Unitarity Cuts of the Worldsheet

Sebastian Mizera (IAS)

Based on [hep-th/2208.12233] and WIP
with Lorenz Eberhardt

Surprisingly little is known about scattering of strings
in flat space beyond the low-energy limit

Veneziano amplitude

Polarization dependence $t_8 = s\, p_1 \cdot \epsilon_2\, p_2 \cdot \epsilon_1\, \epsilon_3 \cdot \epsilon_4 + \ldots$

$$\mathcal{A}_{\text{tree}}^{\text{planar}}(s,t) = -t_8 \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1 - \alpha' s - \alpha' t)}$$

Center of mass energy

Momentum transfer

Inverse string tension

No equivalently-useful expression currently exists at loop level…

But why is the Veneziano amplitude so much better than

$$\mathcal{A}_{\text{tree}}^{\text{planar}}(s,t) = \frac{t_8}{t}\int_0^1 z^{-\alpha' s-1}(1-z)^{-\alpha' t}\mathrm{d}z \ \textbf{?}$$

Doesn't converge in the physical kinematics, e.g., $s > 0,\ t, u < 0$

$\implies$ Have to define it via analytic continuation

A sign of a more general problem

Textbook definition of string amplitudes

Known for $g \lesssim 2,\ n \lesssim 5$

$$\mathcal{A}_{g,n}(p_1, p_2, \ldots, p_n) \stackrel{?}{=} \int_{\mathcal{M}_{g,n}} (\text{correlation function})$$

or $\Gamma \subset \mathcal{M}_{g,n}$

Moduli space of genus-g
Riemann surfaces with n punctures

isn't entirely correct, e.g., not consistent with unitarity
(the integration contour isn't known)

The underlying problem is that we formulate string amplitudes on a *Euclidean* worldsheet, but the target space is *Lorentzian*

(the reason to formulate the theory on a Euclidean worldsheet in the first place is to be able to use CFT technology, manifest UV finiteness, …)

Why hasn't it been a problem before?

Most computations done:

- At tree level
  (meromorphic functions)
- At loop level in the $\alpha' \to 0$ expansion
  (branch cuts fixed by matching with QFT)

[enormous literature: Green, Schwarz, Gross, Veneziano, Di Vecchia, Koba, Nielsen, D'Hoker, Phong, Martinec, Bern, Dixon, Polyakov, Kosower, Vanhove, Schlotterer, Mafra, Stieberger, Brown, Broedel, Hohenegger, Kleinschmidt, Gerken, Roiban, Lipstein, Mason, Monteiro, ...]

where we can get away without being careful about the integration contour

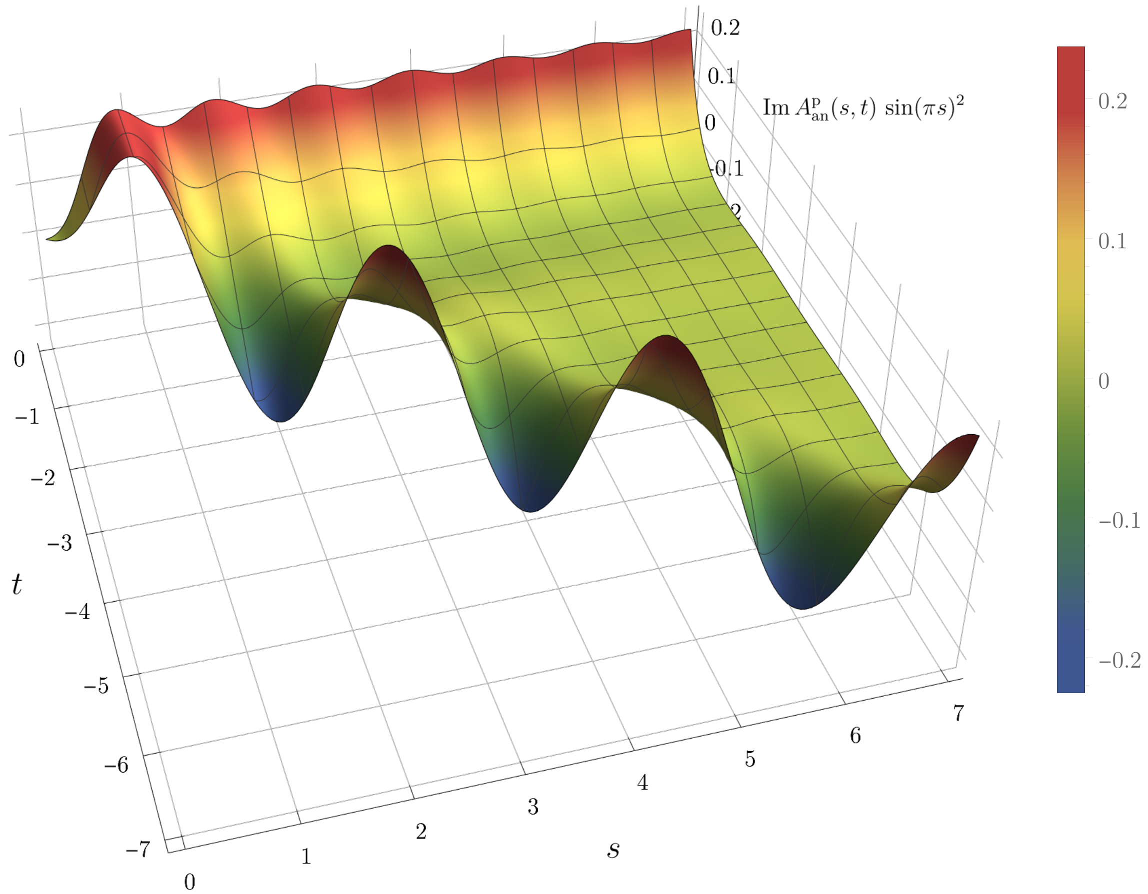
So what does it mean to "compute" an amplitude?

Pragmatic answer:

Be able to efficiently evaluate it numerically

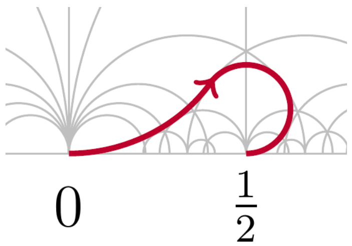
(e.g., known hypergeometric functions, fast convergent integrals, infinite sums, …)

In this talk we'll do it for the imaginary parts of genus-one amplitudes

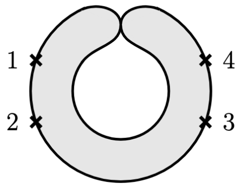


Outline of the talk

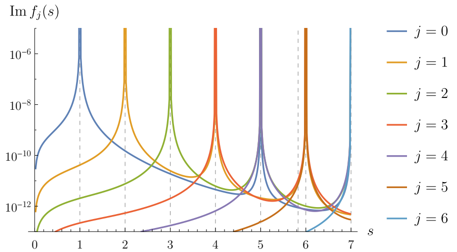
1) Continuation from Euclidean to Lorentzian



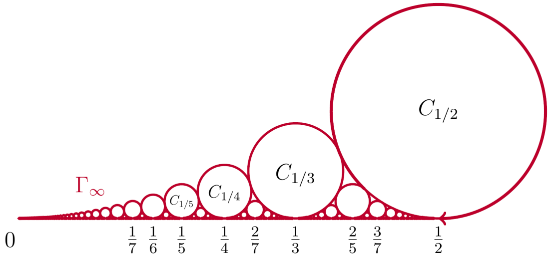
2) Unitarity cuts of the worldsheet



3) Physical properties of the imaginary parts

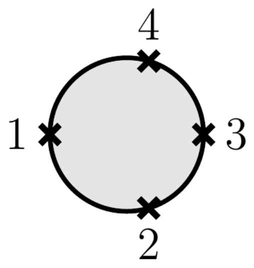


4) Glimpse of the real part (if there's time)



Let’s start at tree level

($\alpha' = 1$ from now on)



$$= \frac{t_8}{t} \int_0^1 z^{-s-1}(1-z)^{-t}\, \mathrm{d}z$$

s-channel poles come from $z \approx 0$, so set $z = e^{-\tau}$ and take $\tau \to \infty$

$$t_8 \int_0^\infty e^{\tau s}\, (\# + \# e^{-\tau} + \# e^{-2\tau} + \ldots)\, \mathrm{d}\tau$$

$-\frac{1}{s}$	$-\frac{\#}{s-1}$	$-\frac{\#}{s-2}$
massless	level-1	level-2

Important distinction

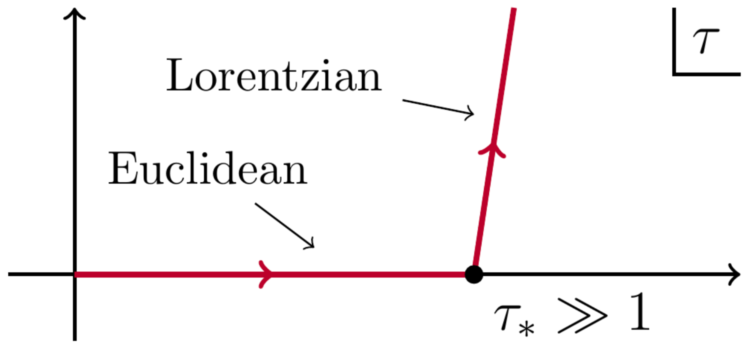
$$\frac{-1}{s-m^2} = \int_0^\infty \mathrm{d}\tau_{\mathrm{E}}\, e^{\tau_{\mathrm{E}}(s-m^2)}$$

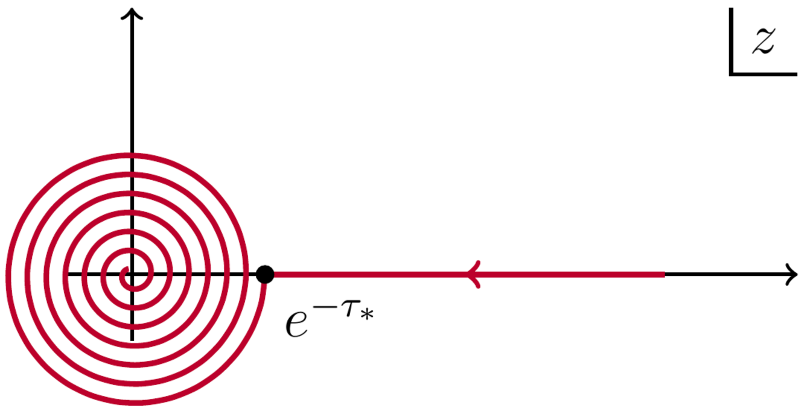
Euclidean proper time

$$\frac{i}{s-m^2} = \lim_{\varepsilon\to 0^+} \int_0^\infty \mathrm{d}\tau_{\mathrm{L}}\, e^{i\tau_{\mathrm{L}}(s-m^2+i\varepsilon)}$$

Lorentzian proper time

This tells us about the correct integration contour





We can resum $\circlearrowright + e^{-2\pi i s} \circlearrowright + e^{-4\pi i s} \circlearrowright + \dots = \dfrac{1}{1 - e^{-2\pi i s}} \circlearrowright$

infinite number of string resonances

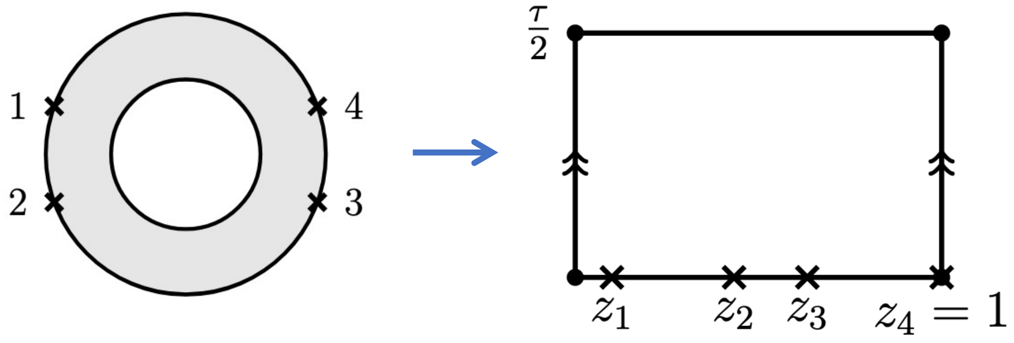
Strategy for finding the contour at higher genus

- Identify local variables $q \sim e^{-(\text{Schwinger parameter})}$
- Continue to Lorentzian signature locally in the moduli space
- Glue everything together

[Witten '13]

Genus-one superstring amplitudes

In this talk we focus on the planar annulus contribution



Modular parameter

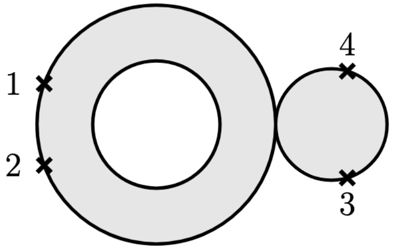
$z_{ij} = z_i - z_j$

$$\mathcal{A}^{\text{planar}}_{\text{annulus}} \overset{?}{=} -i\, t_8 \int_0^{i\infty} \mathrm{d}\tau \int_{0<z_1<z_2<z_3<1} \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3 \left(\frac{\vartheta_1(z_{21},\tau)\vartheta_1(z_{43},\tau)}{\vartheta_1(z_{31},\tau)\vartheta_1(z_{42},\tau)} \right)^{-s} \left(\frac{\vartheta_1(z_{32},\tau)\vartheta_1(z_{41},\tau)}{\vartheta_1(z_{31},\tau)\vartheta_1(z_{42},\tau)} \right)^{-t}$$

Jacobi theta function

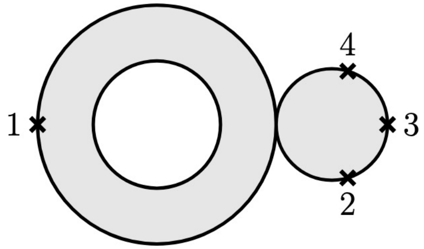
$$\vartheta_1(z,\tau) = i \sum_{n\in\mathbb{Z}} (-1)^n \mathrm{e}^{2\pi i (n-\frac{1}{2})z + \pi i (n-\frac{1}{2})^2 \tau}$$

Various degenerations need the Witten $i\varepsilon$



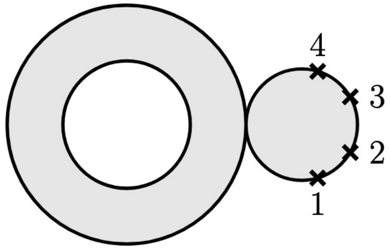
Massive pole exchange

$$q = z_{43}$$



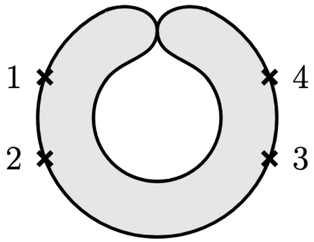
Wave-function renormalization

$$q = z_{42}$$



Tadpole

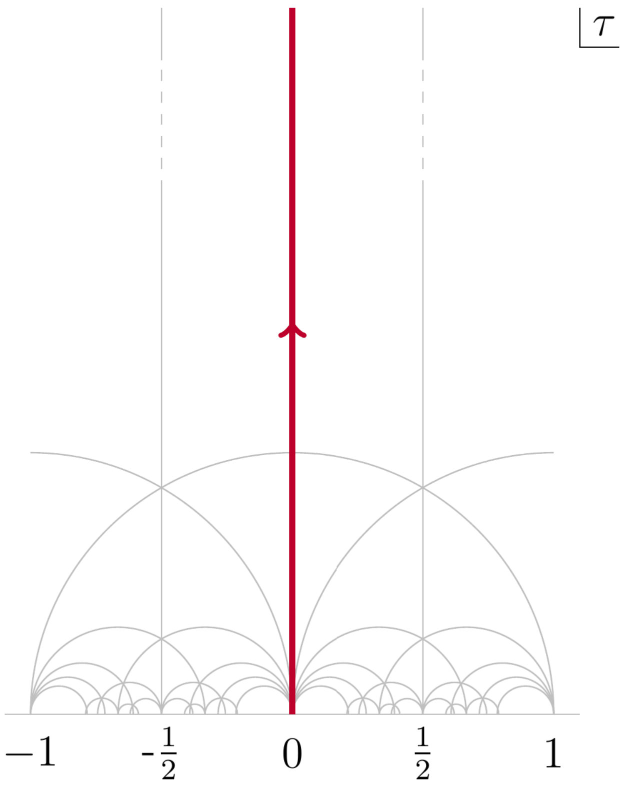
$$q = z_{41}$$

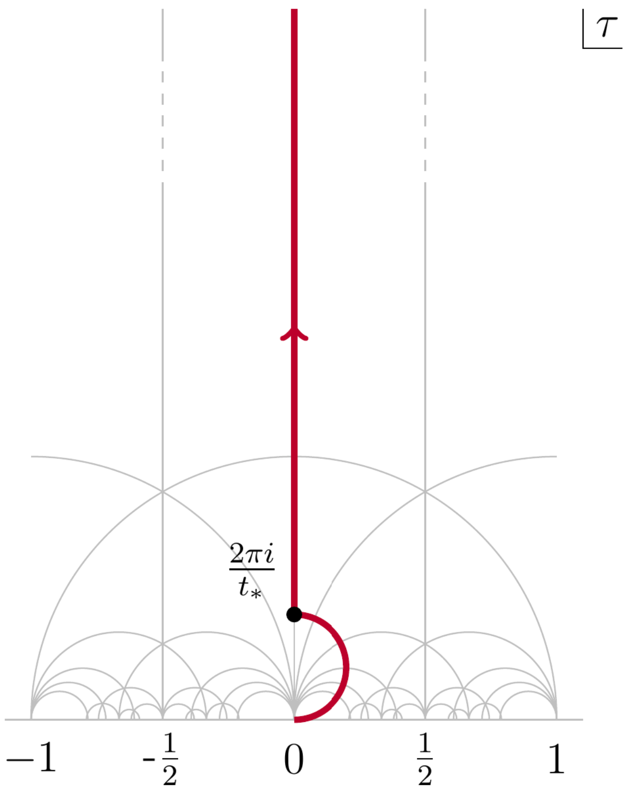


Non-separating degeneration

$$\underbrace{q = e^{-\frac{2\pi i}{\tau}}}_{\text{Unitarity cuts}}$$

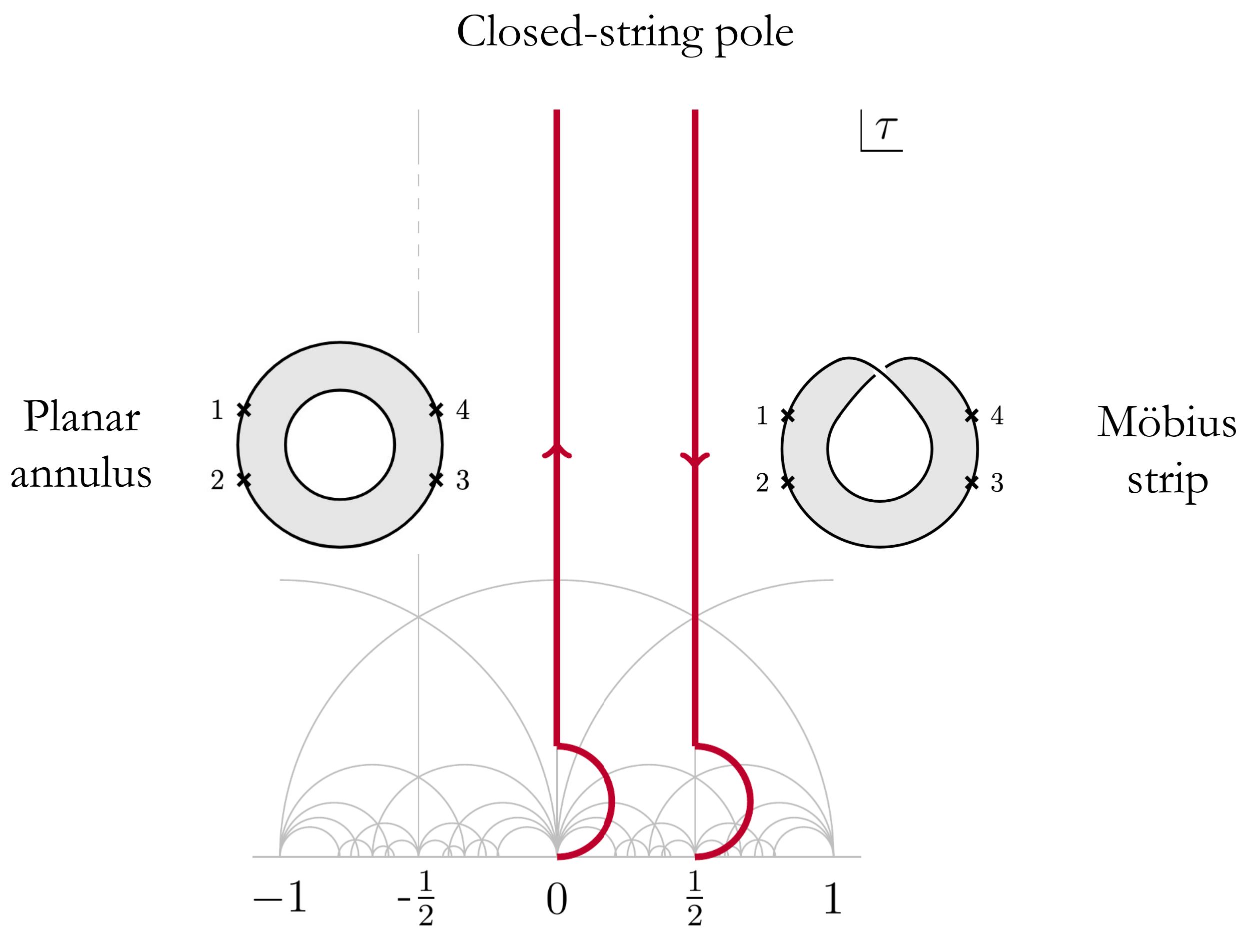
Let's focus on the contour in the fundamental domain, $\tau = \dfrac{2\pi i}{t_* + it}$





Approach the essential singularity from the right

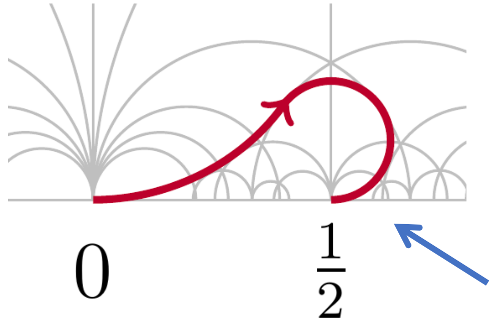
Adding the other planar contribution: Möbius strip



Our proposal for the correct integration contour

(similar for other topologies)

Precise shape
doesn't matter

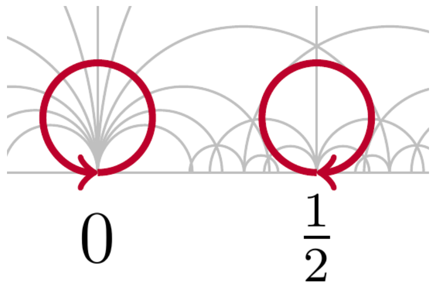


Approach 0 and ½
from the right

We'll come back to it at the end of the talk

For the imaginary part we only need

Size of the circles doesn't matter



They'll give as unitarity cuts of the planar annulus and the Möbius strip

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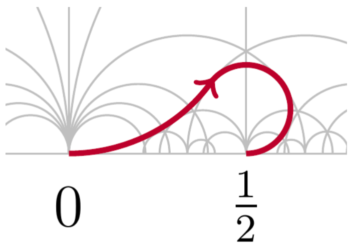

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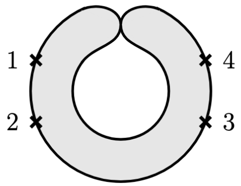

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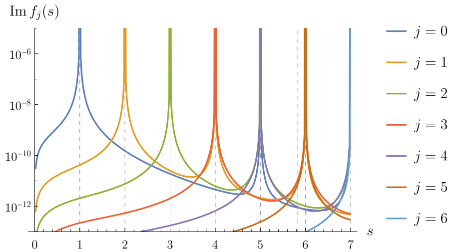

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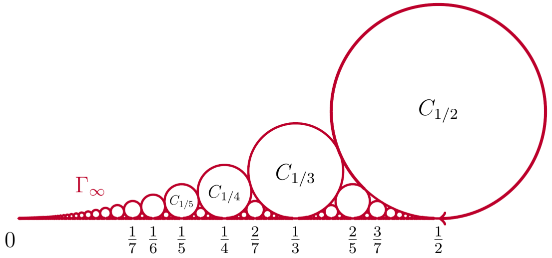


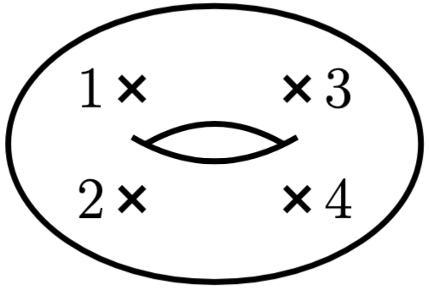
Unitarity cuts

Unitarity, $SS^\dagger = \mathbb{1}$, embodies the physical principle of probability conservation. Using $S = \mathbb{1} + iT$:

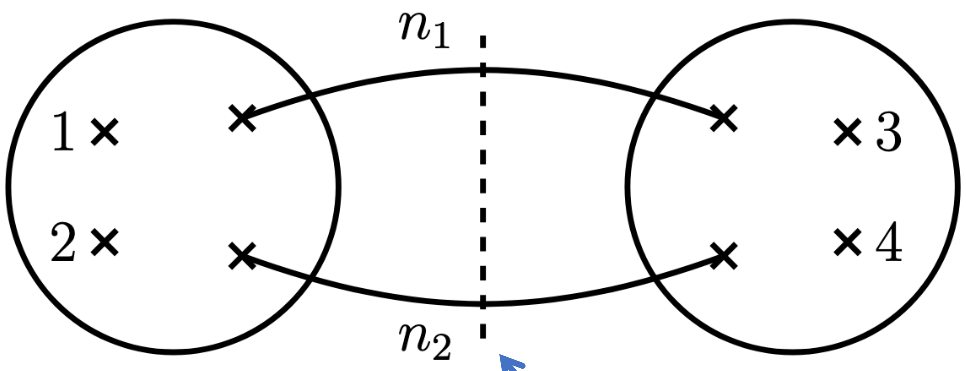
$$\mathrm{Im}\, T = \tfrac{1}{2} T T^\dagger$$

At genus one

Masses squared are integers

Im 

$$= \sum_{\substack{n_1, n_2 \\ \sqrt{n_1}+\sqrt{n_2}\leqslant\sqrt{s} \\ \text{polarizations} \\ \text{degeneracy}}}$$



Thresholds

On-shell states
with positive energy

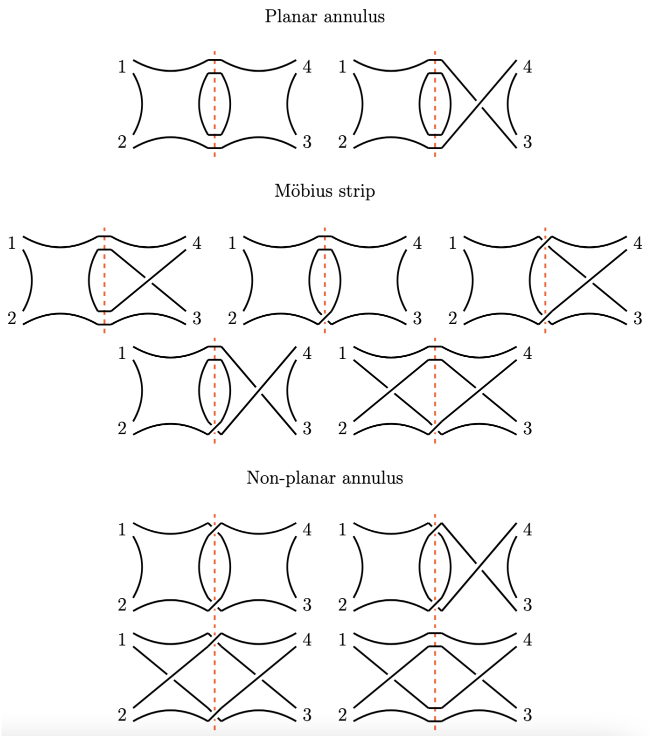
Two options:

- Do unitarity cuts “by hand” just as in field theory
- Let the worldsheet do it for us

First do it by hand

(not feasible beyond the massless cut)

- Color sums



- Polarization sums

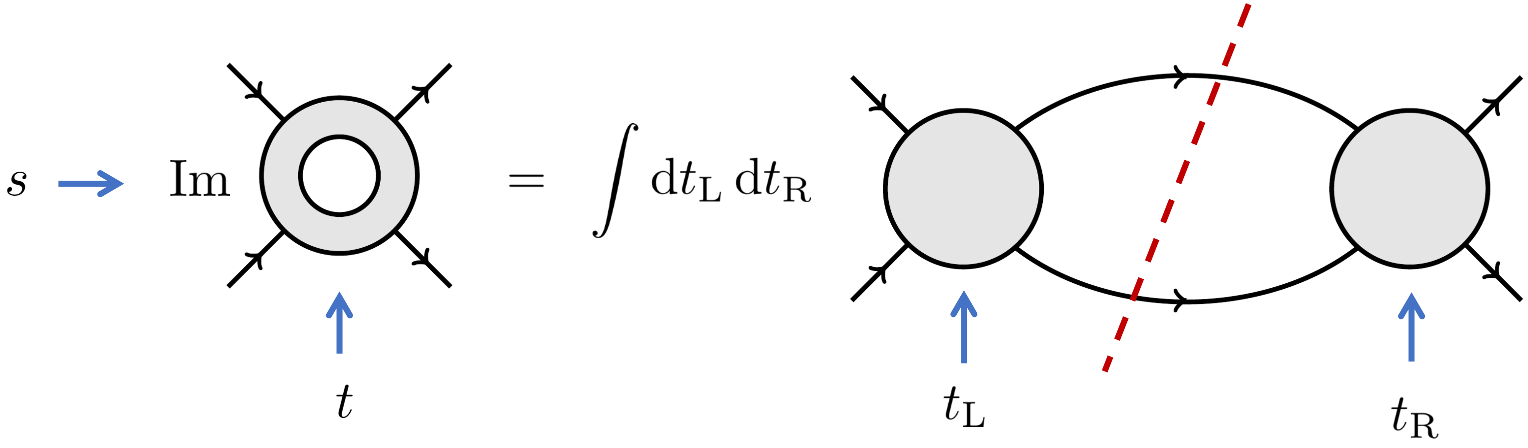
$$\mathcal{P} = \sum_{\text{pol}} \Big[\underbrace{t_8^b(1256)\, t_8^b(345\overline{6})}_{\text{gluons}} - \underbrace{t_8^f(1256)\, t_8^f(345\overline{6})}_{\text{gluinos}} \Big] = \frac{s^2}{2} t_8$$

- Loop integration

$$\int \mathrm{d}^{\mathrm{D}}\ell\, \delta^+[\ell^2]\delta^+[(p_{12}-\ell)^2](\cdots)$$

$$\propto \int_{P>0} \mathrm{d}t_{\mathrm{L}}\, \mathrm{d}t_{\mathrm{R}}\, P^{\frac{\mathrm{D}-5}{2}}(\cdots)$$

After the dust settles



SO$(N = 32)$ gauge group

$$\operatorname{Im} A^{\mathrm{p}}_{\mathrm{an}}\Big|_{s<1} = \frac{N\pi}{60\sqrt{stu}} \int_{P>0} \mathrm{d}t_{\mathrm{L}}\, \mathrm{d}t_{\mathrm{R}}\;\; P(t_{\mathrm{L}}, t_{\mathrm{R}})^{\frac{5}{2}} \frac{\Gamma(1-s)\Gamma(-t_{\mathrm{L}})}{\Gamma(1-s-t_{\mathrm{L}})} \frac{\Gamma(1-s)\Gamma(-t_{\mathrm{R}})}{\Gamma(1-s-t_{\mathrm{R}})}$$

On-shell phase space

$$P(t_{\mathrm{L}}, t_{\mathrm{R}}) = -\frac{s(t^2 + t_{\mathrm{L}}^2 + t_{\mathrm{R}}^2 - 2tt_{\mathrm{L}} - 2tt_{\mathrm{R}} - 2t_{\mathrm{L}}t_{\mathrm{R}}) - 4tt_{\mathrm{L}}t_{\mathrm{R}}}{4tu}$$

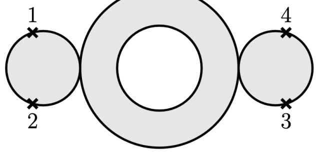
General form after including massive exchanges

New thresholds opening up

$$\mathrm{Im}\, A^{\mathrm{p}}_{\mathrm{an}} = \frac{\pi N}{60}\frac{\Gamma(1-s)^2}{\sqrt{stu}} \sum_{n_1 \geqslant n_2 \geqslant 0} \theta\left[s-(\sqrt{n_1}+\sqrt{n_2})^2\right] \int_{P_{n_1,n_2}>0} \mathrm{d}t_{\mathrm{L}}\, \mathrm{d}t_{\mathrm{R}}\, P_{n_1,n_2}(t_{\mathrm{L}},t_{\mathrm{R}})^{\frac{5}{2}}$$

$$\times Q_{n_1,n_2}(t_{\mathrm{L}},t_{\mathrm{R}}) \frac{\Gamma(-t_{\mathrm{L}})\Gamma(-t_{\mathrm{R}})}{\Gamma(n_1+n_2+1-s-t_{\mathrm{L}})\Gamma(n_1+n_2+1-s-t_{\mathrm{R}})}$$

Double poles at every positive integer



Need a computation to determine the integrand, e.g., $Q_{0,0} = 1$

$$\text{with } P_{n_1,n_2} = -\frac{1}{4stu} \det \begin{bmatrix} 0 & s & u & n_2-s-t_{\mathrm{L}} \\ s & 0 & t & t_{\mathrm{L}}-n_1 \\ u & t & 0 & n_1-t_{\mathrm{R}} \\ n_2-s-t_{\mathrm{L}} & t_{\mathrm{L}}-n_1 & n_1-t_{\mathrm{R}} & 2n_1 \end{bmatrix}$$

Idea

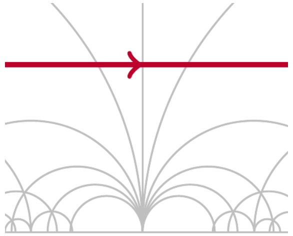
Arrive at the same representation using the newly-discovered moduli space contour,
thus bypassing laborious sums over intermediate states,
spins, degeneracy, colors, polarizations, …

Unitarity cuts of the worldsheet

0

$$\tau \to -\frac{1}{\tau}$$

$$q \to e^{2\pi i \tau}$$



0

q can be *arbitrarily* small

After the modular transformation:

$$\mathrm{Im}\, A^{\mathrm{p}}_{\mathrm{an}} = -\frac{N}{64}\int_{\longrightarrow}\frac{\mathrm{d}\tau}{\tau^2}\int \mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\; \underbrace{q^{s z_{41} z_{32} - t z_{21} z_{43}} \left(\frac{\vartheta_1(z_{21}\tau,\tau)\vartheta_1(z_{43}\tau,\tau)}{\vartheta_1(z_{31}\tau,\tau)\vartheta_1(z_{42}\tau,\tau)}\right)^{-s}\left(\frac{\vartheta_1(z_{41}\tau,\tau)\vartheta_1(z_{32}\tau,\tau)}{\vartheta_1(z_{31}\tau,\tau)\vartheta_1(z_{42}\tau,\tau)}\right)^{-t}}_{\sim\, q^{\mathrm{Trop}(s,t,z_i)} \quad \text{as} \quad q \to 0}$$

Tropical analysis

The integrand goes as q^{Trop} so only terms with $\mathrm{Trop} < 0$ can contribute

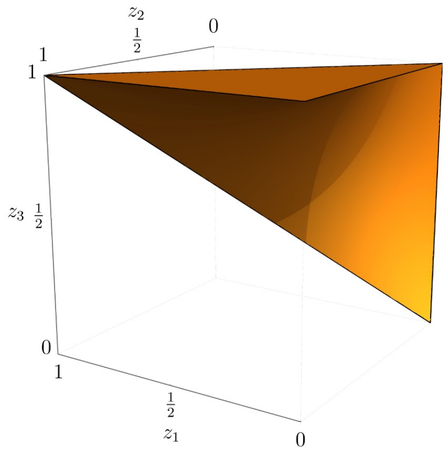
It tells us how many terms in the q-expansion we need to keep, e.g.,

$$\vartheta_1(z\tau,\tau) = i q^{\frac{1}{8}} \Big(\underbrace{q^{-\frac{z}{2}}}_{\text{always dominates}} - \underbrace{q^{\frac{z}{2}}}_{\text{needed near } z \approx 0} - \underbrace{q^{1-\frac{3z}{2}}}_{\text{needed near } z \approx 1} \Big) (1 + \mathcal{O}(q)) \qquad z \in [0,1]$$

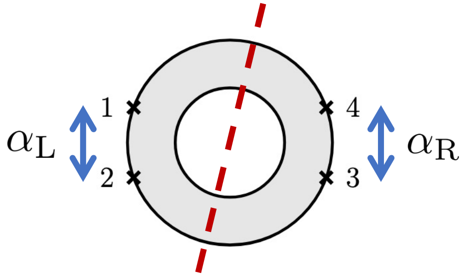
For example, below the first massive threshold

$$q^{sz_{41}z_{32}-tz_{21}z_{43}}\left(\frac{\vartheta_1(z_{21}\tau,\tau)\vartheta_1(z_{43}\tau,\tau)}{\vartheta_1(z_{31}\tau,\tau)\vartheta_1(z_{42}\tau,\tau)}\right)^{-s}\left(\frac{\vartheta_1\vartheta_1}{\vartheta_1\vartheta_1}\right)^{-t}\sim q^{-s(1-z_{41})z_{32}-tz_{21}z_{43}}\;(1-q^{z_{21}})^{-s}(1-q^{z_{43}})^{-s}$$

exact computation supported in



Identify the variables



$$\alpha_{\mathrm{L}}=z_{21}\;,\qquad \alpha_{\mathrm{R}}=z_{43}\;,\qquad t_{\mathrm{L}}=-sz_{32}+tz_{43}$$

and integrate in

$$1=s\sqrt{\frac{-i\tau}{2stu}}\int_{-\infty}^{\infty}\mathrm{d}t_{\mathrm{R}}\,q^{-\frac{1}{4st(s+t)}(st_{\mathrm{R}}-(s+2t)t_{\mathrm{L}}+2t(s+t)\alpha_{\mathrm{R}}-st)^2}$$

Gives exactly the same formula we've derived before from unitarity

$$\operatorname{Im} A_{\mathrm{an}}^{\mathrm{p}}\Big|_{s<1} = \frac{N}{64\sqrt{2stu}} \int_{\longrightarrow} \frac{\mathrm{d}\tau}{(-i\tau)^{\frac{3}{2}}} \int_{\mathcal{R}} \mathrm{d}\alpha_{\mathrm{L}}\, \mathrm{d}\alpha_{\mathrm{R}}\, \mathrm{d}t_{\mathrm{L}}\, \mathrm{d}t_{\mathrm{R}}\; q^{-t_{\mathrm{L}}\alpha_{\mathrm{L}}-t_{\mathrm{R}}\alpha_{\mathrm{R}}-P(t_{\mathrm{L}},t_{\mathrm{R}})}(1-q^{\alpha_{\mathrm{L}}})^{-s}(1-q^{\alpha_{\mathrm{R}}})^{-s}$$

$$= \frac{N\pi}{60\sqrt{stu}} \int_{P>0} \mathrm{d}t_{\mathrm{L}}\, \mathrm{d}t_{\mathrm{R}}\;\; P(t_{\mathrm{L}},t_{\mathrm{R}})^{\frac{5}{2}}\, \frac{\Gamma(1-s)\Gamma(-t_{\mathrm{L}})}{\Gamma(1-s-t_{\mathrm{L}})}\, \frac{\Gamma(1-s)\Gamma(-t_{\mathrm{R}})}{\Gamma(1-s-t_{\mathrm{R}})}$$

Stringy Landau analysis

When does a new contribution to Trop < 0 appear?

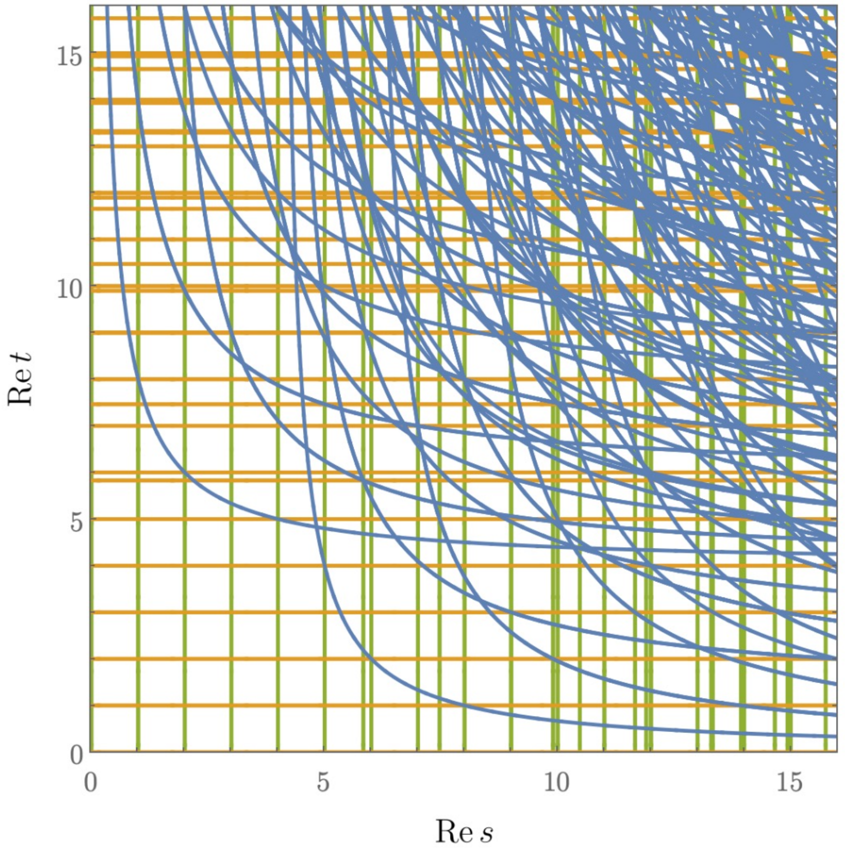
Normal thresholds at

$$s, t, u = (\sqrt{n_1} + \sqrt{n_2})^2$$

Anomalous thresholds at

$$\det \begin{bmatrix} 2n_1 & n_1{+}n_2 & n_1{+}n_3{-}s & n_1{+}n_4 \\ n_1{+}n_2 & 2n_2 & n_2{+}n_3 & n_2{+}n_4{-}t \\ n_1{+}n_3{-}s & n_2{+}n_3 & 2n_3 & n_3{+}n_4 \\ n_1{+}n_4 & n_2{+}n_4{-}t & n_3{+}n_4 & 2n_4 \end{bmatrix} = 0 \qquad n_i \in \mathbb{Z}_{\geqslant 0}$$

Analytic structure away from physical regions is complicated, but consistent with field theory expectations



Branch cuts

Tropical analysis previously featured in

- $\alpha' \to 0$ limit of string amplitudes

[Tourkine '13]

- $\alpha' \to 0$ limit of tree-level amplitudes and loop integrands

[Arkani-Hamed, He, Lam, Frost, Salvatori, Plamondon, Thomas '19-22]

- $\mathcal{N} = 4$ SYM amplitudes

[Drummond, Foster, Gurdogan, Kalousios, Henke, Papathanasiou '19-]

- UV/IR divergences of individual Feynman integrals

[Panzer, Borinsky, Tellander, Helmer, Arkani-Hamed, Hillman, SM '19-22]

But here it plays a different role: we're doing an *exact* computation!

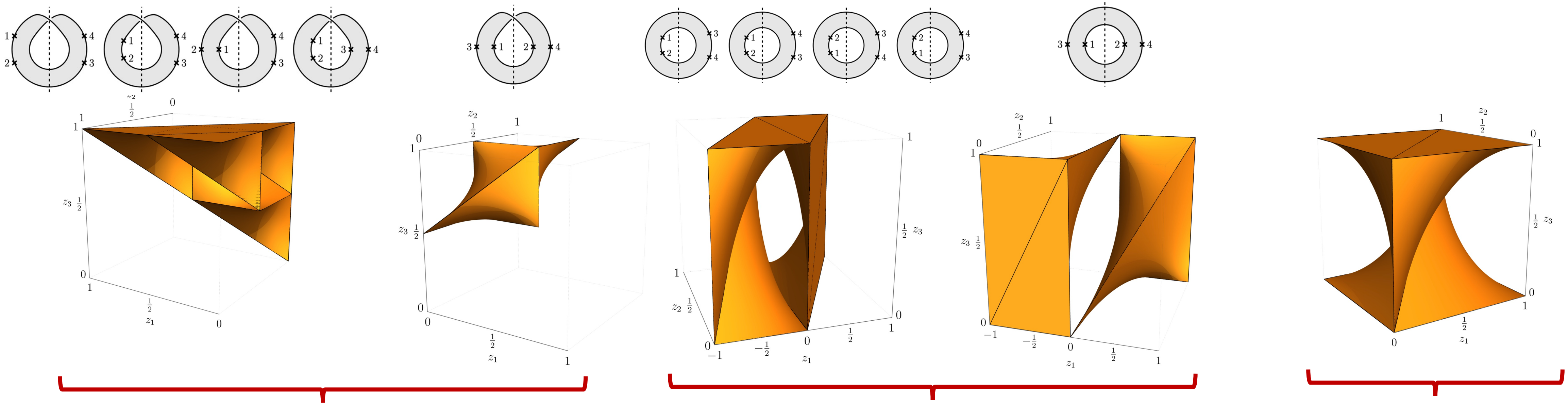
This strategy allows us to go to higher energies bypassing summing over states

$$\operatorname{Im} A_{\text{an}}^{\text{p}} = \frac{\pi N}{60} \frac{\Gamma(1-s)^2}{\sqrt{stu}} \sum_{n_1 \geqslant n_2 \geqslant 0} \theta\left[s - (\sqrt{n_1}+\sqrt{n_2})^2\right] \int_{P_{n_1,n_2}>0} \mathrm{d}t_{\text{L}}\, \mathrm{d}t_{\text{R}}\, P_{n_1,n_2}(t_{\text{L}}, t_{\text{R}})^{\frac{5}{2}}$$
$$\times Q_{n_1,n_2}(t_{\text{L}}, t_{\text{R}}) \frac{\Gamma(-t_{\text{L}})\Gamma(-t_{\text{R}})}{\Gamma(n_1+n_2+1-s-t_{\text{L}})\Gamma(n_1+n_2+1-s-t_{\text{R}})}$$

where the first few polynomials are

$$\begin{aligned}
Q_{0,0} &= 1\ , \\
Q_{1,0} &= 2\left(-2st_{\text{L}}t_{\text{R}} - s^2 t_{\text{L}} + st_{\text{L}} - s^2 t_{\text{R}} + st_{\text{R}} + s^2 t - 2st + t\right)\ , \\
Q_{2,0} &= 2s^4 t_{\text{L}} t_{\text{R}} + 4s^3 t_{\text{L}} t_{\text{R}}^2 + 4s^3 t_{\text{L}}^2 t_{\text{R}} - 4s^3 t t_{\text{L}} t_{\text{R}} - 12 s^3 t_{\text{L}} t_{\text{R}} + 4s^2 t_{\text{L}}^2 t_{\text{R}}^2 - 10 s^2 t_{\text{L}} t_{\text{R}}^2 \\
&\quad - 10 s^2 t_{\text{L}}^2 t_{\text{R}} + 12 s^2 t t_{\text{L}} t_{\text{R}} + 18 s^2 t_{\text{L}} t_{\text{R}} - 2st_{\text{L}}^2 t_{\text{R}}^2 + 4st_{\text{L}} t_{\text{R}}^2 + 4st_{\text{L}}^2 t_{\text{R}} - 12 stt_{\text{L}} t_{\text{R}} \\
&\quad - 6st_{\text{L}} t_{\text{R}} + 4tt_{\text{L}} t_{\text{R}} + s^4 t_{\text{L}}^2 - 2s^4 tt_{\text{L}} - s^4 t_{\text{L}} - 4s^3 t_{\text{L}}^2 + 10 s^3 tt_{\text{L}} + 4s^3 t_{\text{L}} + 5s^2 t_{\text{L}}^2 \\
&\quad - 18 s^2 tt_{\text{L}} - 5s^2 t_{\text{L}} - 2st_{\text{L}}^2 + 14 stt_{\text{L}} + 2st_{\text{L}} - 4tt_{\text{L}} + s^4 t_{\text{R}}^2 - 2s^4 tt_{\text{R}} - s^4 t_{\text{R}} \\
&\quad - 4s^3 t_{\text{R}}^2 + 10 s^3 tt_{\text{R}} + 4s^3 t_{\text{R}} + 5s^2 t_{\text{R}}^2 - 18 s^2 tt_{\text{R}} - 5s^2 t_{\text{R}} - 2st_{\text{R}}^2 + 14 stt_{\text{R}} \\
&\quad + 2st_{\text{R}} - 4tt_{\text{R}} + s^4 t^2 + s^4 t - 6s^3 t^2 - 6s^3 t + 13 s^2 t^2 + 13 s^2 t - 12 st^2 - 12 st \\
&\quad + 4t^2 + 4t\ .
\end{aligned}$$

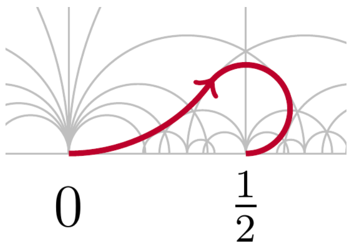
Similar analysis for other genus-one topologies in all kinematic channels



Möbius strip **Non-planar annulus** **Torus**

Outline of the talk

1) Continuation from Euclidean to Lorentzian



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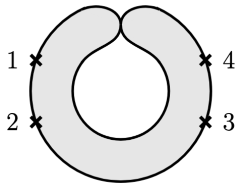
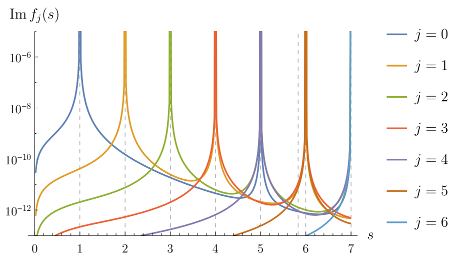


3) Physical properties of the imaginary parts



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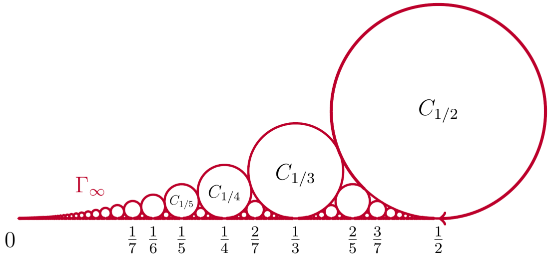
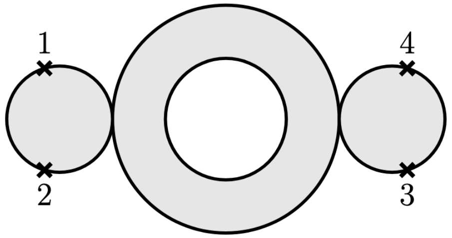


We can now analyze the results

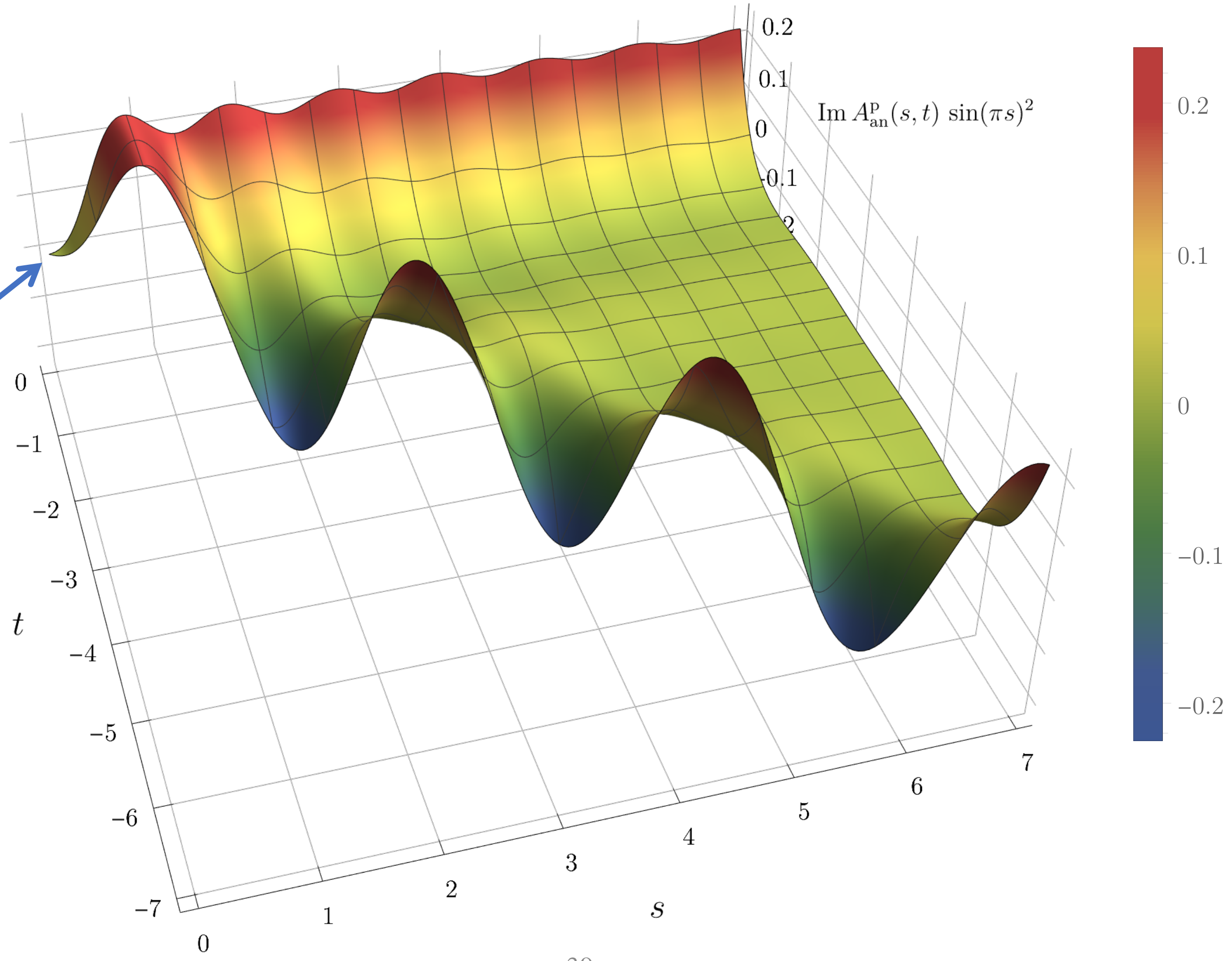
(this talk: planar annulus in the s-channel only)

We often normalize by $\sin(\pi s)^2$ to remove the double poles



$\mathrm{Im}\, A^{\mathrm{p}}_{\mathrm{an}}(s,t)$ does not include the t_8 tensor

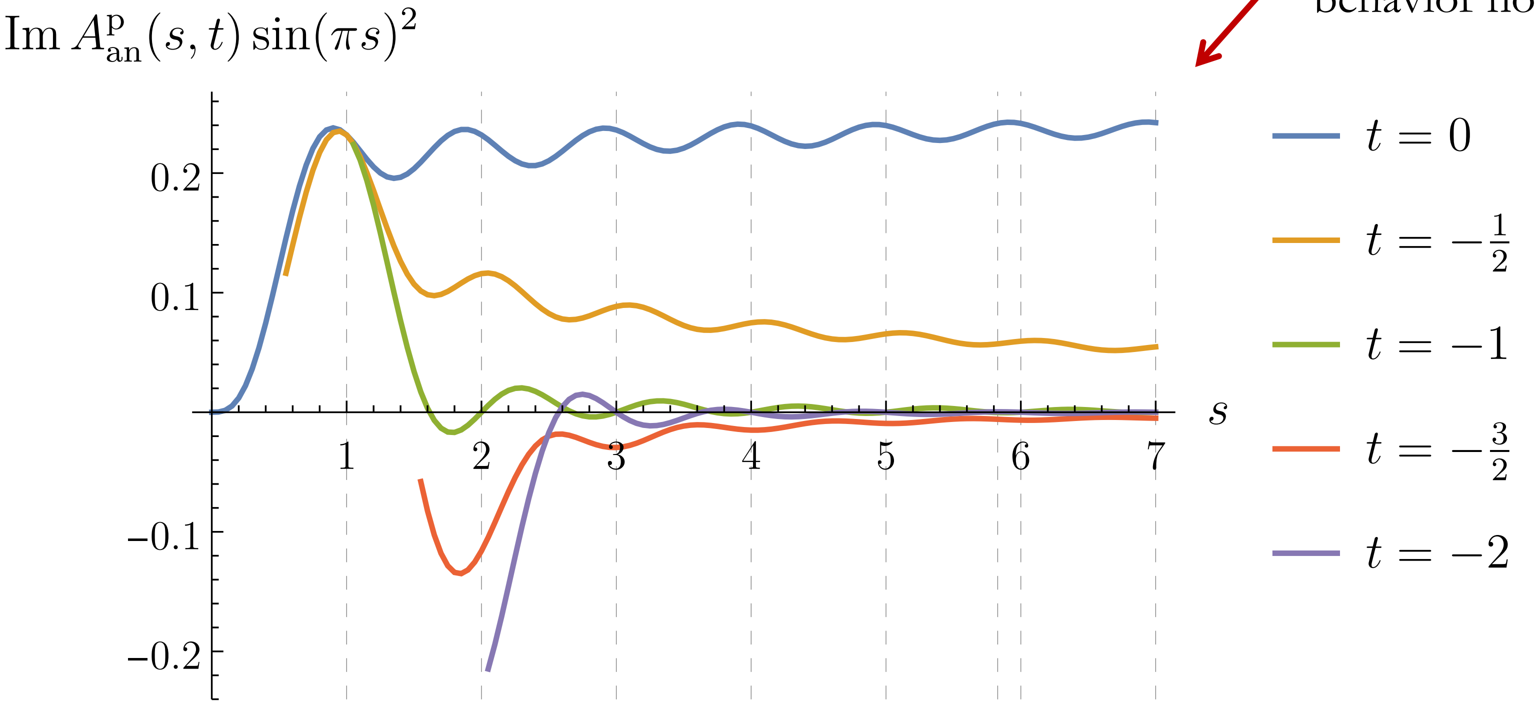
The imaginary part of the planar annulus



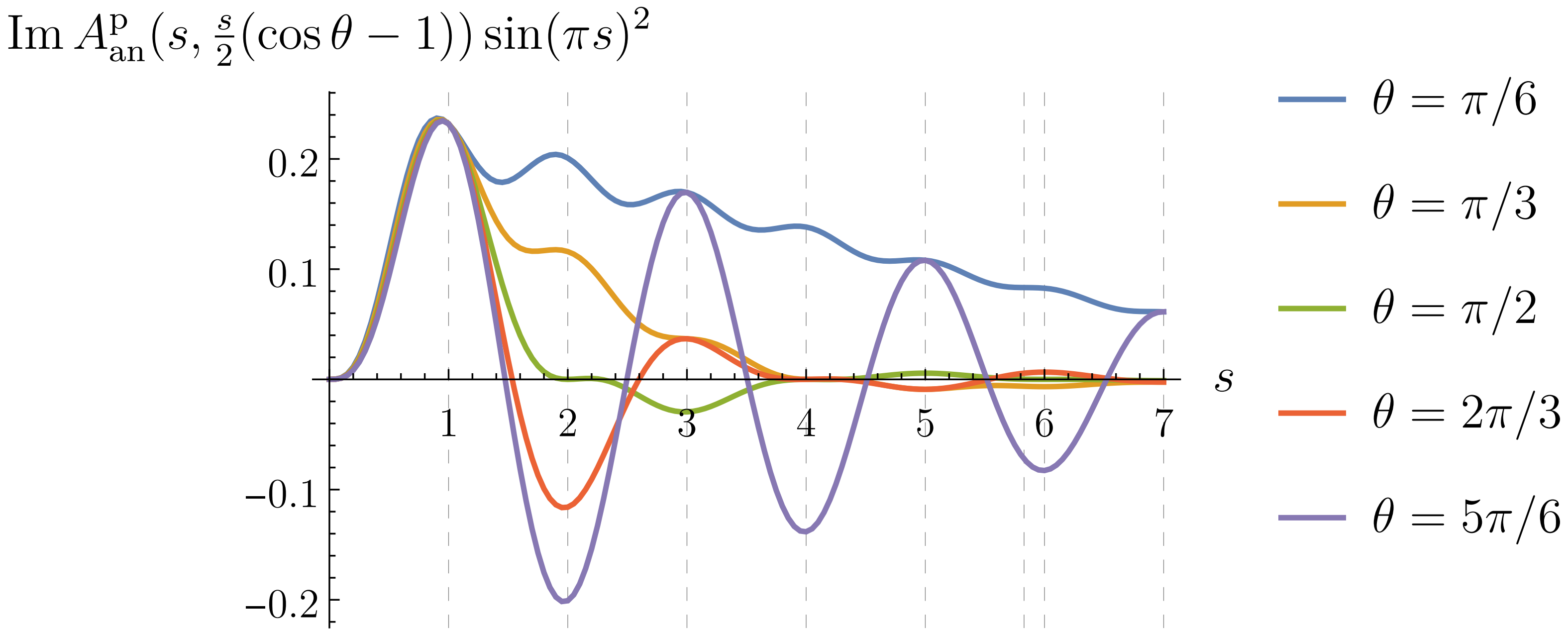
Most of the known results

Fixed momentum transfer

Asymptotic
behavior not reached



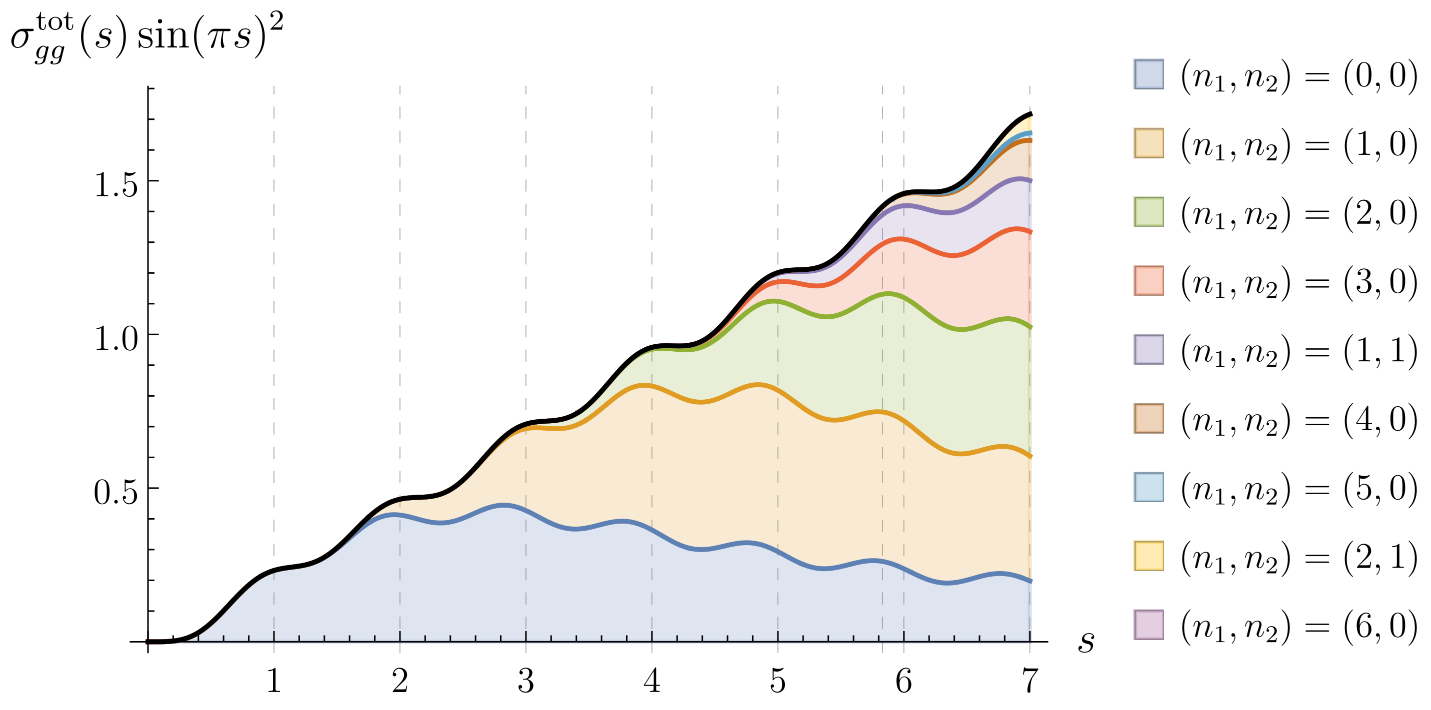
Fixed angle



$\mathrm{Im}\, A^{\mathrm{p}}_{\mathrm{an}}(s, \frac{s}{2}(\cos\theta - 1))\, \sin(\pi s)^2$
0.2
0.1
−0.1
−0.2
1
2
3
4
5
6
7
s
$\theta = \pi/6$
$\theta = \pi/3$
$\theta = \pi/2$
$\theta = 2\pi/3$
$\theta = 5\pi/6$

Total cross section

$\propto \frac{t_8}{s} \mathrm{Im}\, A^{\mathrm{p}}_{\mathrm{an}}(s,0)$

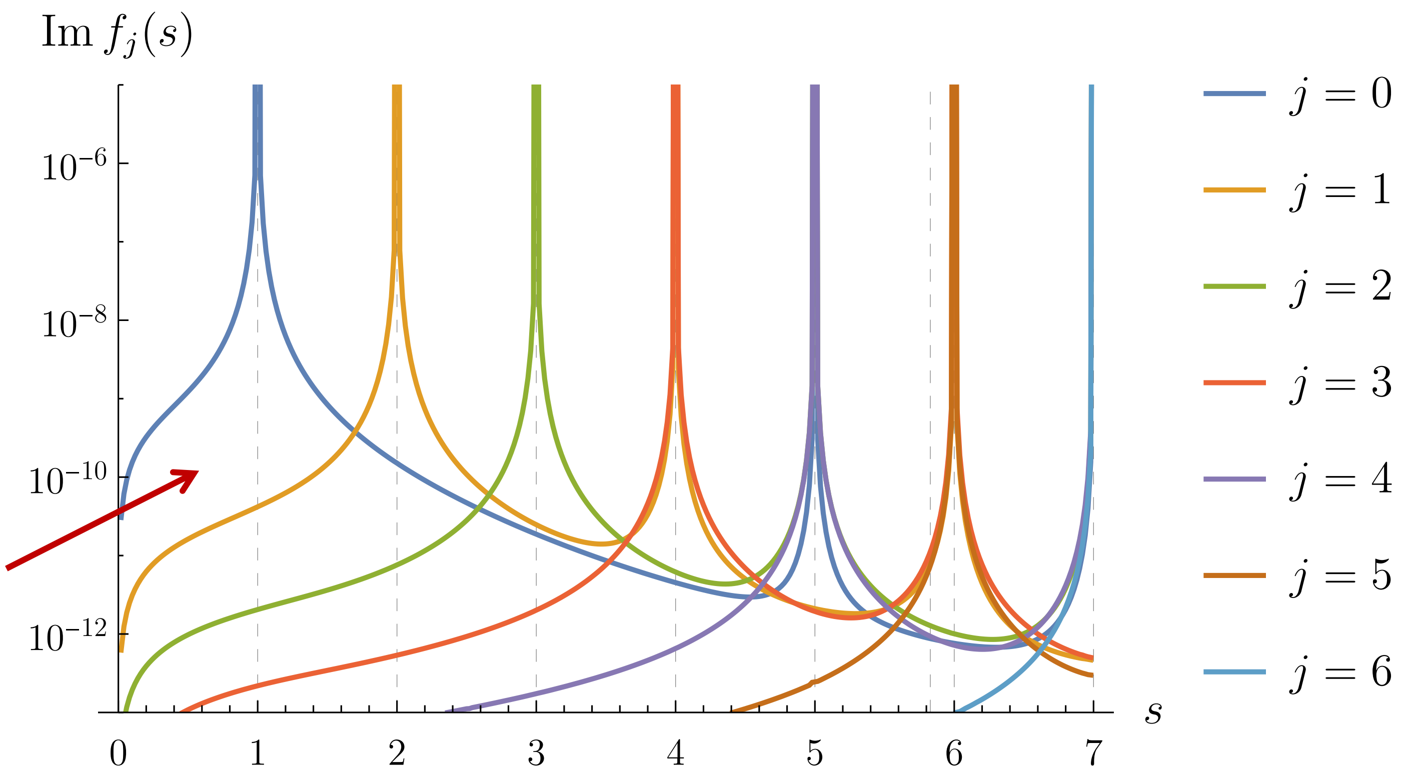


New thresholds opening up very slowly

Low-spin dominance

(cf. [Arkani-Hamed, Huang, Huang '20], [Bern, Kosmopoulos, Zhiboedov '21] at tree level)

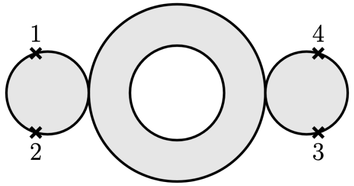
Partial wave coefficients $f_j(s) \propto \int_{-1}^{1} \mathrm{d}z\,(1-z^2)^3\, G_j^{(10)}(z) A^{\mathrm{p}}_{\mathrm{an}}(s,t(z))$



Almost all contributions from spins $j+1 \lesssim s$

Decay widths

Coefficient of the double residue computes decay widths



In agreement with
[Okada, Tsuchiya '89]

$$\underset{s=1}{\mathrm{DRes}}\,\mathrm{Im}\,A^{\mathrm{p}}_{\mathrm{an}} = \frac{\pi^2}{420}\,,$$

$$\underset{s=2}{\mathrm{DRes}}\,\mathrm{Im}\,A^{\mathrm{p}}_{\mathrm{an}} = \frac{\pi^2(t+1)}{420}\,,$$

$$\underset{s=3}{\mathrm{DRes}}\,\mathrm{Im}\,A^{\mathrm{p}}_{\mathrm{an}} = \frac{10883\pi^2(t+1)(t+2)}{8981280}\,,$$

$$\vdots$$

$$\underset{s=10}{\mathrm{DRes}}\,\mathrm{Im}\,A^{\mathrm{p}}_{\mathrm{an}} = 6.8078\cdot 10^{-8}\times(t+1.00045)(t+2.00087)(t+3.0015)(t+4.0028)$$
$$\times(t+5)(t+5.9972)(t+6.9985)(t+7.99913)(t+8.99955)\,.$$

Using decay widths to approximate the amplitude

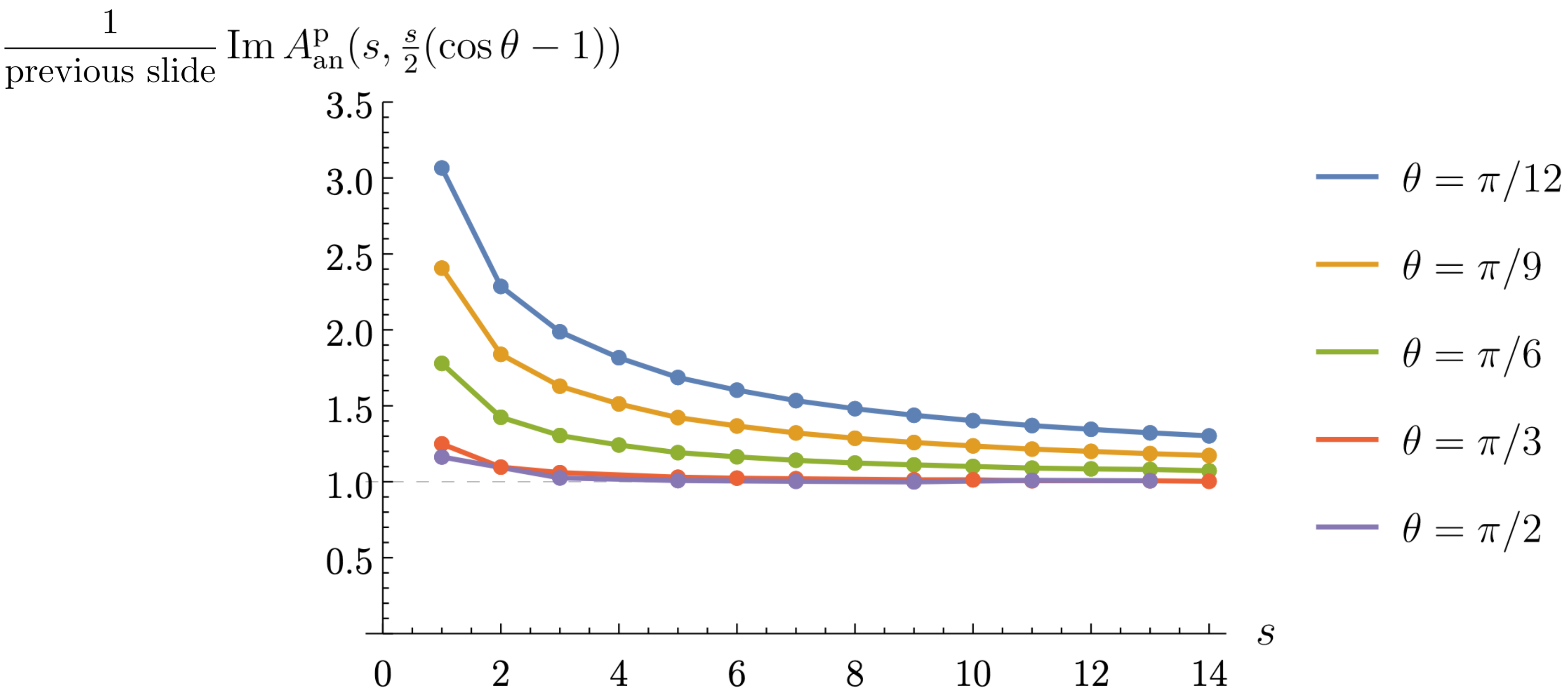
$$\underset{s=n}{\mathrm{DRes}}\, \mathrm{Im}\, A^{\mathrm{p}}_{\mathrm{an}} \sim \frac{1}{4\pi^2} \frac{\Gamma(t+n)}{\Gamma(t+1)\Gamma(n)}$$

In high-energy fixed-angle scattering, this gives

$$\mathrm{Im}\, A^{\mathrm{p}}_{\mathrm{an}}(s,t) \sim -\sqrt{\frac{s}{8\pi t u}\frac{\sin(\pi t)}{\sin(\pi s)^2}}\, \mathrm{e}^{-S_{\mathrm{tree}}} \qquad \text{with} \qquad S_{\mathrm{tree}} = s\log(s) + t\log(-t) + u\log(-u)$$

Exponential decay predicted
by [Gross-Manes ‘89]

Comparison with numerics



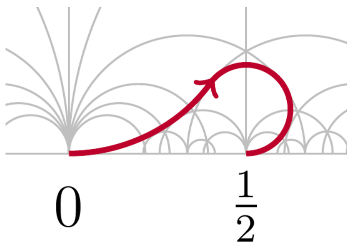
Finally, α' expansion is straightforward

$$
\begin{aligned}
\operatorname{Im} A_{\mathrm{I}} = \pi^2 g_s^4 t_8 \operatorname{tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4}) \Bigg[& \frac{\alpha' \operatorname{Im}\left[(N{-}4)\mathcal{I}_{\mathrm{box}}(s,t) - 2\mathcal{I}_{\mathrm{box}}(s,u)\right]}{120} \\
& + \frac{\zeta_2}{180}\alpha'^3 (N{-}3)s^3 + \frac{\zeta_3}{1260}\alpha'^4 s^3((4N{-}22)s + (N{-}2)t) \\
& + \frac{\zeta_2^2}{50400}\alpha'^5 s^3 \left(2(92N{-}219)s^2 + (15{-}8N)st + (4N{-}9)t^2\right) \\
& + \frac{\zeta_5}{15120}\alpha'^6 s^3 \left((38N{-}208)s^3 + 6(2N{-}5)s^2 t + 3(N{-}4)st^2 + (N{-}2)t^3\right) \\
& + \frac{\zeta_2\zeta_3}{5040}\alpha'^6 s^4 \left(12(N{-}3)s^2 + t((N{-}2)u + t) + st\right) \\
& + \frac{\zeta_3^2}{30240}\alpha'^7 s^4 \left(4(5N{-}28)s^3 + 2(N{+}1)s^2 t - 3(N{-}4)st^2 - (N{-}2)t^3\right) \\
& + \frac{\zeta_2^3}{5292000}\alpha'^7 s^3 \big(70(176N{-}383)s^4 + 25(9{-}11N)s^3 t + 3(119N{-}347)s^2 t^2 \\
& \qquad + 4(17{-}9N)st^3 + 2(16N{-}33)t^4\big) \\
& + \frac{\zeta_7}{831600}\alpha'^8 s^3 \big(20(83N{-}452)s^5 + 5(129N{-}368)s^4 t + 2(148N{-}593)s^3 t^2 \\
& \qquad + (137N{-}362)s^2 t^3 + 4(9N{-}29)st^4 + 10(N{-}2)t^5\big) \\
& + \frac{\zeta_2^2\zeta_3}{756000}\alpha'^8 s^4 \big(60(20N{-}47)s^4 + 5(66{-}23N)s^3 t + 6(33{-}8N)s^2 t^2 \\
& \qquad - (N{-}6)st^3 + 2(9{-}4N)t^4\big) \\
& + \frac{\zeta_2\zeta_5}{37800}(N{-}3)\alpha'^8 s^4 \big(70s^4 - 5s^3 t - 6s^2 t^2 - 2st^3 - t^4\big) + \mathcal{O}(\alpha'^9) \Bigg] + \ldots .
\end{aligned}
$$

Coefficient of N in agreement with [Edison, Guillen, Johansson, Schlotterer, Teng '21]

Outline of the talk

1) Continuation from Euclidean to Lorentzian



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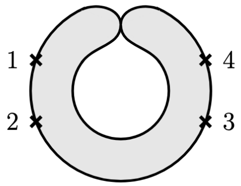

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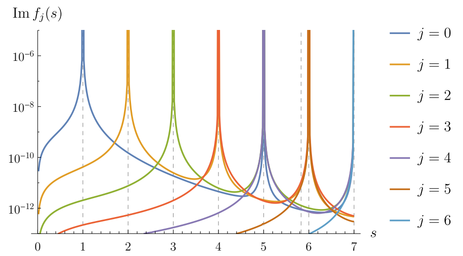

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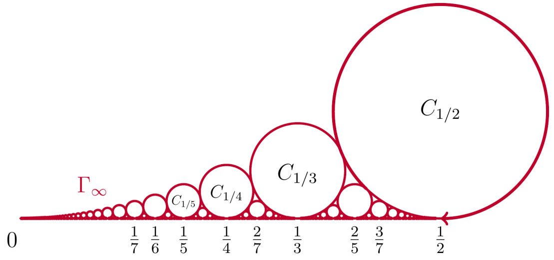
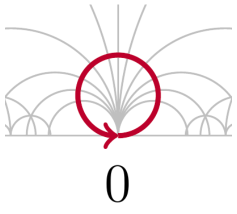


Thank you!

The idea is to recycle the computation of a single circle (infinitely) many times



Great simplification compared with the full, four-dimensional, contour

Farey sequence

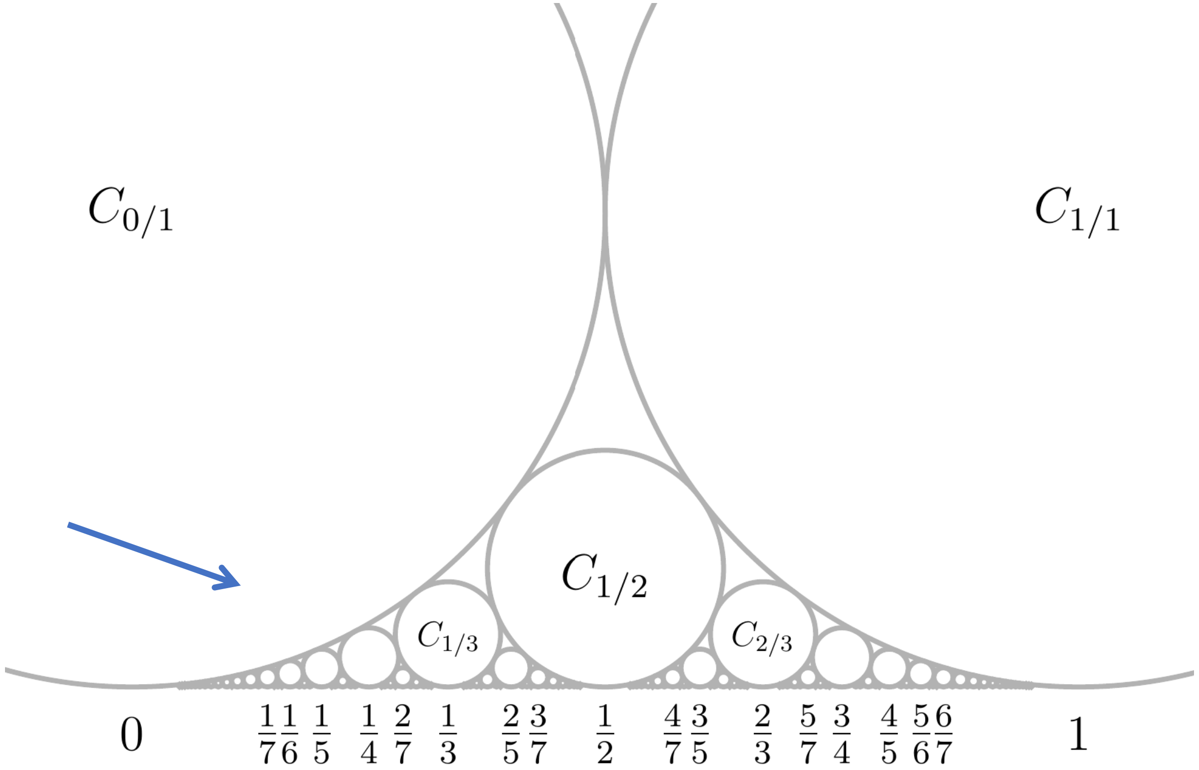
$F_q =$ all irreducible fractions between 0 and 1 with the denominator $\leqslant q$

$$
\begin{aligned}
F_1 &= \left(\tfrac{0}{1}, \tfrac{1}{1}\right) \\
F_2 &= \left(\tfrac{0}{1}, \tfrac{1}{2}, \tfrac{1}{1}\right) \\
F_3 &= \left(\tfrac{0}{1}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{1}{1}\right) \\
F_4 &= \left(\tfrac{0}{1}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{1}{1}\right) \\
F_5 &= \left(\tfrac{0}{1}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{1}{1}\right) \\
&\vdots
\end{aligned}
$$

Ford circles

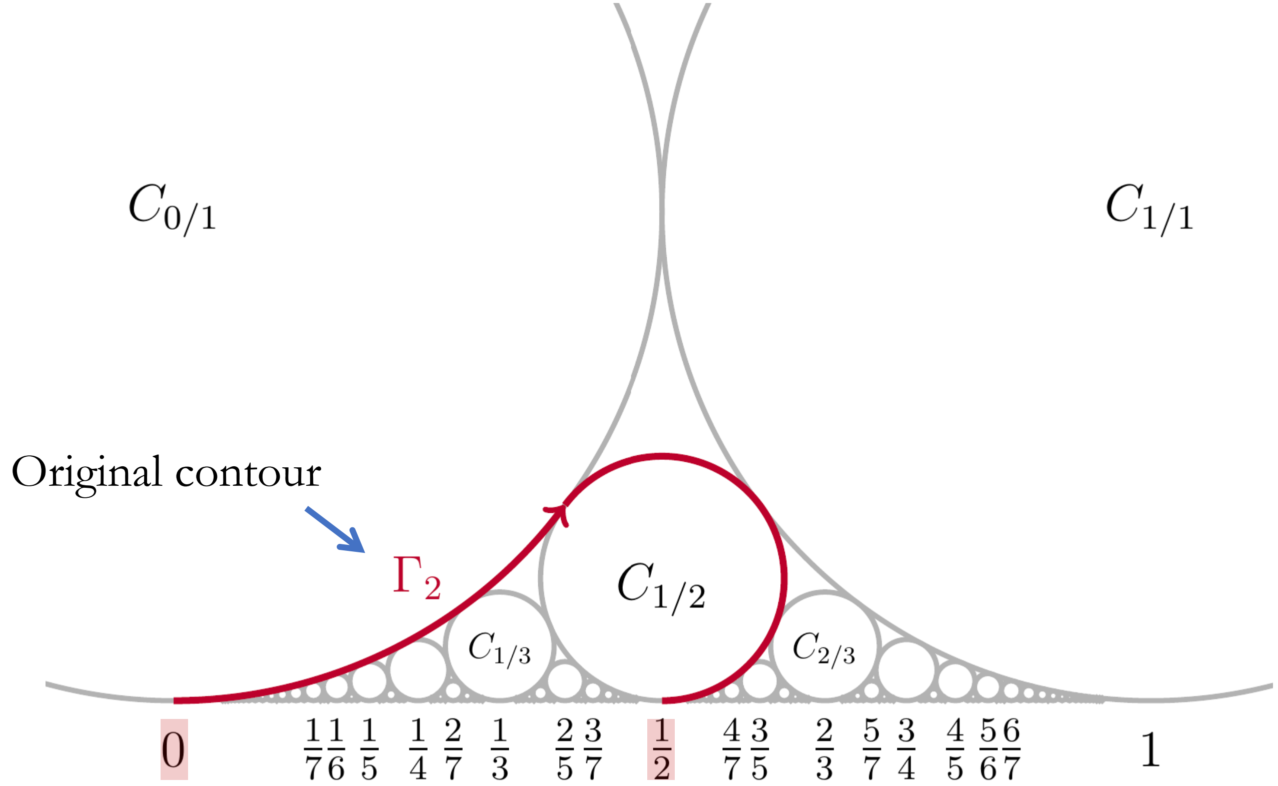
$C_{p/q} =$ circle touching the real axis at $\frac{p}{q}$ with radius $\frac{1}{2q^2}$ in the τ plane

Each one is a modular transform of $C_{0/1}$

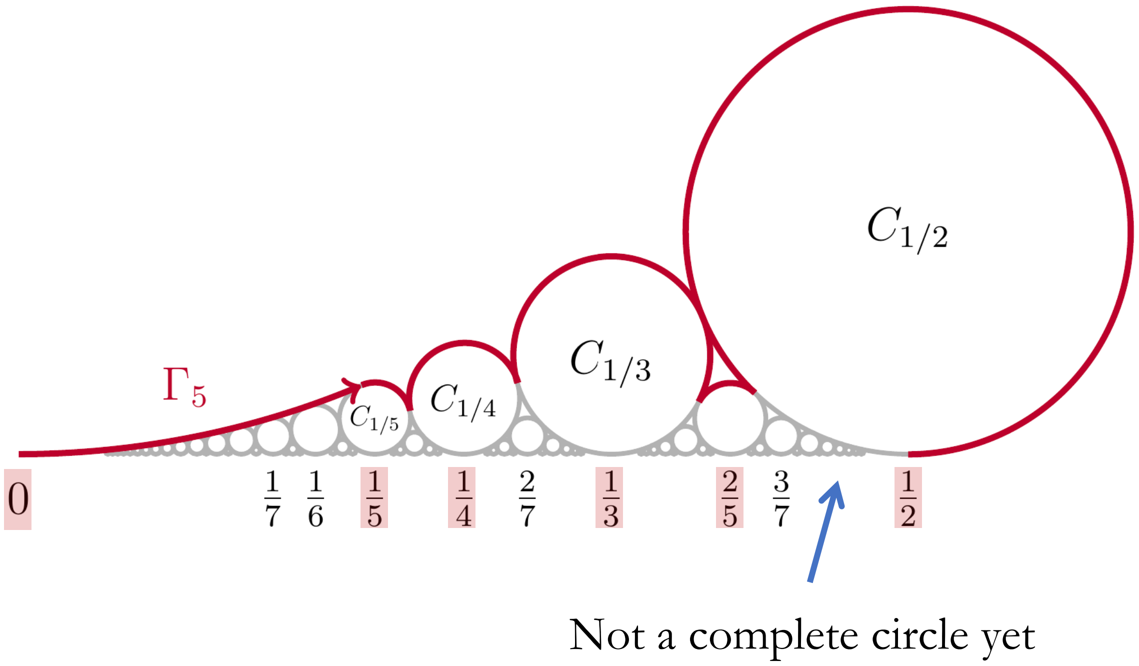


Rademacher contour

$\Gamma_q =$ follow all the Ford circles in the Farey sequence F_q from 0 to ½

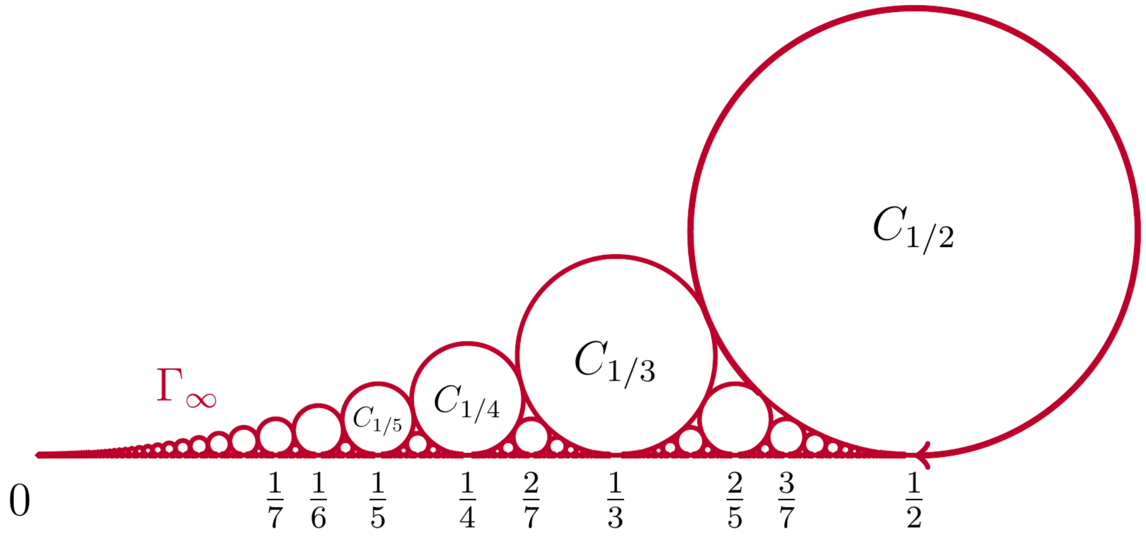


... and so on



$C_{1/2}$
$C_{1/3}$
$C_{1/4}$
$C_{1/5}$
Γ_5
0
$\frac{1}{7}$ $\frac{1}{6}$ $\frac{1}{5}$ $\frac{1}{4}$ $\frac{2}{7}$ $\frac{1}{3}$ $\frac{2}{5}$ $\frac{3}{7}$ $\frac{1}{2}$
Not a complete circle yet

In the limit, we enclose all the circles



In all cases we observed that this series converges!
Stay tuned