# Nonperturbative renormalisation for local composite operators in lattice field theory

**Chris Monahan**

William & Mary

With Anna Hasenfratz, Matthew Rizik, Andrea Shindler, and Oliver Witzel

HU Berlin / NIC DESY Zeuthen Joint Lattice Seminar

WILLIAM & MARY

# Outline

Aim:

1. apply the gradient flow as a nonperturbative renormalisation procedure for local composite operators
2. use relation between gradient flow and real-space renormalisation group to connect low and high energies

1. Renormalisation
2. Block-spin transformation and real-space renormalisation group
3. Gradient flow
4. Our proposal
5. Preliminary numerical results

# Quantum field theories

Quantum field theories are great

- successfully combine quantum mechanics and special relativity
- really successfully describe almost all known subatomic particles and their interactions
- describe many collective phenomena in condensed matter systems
- provided some of the most precise predictions and measurements in the history of Western science

But they come with more than their fair share of issues

- don't describe gravity or explain 80% of the energy density of the universe
- challenging to formulate in a mathematically rigorous way
- strongly-coupled theories generally can't be solved analytically
- quantum fluctuations generate infinities

May not be able to satisfactorily answer the first, but we have the tool to answer the last three - the lattice!

# Running coupling

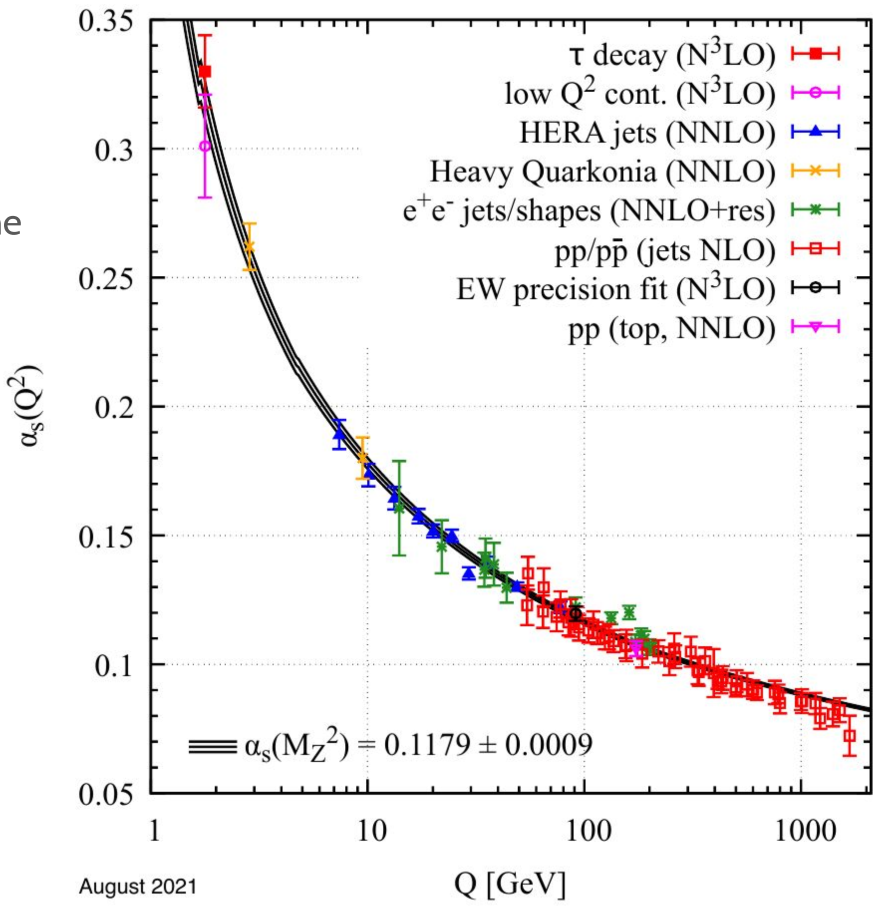
In quantum chromodynamics (QCD), the gauge theory of the strong nuclear force, quantum fluctuations generate the running coupling “constant”

Running is governed by the beta function

$$\beta(\alpha_s) = \mu_R^2 \frac{\mathrm{d}\alpha_s}{\mathrm{d}\mu_R^2} = -\left(b_0\alpha_s^2 + b_1\alpha_s^3 + \cdots\right)$$

Characterised by the QCD Lambda parameter

- fixes overall normalisation
- characterises nonperturbative energy scale at which the strong coupling constant diverges
- fundamental parameter of the standard model




R.L. Workman *et al.* (PDG), PTEP 2022 083C01

# Renormalisation

Quantum fluctuations generate ultraviolet (UV) divergences

According to the textbook story, removing UV divergences is a two-step process

1. Introduce a regulator to remove infinities
2. Apply a renormalisation prescription to enable removal of the regulator

Perturbative calculations typically (but, of course, not always) use

1. Dimensional regularisation
2. MS-bar scheme

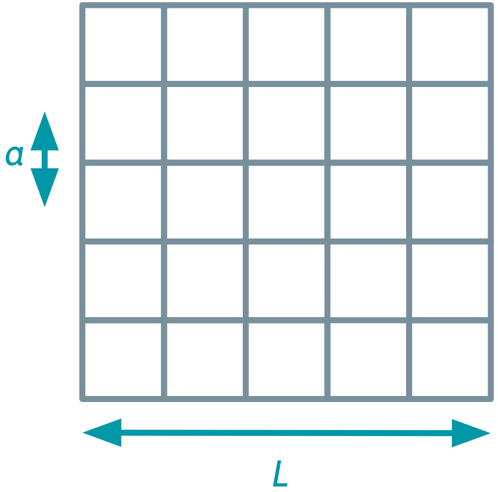
But QCD is nonperturbative at hadronic energies

- Sadly no known nonperturbative implementation of dimensional regularisation
- Introduce a lattice regulator, amenable to nonperturbative (and perturbative) implementation

# The lattice regulator

Lattice field theory is not an approximation

- nonperturbative gauge-invariant ultraviolet (UV) and infrared (IR) regulator




Of course, these advantages come at a cost

- hypercubic lattice breaks many useful symmetries
- finite volume spectra must be connected to infinite volume scattering states
- consistent lattice regularisation of chiral gauge theories remains unclear
- universe does not appear to be discrete at subatomic scales
    - experimental observables must be regulator independent
    - other quantities typically expressed via dimensional regularisation in the MS-bar scheme

One of the “fun” parts of lattice field theory is getting rid of the lattice and connecting to other regularisations

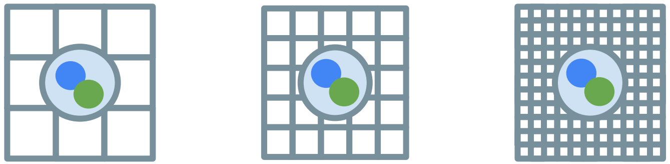
# Renormalisation on the lattice

Nonperturbative renormalisation poses challenges beyond those encountered in perturbation theory

★ Parameters of QCD Lagrangian typically renormalised by tuning physical parameters

$$a_1 m_{u/d}^{(1)} \to a_2 m_{u/d}^{(2)} \to a_3 m_{u/d}^{(3)}$$

$$m_\pi = m_\pi^{\text{phys}}$$



At high precision, this is more challenging than it sounds

- How to define "physical" parameters of QCD in the absence of QED?

See, for example, discussion in "Scale setting", FLAG review 2021

# Renormalisation on the lattice

Nonperturbative renormalisation poses challenges beyond those encountered in perturbation theory

★ Parameters of QCD Lagrangian typically renormalised by tuning physical parameters

★ Composite operators require different approaches

- in some cases, can be fixed by comparison with physical values
- or through symmetries (Ward identities, PCAC...)

★ Ideal scheme:

- Nonperturbative
- Gauge-invariant
- Connect low and high energies
- Amenable to higher order perturbative calculations

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- Connect low and high energies
- Amenable to higher order perturbative calculations

The no-free-lunch theorem applies to all nonperturbative renormalisation schemes (including ours!)

# RI/MOM schemes

"Regularisation independent momentum subtraction" scheme

Martinelli et al., Nucl. Phys. B 445 (1995) 81

- calculate matrix elements of operator between gauge-fixed quark or gluon states
- renormalisation scale directly related to the momentum of the external states

Defined via

$$Z_{\mathcal{O}} \langle p|\mathcal{O}|p\rangle\Big|_{p^2=-\mu^2} = \langle p|\mathcal{O}|p\rangle_{\text{tree}}$$

with various specific choices defining variants of the scheme

Considerations from the no-free lunch theorem:

- Widely used and relatively easy to implement
- Artefacts associated with external state momenta can generally be removed perturbatively
- Breaks gauge invariance (particularly challenging for gluon operators)
- No clear method to move from low to high energies nonperturbatively

Lüscher et al., Nucl. Phys. B 384 (1992) 168
Lüscher et al., Nucl. Phys. B 389 (1993) 247
Sint, Nucl. Phys. B 421 (1994) 135
Jansen et al., Phys. Lett. 372 (1996) 275

# Schrödinger functional

In general:

- "Schrödinger functional" is the propagation kernel for time evolution of fields from time 0 to *T*

For lattice QCD (lattice regularised Schrödinger functional):

- corresponds to a functional integral over all lattice fields that satisfy a specific temporal boundary conditions (classical fields), with periodic spatial boundary conditions
- bulk theory can be probed by inserting fields on the boundaries

For example, combining the PCAC and Schrödinger functional

$$m = \frac{\langle \partial_\mu A_\mu^a(x) O^a \rangle}{2 \langle P^a(x) O^a \rangle}$$

with probe operators defined through boundary quark fields

Considerations from the no-free lunch theorem:

- Gauge invariant
- Amenable to step-scaling from low to high energies
- Enables full control of all uncertainties
- Not so widely used and not so easy to implement

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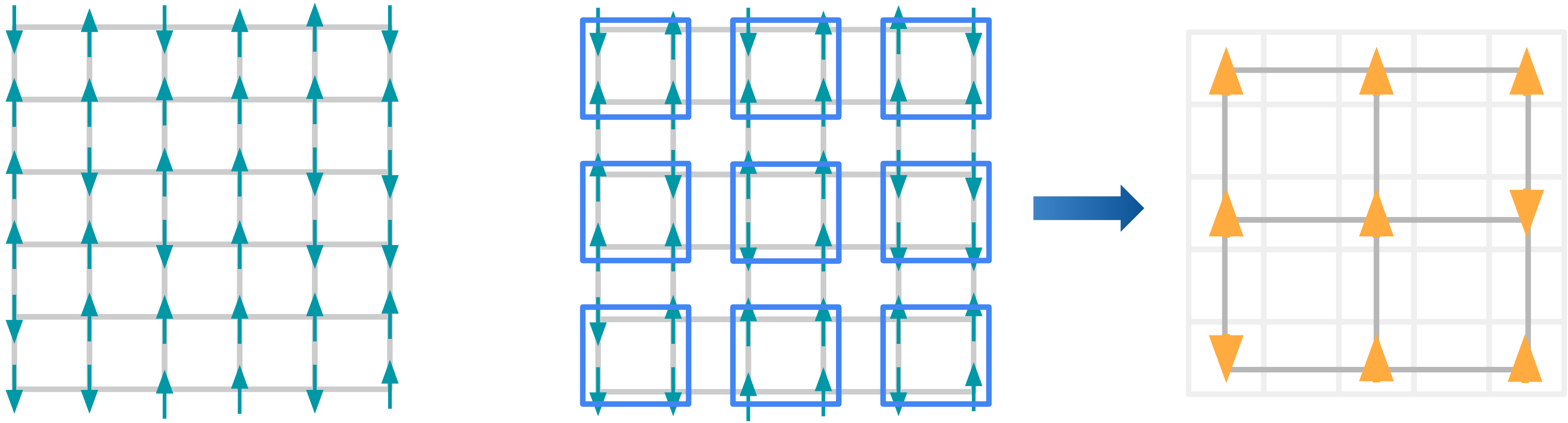
# The block-spin transformation

Intuitive real-space technique from statistical mechanics and low-dimensional spin models

- Originally applied to two-dimensional Ising model near criticality

Kadanoff, Phys Phys Fiz 2 (1966) 263



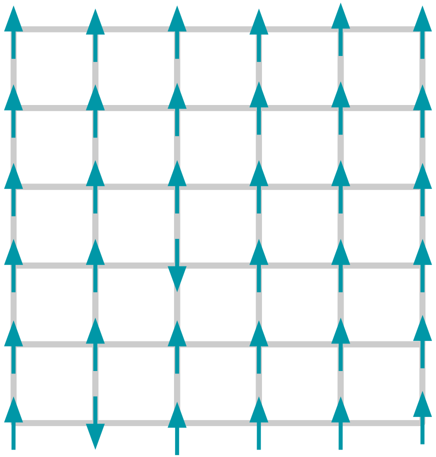
In principle, one can iterate to a single blocked variable - captures long-range correlations

# Fixed points

Real-space renormalisation group flow generated by combining block-spin transformation with a coarse-graining procedure that rescales couplings

- Study dependence of physics (e.g. couplings) along RG flow trajectory

The macroscopic configurations that can be reached from different microscopic configurations are fixed points



For example, in the Ising model with temperature $T$ and (running) coupling $J$

1. $T = 0$ and $J \rightarrow$ infinity - ferromagnetic/ordered phase
2. $T \rightarrow$ infinity and $J = 0$ - thermal/disordered phase
3. $T = T_c$ and $J = J_c$ - critical point with scale-invariant physics

# Scaling, operators and universality

Operators evolve along the renormalisation group trajectory towards a fixed point

1. Operators that always increase - relevant
2. Operators that always decrease - irrelevant
3. Operators that don't do either - marginal

Relevant operators capture the important physics at macroscopic length scales

Near a fixed point, scaling behaviour captured by critical exponents (anomalous dimensions in QFT language)

Different models with the same critical exponents fall into universality classes, with shared relevant operators

- They all behave the same near phase transitions, independent of the microscopic details of the system

A lattice example: lattice artefacts that vanish in the continuum limit are described by irrelevant operators

# Connecting block-spin and QFT

Block-spin transformation provides a real-space averaging or blocking procedure

- leaves partition function invariant
- modifies parameters of the action and expectation values of operators

$$\boxed{\begin{gathered}\Phi_b(x_b) = f(\phi) \\ g_i \to g_i^{(b)}\end{gathered}}$$

In vicinity of a fixed point, scaling operators and two-point functions transform as

$$\Phi_b(x_b) = b^{-d_\phi - \eta/2}\phi(bx_b;\tau)$$

$$\widetilde{\mathcal{O}} = b^{d+\gamma}\mathcal{O} \qquad G_{\mathcal{O}}(g_i, x_0) = b^{-2(d+\gamma)}G_{\mathcal{O}}(g_i^{(b)}, x_0/b)$$

$$\boxed{\begin{gathered}G_{\mathcal{O}}(g_i, x_0) = \langle \mathcal{O}(0)\mathcal{O}(x_0)\rangle \\ x_0 \gg b\end{gathered}}$$

Provides definition of the anomalous dimension of the operator/two-point function

$$\Delta_{\mathcal{O}}(g_i^{(b)}) = b\frac{\mathrm{d}}{\mathrm{d}b}\log G_{\mathcal{O}}(g_i^{(b)}, x_0/b)$$

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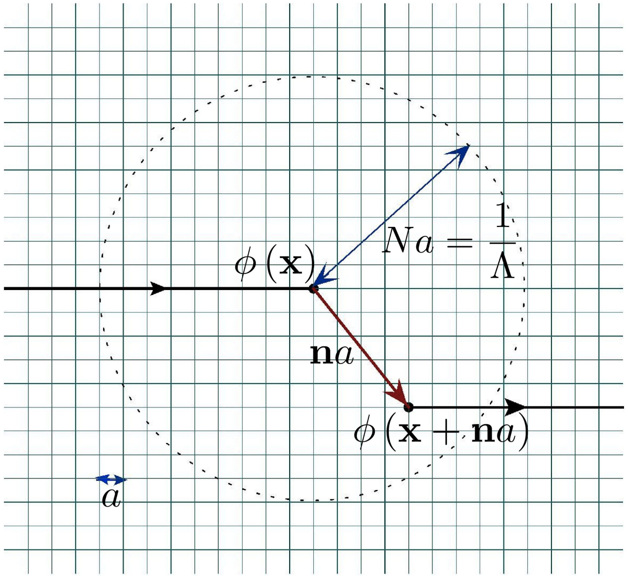
# Smearing

Smearing partially restores rotational symmetry: widely-used lattice technique

- construct operators with improved continuum limits, *i.e.* reduced systematic uncertainties
- suppresses operator mixing
- precisely identify hadronic excited states
- reduce statistical noise



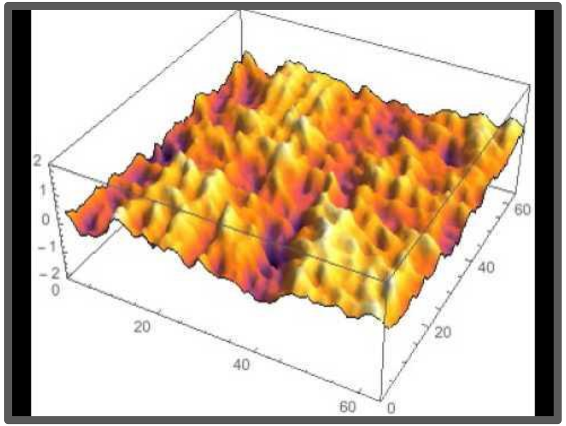

Davoudi & Savage, Phys. Rev. D 86 (2012) 054505

# Gradient flow smearing

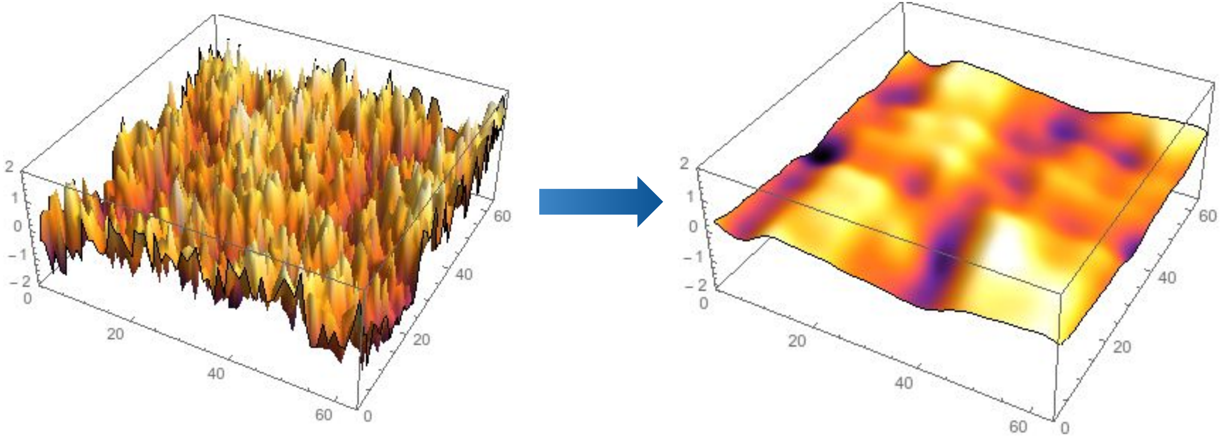
Narayanan & Neuberger, JHEP 0603 064
Lüscher, JHEP 1008 071
Lüscher, JHEP 04 (2013) 123

Gradient flow: deterministic evolution of fields in “flow time” $\tau$ toward classical minimum

Evolution in flow time corresponds to exponential damping of UV modes




Monahan, POS(LATTICE205) 052



Narayanan & Neuberger, JHEP 0603 064
Lüscher, JHEP 1008 071
Lüscher, JHEP 04 (2013) 123

# Gradient flow smearing

Gradient flow: deterministic evolution of fields in “flow time” $\tau$ toward classical minimum

$$\frac{\partial}{\partial\tau}B_\mu(\tau,x) = D_\nu\Big(\partial_\nu B_\mu - \partial_\mu B_\nu + [B_\nu, B_\mu]\Big) \qquad D_\nu = \partial_\nu + [B_\nu, \cdot]$$

$$\frac{\partial}{\partial\tau}\chi(\tau,x) = D_\mu D^\mu \chi(\tau,x) \qquad D_\mu = \partial_\mu + B_\mu$$

Dirichlet boundary conditions

$$B_\mu(\tau=0,x) = A_\mu(x) \qquad \chi(\tau=0,x) = \psi(x)$$

Can be implemented on the lattice and solved nonperturbatively

$$\frac{\partial}{\partial\tau}V_\mu(\tau,x) = -g_0^2\left\{\partial_{x,\mu}S\big[V_\mu(\tau,x)\big]\right\}V_\mu(\tau,x)$$

# Gradient flow smearing

Gradient flow: deterministic evolution of fields in "flow time" $\tau$ toward classical minimum

Solving the flow equations at leading order

$$\widetilde{B}_\mu(p) = e^{-p^2\tau}\widetilde{A}_\mu(p) + \mathcal{O}(g) \qquad B_\mu|_{\tau=0} = A_\mu$$

$$\widetilde{\chi}(p) = e^{-p^2\tau}\widetilde{\psi}(p) + \mathcal{O}(g) \qquad \chi|_{\tau=0} = \psi$$

Provides controlled, continuous smearing

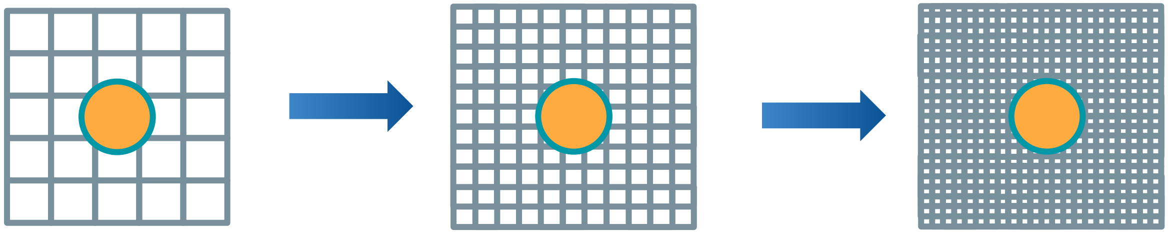
- Gauge invariant
- Nonperturbative
- Renormalised correlation functions remain finite, up to a multiplicative wavefunction renormalisation

Lüscher & Weisz, JHEP 02 (2011) 051
Lüscher, JHEP 04 (2013) 123

# Gradient flow and renormalisation

One approach

- Take continuum limit at fixed flow time in physical units



- Apply small flow-time expansion to relate flowed operators to operators in another scheme

Lüscher, JHEP 08 (2010) 071

Particularly powerful for power-divergent operators (diverge as inverse powers of the lattice spacing)

- Effective CP-violating operators relevant to neutron EDM — Schindler, de Vries & Luu, PoS(LATTICE2014) 251
- Extended Wilson-line operators relevant to x-dependent hadron structure — Monahan & Orginos, JHEP 03 (2017) 116

Alternatively, one can expand renormalised operators in terms of flowed operators

Monahan & Orginos, PRD 91 (2015) 074513
Harlander & Neumann, JHEP 08 (2020) 109

# Gradient flow and real-space renormalisation group

Recall that the block-spin transformation provides a real-space averaging or blocking procedure

- leaves partition function invariant
- modifies parameters of the action and expectation values of operators
- provides definition of the anomalous dimension of operators

Gradient flow provides natural tool for blocking - smears fields over a region ~ $\sqrt{8\tau}$

$$\Phi_b(x_b) = b^{-d_\phi - \eta/2}\phi(bx_b;\tau)$$

Carosso, Hasenfratz & Neil, PRL 121 (2018) 201601

Note:

- Gradient flow is not a renormalisation group transformation (no coarse-graining step)
- Coarse-graining can be included at the level of expectation values

Hasenfratz, Rebbi & Witzel, PRD 106 (2022) 114509
Peterson et al., PoS(LATTICE2021) 174
Hasenfratz & Witzel, PRD 101 (2020) 034514

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# Renormalisation scheme: some preliminaries

Define the bare two-point functions

$$G_{\mathcal{O}}(x_4;\tau) = \int \mathrm{d}^3\mathbf{x}\,\langle \mathcal{O}(\mathbf{x},x_4;\tau)P_{\mathcal{O}}(x)\rangle$$

$$G_V(x_4;\tau) = \frac{1}{3}\sum_{j=1}^{3}\int \mathrm{d}^3\mathbf{x}\,\langle V_j(\mathbf{x},x_4;\tau)P_V(x)\rangle$$

and introduce the bare double ratio

$$\overline{R}_{\mathcal{O}}(x_4;\tau) = \frac{R_{\mathcal{O}}(x_4;\tau=0)}{R_{\mathcal{O}}(x_4;\tau)}$$

$$R_{\mathcal{O}}(x_4;\tau) = \frac{G_{\mathcal{O}}(x_4;\tau)}{G_V(x_4;\tau)}$$

which renormalises as

$$\overline{R}^{\mathrm{R}}_{\mathcal{O}}(x_4;\tau) = \frac{Z_{\mathcal{O}}}{Z_V}\overline{R}_{\mathcal{O}}(x_4;\tau)$$



c.f. Monahan & Orginos, arXiv:1311.2310

# Renormalisation scheme

Define the gradient flow scheme by imposing the renormalisation condition

$$\overline{Z}_{\mathcal{O}}^{\mathrm{GF}}(\mu)\overline{R}_{\mathcal{O}}(x_4;\tau)\Big|_{\substack{\mu^2\tau=c\\ x_4^2\gg\tau/c}} = \overline{R}_{\mathcal{O}}^{(\mathrm{tree})}(x_4;\tau)$$

$$\overline{R}_{\mathcal{O}}(x_4;\tau) = \frac{R_{\mathcal{O}}(x_4;\tau=0)}{R_{\mathcal{O}}(x_4;\tau)}$$

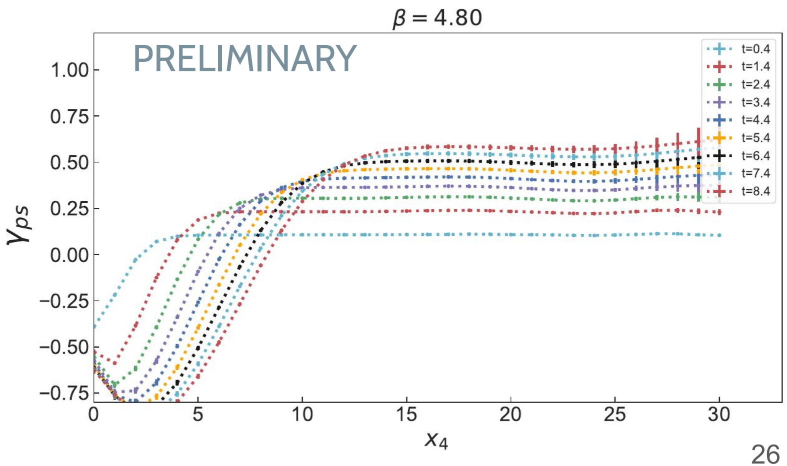
$$R_{\mathcal{O}}(x_4;\tau) = \frac{G_{\mathcal{O}}(x_4;\tau)}{G_V(x_4;\tau)}$$

which allows us to extract

$$\overline{Z}_{\mathcal{O}}^{\mathrm{GF}}(\mu) = \frac{\overline{R}_{\mathcal{O}}^{(\mathrm{tree})}(x_4;\tau)}{\overline{R}_{\mathcal{O}}(x_4;\tau)}\Bigg|_{\substack{\mu^2\tau=c\\ x_4^2\gg\tau/c}}$$

We further define the (nonperturbative) anomalous dimension

$$\gamma_{\mathcal{O}} = -2\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\log Z_{\mathcal{O}}^{\mathrm{GF}}(\mu)$$



# Procedure: a schematic outline

1. Calculate the renormalisation parameter nonperturbatively

$$\overline{Z}_{\mathcal{O}}^{\mathrm{GF}}(\mu) = \left.\frac{\overline{R}_{\mathcal{O}}^{(\mathrm{tree})}(x_4;\tau)}{\overline{R}_{\mathcal{O}}(x_4;\tau)}\right|_{\substack{\mu^2\tau = c \\ x_4^2 \gg \tau/c}}$$

2. Calculate the anomalous dimension nonperturbatively

$$\gamma_{\mathcal{O}} = -2\tau \frac{\mathrm{d}}{\mathrm{d}\tau} \log Z_{\mathcal{O}}^{\mathrm{GF}}(\mu)$$

to move from low to high scales

3. Match to the MS-bar scheme using perturbation theory

$$\overline{R}_{\mathcal{O}}(x_4;\tau) = \frac{R_{\mathcal{O}}(x_4;\tau=0)}{R_{\mathcal{O}}(x_4;\tau)}$$

$$R_{\mathcal{O}}(x_4;\tau) = \frac{G_{\mathcal{O}}(x_4;\tau)}{G_V(x_4;\tau)}$$

# Procedure: in a little more detail

1. Calculate bare two-point functions at fixed bare coupling
2. Calculate renormalisation parameter at fixed bare coupling

$$\overline{Z}_{\mathcal{O}}^{\mathrm{GF}}(\mu) = \left.\frac{\overline{R}_{\mathcal{O}}^{(\mathrm{tree})}(x_4;\tau)}{\overline{R}_{\mathcal{O}}(x_4;\tau)}\right|_{\substack{\mu^2\tau = c \\ x_4^2 \gg \tau/c}}$$

3. Calculate anomalous dimension
   a. Compute ratio of two-point functions at fixed bare coupling

$$R_{\mathcal{O}}(\tau) = \left.\frac{G_{\mathcal{O}}(x_4;\tau)}{G_V(x_4;\tau)}\right|_{x_4^2 \gg \tau}$$

   b. Determine numerical logarithmic derivative at specific flow time (fixes renormalised coupling)

$$\gamma_{\mathcal{O}}^{\mathrm{GF}}\left(g_{\mathrm{GF}}^2(\tau,\beta)\right) = -\frac{2\tau}{\epsilon}\log\frac{R_{\mathcal{O}}(\tau+\epsilon)}{R_{\mathcal{O}}(\tau)}$$

   c. Take chiral limit at fixed bare coupling and infinite volume limit at fixed renormalised coupling
   d. Take continuum limit as limit of infinite flow time at fixed renormalised coupling

# Procedure: in a little more detail

1. Calculate renormalisation parameter at fixed bare coupling

2. Calculate anomalous dimension

3. Determine the beta function on the same ensembles

$$\beta = -\tau \frac{\mathrm{d}g^2_{\mathrm{GF}}}{\mathrm{d}\tau} \qquad g^{\mathrm{GF}}(\tau/a^2, \beta) = \mathcal{N}\tau^2 \langle G_{\mu\nu}(\tau) G^{\mu\nu}(\tau)\rangle \qquad \mathcal{N} = \frac{32\pi^2}{3(N_c^2-1)}$$

4. Numerically integrate renormalisation group equations to relate low and high energy regimes

$$\frac{\overline{Z}^{\mathrm{GF}}_{\mathcal{O}}(\mu_{\mathrm{UV}})}{\overline{Z}^{\mathrm{GF}}_{\mathcal{O}}(\mu_{\mathrm{IR}})} = \exp\left[\int_{g(\mu_{\mathrm{IR}})}^{g(\mu_{\mathrm{UV}})} \mathrm{d}g' \, \frac{\gamma^{\mathrm{GF}}_{\mathcal{O}}(g')}{\beta^{\mathrm{GF}}_{\mathcal{O}}(g')}\right]$$

5. Match results in high energy regime to the MS-bar scheme at fixed order in perturbation theory

$$c^{\overline{\mathrm{MS}}\leftarrow\mathrm{GF}}(\mu_{UV}) = \exp\left\{-\left[\int_0^{\bar{g}_{\mathrm{UV}}} dg' \, \frac{\gamma^{\mathrm{GF}}_{\mathcal{O}}(g')}{\beta^{\mathrm{GF}}(g')} - \int_0^{g_{\mathrm{UV}}} dg' \, \frac{\gamma^{\overline{\mathrm{MS}}}_{\mathcal{O}}(g')}{\beta^{\overline{\mathrm{MS}}}(g')}\right]\right\} \qquad c^{\overline{\mathrm{MS}}\leftarrow\mathrm{GF}}(\mu_{\mathrm{UV}}) = \left(\frac{\bar{g}^2}{g^2}\right)^{-\gamma^{(0)}_{\mathcal{O}}/2b_0}$$

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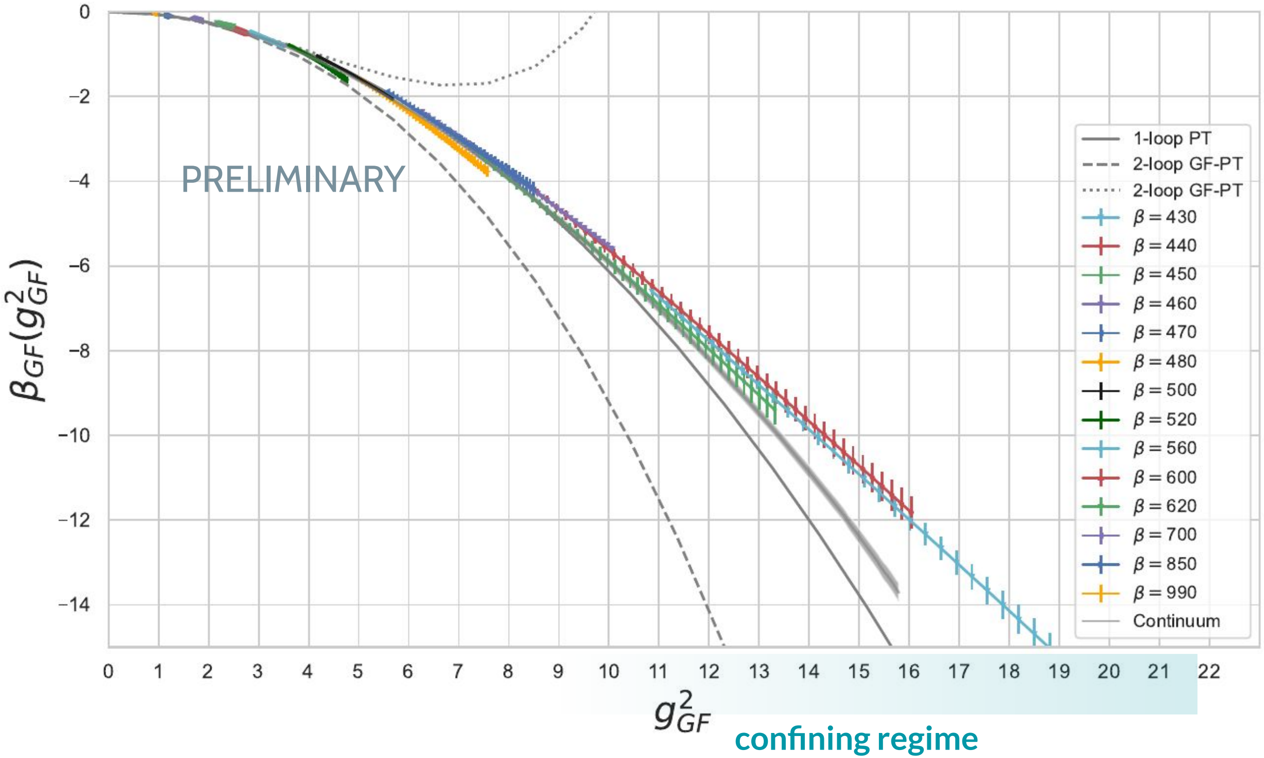
# Preliminary nonperturbative results: pseudoscalar anomalous dimension

Demonstrate the nonperturbative procedure

- Tree-level improved Symanzik gauge configurations with $N_f = 2$ stout-smeared Möbius DWF
- Bare couplings $4.40 < \beta < 9.90$
- "Large volume" simulations at $24^3$x64 with $\beta = 4.40, 4.50$ ($am_q = 0.005, 0.010$) and 4.70 ($am_q = 0.010$)
- "Small volume" simulations at $24^3$x64 and $32^3$x64
- Apply Wilson kernel for gradient flow

# Preliminary nonperturbative results: beta function

Perturbation theory:
Shrock & Rytov, PRD 83 (2011) 056011 and refs. within
Harlander & Neumann, JHEP 06 (2016) 161




c.f nf = 0 study by Schierholz & Nakamura 2201.12875

c.f nf = 2 Hasenfratz & Witzel, PRD 101 (2020) 034514

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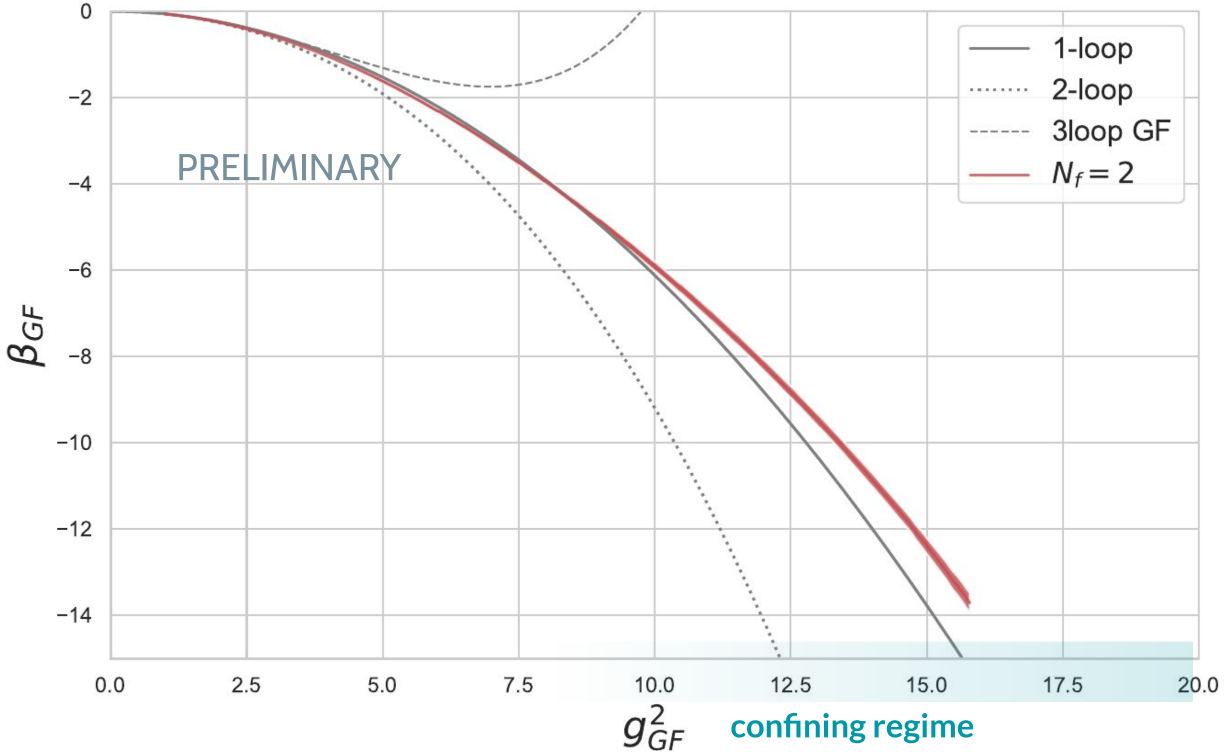


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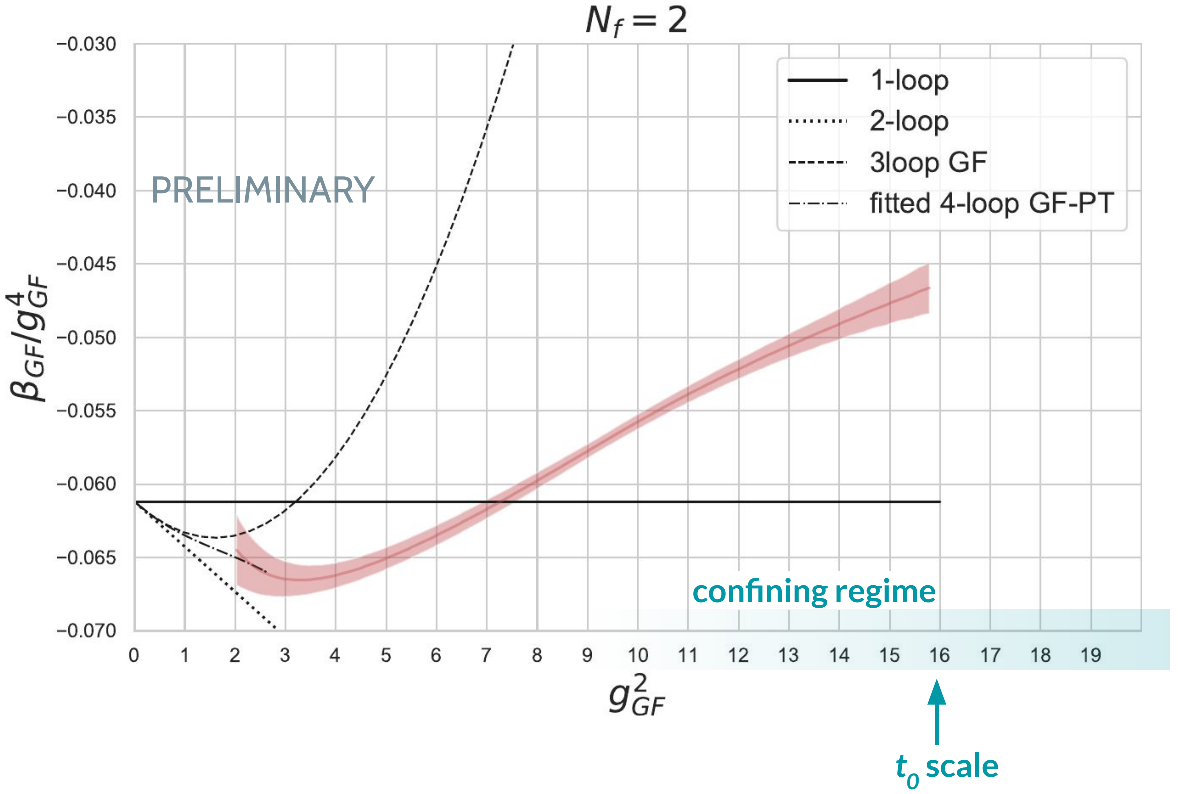



Gradient flow scheme known to be less than optimal in weak coupling regime

See Fodor et al., JHEP 1211 (2012) 007
and, e.g. Bruno et al., PRL 119 (2017) 102001

# $N_f = 2$ Lambda parameter

Reminder: Lambda parameter is a fundamental parameter of QCD

- characterises the nonperturbative energy scale at which the strong coupling constant diverges
- “fixes” the normalisation of the running coupling
- generated by dimensional transmutation
- dominant error in theoretical uncertainty in value of strong coupling constant at $M_Z$

Our nonperturbative calculation of the beta function gives access to the Lambda parameter

$$\Lambda_{\mathrm{QCD}} = \mu \cdot \frac{e^{-1/(2b_0 g_s^2(\mu))}}{(b_0 g_s^2(\mu))^{b_1/(2b_0^2)}} \exp\left[-\int_0^{g_s(\mu)} \mathrm{d}x \left(\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right)\right]$$

$$\beta(x) \sim -b_0 x^3 - b_1 x^5 + \cdots$$

Our preliminary calculation provides proof-of-principle results

# Preliminary nonperturbative results: Lambda parameter

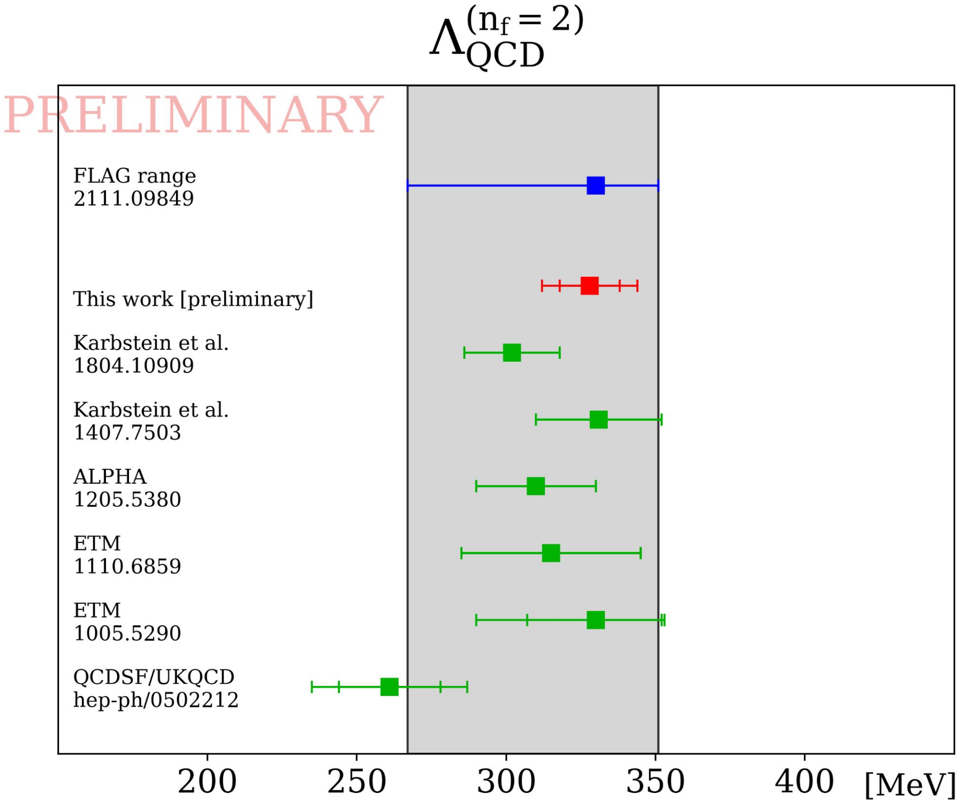
We obtain our **preliminary** value

$$\Lambda_{\mathrm{QCD}}^{(n_f=2)} = 328(13)_{\mathrm{stat.}}(10)_{\mathrm{sys.}}\ \mathrm{MeV}$$

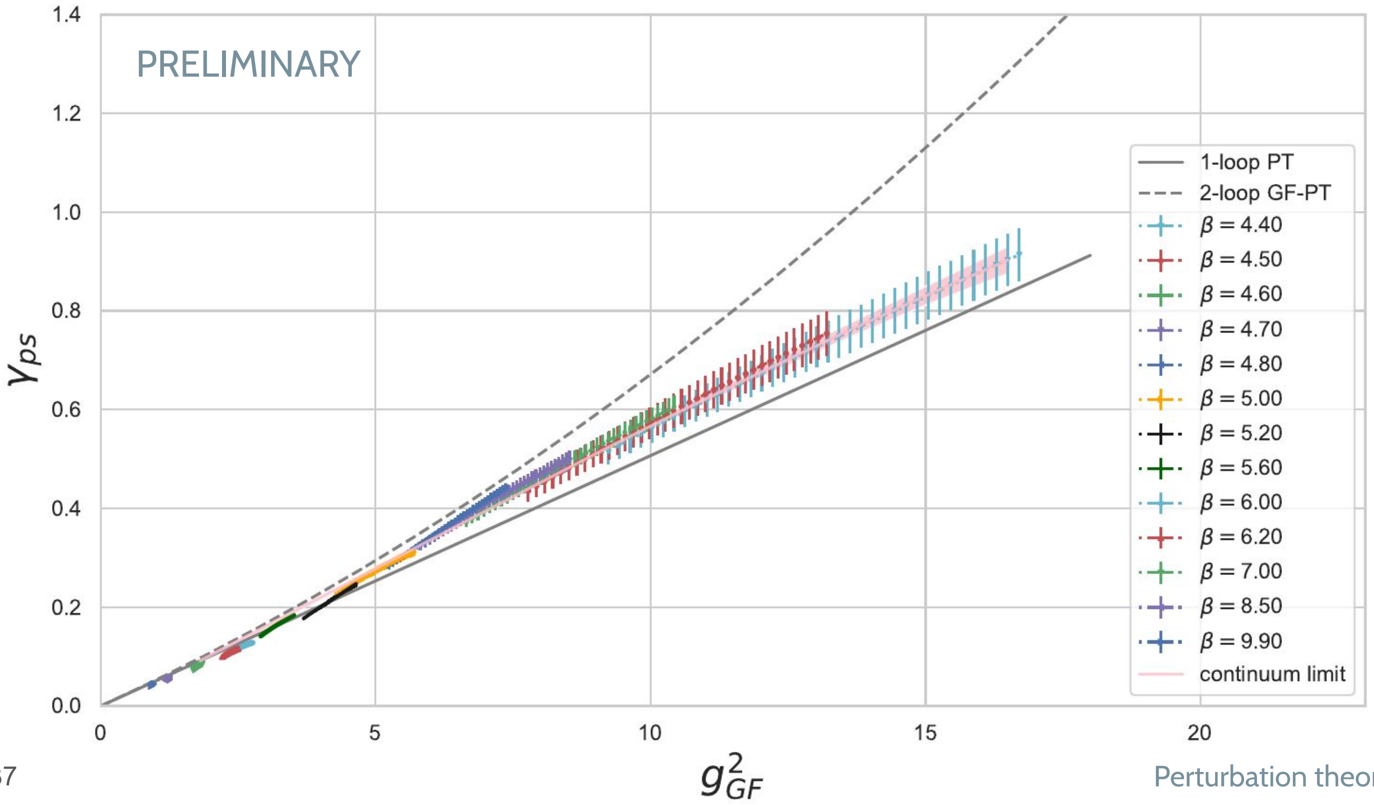
Note:

- Systematic uncertainties dominated by weak coupling regime, where the gradient flow scheme has largest statistical uncertainties
- We do not include (likely small) systematic contributions from the chiral and infinite volume extrapolations

N.B. FLAG assume $r_0$ = 0.472 fm if no $r_0$ scale giv

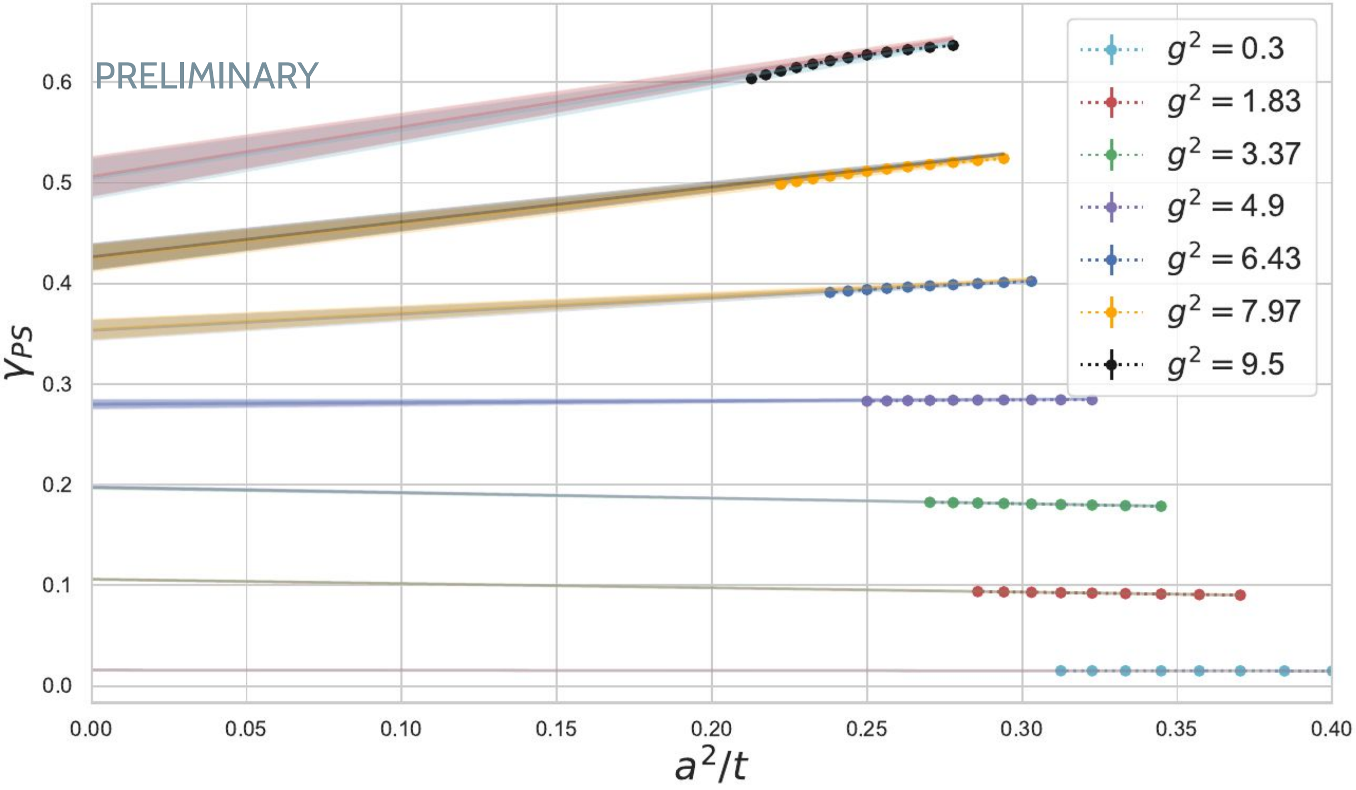


# Preliminary nonperturbative results: pseudoscalar anomalous dimension

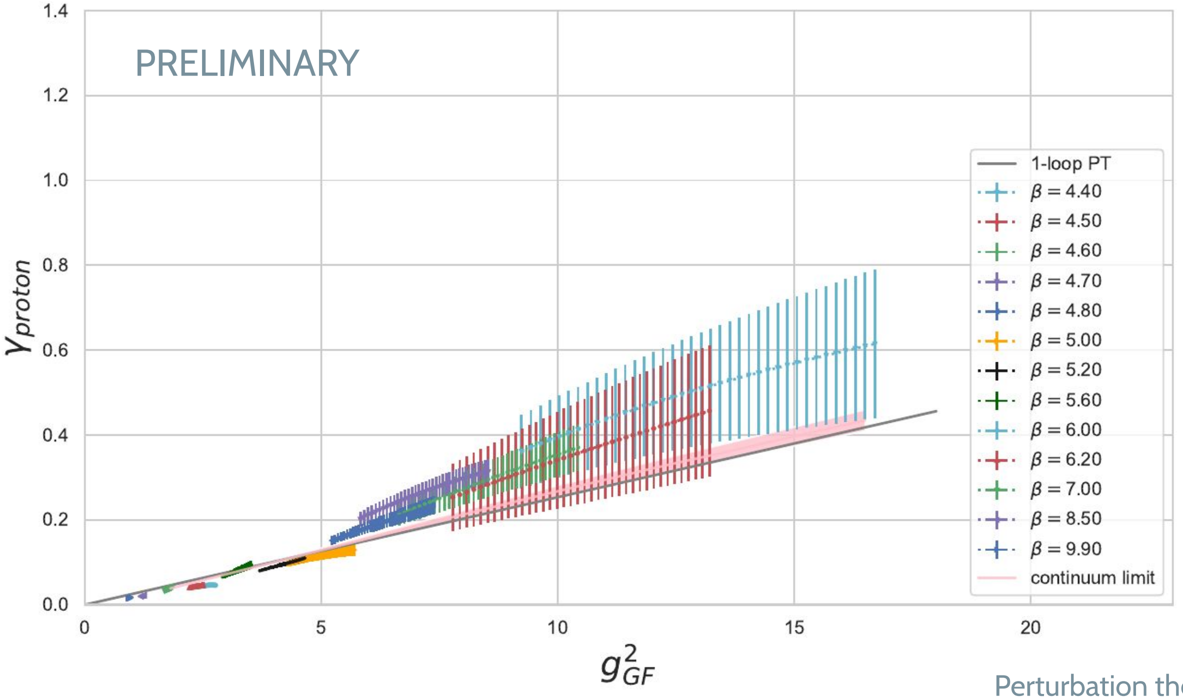



Perturbation theory: Artz et al., JHEP 06 (2019) 121

# Continuum extrapolation



# Preliminary nonperturbative results: proton anomalous dimension




Perturbation theory: Gracey et al., PRD 97 (2018) 116018

N.B. Running of three-quark operators relevant to matching proton decay calculations to phenomenology

# Perturbative matching

Our aim is to match correlation functions in the gradient flow scheme to the MS-bar scheme

$$c^{\overline{\mathrm{MS}}\leftarrow\mathrm{GF}}(\mu_{UV}) = \exp\left\{-\left[\int_0^{\bar{g}_{\mathrm{UV}}} dg' \, \frac{\gamma_{\mathcal{O}}^{\mathrm{GF}}(g')}{\beta^{\mathrm{GF}}(g')} - \int_0^{g_{\mathrm{UV}}} dg' \, \frac{\gamma_{\mathcal{O}}^{\overline{\mathrm{MS}}}(g')}{\beta^{\overline{\mathrm{MS}}}(g')}\right]\right\}$$

At leading order this is

$$c^{\overline{\mathrm{MS}}\leftarrow\mathrm{GF}}(\mu_{\mathrm{UV}}) = \left(\frac{\bar{g}^2}{g^2}\right)^{-\gamma_{\mathcal{O}}^{(0)}/2b_0}$$

$$b_0 = \frac{1}{(4\pi)^2}\left[\frac{11}{3}N_c - \frac{2}{3}N_f\right]$$

$$b_1 = \frac{1}{(4\pi)^4}\left[\frac{34}{3}N_c^2 - \left(\frac{13}{3}N_c - \frac{1}{N_c}\right)N_f\right]$$

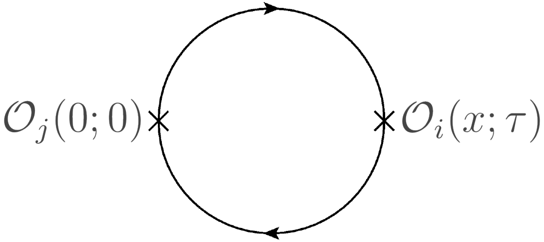
and at next-to-leading order this becomes

$$c^{\overline{\mathrm{MS}}\leftarrow\mathrm{GF}}(\mu_{UV}) = \left(\frac{\bar{g}^2}{g^2}\right)^{-\gamma_{\mathcal{O}}^{(0)}/2b_0}\left[\left(\frac{b_0+b_1\bar{g}^2}{b_0+b_1 g^2}\right)^{-\gamma_{\mathcal{O}}^{(0)}/2b_0}\frac{(b_0+b_1\bar{g}^2)^{\gamma_{\mathcal{O}}^{\mathrm{GF},(1)}/2b_1}}{(b_0+b_1 g^2)^{\gamma_{\mathcal{O}}^{\overline{\mathrm{MS}},(1)}/2b_1}}\right]$$

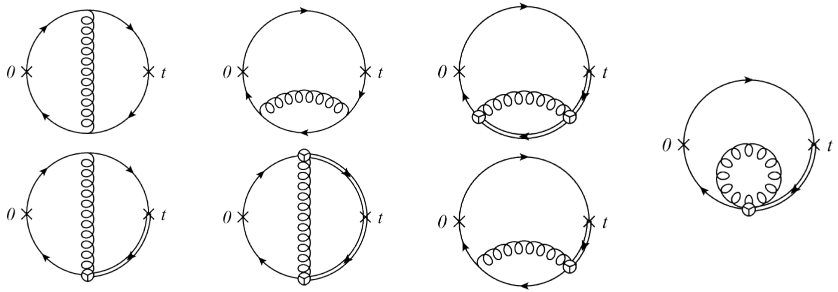
# Perturbative analysis

Perturbative diagrams that need to be calculated are

**Tree level**




**One loop**



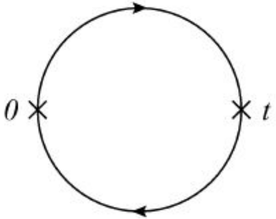
Presence of temporal dependence complicates the calculation

For proof-of-principle calculation, leading log-order matching is sufficient

- Avoids complication from temporal dependence, simplifying calculation

# Perturbative analysis

Leading order contribution




$$= \Gamma_{ij}^{(0,1)} = \begin{cases} 2k_S^2 \frac{\dim(F)}{(4\pi)^2} \frac{1}{t} + \mathcal{O}(m), & i,j = S,S \\ -2k_P^2 \frac{\dim(F)}{(4\pi)^2} \frac{1}{t} + \mathcal{O}(m), & i,j = P,P \\ -k_V^2 \frac{\dim(F)}{(4\pi)^2} \frac{\delta_{\mu\nu}}{t} + \mathcal{O}(m), & i,j = V,V \quad (\gamma_\mu, \gamma_\nu) \\ -k_A^2 \frac{\dim(F)}{(4\pi)^2} \frac{\delta_{\mu\nu}}{t} + \mathcal{O}(m), & i,j = A,A \quad (\gamma_\mu\gamma_5, \gamma_\nu\gamma_5) \\ 0 + \mathcal{O}(m), & i,j = T,T \quad (\sigma_{\mu\nu}, \sigma_{\rho\sigma}) \end{cases}$$

Next-to-leading order pieces

$$\Gamma_{SS}^{(1)}(t,0) = \Gamma_{SS}^{(0)}(t,0) \left\{ 1 + 4C_2(F) \frac{\alpha}{4\pi} \left[\log(4) + 1\right] + \mathcal{O}(m, \alpha^2) \right\},$$

$$\Gamma_{PP}^{(1)}(t,0) = \Gamma_{PP}^{(0)}(t,0) \left\{ 1 + 4C_2(F) \frac{\alpha}{4\pi} \left[\log(4) + 3\right] + \mathcal{O}(m, \alpha^2) \right\},$$

$$\Gamma_{VV}^{(1)}(t,0) = \Gamma_{VV}^{(0)}(t,0) \left\{ 1 - C_2(F) \frac{\alpha}{4\pi} \left[3L - 4\log(4) + \frac{3}{2}\right] + \mathcal{O}(m, \alpha^2) \right\},$$

$$\Gamma_{AA}^{(1)}(t,0) = \Gamma_{AA}^{(0)}(t,0) \left\{ 1 - C_2(F) \frac{\alpha}{4\pi} \left[3L - 4\log(4) + \frac{11}{2}\right] + \mathcal{O}(m, \alpha^2) \right\},$$

$$\Gamma_{TT}^{(1)}(t,0) = k_T^2 \frac{\dim(F)}{(4\pi)^2} \frac{\delta_\mu^{[\rho}\delta_\nu^{\sigma]}}{t} \left\{ 0 + 4C_2(F) \frac{\alpha}{4\pi} + \mathcal{O}(m, \alpha^2) \right\},$$

$$\boxed{L = \frac{1}{\epsilon} + 2\log(8\pi\mu^2 t).}$$

# Perturbative analysis

To extract the renormalisation parameter for the bilinears, we require

$$\Gamma^{R}_{ij}(t,0)\big|_{t=1/8\pi\mu^2} = \Gamma^{(0)}_{ij}(t,0)$$

Fermions at finite flow time require a further field renormalisation, which is scheme dependent (e.g. GF scheme)

$$Z^{\mathrm{GF}}_{\chi} = 1 + C_2(F)\frac{\alpha}{4\pi}\left[\frac{3}{\epsilon}+1\right] + \mathcal{O}(\alpha^2)$$

Rizik, Monahan & Schindler, PRD 102 (2020) 034509

With this choice

$$Z^{\mathrm{GF}}_{S} = 1 - C_2(F)\frac{\alpha}{4\pi}\left[\frac{3}{\epsilon}+8\log(2)+5\right] + \mathcal{O}(\alpha^2),$$

$$Z^{\mathrm{GF}}_{P} = 1 - C_2(F)\frac{\alpha}{4\pi}\left[\frac{3}{\epsilon}+8\log(2)+13\right] + \mathcal{O}(\alpha^2),$$

$$Z^{\mathrm{GF}}_{V} = 1 - C_2(F)\frac{\alpha}{4\pi}\left[8\log(2)-\frac{1}{2}\right] + \mathcal{O}(\alpha^2),$$

$$Z^{\mathrm{GF}}_{A} = 1 - C_2(F)\frac{\alpha}{4\pi}\left[8\log(2)-\frac{9}{2}\right] + \mathcal{O}(\alpha^2).$$

These results reproduce known leading-order anomalous dimensions

# Conclusions

Gradient flow provides controlled, continuous smearing (or blocking procedure) for fields on the lattice

Applied the gradient flow scheme to renormalise local composite operators

- Nonperturbative
- Gauge-invariant
- Provides nonperturbative step-scaling procedure
- Defined for both small- and large-volume regimes

Determined mass and proton anomalous dimensions in the continuum

Calculated

$$\Lambda_{\mathrm{QCD}}^{(n_f=2)} = 328(13)_{\mathrm{stat.}}(10)_{\mathrm{sys.}}\ \mathrm{MeV}$$

# Thank you!

Chris Monahan

cjmonahan@wm.edu

This work supported in part by DOE Early Career Award DE-SC0023047

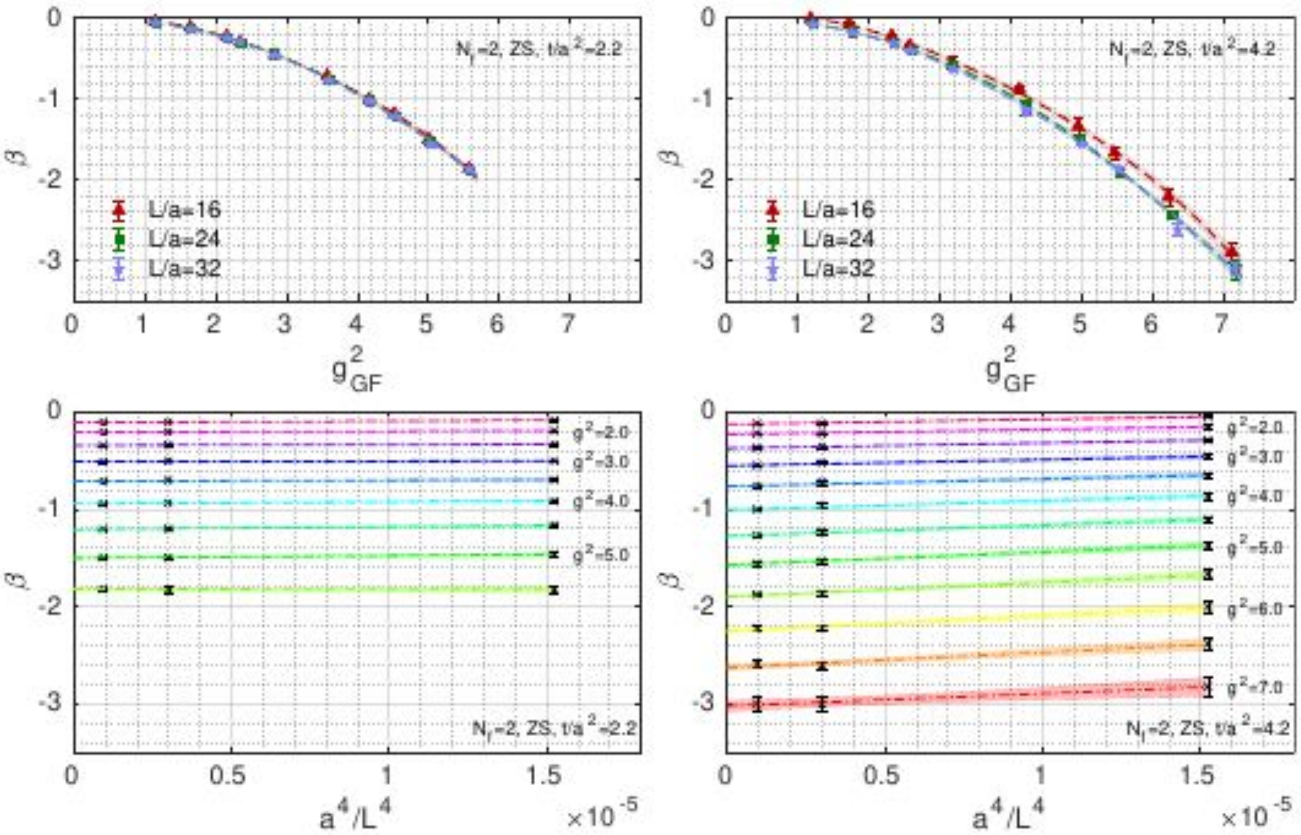
# The beta function in even more detail

Hasenfratz & Witzel, PRD 101 (2020) 034514

1. Calculate GF coupling and its derivative on every ensemble
   a. At a given bare coupling $\beta$, a specific value of $\tau/a^2$ will determine the renormalised coupling $g^{\mathrm{GF}}(\tau/a^2, \beta)$
2. By fixing $\tau/a^2$ across ensembles (i.e. different $\beta$), this will map out a set of points of the beta function as a function of the renormalised coupling
   a. Interpolate these points to obtain the beta function at arbitrary renormalised coupling
3. Now fix both $\tau/a^2$ and the renormalised coupling, and take the infinite volume limit via $A + B/L^4$
   a. This gives the infinite volume beta function at finite lattice spacing
4. Now take the continuum limit at fixed renormalised coupling
   a. Near the fixed point (i.e. the continuum), expect $a^2/\tau$ dependence
   b. Every value of $\tau/a^2$ generates curve representing the infinite volume beta function as a function of renormalised coupling (there will be an infinite family of such curves, because $\tau/a^2$ is continuous, but in practice discrete values are taken)
   c. Fix the renormalised coupling and extrapolate to infinite $\tau/a^2$ to obtain the infinite volume, continuum continuous beta function
   d. Maximum $\tau/a^2$ is constrained by finite volume effects and the minimum $\tau/a^2$ by cutoff effects

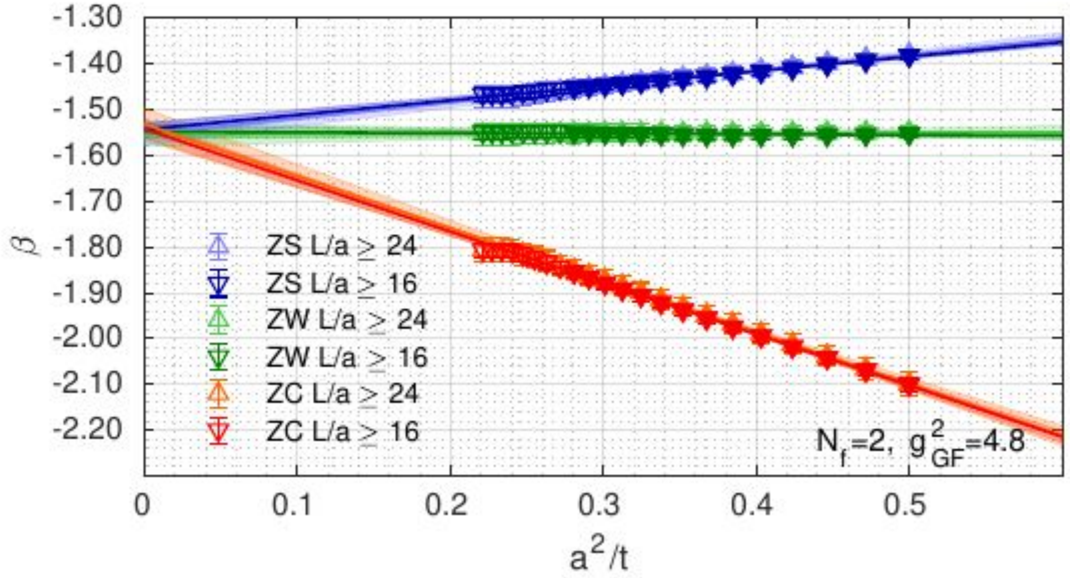
# Determining the beta function

Hasenfratz & Witzel, PRD 101 (2020) 034514



# Determining the beta function

Hasenfratz & Witzel, PRD 101 (2020) 034514



# Perturbative matching

Our aim is to match correlation functions in the gradient flow scheme to the MS-bar scheme

$$G'_{\mathcal{O}}(g', \mu') = \Delta_{Z_{\mathcal{O}}} G_{\mathcal{O}}(g, \mu)$$

$$Z'_{\mathcal{O}}(g', \mu') = \Delta_{Z_{\mathcal{O}}} Z_{\mathcal{O}}(g, \mu)$$

$$(g')^2 = \Delta_{\mathrm{g}}(g) g^2$$

Anomalous dimensions related via

$$\gamma_{\mathcal{O}}^{\mathrm{GF},(1)} = \gamma_{\mathcal{O}}^{\overline{\mathrm{MS}},(1)} - \gamma_{\mathcal{O}}^{(0)} \Delta_{\mathrm{g}}^{(1)} + 2 b_0 \Delta Z_{\mathcal{O}}^{(1)}$$

$$g_{\mathrm{GF}}^2 = g_{\overline{\mathrm{MS}}}^2 + \Delta_{\mathrm{g}}^{(1)} g_{\overline{\mathrm{MS}}}^4$$

where

$$b_0 = \frac{1}{(4\pi)^2}\left[\frac{11}{3}N_c - \frac{2}{3}N_f\right]$$

$$\Delta_{\mathrm{g}}^{(1)} = \frac{1}{(4\pi)^2}\left[N_c\left(\frac{11}{3}L + \frac{52}{9} - 3\ln 3\right) - N_f\left(\frac{2}{3}L + \frac{4}{9} - \frac{4}{3}\ln 2\right)\right]$$

$$L = \ln\left(8\mu^2 t\right) + \gamma_{\mathrm{E}}$$

Lüscher, JHEP 08 (2010) 071