- ▶ 120 gauge configurations, 8 sources;
- ▶ 8+2 momenta (Twisted BC).

## Simulation:

- ▶ RHQ action for  $b$  quark [El-Khadra et al. (1997)<sup>13</sup>, Christ et al. (2007)<sup>14</sup>, Lin and Christ (2007)<sup>15</sup>]:
  - ▶ based on clover action with anisotropic terms;
  - ▶ 3 parameters non-perturbatively tuned to remove higher order discretization errors;
  - ▶  $b$  quark simulated at its **physical** mass;
- ▶ DWF action for  $s, c$  quarks with **near-to-physical** mass.

# Results and comparison



Key points:

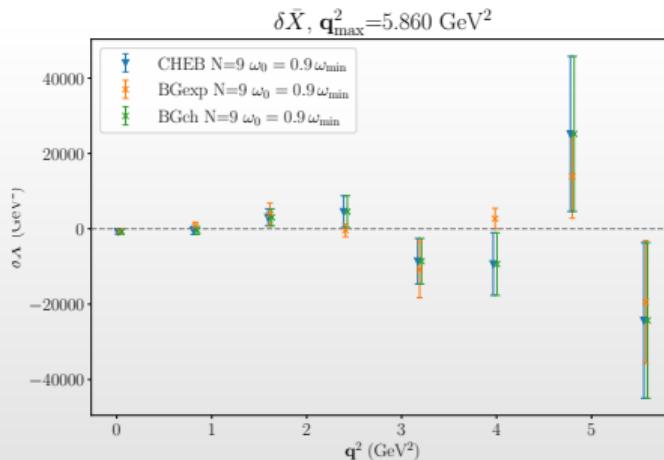
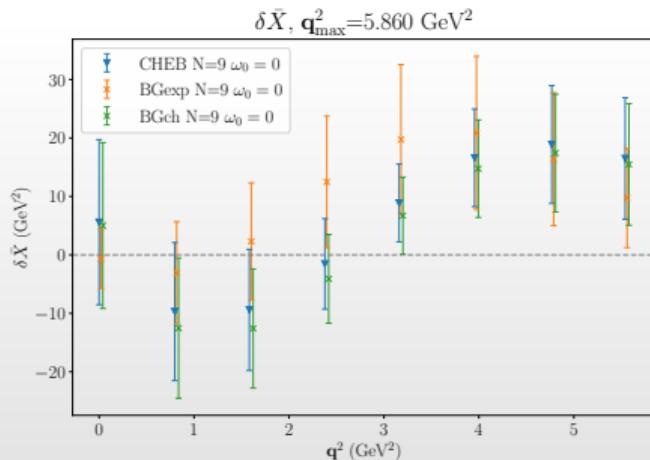
- Chebyshev and Backus-Gilbert approaches are fully compatible;
- pilot study:
  - values are in the right ballpark (compared to  $B$  decay rate, based on  $SU(3)$  flavour symmetry);
  - low statistics, roughly 5 – 10% error.

# First interpretation of the two methods

Recalling  $\bar{X}(\mathbf{q}^2) = C^{\mu\nu}(2t_0)\bar{X}_{\bar{C}\mu\nu}$  we can interpret the two methods as

$$\bar{X}_{\bar{C}\mu\nu} = \bar{X}_{\bar{C}\mu\nu}^{\text{naive}} + \delta\bar{X}_{\bar{C}\mu\nu}, \quad \begin{cases} \delta\bar{X}_{\bar{C}\mu\nu}^{CHEB} &= \sum_{k=0}^N c_{\mu\nu,k} \delta C_{\mu\nu}(k), \\ \delta\bar{X}_{\bar{C}\mu\nu}^{BG} &= \sum_{k=0}^N \delta g_{\mu\nu,k} C_{\mu\nu}(k) \end{cases}$$

i.e. a “naive” piece, where we just blindly apply the polynomial approximation, and a correction term, which is essentially a noisy zero that takes care of the variance reduction.



## Ground state limit

As a useful cross-check of the inclusive analysis, we can consider the limit where only the ground state dominates, i.e.

$$W_{\mu\nu} \rightarrow \delta(\omega - E_{D_s}) \frac{1}{4M_{B_s}E_{D_s}} \langle B_s | J_\mu^\dagger | D_s^{(*)} \rangle \langle D_s^{(*)} | J_\nu | B_s \rangle$$

If we decompose

$$\bar{X} = \bar{X}^{\parallel} + \bar{X}^{\perp}$$

and restrict to the **vector currents (VV)** the matrix element can be decomposed as

$$\langle D_s | V_\mu | B_s \rangle = f_+(q^2)(p_{B_s} + p_{D_s})_\mu + f_-(q^2)(p_{B_s} - p_{D_s})_\mu$$

and we can show that

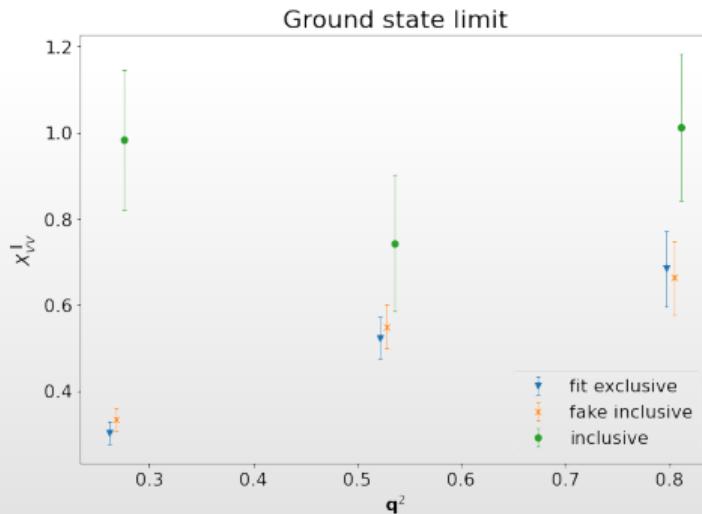
$$\bar{X}_{VV}^{\parallel} \rightarrow \frac{M_{B_s}}{E_{D_s}} \mathbf{q}^2 |f_+(q^2)|^2$$

## Exclusive decay limit

The matrix element and form factors can be extracted from **3pt-correlation functions**. From that we can generate mock data for the 4pt functions (where only the ground state contributes)

$$C_{\mu\nu}^G = \frac{1}{4M_{B_s}E_{D_s}} \langle B_s | V_\mu^\dagger | D_s \rangle \langle D_s | V_\nu | B_s \rangle e^{-E_{D_s}t}$$

and run the analysis!



# Summary and outlook

## Summary:

- ▶ promising prospects for inclusive decays on the lattice;
  - ▶ solid approach for the analysis: Chebyshev and Backus-Gilbert approaches compatible within error.
- ⇒ first publication on the way.

## Coming next:

- ▶ continue to work towards understanding the systematics involved in solving the inverse problem;
  - ▶ dedicated simulations to address the systematics for polynomial approximation, finite volume effects, continuum limit,...;
  - ▶ understand better the ground state limit (compare with form factors);
  - ▶ find better observables to compare with experiments (LHCb, Belle II).
- ⇒ prepare for a full study  $B_s/B$  (and in parallel also  $D_s/D$ ).

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THANK YOU!

# References |

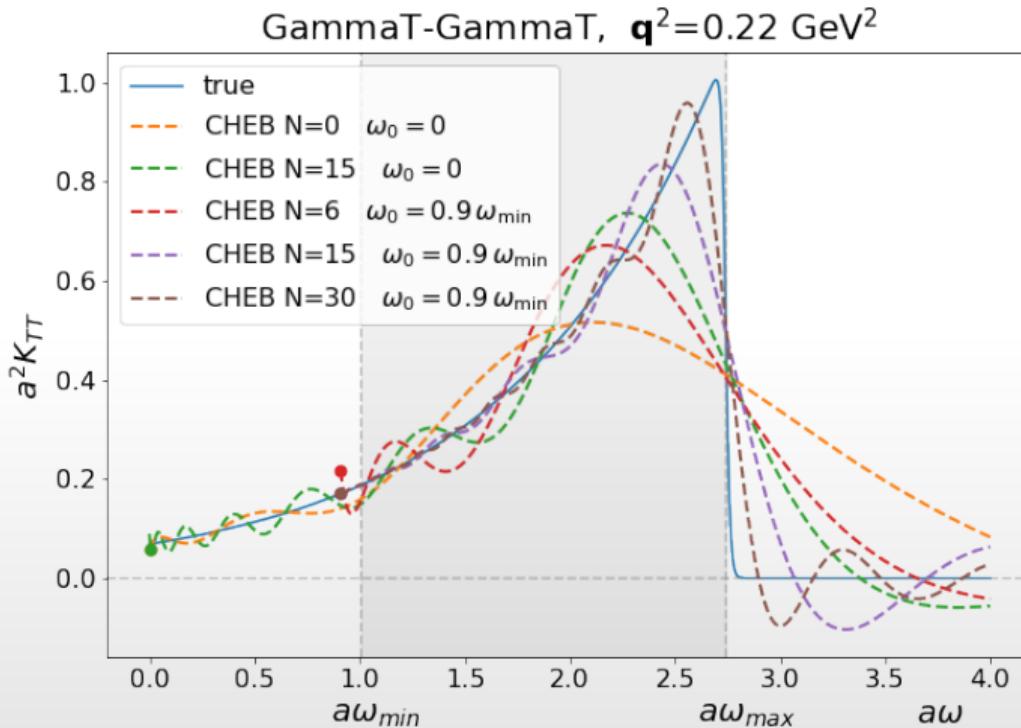
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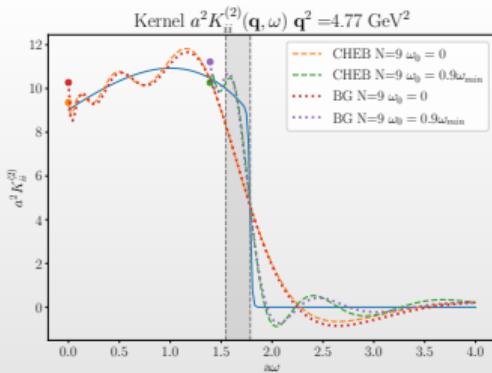
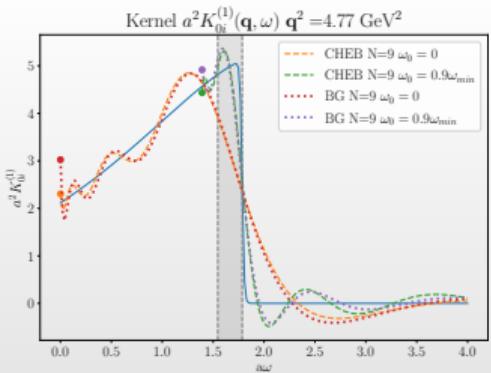
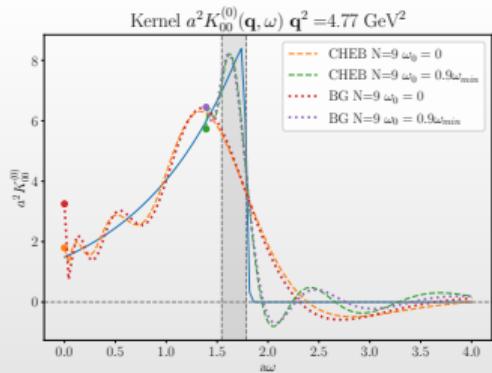
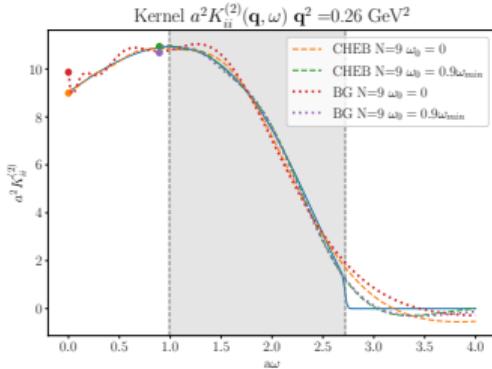
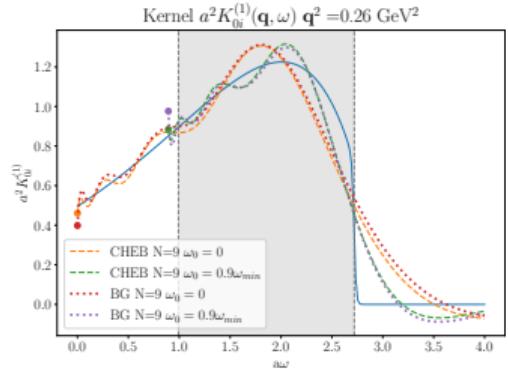
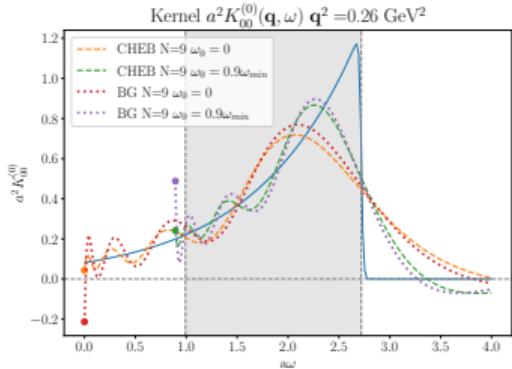
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# BACKUP

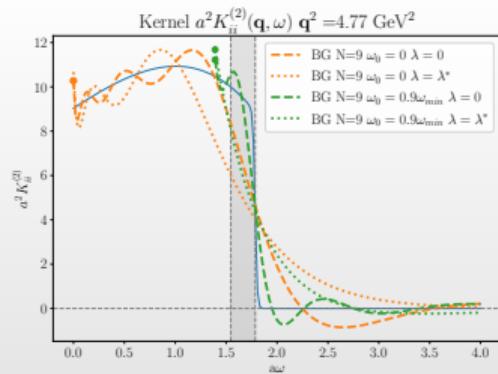
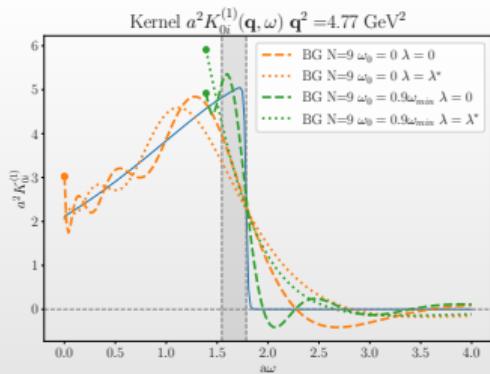
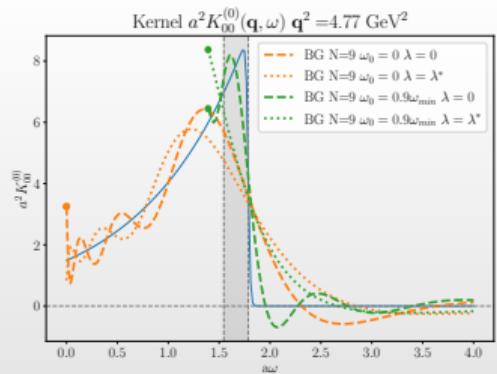
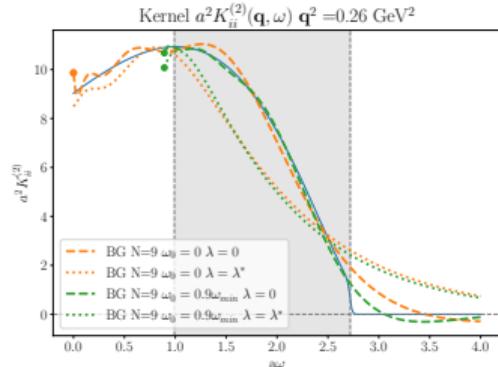
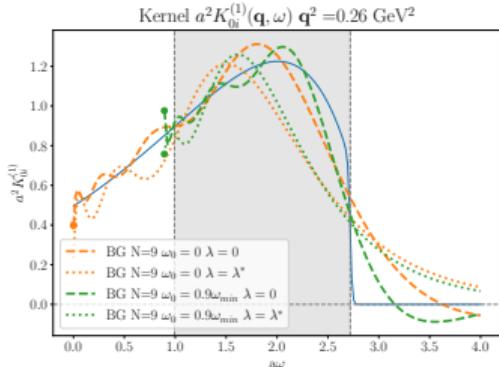
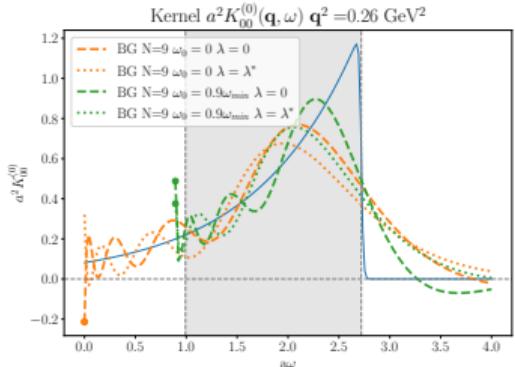
# Chebyshev polynomial approximation: more



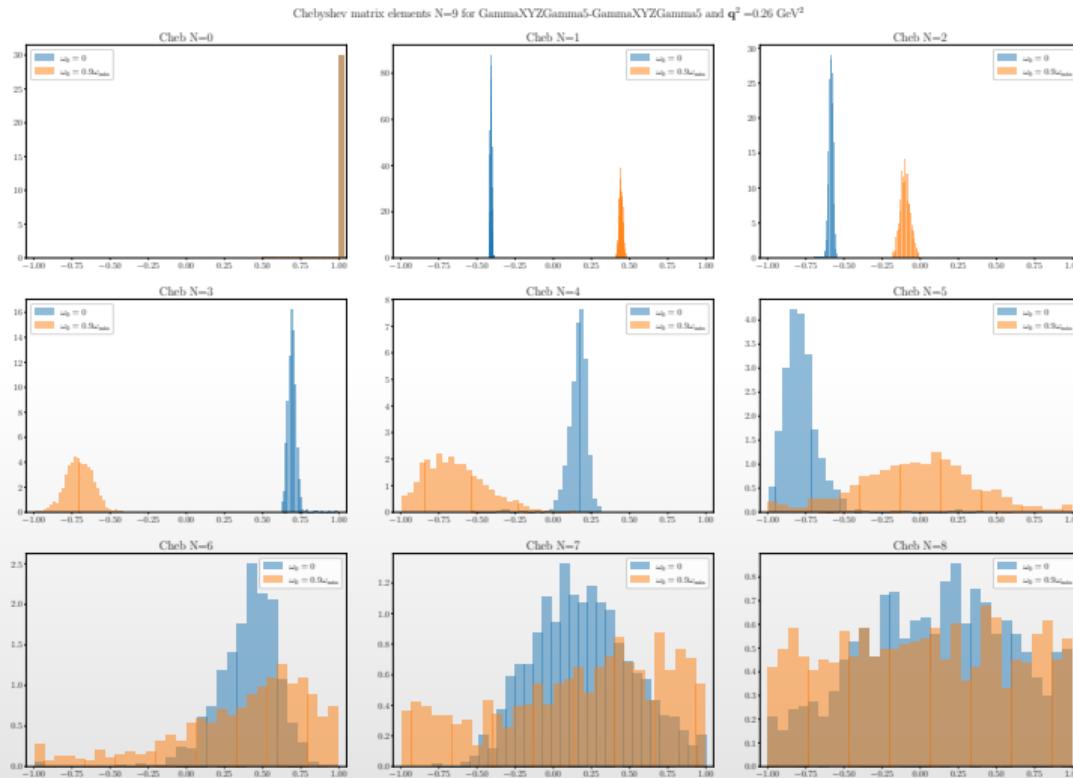
# Kernels



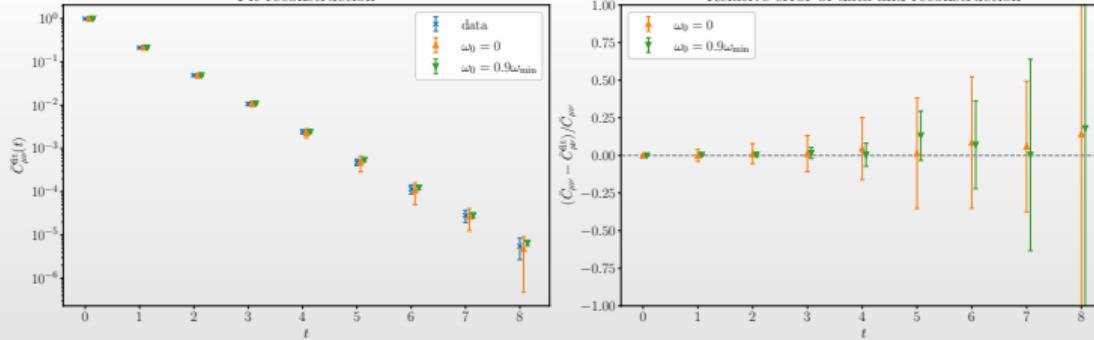
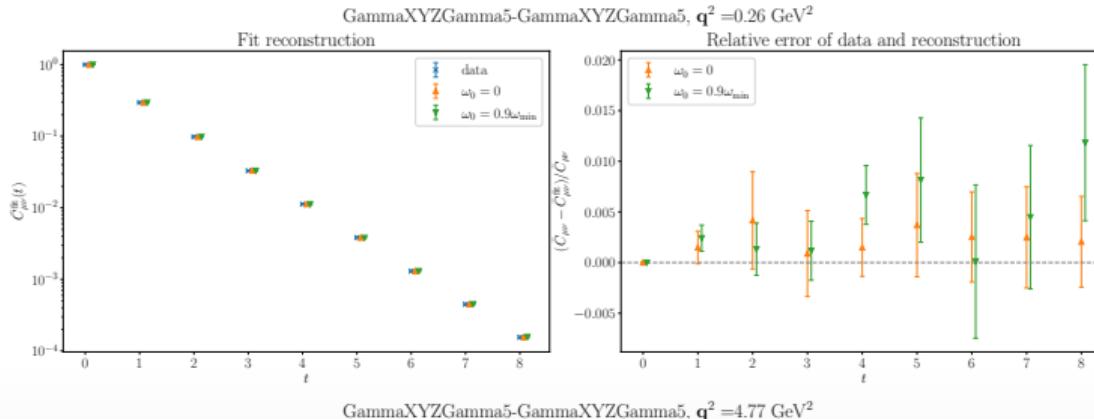
# Kernels Backus-Gilbert



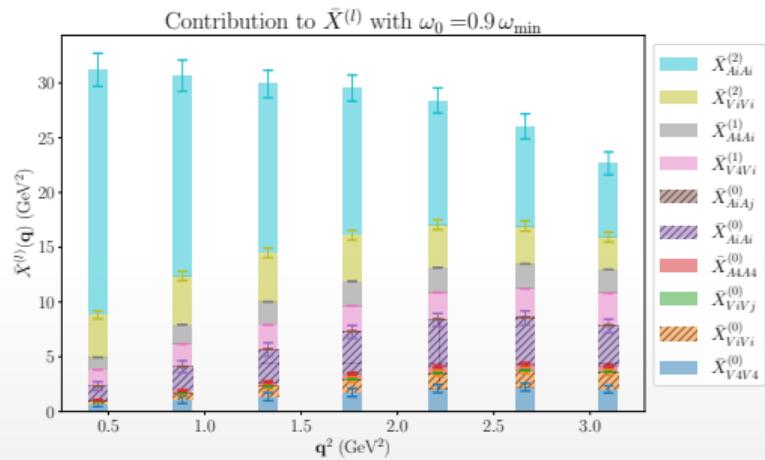
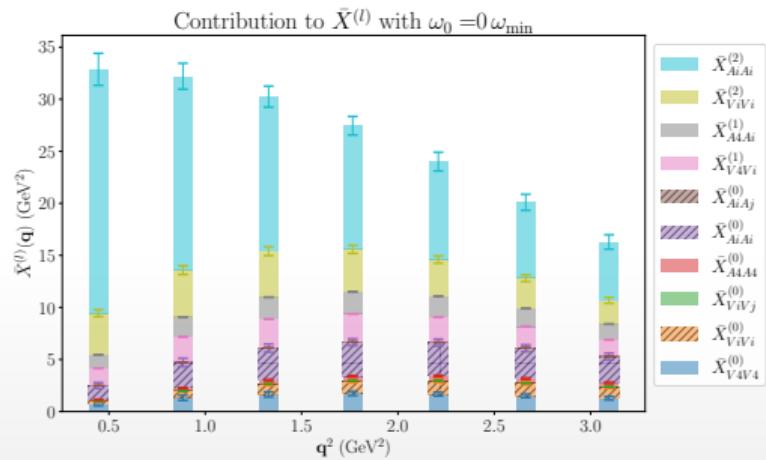
# Chebyshev data reconstruction - distribution



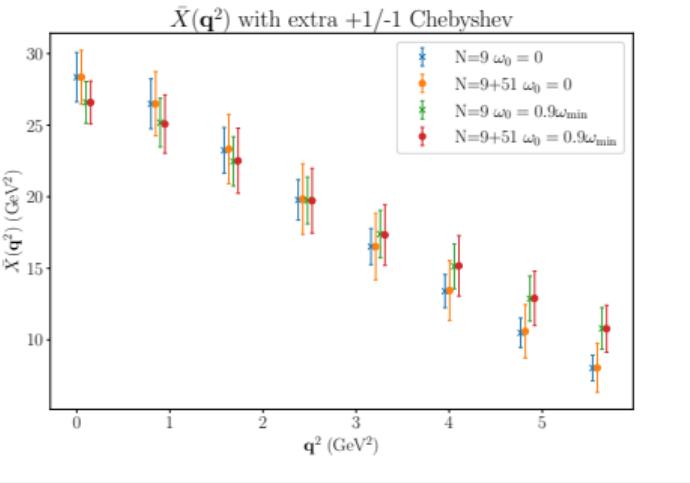
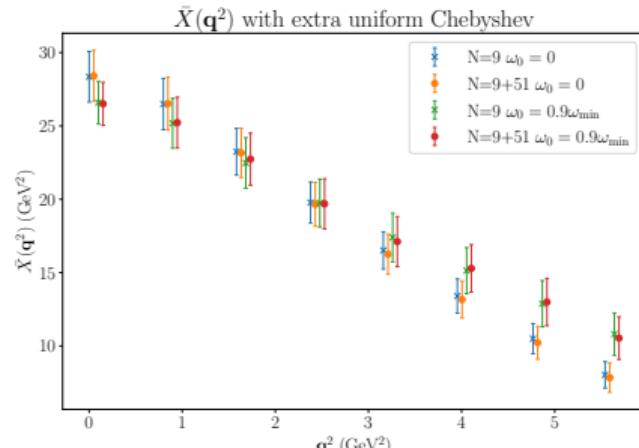
# Chebyshev data reconstruction - data



# $\bar{X}$ contributions



# Systematics from Chebyshev



# Scan over $\lambda$ (Backus-Gilbert)

