How far can the Coulomb gas go?

Connor Behan *

2022-10-10
DESY seminar

* Oxford University

Related: 2xxx.xxxxx with A. Antunes

Review of minimal models

Chiral algebra is the Virasoro algebra

$$[L_m, L_n] = L_{m+n} + \frac{c}{12} m(m^2 - 1)\delta_{m+n,0}.$$

Primary operators satisfy $[L_0, \phi] = h\phi$ and $[L_n, \phi] = 0$ for $n > 0$.

Review of minimal models

Chiral algebra is the Virasoro algebra

$$[L_m, L_n] = L_{m+n} + \frac{c}{12} m(m^2 - 1)\delta_{m+n,0}.$$

Primary operators satisfy $[L_0, \phi] = h\phi$ and $[L_n, \phi] = 0$ for $n > 0$. Maximally degenerate unitary irreps with $r, r' \geq 1$ and

$$c = 1 - \frac{6}{m(m+1)}, \quad h_{(r;r')} = \beta_{(r;r')} \left(\beta_{(r;r')} - \sqrt{\frac{1-c}{6}} \right)$$

$$\beta_{(r;r')} = \sqrt{\frac{1}{2}\frac{m+1}{m}}\frac{r-1}{\sqrt{2}} - \sqrt{\frac{1}{2}\frac{m}{m+1}}\frac{r'-1}{\sqrt{2}},$$

can be used to build models.

Review of minimal models

Chiral algebra is the Virasoro algebra

$$[L_m, L_n] = L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}.$$

Primary operators satisfy $[L_0, \phi] = h\phi$ and $[L_n, \phi] = 0$ for $n > 0$. Maximally degenerate unitary irreps with $r, r' \geq 1$ and

$$c = 1 - \frac{6}{m(m+1)}, \quad h_{(r;r')} = \beta_{(r;r')}\left(\beta_{(r;r')} - \sqrt{\frac{1-c}{6}}\right)$$

$$\beta_{(r;r')} = \sqrt{\frac{1}{2}\frac{m+1}{m}}\frac{r-1}{\sqrt{2}} - \sqrt{\frac{1}{2}\frac{m}{m+1}}\frac{r'-1}{\sqrt{2}},$$

can be used to build models. Null state conditions on $\left\langle \phi_{(r_1;r_1')}\phi_{(r_2;r_2')}\mathcal{O}\right\rangle$ constrain $\mathcal{O}$ leading to fusion rules

$$\phi_{(r_1;r_1')} \times \phi_{(r_2;r_2')} = \sum_{r_3=|r_{12}|+1}^{r_1+r_2-1} \sum_{r_3'=|r_{12}'|+1}^{r_1'+r_2'-1} \phi_{(r_3;r_3')}.$$

Review of minimal models

Chiral algebra is the Virasoro algebra

$$[L_m, L_n] = L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}.$$

Primary operators satisfy $[L_0, \phi] = h\phi$ and $[L_n, \phi] = 0$ for $n > 0$. Maximally degenerate unitary irreps with $\lambda, \lambda' \geq 0$ and

$$c = 1 - \frac{6}{m(m+1)}, \quad h_{(r;r')} = \beta_{(r;r')} \left(\beta_{(r;r')} - \sqrt{\frac{1-c}{6}} \right)$$

$$\beta_{(r;r')} = \sqrt{\frac{1}{2}\frac{m+1}{m}}\frac{r-1}{\sqrt{2}} - \sqrt{\frac{1}{2}\frac{m}{m+1}}\frac{r'-1}{\sqrt{2}},$$

can be used to build models. Null state conditions on $\left\langle \phi_{(r_1;r_1')}\phi_{(r_2;r_2')}\mathcal{O} \right\rangle$ constrain $\mathcal{O}$ leading to fusion rules

$$\phi_{(r_1;r_1')} \times \phi_{(r_2;r_2')} = \sum_{r_3=|r_{12}|+1}^{r_1+r_2-1} \sum_{r_3'=|r_{12}'|+1}^{r_1'+r_2'-1} \phi_{(r_3;r_3')}.$$

Review of minimal models

Chiral algebra is the Virasoro algebra

$$[L_m, L_n] = L_{m+n} + \frac{c}{12} m(m^2 - 1)\delta_{m+n,0}.$$

Primary operators satisfy $[L_0, \phi] = h\phi$ and $[L_n, \phi] = 0$ for $n > 0$. Maximally degenerate unitary irreps with $\lambda, \lambda' \geq 0$ and

$$c = 1 - \frac{6}{m(m+1)}, \quad h_{(\lambda;\lambda')} = \beta_{(\lambda;\lambda')} \left(\beta_{(\lambda;\lambda')} - \sqrt{\frac{1-c}{6}} \right)$$

$$\beta_{(\lambda;\lambda')} = \sqrt{\frac{1}{2}\frac{m+1}{m}}\frac{\lambda}{\sqrt{2}} - \sqrt{\frac{1}{2}\frac{m}{m+1}}\frac{\lambda'}{\sqrt{2}},$$

can be used to build models. Null state conditions on $\left\langle \phi_{(r_1;r_1')}\phi_{(r_2;r_2')}\mathcal{O} \right\rangle$ constrain $\mathcal{O}$ leading to fusion rules

$$\phi_{(r_1;r_1')} \times \phi_{(r_2;r_2')} = \sum_{r_3=|r_{12}|+1}^{r_1+r_2-1} \sum_{r_3'=|r_{12}'|+1}^{r_1'+r_2'-1} \phi_{(r_3;r_3')}.$$

Review of minimal models

Chiral algebra is the Virasoro algebra

$$[L_m, L_n] = L_{m+n} + \frac{c}{12} m(m^2 - 1)\delta_{m+n,0}.$$

Primary operators satisfy $[L_0, \phi] = h\phi$ and $[L_n, \phi] = 0$ for $n > 0$. Maximally degenerate unitary irreps with $\lambda, \lambda' \geq 0$ and

$$c = 1 - \frac{6}{m(m+1)}, \quad h_{(\lambda;\lambda')} = \beta_{(\lambda;\lambda')} \left(\beta_{(\lambda;\lambda')} - \sqrt{\frac{1-c}{6}} \right)$$

$$\beta_{(\lambda;\lambda')} = \sqrt{\frac{1}{2}\frac{m+1}{m}}\frac{\lambda}{\sqrt{2}} - \sqrt{\frac{1}{2}\frac{m}{m+1}}\frac{\lambda'}{\sqrt{2}},$$

can be used to build models. Null state conditions on $\left\langle \phi_{(r_1;r_1')}\phi_{(r_2;r_2')}\mathcal{O} \right\rangle$ constrain $\mathcal{O}$ leading to fusion rules

$$\phi_{(\lambda_1;\lambda_1')} \times \phi_{(\lambda_2;\lambda_2')} = \sum_{\lambda_3 \in \lambda_1 \times \lambda_2} \sum_{\lambda_3' \in \lambda_1' \times \lambda_2'} \phi_{(\lambda_3;\lambda_3')}.$$

Review of minimal models

Chiral algebra is the Virasoro algebra

$$[L_m, L_n] = L_{m+n} + \frac{c}{12} m(m^2 - 1)\delta_{m+n,0}.$$

Primary operators satisfy $[L_0, \phi] = h\phi$ and $[L_n, \phi] = 0$ for $n > 0$. Maximally degenerate unitary irreps with $\lambda, \lambda' \geq 0$ and

$$c = 1 - 12\alpha_0^2, \quad h_{(\lambda;\lambda')} = \beta_{(\lambda;\lambda')}(\beta_{(\lambda;\lambda')} - \sqrt{2}\alpha_0),$$
$$\beta_{(\lambda;\lambda')} = \sqrt{\frac{1}{2}\frac{m+1}{m}}\frac{\lambda}{\sqrt{2}} - \sqrt{\frac{1}{2}\frac{m}{m+1}}\frac{\lambda'}{\sqrt{2}}, \quad \alpha_0 = -\frac{1}{\sqrt{2}}\beta_{(2;2)}$$

can be used to build models. Null state conditions on $\left\langle \phi_{(r_1;r_1')}\phi_{(r_2;r_2')}\mathcal{O} \right\rangle$ constrain $\mathcal{O}$ leading to fusion rules

$$\phi_{(\lambda_1;\lambda_1')} \times \phi_{(\lambda_2;\lambda_2')} = \sum_{\lambda_3 \in \lambda_1 \times \lambda_2} \sum_{\lambda_3' \in \lambda_1' \times \lambda_2'} \phi_{(\lambda_3;\lambda_3')}.$$

Enter the Coulomb gas

Consider free CFT with $c = 1$ defined by

$$S = \frac{1}{16\pi} \int \mathrm{d}^2 z \, \sqrt{g} (\partial_\alpha X)^2.$$

Enter the Coulomb gas

Consider free CFT with $c = 1$ defined by

$$S = \frac{1}{16\pi} \int d^2 z \sqrt{g} (\partial_\alpha X)^2.$$

Primaries with respect to $U(1)$ are given by

$$V_\beta(z, \bar{z}) = e^{i\beta X(z,\bar{z})}, \quad h_\beta = \beta^2.$$

Enter the Coulomb gas

Consider free CFT with $c = 1$ defined by

$$S = \frac{1}{16\pi} \int \mathrm{d}^2 z \, \sqrt{g} (\partial_\alpha X)^2.$$

Primaries with respect to $U(1)$ are given by

$$V_\beta(z, \bar{z}) = e^{i\beta X(z,\bar{z})}, \quad h_\beta = \beta^2.$$

Use $\langle X(z, \bar{z}) X(0,0) \rangle = -4 \log |z|$ and Wick's theorem to derive

$$\langle V_{\beta_1}(z_1, \bar{z}_1) \ldots V_{\beta_n}(z_n, \bar{z}_n) \rangle = \begin{cases} \prod_{i<j} |z_{ij}|^{4\beta_i \beta_j} & \sum_i \beta_i = 0 \\ 0 & \textit{otherwise} \end{cases}$$

Enter the Coulomb gas

Consider free CFT with $c = 1 - 12\alpha_0^2$ defined by

$$S = \frac{1}{16\pi} \int \mathrm{d}^2 z \sqrt{g} \left[(\partial_\alpha X)^2 + 2\sqrt{2} i \alpha_0 R X \right].$$

Primaries with respect to $U(1)$ are given by

$$V_\beta(z, \bar{z}) = e^{i\beta X(z,\bar{z})}, \quad h_\beta = \beta^2.$$

Use $\langle X(z,\bar{z}) X(0,0) \rangle = -4 \log |z|$ and Wick's theorem to derive

$$\langle V_{\beta_1}(z_1, \bar{z}_1) \ldots V_{\beta_n}(z_n, \bar{z}_n) \rangle = \begin{cases} \prod_{i<j} |z_{ij}|^{4\beta_i\beta_j} & \sum_i \beta_i = 0 \\ 0 & \textit{otherwise} \end{cases}$$

Enter the Coulomb gas

Consider free CFT with $c = 1 - 12\alpha_0^2$ defined by

$$S = \frac{1}{16\pi} \int \mathrm{d}^2 z \sqrt{g} \left[(\partial_\alpha X)^2 + 2\sqrt{2} i \alpha_0 R X \right].$$

Primaries with respect to $U(1)$ are given by

$$V_\beta(z, \bar{z}) = e^{i\beta X(z,\bar{z})}, \quad h_\beta = \beta(\beta - \sqrt{2}\alpha_0).$$

Use $\langle X(z,\bar{z}) X(0,0) \rangle = -4 \log |z|$ and Wick's theorem to derive

$$\langle V_{\beta_1}(z_1, \bar{z}_1) \ldots V_{\beta_n}(z_n, \bar{z}_n) \rangle = \begin{cases} \prod_{i<j} |z_{ij}|^{4\beta_i \beta_j} & \sum_i \beta_i = \sqrt{2}\alpha_0 \\ 0 & \textit{otherwise} \end{cases}$$

Enter the Coulomb gas

Consider free CFT with $c = 1 - 12\alpha_0^2$ defined by

$$S = \frac{1}{16\pi} \int \mathrm{d}^2 z \, \sqrt{g} \left[(\partial_\alpha X)^2 + 2\sqrt{2} i \alpha_0 R X \right].$$

Primaries with respect to $U(1)$ are given by

$$V_\beta(z, \bar{z}) = e^{i\beta X(z,\bar{z})}, \quad h_\beta = \beta(\beta - \sqrt{2}\alpha_0).$$

Use $\langle X(z,\bar{z}) X(0,0) \rangle = -4 \log |z|$ and Wick's theorem to derive

$$\langle V_{\beta_1}(z_1, \bar{z}_1) \ldots V_{\beta_n}(z_n, \bar{z}_n) \rangle = \begin{cases} \prod_{i<j} |z_{ij}|^{4\beta_i \beta_j} & \sum_i \beta_i = \sqrt{2}\alpha_0 \\ 0 & \textit{otherwise} \end{cases}$$

E.g. for $\alpha_0 = \frac{1}{2\sqrt{6}}$ (Ising model), ϵ is either $\beta = -\frac{1}{\sqrt{3}}$ or $\beta = \frac{\sqrt{3}}{2}$.

Enter the Coulomb gas

Consider free CFT with $c = 1 - 12\alpha_0^2$ defined by

$$S = \frac{1}{16\pi} \int \mathrm{d}^2 z \, \sqrt{g} \left[(\partial_\alpha X)^2 + 2\sqrt{2} i \alpha_0 R X + V_{\sqrt{2}\alpha_+} + V_{\sqrt{2}\alpha_-} \right].$$

Primaries with respect to $U(1)$ are given by

$$V_\beta(z, \bar{z}) = e^{i\beta X(z,\bar{z})}, \quad h_\beta = \beta(\beta - \sqrt{2}\alpha_0).$$

Use $\langle X(z,\bar{z}) X(0,0) \rangle = -4 \log |z|$ and Wick's theorem to derive

$$\langle V_{\beta_1}(z_1, \bar{z}_1) \ldots V_{\beta_n}(z_n, \bar{z}_n) \rangle = \begin{cases} \prod_{i<j} |z_{ij}|^{4\beta_i \beta_j} & \sum_i \beta_i = \sqrt{2}\alpha_0 \\ 0 & \textit{otherwise} \end{cases}$$

E.g. for $\alpha_0 = \frac{1}{2\sqrt{6}}$ (Ising model), ϵ is either $\beta = -\frac{1}{\sqrt{3}}$ or $\beta = \frac{\sqrt{3}}{2}$.

Enter the Coulomb gas

Consider free CFT with $c = 1 - 12\alpha_0^2$ defined by

$$S = \frac{1}{16\pi}\int \mathrm{d}^2 z\,\sqrt{g}\left[(\partial_\alpha X)^2 + 2\sqrt{2}i\alpha_0 RX + V_{\sqrt{2}\alpha_+} + V_{\sqrt{2}\alpha_-}\right].$$

Primaries with respect to $U(1)$ are given by

$$V_\beta(z,\bar{z}) = e^{i\beta X(z,\bar{z})}, \quad h_\beta = \beta(\beta - \sqrt{2}\alpha_0).$$

Use $\langle X(z,\bar{z})X(0,0)\rangle = -4\log|z|$ and Wick's theorem to derive

$$\langle V_{\beta_1}(z_1,\bar{z}_1)\dots V_{\beta_n}(z_n,\bar{z}_n)\rangle = \begin{cases} \prod_{i<j}|z_{ij}|^{4\beta_i\beta_j} & \sum_i \beta_i = \sqrt{2}\alpha_0 \\ 0 & \textit{otherwise} \end{cases}$$

E.g. for $\alpha_0 = \frac{1}{2\sqrt{6}}$ (Ising model), ϵ is either $\beta = -\frac{1}{\sqrt{3}}$ or $\beta = \frac{\sqrt{3}}{2}$.
Works if and only if $\alpha_+ = -\sqrt{\frac{1}{2}\frac{m+1}{m}}$ and $\alpha_- = \sqrt{\frac{1}{2}\frac{m}{m+1}}$.

W-minimal models

Simplest W-algebra in [Zamolodchikov; 86] is part of an ADE family. Can define using coset construction $W[\hat{\mathfrak{g}}] = \frac{\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_1}{\hat{\mathfrak{g}}_{k+1}}$, Casimir construction

$$T(z) \equiv W^{(2)}(z) = \delta_{ab}(J^a J^b)(z), \quad W^{(3)}(z) = d_{abc}(J^a(J^b J^c))(z), \quad \ldots$$

or quantum Drinfeld Sokolov reduction.

W-minimal models

Simplest W-algebra in [Zamolodchikov; 86] is part of an ADE family. Can define using coset construction $W[\hat{\mathfrak{g}}] = \frac{\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_1}{\hat{\mathfrak{g}}_{k+1}}$, Casimir construction

$$T(z) \equiv W^{(2)}(z) = \delta_{ab}(J^a J^b)(z), \quad W^{(3)}(z) = d_{abc}(J^a(J^b J^c))(z), \quad \ldots$$

or quantum Drinfeld Sokolov reduction.

- Admit integrable deformations as in [Zamolodchikov; 87] .
- Describe critical points of lattice IRF models [Date, Jimbo, Miwa, Okado; 86] .
- Appear in examples of holographic duality [Gaberdiel, Gopakumar; 1011.2986] .

W-minimal models

Simplest W-algebra in [Zamolodchikov; 86] is part of an ADE family. Can define using coset construction $W[\hat{\mathfrak{g}}] = \frac{\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_1}{\hat{\mathfrak{g}}_{k+1}}$, Casimir construction

$$T(z) \equiv W^{(2)}(z) = \delta_{ab}(J^a J^b)(z), \quad W^{(3)}(z) = d_{abc}(J^a(J^b J^c))(z), \quad \ldots$$

or quantum Drinfeld Sokolov reduction.

- Admit integrable deformations as in [Zamolodchikov; 87] .
- Describe critical points of lattice IRF models [Date, Jimbo, Miwa, Okado; 86] .
- Appear in examples of holographic duality [Gaberdiel, Gopakumar; 1011.2986] .

For charges solving $\alpha_\pm^2 - \alpha_0\alpha_\pm = \frac{1}{2}$, we had $c = 1 - 12\alpha_0^2$ and

$$h_{(\lambda;\lambda')} = (\beta_{(\lambda;\lambda')}, \beta_{(\lambda;\lambda')} - \sqrt{2}\alpha_0), \quad \beta_{(\lambda;\lambda')} = -\left[\alpha_+\lambda + \alpha_-\lambda'\right]\frac{1}{\sqrt{2}}$$

$$\phi_{(\lambda_1;\lambda_1')} \times \phi_{(\lambda_2;\lambda_2')} = \sum_{\lambda_3 \in \lambda_1 \times \lambda_2} \sum_{\lambda_3' \in \lambda_1' \times \lambda_2'} \phi_{(\lambda_3;\lambda_3')}.$$

W-minimal models

Simplest W-algebra in [Zamolodchikov; 86] is part of an ADE family. Can define using coset construction $W[\hat{\mathfrak{g}}] = \frac{\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_1}{\hat{\mathfrak{g}}_{k+1}}$, Casimir construction

$$T(z) \equiv W^{(2)}(z) = \delta_{ab}(J^a J^b)(z), \quad W^{(3)}(z) = d_{abc}(J^a(J^b J^c))(z), \quad \dots$$

or quantum Drinfeld Sokolov reduction.

- Admit integrable deformations as in [Zamolodchikov; 87] .
- Describe critical points of lattice IRF models [Date, Jimbo, Miwa, Okado; 86] .
- Appear in examples of holographic duality [Gaberdiel, Gopakumar; 1011.2986] .

For charges solving $\alpha_\pm^2 - \alpha_0 \alpha_\pm = \frac{1}{2}$, now $c = rank(\mathfrak{g}) - 24\alpha_0^2 \rho^2$ and

$$h_{(\lambda;\lambda')} = (\beta_{(\lambda;\lambda')}, \beta_{(\lambda;\lambda')} - 2\alpha_0 \rho), \quad \beta_{(\lambda;\lambda')} = -\sum_i [\alpha_+ \lambda_i + \alpha_- \lambda'_i]\, \omega_i$$

$$\phi_{(\lambda_1;\lambda'_1)} \times \phi_{(\lambda_2;\lambda'_2)} = \sum_{\lambda_3 \in \lambda_1 \times \lambda_2} \sum_{\lambda'_3 \in \lambda'_1 \times \lambda'_2} \phi_{(\lambda_3;\lambda'_3)}.$$

Multi-component Coulomb gas

Free field realization of Virasoro minimal models looked like

$$S = \frac{1}{16\pi} \int \mathrm{d}^2 z \, \sqrt{g} \left[(\partial_\alpha X)^2 + 2\sqrt{2} i \alpha_0 R X + V_{\sqrt{2}\alpha_+} + V_{\sqrt{2}\alpha_-} \right]$$

with the correspondence

$$\phi_{(\lambda;\lambda')}(z,\bar{z}) \leftrightarrow N^{-1}_{(\lambda;\lambda')} V_{\beta_{(\lambda;\lambda')}}(z,\bar{z}) \equiv N^{-1}_{(\lambda;\lambda')} e^{i(\beta_{(\lambda;\lambda')}, X)(z,\bar{z})}.$$

Multi-component Coulomb gas

Free field realization of $W[\hat{\mathfrak{g}}]$ minimal models looks like

$$S = \frac{1}{16\pi} \int \mathrm{d}^2 z \sqrt{g} \left[(\partial_\alpha X_i)^2 + 4 i \alpha_0 R(\rho, X) + \sum_i (V_{\alpha_+ \alpha_i} + V_{\alpha_- \alpha_i}) \right]$$

with the correspondence

$$\phi_{(\lambda;\lambda')}(z,\bar{z}) \leftrightarrow N^{-1}_{(\lambda;\lambda')} V_{\beta_{(\lambda;\lambda')}}(z,\bar{z}) \equiv N^{-1}_{(\lambda;\lambda')} e^{i(\beta_{(\lambda;\lambda')}, X)(z,\bar{z})}.$$

Multi-component Coulomb gas

Free field realization of $W[\hat{\mathfrak{g}}]$ minimal models looks like

$$S = \frac{1}{16\pi}\int \mathrm{d}^2 z\,\sqrt{g}\left[(\partial_\alpha X_i)^2 + 4i\alpha_0 R(\rho, X) + \sum_i (V_{\alpha_+\alpha_i} + V_{\alpha_-\alpha_i})\right]$$

with the correspondence

$$\phi_{(\lambda;\lambda')}(z,\bar{z}) \leftrightarrow N^{-1}_{(\lambda;\lambda')} V_{\beta_{(\lambda;\lambda')}}(z,\bar{z}) \equiv N^{-1}_{(\lambda;\lambda')} e^{i(\beta_{(\lambda;\lambda')},X)(z,\bar{z})}.$$

Note that $V_{\beta_{(\lambda;\lambda')}}$ and $V_{2\alpha_0\rho-\beta^*_{(\lambda;\lambda')}}$ map to same primary with inverse normalizations.

Multi-component Coulomb gas

Free field realization of $W[\hat{\mathfrak{g}}]$ minimal models looks like

$$S = \frac{1}{16\pi} \int \mathrm{d}^2 z \sqrt{g} \left[(\partial_\alpha X_i)^2 + 4 i \alpha_0 R(\rho, X) + \sum_i (V_{\alpha_+ \alpha_i} + V_{\alpha_- \alpha_i}) \right]$$

with the correspondence

$$\phi_{(\lambda;\lambda')}(z,\bar{z}) \leftrightarrow N^{-1}_{(\lambda;\lambda')} V_{\beta_{(\lambda;\lambda')}}(z,\bar{z}) \equiv N^{-1}_{(\lambda;\lambda')} e^{i(\beta_{(\lambda;\lambda')},X)(z,\bar{z})}.$$

Note that $V_{\beta_{(\lambda;\lambda')}}$ and $V_{2\alpha_0\rho - \beta^*_{(\lambda;\lambda')}}$ map to same primary with inverse normalizations. In order to compute $C_{1,2,3}$,

$$\begin{aligned}
N_1 N_2 N_3^{-1} C^3_{1,2} &= \left\langle V_{\beta_1} V_{\beta_2} V_{2\alpha_0\rho - \beta_3^*} \ldots \right\rangle & N_1^2 &= \left\langle V_1 V_1 V_{2\alpha_0\rho} \ldots \right\rangle \\
N_2 N_3 N_1^{-1} C^1_{2,3} &= \left\langle V_{\beta_2} V_{\beta_3} V_{2\alpha_0\rho - \beta_1^*} \ldots \right\rangle & N_2^2 &= \left\langle V_2 V_2 V_{2\alpha_0\rho} \ldots \right\rangle \\
N_3 N_1 N_2^{-1} C^2_{3,1} &= \left\langle V_{\beta_3} V_{\beta_1} V_{2\alpha_0\rho - \beta_2^*} \ldots \right\rangle & N_3^2 &= \left\langle V_3 V_3 V_{2\alpha_0\rho} \ldots \right\rangle
\end{aligned}$$

Useful integrals

Building blocks in terms of $\gamma(z) \equiv \Gamma(z)/\Gamma(1-z)$ are

$$K_1(\alpha,\beta) = \int d^2z\, |z|^{2\alpha}|z-1|^{2\beta} = \pi \frac{\gamma(\alpha+1)\gamma(\beta+1)}{\gamma(\alpha+\beta+2)}$$

$$K_2(\alpha,\beta) = \int d^2z_1 d^2z_2\, (|z_1||z_1-1||z_2||z_2-1|)^{2\alpha}|z_{12}|^{4\beta}$$

$$= 2\pi^2 \frac{\gamma(2\beta)}{\gamma(\beta)} \frac{\gamma(\alpha+1)^2}{\gamma(2\alpha+\beta+2)} \frac{\gamma(\alpha+\beta+1)^2}{\gamma(2\alpha+2\beta+2)}$$

[Symanzik; 72] [Paulos, Rychkov, van Rees, Zan; 1509.00008] .

Useful integrals

Building blocks in terms of $\gamma(z) \equiv \Gamma(z)/\Gamma(1-z)$ are

$$K_1(\alpha,\beta) = \int d^2z\,|z|^{2\alpha}|z-1|^{2\beta} = \pi\frac{\gamma(\alpha+1)\gamma(\beta+1)}{\gamma(\alpha+\beta+2)}$$

$$K_2(\alpha,\beta) = \int d^2z_1 d^2z_2\,(|z_1||z_1-1||z_2||z_2-1|)^{2\alpha}|z_{12}|^{4\beta}$$

$$= 2\pi^2\frac{\gamma(2\beta)}{\gamma(\beta)}\frac{\gamma(\alpha+1)^2}{\gamma(2\alpha+\beta+2)}\frac{\gamma(\alpha+\beta+1)^2}{\gamma(2\alpha+2\beta+2)}$$

[Symanzik; 72] [Paulos, Rychkov, van Rees, Zan; 1509.00008] . Directly useful when each screening charge appears at most twice. Indirectly through chaining.

$$N_{(0,0;1,1)}\,C^{(0,0;1,1)}_{(0,0;1,1)(0,0;1,1)}\cdots \leq 2, \quad N^2_{(0,0;1,1)}\cdots > 2$$

Useful integrals

Building blocks in terms of $\gamma(z) \equiv \Gamma(z)/\Gamma(1-z)$ are

$$K_1(\alpha,\beta) = \int d^2z\, |z|^{2\alpha}|z-1|^{2\beta} = \pi\frac{\gamma(\alpha+1)\gamma(\beta+1)}{\gamma(\alpha+\beta+2)}$$

$$K_2(\alpha,\beta) = \int d^2z_1 d^2z_2\, (|z_1||z_1-1||z_2||z_2-1|)^{2\alpha}|z_{12}|^{4\beta}$$
$$= 2\pi^2\frac{\gamma(2\beta)}{\gamma(\beta)}\frac{\gamma(\alpha+1)^2}{\gamma(2\alpha+\beta+2)}\frac{\gamma(\alpha+\beta+1)^2}{\gamma(2\alpha+2\beta+2)}$$

[Symanzik; 72] [Paulos, Rychkov, van Rees, Zan; 1509.00008] . Directly useful when each screening charge appears at most twice. Indirectly through chaining.

$$N_{(0,0;1,1)}\, C^{(0,0;1,1)}_{(0,0;1,1)(0,0;1,1)} \cdots \leq 2, \quad N^2_{(0,0;1,1)} \cdots > 2$$

$$N^2_{(0,0;0,1)},\; N_{(0,0;1,1)}\, C^{(0,0;1,1)}_{(0,0;0,1)(0,0;1,0)},\; N^2_{(0,0;0,1)} N^{-1}_{(0,0;1,1)}\, C^{(0,0;1,1)}_{(0,0;0,1)(0,0;1,0)} \cdots \leq 2!$$

Limits to this method

Despite some generalizations ([Fukuda, Hosomichi; hep-th/0105217] [Fateev, Litvinov; 0709.3806]), three basic families are out of reach.

1. Weights too large and too far from extremality. E.g. if $C^{\beta}_{\beta,\beta}$ is non-trivial, $C^{N\beta}_{N\beta,N\beta}$ requires N times as many of each screening charge.
2. Norm is already too complicated for all fundamental representations. Happens for $\mathfrak{e}_6$ where simplest norm $N^2_{(\mathbf{0};1,0,0,0,0,0)} = N^2_{(\mathbf{0};0,0,0,0,1,0)}$ requires multiplicities of 2, 3, 4, 3, 2, 2.
3. Norms which are not too complicated might form a closed subsector. Happens for tensor representations of $\mathfrak{d}_n$.

Are we stuck?

At small central charge, large weights can be equivalent to small weights or outside the Kac table. At large central charge, consider ϵ-expansion of integral where $\epsilon = 2\alpha_-^2 - 1$.

Are we stuck?

At small central charge, large weights can be equivalent to small weights or outside the Kac table. At large central charge, consider ϵ-expansion of integral where $\epsilon = 2\alpha_-^2 - 1$.

These are conformal integrals ($\sum_i \Delta_i = d$). After using

$$\int \frac{\mathrm{d}^d z}{\pi^{\frac{d}{2}}} \prod_{i=1}^{n} \frac{\Gamma(\Delta_i)}{|z - z_i|^{2\Delta_i}} = \prod_{i<j} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}\delta_{ij}}{2\pi i} \Gamma(\delta_{ij}) |z_{ij}|^{-2\delta_{ij}}, \quad \sum_{j \neq i} \delta_{ij} = \Delta_i,$$

expansion of Mellin-Barnes integral can be done with `MB.m` package [Czakon; hep-ph/0511200] . See also [Yuan; 1801.07283] .

Are we stuck?

At small central charge, large weights can be equivalent to small weights or outside the Kac table. At large central charge, consider ϵ-expansion of integral where $\epsilon = 2\alpha_-^2 - 1$.

These are conformal integrals ($\sum_i \Delta_i = d$). After using

$$\int \frac{d^d z}{\pi^{\frac{d}{2}}} \prod_{i=1}^{n} \frac{\Gamma(\Delta_i)}{|z - z_i|^{2\Delta_i}} = \prod_{i<j} \int_{-i\infty}^{i\infty} \frac{d\delta_{ij}}{2\pi i} \Gamma(\delta_{ij}) |z_{ij}|^{-2\delta_{ij}}, \quad \sum_{j \neq i} \delta_{ij} = \Delta_i,$$

expansion of Mellin-Barnes integral can be done with `MB.m` package [Czakon; hep-ph/0511200] . See also [Yuan; 1801.07283] .

$$\int d^8 \vec{z} (|z_1||z_2||z_3 - 1||z_4 - 1|)^{-4\alpha_-^2} (|z_{12}||z_{34}|)^2 (|z_{13}||z_{14}||z_{23}||z_{24}|)^{-4\alpha_-^2}$$

$$= 10\pi^4 (2\alpha_-^2 - 1)^{-4} + \ldots \quad (1 \textit{ hour})$$

$$\int d^{10} \vec{z} (|z_1||z_2||z_1 - 1||z_2 - 1|)^{-4\alpha_-^2} |z_{12}|^{8\alpha_-^2} |z_{34}|^2 |z_5 - 1|^{4 - 12\alpha_-^2}$$

$$(|z_{13}||z_{14}||z_{23}||z_{24}||z_{35}||z_{45}|)^{-4\alpha_-^2} = 28\pi^5 (2\alpha_-^2 - 1)^{-5} + \ldots \quad (10 \textit{ hours})$$

Are we stuck?

At small central charge, large weights can be equivalent to small weights or outside the Kac table. At large central charge, consider ϵ-expansion of integral where $\epsilon = 2\alpha_-^2 - 1$.

These are conformal integrals ($\sum_i \Delta_i = d$). After using

$$\int \frac{d^d z}{\pi^{\frac{d}{2}}} \prod_{i=1}^{n} \frac{\Gamma(\Delta_i)}{|z - z_i|^{2\Delta_i}} = \prod_{i<j} \int_{-i\infty}^{i\infty} \frac{d\delta_{ij}}{2\pi i} \Gamma(\delta_{ij}) |z_{ij}|^{-2\delta_{ij}}, \quad \sum_{j \neq i} \delta_{ij} = \Delta_i,$$

expansion of Mellin-Barnes integral can be done with `MB.m` package [Czakon; hep-ph/0511200] . See also [Yuan; 1801.07283] .

$$\int d^8 \vec{z} (|z_1||z_2||z_3 - 1||z_4 - 1|)^{-4\alpha_-^2} (|z_{12}||z_{34}|)^{2+8\alpha_-^2 - 8\beta_-^2} (|z_{13}||z_{14}||z_{23}||z_{24}|)^{-4\alpha_-^2}$$

$$= 10\pi^4 (2\alpha_-^2 - 1)^{-4} + \dots \quad (1 \textit{ hour})$$

$$\int d^{10} \vec{z} (|z_1||z_2||z_1 - 1||z_2 - 1|)^{-4\alpha_-^2} |z_{12}|^{8\beta_-^2} |z_{34}|^{2+8\alpha_-^2 - 8\beta_-^2} |z_5 - 1|^{-2}$$

$$(|z_{13}||z_{14}||z_{23}||z_{24}||z_{35}||z_{45}|)^{-4\alpha_-^2} = 28\pi^5 (2\alpha_-^2 - 1)^{-5} + \dots \quad (10 \textit{ hours})$$

Screening 4pt functions instead

With 4pt functions of the form

$$\left\langle \phi_{(\lambda_1;\lambda_1')}(0)\phi_{(\lambda_1;\lambda_1')}(z,\bar{z})\phi_{(\lambda_2;\lambda_2')}(1)\phi_{(\lambda_2;\lambda_2')}(\infty)\right\rangle = \sum_{j=1}^{M} X_j \left|I_j(z)\right|^2$$

$$I_j(z) = N_j z^{h_j - 2h_{(\lambda_1;\lambda_1')}}\left[1 + O(z)\right]$$

we can use [Dotsenko, Fateev; 84] [Dotsenko, Fateev; 85] .

Screening 4pt functions instead

With 4pt functions of the form

$$\left\langle \phi_{(\lambda_1;\lambda_1')}(0)\phi_{(\lambda_1;\lambda_1')}(z,\bar{z})\phi_{(\lambda_2;\lambda_2')}(1)\phi_{(\lambda_2;\lambda_2')}(\infty)\right\rangle = \sum_{j=1}^{M} X_j \left|I_j(z)\right|^2$$

$$I_j(z) = N_j z^{h_j - 2h_{(\lambda_1;\lambda_1')}}\left[1 + O(z)\right]$$

we can use [Dotsenko, Fateev; 84] [Dotsenko, Fateev; 85] . If we can compute all N_j and express

$$I_j(z) = \sum_k F_{jk}\tilde{I}_k(z), \quad \tilde{I}_k(1-z) = \tilde{N}_k z^{h_k - h_{(\lambda_1;\lambda_1')} - h_{(\lambda_2;\lambda_2')}}\left[1 + O(z)\right],$$

1. Killing $(1-z)^{h_j}(1-\bar{z})^{h_k}$ terms fixes $\frac{X_j}{X_k} = \frac{F^*_{kk}(F^{-1})_{kj}}{F^*_{jk}(F^{-1})_{kk}}$.
2. Norms allow us to write $\frac{C^j_{(\lambda_1;\lambda_1')(\lambda_1;\lambda_1')}C^j_{(\lambda_2;\lambda_2')(\lambda_2;\lambda_2')}}{C^k_{(\lambda_1;\lambda_1')(\lambda_1;\lambda_1')}C^k_{(\lambda_2;\lambda_2')(\lambda_2;\lambda_2')}} = \frac{N_j^2}{N_k^2}\frac{X_j}{X_k}$.
3. Identity term $C^{(\mathbf{0};\mathbf{0})}_{(\lambda_1;\lambda_1')(\lambda_1;\lambda_1')}C^{(\mathbf{0};\mathbf{0})}_{(\lambda_2;\lambda_2')(\lambda_2;\lambda_2')} = 1$ completes solution.

Screening 4pt functions instead

Build integrals using $\langle V_{\beta_1}(z_1)\dots V_{\beta_n}(z_n)\rangle = \prod_{i<j} z_{ij}^{2\beta_i\beta_j}$.

Screening 4pt functions instead

Build integrals using $\langle V_{\beta_1}(z_1)\dots V_{\beta_n}(z_n)\rangle = \prod_{i<j} z_{ij}^{2\beta_i\beta_j}$. For operators that require only $V_{\sqrt{2}\alpha_-}(t)$,

$$I_*(z) = \int_0^z \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n$$
$$\prod_{i=1}^n t_i^a (z - t_i)^b (1 - t_i)^c \prod_{i<j} t_{ij}^{2q}$$

can be expressed in the t-channel with contour deformation.

Screening 4pt functions instead

Build integrals using $\langle V_{\beta_1}(z_1)\dots V_{\beta_n}(z_n)\rangle = \prod_{i<j} z_{ij}^{2\beta_i\beta_j}$. For operators that require only $V_{\sqrt{2}\alpha_-}(t)$,

$$I_*(z) = \int_0^z \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n$$

$$\prod_{i=1}^{n} t_i^a (z - t_i)^b (1 - t_i)^c \prod_{i<j} t_{ij}^{2q}$$

can be expressed in the t-channel with contour deformation. After $t_i \mapsto zt_i$, norm comes from Selberg integral.

$$\int_0^1 \mathrm{d}t_1 \cdots \int_0^1 \mathrm{d}t_n \prod_{i=1}^{n} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{i<j} |t_{ij}|^{2\gamma}$$

$$= \prod_{j=0}^{n-1} \frac{\Gamma[\alpha + j\gamma]\Gamma[\beta + j\gamma]\Gamma[1 + (j+1)\gamma]}{\Gamma[\alpha + \beta + (n+j-1)\gamma]\Gamma[1+\gamma]}.$$

Screening 4pt functions instead

Build integrals using $\langle V_{\beta_1}(z_1)\dots V_{\beta_n}(z_n)\rangle = \prod_{i<j} z_{ij}^{2\beta_i\beta_j}$. For operators that require only $V_{\sqrt{2}\alpha_-}(t)$,

$$I_*(z) = \int_0^z \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n$$

$$\prod_{i=1}^{n} t_i^a (z-t_i)^b (1-t_i)^c \prod_{i<j} t_{ij}^{2q}$$

can be expressed in the t-channel with contour deformation. After $t_i \mapsto z t_i$, norm comes from Selberg integral.

$$\int_0^1 \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \prod_{i=1}^{n} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{i<j} |t_{ij}|^{2\gamma}$$

$$= \frac{1}{n!} \prod_{j=0}^{n-1} \frac{\Gamma[\alpha+j\gamma]\Gamma[\beta+j\gamma]\Gamma[1+(j+1)\gamma]}{\Gamma[\alpha+\beta+(n+j-1)\gamma]\Gamma[1+\gamma]}.$$

Screening 4pt functions instead

Build integrals using $\langle V_{\beta_1}(z_1)\dots V_{\beta_n}(z_n)\rangle = \prod_{i<j} z_{ij}^{2\beta_i\beta_j}$. For operators that require only $V_{\sqrt{2}\alpha_-}(t)$,

$$I_*(z) = \int_1^\infty \mathrm{d}t_1 \cdots \int_1^{t_{m-1}} \mathrm{d}t_m \int_0^z \mathrm{d}t_{m+1} \cdots \int_0^{t_{n-1}} \mathrm{d}t_n$$

$$\prod_{i=1}^{m} t_i^a (t_i - z)^b (t_i - 1)^c \prod_{i=m+1}^{n} t_i^a (z - t_i)^b (1 - t_i)^c \prod_{i<j} t_{ij}^{2q}$$

can be expressed in the t-channel with contour deformation. After $t_i \mapsto z t_i$, norm comes from Selberg integral.

$$\int_0^1 \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \prod_{i=1}^{n} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{i<j} |t_{ij}|^{2\gamma}$$

$$= \frac{1}{n!} \prod_{j=0}^{n-1} \frac{\Gamma[\alpha + j\gamma]\Gamma[\beta + j\gamma]\Gamma[1 + (j+1)\gamma]}{\Gamma[\alpha + \beta + (n + j - 1)\gamma]\Gamma[1+\gamma]}.$$

Ordered vs unordered integrals

Switch between

$$I_*(z) = \int_1^\infty \mathrm{d}t_1 \cdots \int_1^{t_{m-1}} \mathrm{d}t_m \int_0^z \mathrm{d}t_{m+1} \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \prod_{i<j} t_{ij}^{2q} \ldots$$

$$J_*(z) = \int_1^\infty \mathrm{d}t_1 \cdots \int_1^\infty \mathrm{d}t_m \int_0^z \mathrm{d}t_{m+1} \cdots \int_0^z \mathrm{d}t_n \prod_{i<j} t_{ij}^{2q} \ldots$$

with $J_*(z) = \prod_{l=1}^m \frac{\sin(l\pi q)}{\sin(\pi q)} \prod_{l=1}^{n-m} \frac{\sin(l\pi q)}{\sin(\pi q)} I_*(z)$.

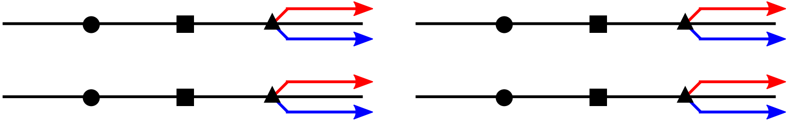
Ordered vs unordered integrals

Switch between

$$I_*(z) = \int_1^\infty \mathrm{d}t_1 \cdots \int_1^{t_{m-1}} \mathrm{d}t_m \int_0^z \mathrm{d}t_{m+1} \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \prod_{i<j} t_{ij}^{2q} \ldots$$

$$J_*(z) = \int_1^\infty \mathrm{d}t_1 \cdots \int_1^\infty \mathrm{d}t_m \int_0^z \mathrm{d}t_{m+1} \cdots \int_0^z \mathrm{d}t_n \prod_{i<j} t_{ij}^{2q} \ldots$$

with $J_*(z) = \prod_{l=1}^m \frac{\sin(l\pi q)}{\sin(\pi q)} \prod_{l=1}^{n-m} \frac{\sin(l\pi q)}{\sin(\pi q)} I_*(z)$.



Circle, square, triangle are $0, z, 1$.

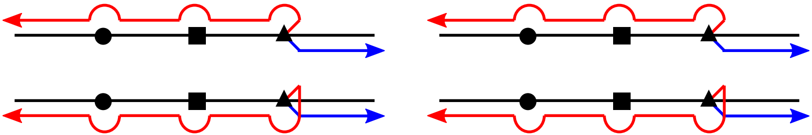
Ordered vs unordered integrals

Switch between

$$I_*(z) = \int_1^\infty \mathrm{d}t_1 \cdots \int_1^{t_{m-1}} \mathrm{d}t_m \int_0^z \mathrm{d}t_{m+1} \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \prod_{i<j} t_{ij}^{2q} \ldots$$

$$J_*(z) = \int_1^\infty \mathrm{d}t_1 \cdots \int_1^{\infty} \mathrm{d}t_m \int_0^z \mathrm{d}t_{m+1} \cdots \int_0^{z} \mathrm{d}t_n \prod_{i<j} t_{ij}^{2q} \ldots$$

with $J_*(z) = \prod_{l=1}^{m} \frac{\sin(l\pi q)}{\sin(\pi q)} \prod_{l=1}^{n-m} \frac{\sin(l\pi q)}{\sin(\pi q)} I_*(z)$.



Circle, square, triangle are $0, z, 1$.

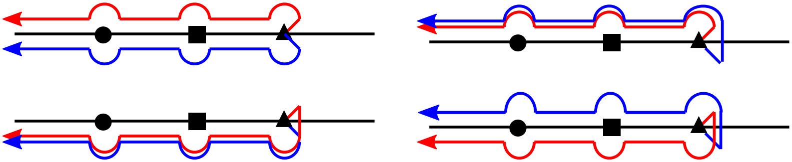
Ordered vs unordered integrals

Switch between

$$I_*(z) = \int_1^{\infty} \mathrm{d}t_1 \cdots \int_1^{t_{m-1}} \mathrm{d}t_m \int_0^{z} \mathrm{d}t_{m+1} \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \prod_{i<j} t_{ij}^{2q} \ldots$$

$$J_*(z) = \int_1^{\infty} \mathrm{d}t_1 \cdots \int_1^{\infty} \mathrm{d}t_m \int_0^{z} \mathrm{d}t_{m+1} \cdots \int_0^{z} \mathrm{d}t_n \prod_{i<j} t_{ij}^{2q} \ldots$$

with $J_*(z) = \prod_{l=1}^{m} \frac{\sin(l\pi q)}{\sin(\pi q)} \prod_{l=1}^{n-m} \frac{\sin(l\pi q)}{\sin(\pi q)} I_*(z)$.



Circle, square, triangle are $0, z, 1$.

From Virasoro to W-algebras

Complete Virasoro solution allows us to go back and compute

$$\int \mathrm{d}^2 z_1 \ldots \mathrm{d}^2 z_n \prod_{i=1}^{n} |z_i|^{2\alpha} |z_i - 1|^{2\beta} \prod_{i<j} |z_{ij}|^{4\gamma}.$$

(B.10), are divergent, in general, and have to be regularized somehow. We defined and calculated them by actually reducing them back to the contour integrals of the preceding sections and appendix A.

From Virasoro to W-algebras

Complete Virasoro solution allows us to go back and compute

$$\int \mathrm{d}^2 z_1 \ldots \mathrm{d}^2 z_n \prod_{i=1}^{n} |z_i|^{2\alpha} |z_i - 1|^{2\beta} \prod_{i<j} |z_{ij}|^{4\gamma}.$$

(B.10), are divergent, in general, and have to be regularized somehow. We defined and calculated them by actually reducing them back to the contour integrals of the preceding sections and appendix A.

Perhaps norms for W-algebras are simple [Mukhin, Varchenko; 2000] .
A few generalized Selberg integrals are in [Warnaar; 0708.1193] [Warnaar; 0901.4176] .

From Virasoro to W-algebras

Complete Virasoro solution allows us to go back and compute

$$\int \mathrm{d}^2 z_1 \ldots \mathrm{d}^2 z_n \prod_{i=1}^{n} |z_i|^{2\alpha} |z_i - 1|^{2\beta} \prod_{i<j} |z_{ij}|^{4\gamma}.$$

> (B.10), are divergent, in general, and have to be regularized somehow. We defined and calculated them by actually reducing them back to the contour integrals of the preceding sections and appendix A.

Perhaps norms for W-algebras are simple [Mukhin, Varchenko; 2000] .
A few generalized Selberg integrals are in [Warnaar; 0708.1193] [Warnaar; 0901.4176] .

$$\frac{\Gamma(\Delta)}{(t_i - t_j)^{\Delta}} = \int_{-i\infty}^{i\infty} \frac{\mathrm{d}s}{2\pi i} \Gamma(-s)\Gamma(\Delta + s) t_i^s (-t_j)^{-\Delta - s}$$

Use `MB.m` to extract most singular term in $\epsilon = 2\alpha_-^2 - 1$.

Deforming different types of screening charges

Explicit examples seem to be quite recent

[Belavin, Estienne, Foda, Santachiara; 1602.03870] [Coman, Pomoni, Teschner; 1712.10225] .

Deforming different types of screening charges

Explicit examples seem to be quite recent

[Belavin, Estienne, Foda, Santachiara; 1602.03870] [Coman, Pomoni, Teschner; 1712.10225] .

$$\left\langle \phi_{(\mathbf{0};1,1)}\phi_{(\mathbf{0};1,0)}\phi_{(\mathbf{0};1,1)}\phi_{(\mathbf{0};1,2)} \right\rangle = \oint \mathrm{d}t_1 \mathrm{d}t_2 \left[t_1(t_1 - z)(t_1 - 1)t_2(t_2 - 1)t_{12}\right]^{-2\alpha_-^2}$$

Deforming different types of screening charges

Explicit examples seem to be quite recent

[Belavin, Estienne, Foda, Santachiara; 1602.03870] [Coman, Pomoni, Teschner; 1712.10225] .

$$\left\langle \phi_{(\mathbf{0};1,1)}\phi_{(\mathbf{0};1,0)}\phi_{(\mathbf{0};1,1)}\phi_{(\mathbf{0};1,2)} \right\rangle = \oint \mathrm{d}t_1 \mathrm{d}t_2 \left[t_1(t_1 - z)(t_1 - 1)t_2(t_2 - 1)t_{12} \right]^{-2\alpha_-^2}$$

t_1 interval	t_2 interval
$(-\infty, 0)$	$(-\infty, t_1)$, $(t_1, 0)$, $(0, 1)$, $(1, \infty)$
$(0, z)$	$(-\infty, 0)$, $(0, t_1)$, $(t_1, 1)$, $(1, \infty)$
$(z, 1)$	$(-\infty, 0)$, $(0, t_1)$, $(t_1, 1)$, $(1, \infty)$
$(1, \infty)$	$(-\infty, 0)$, $(0, 1)$, $(1, t_1)$, (t_1, ∞)

Deforming different types of screening charges

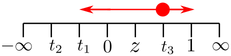
Explicit examples seem to be quite recent

[Belavin, Estienne, Foda, Santachiara; 1602.03870] [Coman, Pomoni, Teschner; 1712.10225] .

$$\left\langle \phi_{(\mathbf{0};1,1)}\phi_{(\mathbf{0};1,0)}\phi_{(\mathbf{0};1,1)}\phi_{(\mathbf{0};1,2)} \right\rangle = \oint \mathrm{d}t_1 \mathrm{d}t_2 \left[t_1(t_1 - z)(t_1 - 1) t_2 (t_2 - 1) t_{12} \right]^{-2\alpha_-^2}$$

t_1 interval	t_2 interval
$(-\infty, 0)$	$(-\infty, t_1)$, $(t_1, 0)$, $(0, 1)$, $(1, \infty)$
$(0, z)$	$(-\infty, 0)$, $(0, t_1)$, $(t_1, 1)$, $(1, \infty)$
$(z, 1)$	$(-\infty, 0)$, $(0, t_1)$, $(t_1, 1)$, $(1, \infty)$
$(1, \infty)$	$(-\infty, 0)$, $(0, 1)$, $(1, t_1)$, (t_1, ∞)

Imagine our integrand is $\left[t_1(t_1 - z)(t_3 - 1) t_{12} t_{13} t_{23} \right]^{-2\alpha_-^2}$.



$\int_{t_1}^{1} \mathrm{d}t_3$

Deforming different types of screening charges

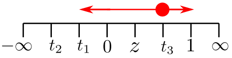
Explicit examples seem to be quite recent

[Belavin, Estienne, Foda, Santachiara; 1602.03870] [Coman, Pomoni, Teschner; 1712.10225] .

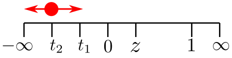
$$\left\langle \phi_{(\mathbf{0};1,1)}\phi_{(\mathbf{0};1,0)}\phi_{(\mathbf{0};1,1)}\phi_{(\mathbf{0};1,2)}\right\rangle = \oint \mathrm{d}t_1\mathrm{d}t_2\,[t_1(t_1-z)(t_1-1)t_2(t_2-1)t_{12}]^{-2\alpha_-^2}$$

t_1 interval	t_2 interval
$(-\infty, 0)$	$(-\infty, t_1)$, $(t_1, 0)$, $(0, 1)$, $(1, \infty)$
$(0, z)$	$(-\infty, 0)$, $(0, t_1)$, $(t_1, 1)$, $(1, \infty)$
$(z, 1)$	$(-\infty, 0)$, $(0, t_1)$, $(t_1, 1)$, $(1, \infty)$
$(1, \infty)$	$(-\infty, 0)$, $(0, 1)$, $(1, t_1)$, (t_1, ∞)

Imagine our integrand is $[t_1(t_1-z)(t_3-1)t_{12}t_{13}t_{23}]^{-2\alpha_-^2}$.



$\int_{t_1}^{1} \mathrm{d}t_3$



$\int_{-\infty}^{t_1} \mathrm{d}t_2 \int_{t_1}^{1} \mathrm{d}t_3$

Deforming different types of screening charges

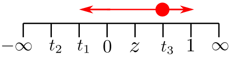
Explicit examples seem to be quite recent

[Belavin, Estienne, Foda, Santachiara; 1602.03870] [Coman, Pomoni, Teschner; 1712.10225] .

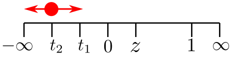
$$\left\langle \phi_{(\mathbf{0};1,1)}\phi_{(\mathbf{0};1,0)}\phi_{(\mathbf{0};1,1)}\phi_{(\mathbf{0};1,2)}\right\rangle = \oint \mathrm{d}t_1\mathrm{d}t_2\,[t_1(t_1-z)(t_1-1)t_2(t_2-1)t_{12}]^{-2\alpha_-^2}$$

t_1 interval	t_2 interval
$(-\infty,0)$	$(-\infty,t_1)$, $(t_1,0)$, $(0,1)$, $(1,\infty)$
$(0,z)$	$(-\infty,0)$, $(0,t_1)$, $(t_1,1)$, $(1,\infty)$
$(z,1)$	$(-\infty,0)$, $(0,t_1)$, $(t_1,1)$, $(1,\infty)$
$(1,\infty)$	$(-\infty,0)$, $(0,1)$, $(1,t_1)$, (t_1,∞)

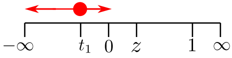
Imagine our integrand is $[t_1(t_1-z)(t_3-1)t_{12}t_{13}t_{23}]^{-2\alpha_-^2}$.



$\int_{t_1}^{1}\mathrm{d}t_3$



$\int_{-\infty}^{t_1}\mathrm{d}t_2\int_{t_1}^{1}\mathrm{d}t_3$



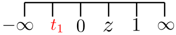
$\int_{-\infty}^{0}\mathrm{d}t_1\int_{-\infty}^{t_1}\mathrm{d}t_2\int_{t_1}^{1}\mathrm{d}t_3$

Contour validation

The integrand $[t_1(t_1 - z)(t_3 - 1)t_{12}t_{13}t_{23}]^{-2\alpha_-^2}$ has a problem with $\int_{-\infty}^{0} \mathrm{d}t_1 \int_{t_1}^{\infty} \mathrm{d}t_2 \int_{t_1}^{1} \mathrm{d}t_3$.

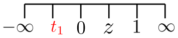
Contour validation

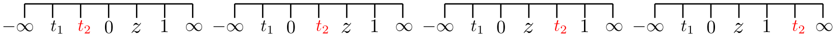
The integrand $[t_1(t_1 - z)(t_3 - 1)t_{12}t_{13}t_{23}]^{-2\alpha_-^2}$ has a problem with $\int_{-\infty}^{0} \mathrm{d}t_1 \int_{t_1}^{\infty} \mathrm{d}t_2 \int_{t_1}^{1} \mathrm{d}t_3$.



Contour validation

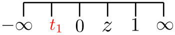
The integrand $[t_1(t_1 - z)(t_3 - 1)t_{12}t_{13}t_{23}]^{-2\alpha_-^2}$ has a problem with $\int_{-\infty}^{0} \mathrm{d}t_1 \int_{t_1}^{\infty} \mathrm{d}t_2 \int_{t_1}^{1} \mathrm{d}t_3$.

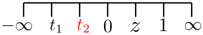




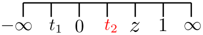
Contour validation

The integrand $[t_1(t_1 - z)(t_3 - 1)t_{12}t_{13}t_{23}]^{-2\alpha_-^2}$ has a problem with $\int_{-\infty}^{0} \mathrm{d}t_1 \int_{t_1}^{\infty} \mathrm{d}t_2 \int_{t_1}^{1} \mathrm{d}t_3$.

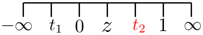




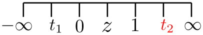
Bad



Bad



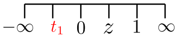
Bad

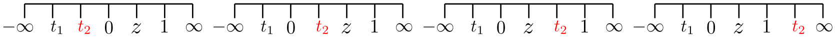


Good

Contour validation

The integrand $[t_1(t_1 - z)(t_3 - 1)t_{12}t_{13}t_{23}]^{-2\alpha_-^2}$ has a problem with $\int_{-\infty}^{0} \mathrm{d}t_1 \int_{t_1}^{\infty} \mathrm{d}t_2 \int_{t_1}^{1} \mathrm{d}t_3$.





Bad Bad Bad Good

t_1 interval	t_2 interval
$(-\infty, 0)$	$(-\infty, t_1)$, $(t_1, 0)$, $(0, 1)$, $(1, \infty)$
$(0, z)$	$(-\infty, 0)$, $(0, t_1)$, $(t_1, 1)$, $(1, \infty)$
$(z, 1)$	$(-\infty, 0)$, $(0, t_1)$, $(t_1, 1)$, $(1, \infty)$
$(1, \infty)$	$(-\infty, 0)$, $(0, 1)$, $(1, t_1)$, (t_1, ∞)

Returning to the 16 contour problem, 8 relations are easy.

$$\int_{-\infty}^{0} \mathrm{d}t_1 \left[\int_{-\infty}^{t_1} + e^{\pm 2\pi i \alpha_-^2} \int_{t_1}^{0} + e^{\pm 4\pi i \alpha_-^2} \int_{0}^{1} + e^{\pm 6\pi i \alpha_-^2} \int_{1}^{\infty} \right] \mathrm{d}t_2 = 0$$

Finishing the example

Trying $[t_1(t_1 - z)(t_1 - 1)t_2(t_2 - 1)t_{12}]^{-2\alpha_-^2}$ with t_2 first,

$$\int_0^1 \mathrm{d}t_2 \left[\int_{-\infty}^0 \mathrm{d}t_1 + \dots \right] = 0.$$

Finishing the example

Trying $[t_1(t_1 - z)(t_1 - 1)t_2(t_2 - 1)t_{12}]^{-2\alpha_-^2}$ with t_2 first,

$$\int_0^1 \mathrm{d}t_2 \left[\int_{-\infty}^{0} \mathrm{d}t_1 + e^{\pm 2\pi i \alpha_-^2} \int_0^{t_2 ? z ?} \mathrm{d}t_1 + \dots \right] = 0.$$

Finishing the example

Trying $[t_1(t_1 - z)(t_1 - 1)t_2(t_2 - 1)t_{12}]^{-2\alpha_-^2}$ with t_2 first,

$$\int_0^1 \mathrm{d}t_2 \left[\int_{-\infty}^{0} \mathrm{d}t_1 + e^{\pm 2\pi i \alpha_-^2} \int_0^{t_2?z?} \mathrm{d}t_1 + \dots \right] = 0.$$

Split $(0,1) = (0,z) \cup (z,1)$ even though $t_2 \to z$ is non-singular.

$$\int_0^1 \mathrm{d}t_2 \int_{-\infty}^{0} \mathrm{d}t_1 + \int_0^z \mathrm{d}t_2 \left[e^{\pm 2\pi i \alpha_-^2} \int_0^{t_2} + e^{\pm 4\pi i \alpha_-^2} \int_{t_2}^{z} + e^{\pm 6\pi i \alpha_-^2} \int_z^1 \right] \mathrm{d}t_1 +$$

$$\int_z^1 \mathrm{d}t_2 \left[e^{\pm 2\pi i \alpha_-^2} \int_0^z + e^{\pm 4\pi i \alpha_-^2} \int_z^{t_2} + e^{\pm 6\pi i \alpha_-^2} \int_{t_2}^1 \right] \mathrm{d}t_1 + e^{\pm 8\pi i \alpha_-^2} \int_0^1 \mathrm{d}t_2 \int_1^{\infty} \mathrm{d}t_1 = 0$$

Finishing the example

Trying $[t_1(t_1 - z)(t_1 - 1)t_2(t_2 - 1)t_{12}]^{-2\alpha_-^2}$ with t_2 first,

$$\int_0^1 dt_2 \left[\int_{-\infty}^0 dt_1 + e^{\pm 2\pi i \alpha_-^2} \int_0^{t_2 ? z ?} dt_1 + \dots \right] = 0.$$

Split $(0,1) = (0,z) \cup (z,1)$ even though $t_2 \to z$ is non-singular.

$$\int_0^1 dt_2 \int_{-\infty}^0 dt_1 + \int_0^z dt_2 \left[e^{\pm 2\pi i \alpha_-^2} \int_0^{t_2} + e^{\pm 4\pi i \alpha_-^2} \int_{t_2}^z + e^{\pm 6\pi i \alpha_-^2} \int_z^1 \right] dt_1 +$$

$$\int_z^1 dt_2 \left[e^{\pm 2\pi i \alpha_-^2} \int_0^z + e^{\pm 4\pi i \alpha_-^2} \int_z^{t_2} + e^{\pm 6\pi i \alpha_-^2} \int_{t_2}^1 \right] dt_1 + e^{\pm 8\pi i \alpha_-^2} \int_0^1 dt_2 \int_1^\infty dt_1 = 0$$

Change of order makes this

$$\int_{-\infty}^0 dt_1 \int_0^1 dt_2 + e^{\pm 8\pi i \alpha_-^2} \int_1^\infty dt_1 \int_0^1 dt_2 +$$

$$e^{\pm 2\pi i \alpha_-^2} \int_0^z dt_1 \int_{t_1}^z dt_2 + e^{\pm 4\pi i \alpha_-^2} \int_0^z dt_1 \int_0^{t_1} dt_2 + e^{\pm 6\pi i \alpha_-^2} \int_z^1 dt_1 \int_0^z dt_2 +$$

$$e^{\pm 2\pi i \alpha_-^2} \int_0^z dt_1 \int_z^1 dt_2 + e^{\pm 4\pi i \alpha_-^2} \int_z^1 dt_1 \int_{t_1}^1 dt_2 + e^{\pm 6\pi i \alpha_-^2} \int_z^1 dt_1 \int_z^{t_1} dt_2 = 0.$$

Finishing the example

Trying $[t_1(t_1-z)(t_1-1)t_2(t_2-1)t_{12}]^{-2\alpha_-^2}$ with t_2 first,

$$\int_0^1 dt_2 \left[\int_{-\infty}^0 dt_1 + e^{\pm 2\pi i\alpha_-^2}\int_0^{t_2?z?} dt_1 + \dots\right] = 0.$$

Split $(0,1) = (0,z)\cup(z,1)$ even though $t_2 \to z$ is non-singular.

$$\int_0^1 dt_2 \int_{-\infty}^0 dt_1 + \int_0^z dt_2 \left[e^{\pm 2\pi i\alpha_-^2}\int_0^{t_2} + e^{\pm 4\pi i\alpha_-^2}\int_{t_2}^z + e^{\pm 6\pi i\alpha_-^2}\int_z^1\right] dt_1 +$$

$$\int_z^1 dt_2 \left[e^{\pm 2\pi i\alpha_-^2}\int_0^z + e^{\pm 4\pi i\alpha_-^2}\int_z^{t_2} + e^{\pm 6\pi i\alpha_-^2}\int_{t_2}^1\right] dt_1 + e^{\pm 8\pi i\alpha_-^2}\int_0^1 dt_2 \int_1^\infty dt_1 = 0$$

Change of order makes this

$$\int_{-\infty}^0 dt_1 \int_0^1 dt_2 + e^{\pm 8\pi i\alpha_-^2}\int_1^\infty dt_1 \int_0^1 dt_2 +$$

$$e^{\pm 2\pi i\alpha_-^2}\int_0^z dt_1 \int_{t_1}^z dt_2 + e^{\pm 4\pi i\alpha_-^2}\int_0^z dt_1 \int_0^{t_1} dt_2 + e^{\pm 6\pi i\alpha_-^2}\int_z^1 dt_1 \int_0^z dt_2 +$$

$$e^{\pm 2\pi i\alpha_-^2}\int_0^z dt_1 \int_z^1 dt_2 + e^{\pm 4\pi i\alpha_-^2}\int_z^1 dt_1 \int_{t_1}^1 dt_2 + e^{\pm 6\pi i\alpha_-^2}\int_z^1 dt_1 \int_z^{t_1} dt_2 = 0.$$

Finishing the example

Trying $[t_1(t_1-z)(t_1-1)t_2(t_2-1)t_{12}]^{-2\alpha_-^2}$ with t_2 first,

$$\int_0^1 \mathrm{d}t_2 \left[\int_{-\infty}^0 \mathrm{d}t_1 + e^{\pm 2\pi i\alpha_-^2}\int_0^{t_2?z?} \mathrm{d}t_1 + \dots\right] = 0.$$

Split $(0,1) = (0,z) \cup (z,1)$ even though $t_2 \to z$ is non-singular.

$$\int_0^1 \mathrm{d}t_2 \int_{-\infty}^0 \mathrm{d}t_1 + \int_0^z \mathrm{d}t_2 \left[e^{\pm 2\pi i\alpha_-^2}\int_0^{t_2} + e^{\pm 4\pi i\alpha_-^2}\int_{t_2}^z + e^{\pm 6\pi i\alpha_-^2}\int_z^1\right] \mathrm{d}t_1 +$$
$$\int_z^1 \mathrm{d}t_2 \left[e^{\pm 2\pi i\alpha_-^2}\int_0^z + e^{\pm 4\pi i\alpha_-^2}\int_z^{t_2} + e^{\pm 6\pi i\alpha_-^2}\int_{t_2}^1\right] \mathrm{d}t_1 + e^{\pm 8\pi i\alpha_-^2}\int_0^1 \mathrm{d}t_2 \int_1^\infty \mathrm{d}t_1 = 0$$

Change of order makes this

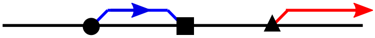
$$\int_{-\infty}^0 \mathrm{d}t_1 \int_0^1 \mathrm{d}t_2 + e^{\pm 8\pi i\alpha_-^2}\int_1^\infty \mathrm{d}t_1 \int_0^1 \mathrm{d}t_2 +$$
$$e^{\pm 2\pi i\alpha_-^2}\int_0^z \mathrm{d}t_1 \int_{t_1}^1 \mathrm{d}t_2 + e^{\pm 4\pi i\alpha_-^2}\int_0^z \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 + e^{\pm 6\pi i\alpha_-^2}\int_z^1 \mathrm{d}t_1 \int_0^{t1} \mathrm{d}t_2 +$$
$$e^{\pm 4\pi i\alpha_-^2}\int_z^1 \mathrm{d}t_1 \int_{t_1}^1 \mathrm{d}t_2 \qquad = 0.$$

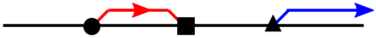
Symmetry relations

The ways to reorder n integrals are elements of $G = S_n$.
Screening charge assignment of $V_{\alpha_{-}\alpha_1}^{n_1} \dots V_{\alpha_{-}\alpha_{rank(\mathfrak{g})}}^{n_{rank(\mathfrak{g})}}$ has symmetry group $H = S_{n_1} \times \cdots \times S_{n_{rank(\mathfrak{g})}}$.

Symmetry relations

The ways to reorder n integrals are elements of $G = S_n$.
Screening charge assignment of $V_{\alpha-\alpha_1}^{n_1} \dots V_{\alpha-\alpha_{rank(\mathfrak{g})}}^{n_{rank(\mathfrak{g})}}$ has symmetry group $H = S_{n_1} \times \cdots \times S_{n_{rank(\mathfrak{g})}}$.





Symmetry relations

The ways to reorder n integrals are elements of $G = S_n$.
Screening charge assignment of $V^{n_1}_{\alpha_-\alpha_1} \dots V^{n_{rank(\mathfrak{g})}}_{\alpha_-\alpha_{rank(\mathfrak{g})}}$ has symmetry group $H = S_{n_1} \times \dots \times S_{n_{rank(\mathfrak{g})}}$.

$$\left\langle \phi_{(\mathbf{0};0,1,0)}(0)\phi_{(\mathbf{0};0,1,0)}(z)\phi_{(\mathbf{0};0,1,0)}(1)\phi_{(\mathbf{0};0,1,0)}(\infty)\right\rangle = \oint \mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4$$

$$[t_1 t_2 (t_1 - z)(t_2 - z)(t_1 - 1)(t_2 - 1)]^{-2\alpha_-^2} t_{12}^{4\alpha_-^2} [t_{13} t_{14} t_{23} t_{24}]^{-2\alpha_-^2}$$

Algorithm generates 180 contours, 90 after symmetries, 10 after easy relations, 3 after hard relations.

Symmetry relations

The ways to reorder n integrals are elements of $G = S_n$.
Screening charge assignment of $V_{\alpha_-\alpha_1}^{n_1} \dots V_{\alpha_-\alpha_{rank(\mathfrak{g})}}^{n_{rank(\mathfrak{g})}}$ has symmetry group $H = S_{n_1} \times \dots \times S_{n_{rank(\mathfrak{g})}}$.

$$\left\langle \phi_{(\mathbf{0};0,1,0)}(0)\phi_{(\mathbf{0};0,1,0)}(z)\phi_{(\mathbf{0};0,1,0)}(1)\phi_{(\mathbf{0};0,1,0)}(\infty)\right\rangle = \oint \mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4$$

$$[t_1 t_2(t_1 - z)(t_2 - z)(t_1 - 1)(t_2 - 1)]^{-2\alpha_-^2} t_{12}^{4\alpha_-^2} [t_{13} t_{14} t_{23} t_{24}]^{-2\alpha_-^2}$$

Algorithm generates 180 contours, 90 after symmetries, 10 after easy relations, 3 after hard relations.

- $\int_0^z \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_{t_2}^{t_1} \mathrm{d}t_3 \int_{t_2}^{t_1} \mathrm{d}t_4 \Rightarrow \phi_{(\mathbf{0};0,0,0)}$
- $\int_0^z \mathrm{d}t_1 \int_1^{\infty} \mathrm{d}t_2 \int_{t_1}^{t_2} \mathrm{d}t_3 \int_{t_1}^{t_2} \mathrm{d}t_4 \Rightarrow \phi_{(\mathbf{0};1,0,1)}$
- $\int_1^{\infty} \mathrm{d}t_1 \int_1^{t_1} \mathrm{d}t_2 \int_{t_2}^{t_1} \mathrm{d}t_3 \int_{t_2}^{t_1} \mathrm{d}t_4 \Rightarrow \phi_{(\mathbf{0};0,2,0)}$

Good and bad mixed correlators

- We can extract C_{HHL} from $\langle HHHH \rangle$ but $\langle HHLL \rangle$ is better.
- To get different powers of z, at least one $V_{\alpha_{-\alpha_i}}(t)$ must be integrated from 0 to z.
- Consider $\left\langle \phi_{(\mathbf{0};\lambda_1)}(0)\phi_{(\mathbf{0};\lambda_2)}(z)\phi_{(\mathbf{0};\lambda_3)}(1)\phi_{(\mathbf{0};\lambda_4)}(\infty) \right\rangle$. Left end on $0 \Rightarrow (\lambda_1, \alpha_i) \neq 0$. Right end on $z \Rightarrow (\lambda_2, \alpha_i) \neq 0$.

Good and bad mixed correlators

- We can extract C_{HHL} from $\langle HHHH \rangle$ but $\langle HHLL \rangle$ is better.
- To get different powers of z, at least one $V_{\alpha_{-\alpha_i}}(t)$ must be integrated from 0 to z.
- Consider $\left\langle \phi_{(\mathbf{0};\lambda_1)}(0)\phi_{(\mathbf{0};\lambda_2)}(z)\phi_{(\mathbf{0};\lambda_3)}(1)\phi_{(\mathbf{0};\lambda_4)}(\infty)\right\rangle$. Left end on $0 \Rightarrow (\lambda_1, \alpha_i) \neq 0$. Right end on $z \Rightarrow (\lambda_2, \alpha_i) \neq 0$.

We still need the matrix elements $\alpha_{km}^{(m)}$, to use formula (2.13) and find the structure constants X_k of the four-point correlation functions. One can check that the technique, which is used above, does not lead to easy calculations in the case of the matrix elements α_{km}. So, we use an alternative way.

The other crossing matrix derivation

Virasoro basis element with $y = 1 - z$.

$$I_*(y) = \int_1^\infty \mathrm{d}t_1 \cdots \int_1^{t_{m-1}} \mathrm{d}t_m \int_0^{1-y} \mathrm{d}t_{m+1} \cdots \int_0^{t_{n-1}} \mathrm{d}t_n$$

$$\prod_{i=1}^{m} t_i^a (t_i - 1 + y)^b (t_i - 1)^c \prod_{i=m+1}^{n} t_i^a (1 - y - t_i)^b (1 - t_i)^c \prod_{i<j} t_{ij}^{2q}$$

The other crossing matrix derivation

Virasoro basis element with $y = 1 - z$.

$$I_*(y) = \int_1^\infty \mathrm{d}t_1 \cdots \int_1^{t_{m-1}} \mathrm{d}t_m \int_0^{1-y} \mathrm{d}t_{m+1} \cdots \int_0^{t_{n-1}} \mathrm{d}t_n$$
$$\prod_{i=1}^{m} t_i^a (t_i - 1 + y)^b (t_i - 1)^c \prod_{i=m+1}^{n} t_i^a (1 - y - t_i)^b (1 - t_i)^c \prod_{i<j} t_{ij}^{2q}$$

With $t_1, \ldots t_m$, perform $t_i \mapsto 1 + t_i y$ on last k.
With $t_{m+1}, \ldots t_n$, perform $t_i \mapsto 1 - t_i y$ on first l.

The other crossing matrix derivation

Virasoro basis element with $y = 1 - z$.

$$I_*(y) = \int_1^\infty \mathrm{d}t_1 \cdots \int_1^{t_{m-1}} \mathrm{d}t_m \int_0^{1-y} \mathrm{d}t_{m+1} \cdots \int_0^{t_{n-1}} \mathrm{d}t_n$$

$$\prod_{i=1}^{m} t_i^a (t_i - 1 + y)^b (t_i - 1)^c \prod_{i=m+1}^{n} t_i^a (1 - y - t_i)^b (1 - t_i)^c \prod_{i<j} t_{ij}^{2q}$$

With $t_1, \ldots t_m$, perform $t_i \mapsto 1 + t_i y$ on all of them.
With $t_{m+1}, \ldots t_n$, perform $t_i \mapsto 1 - t_i y$ on all of them.

The other crossing matrix derivation

Virasoro basis element with $y = 1 - z$.

$$I_*(y) = y^{n[(n-1)q+b+c+1]} \int_0^\infty \mathrm{d}t_1 \cdots \int_0^{t_{m-1}} \mathrm{d}t_m \int_1^{1/y} \mathrm{d}t_{m+1} \cdots \int_{t_{n-1}}^{1/y} \mathrm{d}t_n$$

$$\prod_{i=1}^{m} (1 + t_i y)^a (t_i + 1)^b t_i^c \prod_{i=m+1}^{n} (1 - t_i y)^a (t_i - 1)^b t_i^c$$

$$\prod_{\substack{i,j=1 \\ i<j}}^{m} t_{ij}^{2q} \prod_{\substack{i,j=m+1 \\ i<j}}^{n} t_{ji}^{2q} \prod_{i=1}^{m} \prod_{j=m+1}^{n} (t_i + t_j)^{2q}$$

With $t_1, \ldots t_m$, perform $t_i \mapsto 1 + t_i y$ on all of them.
With $t_{m+1}, \ldots t_n$, perform $t_i \mapsto 1 - t_i y$ on all of them.

The other crossing matrix derivation

Virasoro basis element with $y = 1 - z$.

$$I_*(y) \approx y^{n[(n-1)q+b+c+1]} \int_0^\infty \mathrm{d}t_1 \cdots \int_0^{t_{m-1}} \mathrm{d}t_m \int_1^\infty \mathrm{d}t_{m+1} \cdots \int_{t_{n-1}}^\infty \mathrm{d}t_n$$

$$\prod_{i=1}^{m} (t_i + 1)^b t_i^c \prod_{i=m+1}^{n} (t_i - 1)^b t_i^c$$

$$\prod_{\substack{i,j=1 \\ i<j}}^{m} t_{ij}^{2q} \prod_{\substack{i,j=m+1 \\ i<j}}^{n} t_{ji}^{2q} \prod_{i=1}^{m} \prod_{j=m+1}^{n} (t_i + t_j)^{2q}$$

With $t_1, \ldots t_m$, perform $t_i \mapsto 1 + t_i y$ on all of them.
With $t_{m+1}, \ldots t_n$, perform $t_i \mapsto 1 - t_i y$ on all of them.
We can now zoom in on a single power of y.

The other crossing matrix derivation

Virasoro basis element with $y = 1 - z$.

$$I_*(y) \approx y^{n[(n-1)q+b+c+1]} \int_0^\infty \mathrm{d}t_1 \cdots \int_0^{t_{m-1}} \mathrm{d}t_m \int_1^\infty \mathrm{d}t_{m+1} \cdots \int_{t_{n-1}}^\infty \mathrm{d}t_n$$

$$\prod_{i=1}^{m} (t_i+1)^b t_i^c \prod_{i=m+1}^{n} (t_i-1)^b t_i^c$$

$$\prod_{\substack{i,j=1 \\ i<j}}^{m} t_{ij}^{2q} \prod_{\substack{i,j=m+1 \\ i<j}}^{n} t_{ji}^{2q} \prod_{i=1}^{m} \prod_{j=m+1}^{n} (t_i+t_j)^{2q}$$

With $t_1, \ldots t_m$, perform $t_i \mapsto 1 + t_i y$ on all of them.
With $t_{m+1}, \ldots t_n$, perform $t_i \mapsto 1 - t_i y$ on all of them.
We can now zoom in on a single power of y.
Does not increase difficulty of $\epsilon = 2\alpha_-^2 - 1$ expansion.

A bad mixed correlator

Two exchanged blocks but only one possible local exponent.

$$\left\langle \phi_{(\mathbf{0};1,0)}\phi_{(\mathbf{0};0,1)}\phi_{(\mathbf{0};1,0)}\phi_{(\mathbf{0};0,1)} \right\rangle = \oint \mathrm{d}t_1 \mathrm{d}t_2 \left[t_1(t_2 - z)(t_1 - 1)t_{12} \right]^{-2\alpha_-^2}$$

A bad mixed correlator

Two exchanged blocks but only one possible local exponent.

$$\left\langle \phi_{(\mathbf{0};1,0)}\phi_{(\mathbf{0};0,1)}\phi_{(\mathbf{0};1,0)}\phi_{(\mathbf{0};0,1)} \right\rangle = \oint \mathrm{d}t_1 \mathrm{d}t_2 \left[t_1(t_2 - z)(t_1 - 1)t_{12} \right]^{-2\alpha_-^2}$$

Monodromy relations narrow the basis down to two integrals.

$$I_1(z) = \int_{-\infty}^{0} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \ldots, \quad I_2(z) = \int_{1}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{z} \mathrm{d}t_2 \ldots$$

A bad mixed correlator

Two exchanged blocks but only one possible local exponent.

$$\left\langle \phi_{(\mathbf{0};1,0)}\phi_{(\mathbf{0};0,1)}\phi_{(\mathbf{0};1,0)}\phi_{(\mathbf{0};0,1)} \right\rangle = \oint \mathrm{d}t_1 \mathrm{d}t_2 \left[t_1(t_2 - z)(t_1 - 1)t_{12} \right]^{-2\alpha_-^2}$$

Monodromy relations narrow the basis down to two integrals.

$$I_1(z) = \int_{-\infty}^{0} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \ldots, \quad I_2(z) = \int_{1}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{z} \mathrm{d}t_2 \ldots$$

Crossing symmetry only fixes $X_2 = X_1$.

$$\frac{G(z,\bar{z})}{|z|^{\frac{32}{3}\alpha_-^2 - 4}} = X_0[I_1(z)I_2(z)^* + I_1(z)^* I_2(z)] + X_1|I_1(z)|^2 + X_2|I_2(z)|^2$$

Exchanges of $\phi_{(\mathbf{0};1,1)}$ and $\phi_{(\mathbf{0};0,0)}$ correspond to z^0 and $z^{2-6\alpha_-^2}$.

A bad mixed correlator

For small z in s-channel and small y in t-channel,

$$I_1(z) = -\frac{1}{3(2\alpha_-^2 - 1)^2}[1 + O(z)], \quad I_2(z) = \frac{1}{(2\alpha_-^2 - 1)^2}[1 + O(z)].$$

A bad mixed correlator

For small z in s-channel and small y in t-channel,

$$I_1(z) = -\frac{1}{3(2\alpha_-^2 - 1)^2}[1 + O(z)], \quad I_2(z) = \frac{1}{(2\alpha_-^2 - 1)^2}[1 + O(z)].$$

$$I_1(y) = \frac{1}{(2\alpha_-^2 - 1)^2}[1 + O(y)]$$

$$I_2(y) = -\frac{1}{3(2\alpha_-^2 - 1)^2}[1 + O(y)] + \frac{1}{(2\alpha_-^2 - 1)^2} y^{2-6\alpha_-^2}[1 + O(y)].$$

A bad mixed correlator

For small z in s-channel and small y in t-channel,

$$I_1(z) = -\frac{1}{3(2\alpha_-^2 - 1)^2}[1 + O(z)], \quad I_2(z) = \frac{1}{(2\alpha_-^2 - 1)^2}[1 + O(z)].$$

$$I_1(y) = \frac{1}{(2\alpha_-^2 - 1)^2}[1 + O(y)]$$

$$I_2(y) = -\frac{1}{3(2\alpha_-^2 - 1)^2}[1 + O(y)] + \frac{1}{(2\alpha_-^2 - 1)^2} y^{2-6\alpha_-^2}[1 + O(y)].$$

Trivial monodromy in y now fixes $X_0 = \frac{1}{3}X_2$.

$$\frac{G(z,\bar{z})}{|z|^{\frac{32}{3}\alpha_-^2 - 4}} = (2\alpha_-^2 - 1)^4 \left[|I_1(z)|^2 + |I_2(z)|^2 + \frac{2}{3} Re I_1(z) I_2(z)^* \right]$$

A bad mixed correlator

For small z in s-channel and small y in t-channel,

$$I_1(z) = -\frac{1}{3(2\alpha_-^2 - 1)^2}[1 + O(z)], \quad I_2(z) = \frac{1}{(2\alpha_-^2 - 1)^2}[1 + O(z)].$$

$$I_1(y) = \frac{1}{(2\alpha_-^2 - 1)^2}[1 + O(y)]$$

$$I_2(y) = -\frac{1}{3(2\alpha_-^2 - 1)^2}[1 + O(y)] + \frac{1}{(2\alpha_-^2 - 1)^2} y^{2-6\alpha_-^2}[1 + O(y)].$$

Trivial monodromy in y now fixes $X_0 = \frac{1}{3}X_2$.

$$\frac{G(z,\bar{z})}{|z|^{\frac{32}{3}\alpha_-^2 - 4}} = (2\alpha_-^2 - 1)^4 \left[|I_1(z)|^2 + |I_2(z)|^2 + \frac{2}{3} Re I_1(z) I_2(z)^* \right]$$

$$\supset |z|^{16\alpha_-^2 - 4} \left[\frac{1}{4} + \frac{1}{4} + \frac{2}{3}\frac{1}{4} \right], \quad z \to \infty?$$

Expanding in the u-channel

First step in verifying $C^{(\mathbf{0};0,2)}_{(\mathbf{0};0,1)(\mathbf{0};0,1)} = \sqrt{\frac{2}{3}}$ comes from $t_i \mapsto (zt_i)^{-1}$.

$$I_1\left(\frac{1}{z}\right) = z^{8\alpha_-^2-2}\int_{-\infty}^{0}\frac{\mathrm{d}t_1}{t_1^2}\int_{t_1}^{0}\frac{\mathrm{d}t_2}{t_2^2}\left[\frac{(1-t_2)(t_2-t_1)(1-t_1 z)}{(-t_3)^3(-t_2)^2}\right]^{-2\alpha_-^2}$$

$$= \frac{-z^{8\alpha_-^2-2}}{2(2\alpha_-^2-1)^2}[1+O(z)]$$

Expanding in the u-channel

First step in verifying $C^{(\mathbf{0};0,2)}_{(\mathbf{0};0,1)(\mathbf{0};0,1)} = \sqrt{\frac{2}{3}}$ comes from $t_i \mapsto (zt_i)^{-1}$.

$$I_1\left(\frac{1}{z}\right) = z^{8\alpha_-^2-2}\int_{-\infty}^{0}\frac{dt_1}{t_1^2}\int_{t_1}^{0}\frac{dt_2}{t_2^2}\left[\frac{(1-t_2)(t_2-t_1)(1-t_1z)}{(-t_3)^3(-t_2)^2}\right]^{-2\alpha_-^2}$$

$$= \frac{-z^{8\alpha_-^2-2}}{2(2\alpha_-^2-1)^2}[1+O(z)]$$

For the next integral, try $t_i \mapsto t_i/z$.

$$I_2\left(\frac{1}{z}\right) = z^{-2}\int_{z}^{\infty}dt_1\int_{-\infty}^{1}dt_2\dots$$

Expanding in the u-channel

First step in verifying $C^{(\mathbf{0};0,2)}_{(\mathbf{0};0,1)(\mathbf{0};0,1)} = \sqrt{\frac{2}{3}}$ comes from $t_i \mapsto (zt_i)^{-1}$.

$$I_1\left(\frac{1}{z}\right) = z^{8\alpha_-^2-2}\int_{-\infty}^{0}\frac{\mathrm{d}t_1}{t_1^2}\int_{t_1}^{0}\frac{\mathrm{d}t_2}{t_2^2}\left[\frac{(1-t_2)(t_2-t_1)(1-t_1 z)}{(-t_3)^3(-t_2)^2}\right]^{-2\alpha_-^2}$$

$$= \frac{-z^{8\alpha_-^2-2}}{2(2\alpha_-^2-1)^2}[1+O(z)]$$

For the next integral, try $t_i \mapsto t_i/z$.

$$I_2\left(\frac{1}{z}\right) = z^{-2}\left[\int_z^1 \mathrm{d}t_1\int_{-\infty}^{t_1}\mathrm{d}t_2 + \int_z^1 \mathrm{d}t_1\int_{t_1}^1 \mathrm{d}t_2 + \int_1^\infty \mathrm{d}t_1\int_{-\infty}^1 \mathrm{d}t_2\right]\dots$$

Expanding in the u-channel

First step in verifying $C^{(\mathbf{0};0,2)}_{(\mathbf{0};0,1)(\mathbf{0};0,1)} = \sqrt{\frac{2}{3}}$ comes from $t_i \mapsto (zt_i)^{-1}$.

$$I_1\left(\frac{1}{z}\right) = z^{8\alpha_-^2-2}\int_{-\infty}^{0}\frac{\mathrm{d}t_1}{t_1^2}\int_{t_1}^{0}\frac{\mathrm{d}t_2}{t_2^2}\left[\frac{(1-t_2)(t_2-t_1)(1-t_1z)}{(-t_3)^3(-t_2)^2}\right]^{-2\alpha_-^2}$$
$$= \frac{-z^{8\alpha_-^2-2}}{2(2\alpha_-^2-1)^2}[1+O(z)]$$

For the next integral, try $t_i \mapsto t_i/z$.

$$I_2\left(\frac{1}{z}\right) = z^{-2}\left[\int_z^1\mathrm{d}t_1\int_{-\infty}^{t_1}\mathrm{d}t_2 + \int_z^1\mathrm{d}t_1\int_{t_1}^{1}\mathrm{d}t_2 + \int_1^{\infty}\mathrm{d}t_1\int_{-\infty}^{1}\mathrm{d}t_2\right]\dots$$

Write integrand as $\left(\frac{t_{12}}{z}\right)^{-2\alpha_-^2}\left[\frac{t_1}{z}\frac{1-t_2}{z}\frac{t_1-z}{z}\right]^{-2\alpha_-^2}$ or
$e^{-2\pi i\alpha_-^2}\left(\frac{t_{21}}{z}\right)^{-2\alpha_-^2}\left[\frac{t_1}{z}\frac{1-t_2}{z}\frac{t_1-z}{z}\right]^{-2\alpha_-^2}$ to get

$$I_2\left(\frac{1}{z}\right) = \frac{z^{8\alpha_-^2-2}}{(2\alpha_-^2-1)^2}\left(1-2+\frac{1}{2}\right)[1+O(z)].$$

The real test

Extending preturbative results of [Dotsenko, Nguyen, Santachiara; hep-th/0104197] requires $C^{(\mathbf{0};0,1,0,0,0,0)}_{(\mathbf{0};0,0,0,0,0,1)(\mathbf{0};0,0,0,0,0,1)}$ in $W[\hat{\mathfrak{d}}_6]$. Start with $W[\hat{\mathfrak{d}}_4]$.

The real test

Extending preturbative results of [Dotsenko, Nguyen, Santachiara; hep-th/0104197] requires $C^{(\mathbf{0};0,1,0,0,0,0)}_{(\mathbf{0};0,0,0,0,0,1)(\mathbf{0};0,0,0,0,0,1)}$ in $W[\hat{\mathfrak{d}}_6]$. Start with $W[\hat{\mathfrak{d}}_4]$.

- $\langle V(0)S(z)V(1)S(\infty)\rangle$ has 24 integrals and 22 relations.
- Exchanged $\phi_{(\mathbf{0};0,0,1,0)}$ is only in $I_1(z)$.
- Exchanged $\phi_{(\mathbf{0};1,0,0,1)}$ lives in $I_1(z)$ and image under $z \leftrightarrow 1-z$.
- $\langle V(0)V(z)S(1)S(\infty)\rangle$ has 648 integrals but only 632 relations.

The real test

Extending preturbative results of [Dotsenko, Nguyen, Santachiara; hep-th/0104197] requires $C^{(\mathbf{0};0,1,0,0,0,0)}_{(\mathbf{0};0,0,0,0,0,1)(\mathbf{0};0,0,0,0,0,1)}$ in $W[\hat{\mathfrak{d}}_6]$. Start with $W[\hat{\mathfrak{d}}_4]$.

- $\langle V(0)S(z)V(1)S(\infty)\rangle$ has 24 integrals and 22 relations.
- Exchanged $\phi_{(\mathbf{0};0,0,1,0)}$ is only in $I_1(z)$.
- Exchanged $\phi_{(\mathbf{0};1,0,0,1)}$ lives in $I_1(z)$ and image under $z \leftrightarrow 1-z$.
- $\langle V(0)V(z)S(1)S(\infty)\rangle$ has 648 integrals but only 632 relations.

One integral has $z^{6-16\alpha_-^2}$ for $\phi_{(\mathbf{0};0,0,0,0)}$.

$$I_1(z) = \int_0^z \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_{t_2}^{t_1} \mathrm{d}t_3 \int_{t_2}^{t_3} \mathrm{d}t_4 \int_{t_4}^{t_3} \mathrm{d}t_5 \int_{t_4}^{t_3} \mathrm{d}t_6 \dots$$

The real test

Extending preturbative results of [Dotsenko, Nguyen, Santachiara; hep-th/0104197] requires $C^{(\mathbf{0};0,1,0,0,0,0)}_{(\mathbf{0};0,0,0,0,0,1)(\mathbf{0};0,0,0,0,0,1)}$ in $W[\hat{\mathfrak{d}}_6]$. Start with $W[\hat{\mathfrak{d}}_4]$.

- $\langle V(0)S(z)V(1)S(\infty)\rangle$ has 24 integrals and 22 relations.
- Exchanged $\phi_{(\mathbf{0};0,0,1,0)}$ is only in $I_1(z)$.
- Exchanged $\phi_{(\mathbf{0};1,0,0,1)}$ lives in $I_1(z)$ and image under $z \leftrightarrow 1-z$.
- $\langle V(0)V(z)S(1)S(\infty)\rangle$ has 648 integrals but only 632 relations.

One integral has $z^{6-16\alpha_-^2}$ for $\phi_{(\mathbf{0};0,0,0,0)}$.

$$I_1(z) = \int_0^z \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_{t_2}^{t_1} \mathrm{d}t_3 \int_{t_2}^{t_3} \mathrm{d}t_4 \int_{t_4}^{t_3} \mathrm{d}t_5 \int_{t_4}^{t_3} \mathrm{d}t_6 \dots$$

Eight have $z^{1-4\alpha_-^2}$ for $\phi_{(\mathbf{0};0,1,0,0)}$, four can help us cancel $y^0\bar{y}^{3-8\alpha_-^2}$.

$$I_2(z) = \int_{-\infty}^0 \mathrm{d}t_1 \int_0^z \mathrm{d}t_2 \int_{-\infty}^{t_1} \mathrm{d}t_3 \int_{t_1}^{t_2} \mathrm{d}t_4 \int_{t_3}^{t_4} \mathrm{d}t_5 \int_{t_3}^{t_4} \mathrm{d}t_6 \dots ?$$

Others lead to “weird” numbers including $(2\alpha_-^2-1)^{-4}$ or $(2\alpha_-^2-1)^{-5}$.

The real test

Extending preturbative results of [Dotsenko, Nguyen, Santachiara; hep-th/0104197] requires $C^{(\mathbf{0};0,1,0,0,0,0)}_{(\mathbf{0};0,0,0,0,0,1)(\mathbf{0};0,0,0,0,0,1)}$ in $W[\hat{\mathfrak{d}}_6]$. Start with $W[\hat{\mathfrak{d}}_4]$.

- $\langle V(0)S(z)V(1)S(\infty)\rangle$ has 1536 integrals and 1528 relations.
- Exchanged $\phi_{(\mathbf{0};0,0,0,0,1,0)}$ is only in $I_1(z)$.
- Exchanged $\phi_{(\mathbf{0};1,0,0,0,0,1)}$ lives in $I_1(z)$ and image under $z \leftrightarrow 1-z$.
- $\langle V(0)V(z)S(1)S(\infty)\rangle$ has 40992 integrals, not enough relations.

One integral has $z^{10-24\alpha_-^2}$ for $\phi_{(\mathbf{0};0,0,0,0,0,0)}$.

$$I_1(z) = \int_0^z dt_1 \int_0^{t_1} dt_2 \int_{t_2}^{t_1} dt_3 \int_{t_2}^{t_3} dt_4 \int_{t_4}^{t_3} dt_5 \int_{t_4}^{t_5} dt_6 \int_{t_6}^{t_5} dt_7 \int_{t_6}^{t_7} dt_8 \int_{t_8}^{t_7} dt_9 \int_{t_8}^{t_7} dt_{10} \,.$$

Many have $z^{1-4\alpha_-^2}$ for $\phi_{(\mathbf{0};0,1,0,0,0,0)}$, fewer can help us cancel $y^0\bar{y}^{5-12\alpha_-^2}$.

$$I_2(z) = \int_{-\infty}^0 dt_1 \int_0^z dt_2 \int_{-\infty}^{t_1} dt_3 \int_{t_1}^{t_2} dt_4 \int_{-\infty}^{t_3} dt_5 \int_{t_3}^{t_4} dt_6 \int_{-\infty}^{t_5} dt_7 \int_{t_5}^{t_6} dt_8 \int_{t_7}^{t_8} dt_9 \int_{t_7}^{t_8} d$$

Others lead to “weird” numbers including $(2\alpha_-^2 - 1)^{-8}$ or $(2\alpha_-^2 - 1)^{-9}$.

The real test

To reproduce $C_{(\mathbf{0};1,0,0,0)(\mathbf{0};0,0,0,1)}^{(\mathbf{0};0,0,1,0)} = \sqrt{2}$ in $W[\hat{\mathfrak{d}}_4]$,

$$N_1^{-2}|I_1(z)|^2 + XN_2^{-2}|I_2(z)|^2 \Rightarrow X = \frac{7}{4}.$$

Notice that $X = C_{(\mathbf{0};1,0,0,0)(\mathbf{0};1,0,0,0)}^{(\mathbf{0};0,1,0,0)} C_{(\mathbf{0};0,0,0,1)(\mathbf{0};0,0,0,1)}^{(\mathbf{0};0,1,0,0)}$!

The real test

To reproduce $C^{(\mathbf{0};0,0,1,0)}_{(\mathbf{0};1,0,0,0)(\mathbf{0};0,0,0,1)} = \sqrt{2}$ in $W[\hat{\mathfrak{d}}_4]$,

$$N_1^{-2}|I_1(z)|^2 + XN_2^{-2}|I_2(z)|^2 \Rightarrow X = \frac{7}{4}.$$

Notice that $X = C^{(\mathbf{0};0,1,0,0)}_{(\mathbf{0};1,0,0,0)(\mathbf{0};1,0,0,0)} C^{(\mathbf{0};0,1,0,0)}_{(\mathbf{0};0,0,0,1)(\mathbf{0};0,0,0,1)}$!

To reproduce $C^{(\mathbf{0};0,0,0,0,1,0)}_{(\mathbf{0};1,0,0,0,0,0)(\mathbf{0};0,0,0,0,0,1)} = \sqrt{3}$ in $W[\hat{\mathfrak{d}}_6]$,

$$N_1^{-2}|I_1(z)|^2 + YN_2^{-2}|I_2(z)|^2 \Rightarrow Y = 11.$$

If Y is similarly special, $C^{(\mathbf{0};0,1,0,0,0,0)}_{(\mathbf{0};0,0,0,0,0,1)(\mathbf{0};0,0,0,0,0,1)} = \frac{11}{10}\sqrt{\frac{33}{2}}$.

The real test

To reproduce $C^{(\mathbf{0};0,0,1,0)}_{(\mathbf{0};1,0,0,0)(\mathbf{0};0,0,0,1)} = \sqrt{2}$ in $W[\hat{\mathfrak{d}}_4]$,

$$N_1^{-2}|I_1(z)|^2 + XN_2^{-2}|I_2(z)|^2 \Rightarrow X = \frac{7}{4}.$$

Notice that $X = C^{(\mathbf{0};0,1,0,0)}_{(\mathbf{0};1,0,0,0)(\mathbf{0};1,0,0,0)} C^{(\mathbf{0};0,1,0,0)}_{(\mathbf{0};0,0,0,1)(\mathbf{0};0,0,0,1)}$!

To reproduce $C^{(\mathbf{0};0,0,0,0,1,0)}_{(\mathbf{0};1,0,0,0,0,0)(\mathbf{0};0,0,0,0,0,1)} = \sqrt{3}$ in $W[\hat{\mathfrak{d}}_6]$,

$$N_1^{-2}|I_1(z)|^2 + YN_2^{-2}|I_2(z)|^2 \Rightarrow Y = 11.$$

If Y is similarly special, $C^{(\mathbf{0};0,1,0,0,0,0)}_{(\mathbf{0};0,0,0,0,0,1)(\mathbf{0};0,0,0,0,0,1)} = \frac{11}{10}\sqrt{\frac{33}{2}}$.

- The "Coulomb gas method" is really three.
- All still needed pending a breakthrough in special functions.
- Most parts are now algorithmic but some puzzles remain.