On the contribution of the electromagnetic dipole operator to the $B_s \to \mu^+\mu^-$ decay amplitude

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TRR 257 - Particle Physics Phenomenology after the Higgs Discovery

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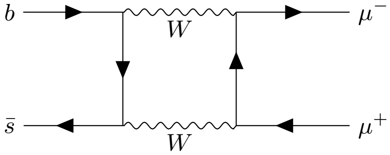
- Flavour changing neutral current (loop-induced)
- Rare decay $\leftrightarrow$ sensitive to New Physics
- complementary to other decay modes ($\bar{B} \to X_s\gamma$, $B \to K^{(*)}\ell\ell, \ldots$)



Integrating out heavy W, Z, t, H leads to *effective weak Hamiltonian*.

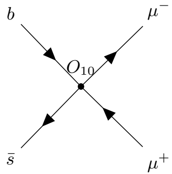
Most important operators for $b \to s\ell^+\ell^-$ transitions:

semi-leptonic $(V-A)\times A$:	$\mathcal{O}_{10}$	$=$	$(\bar{s}\gamma_\mu P_L b)(\bar{\mu}\gamma^\mu\gamma_5\mu)$
semi-leptonic $(V-A)\times V$:	$\mathcal{O}_9$	$=$	$(\bar{s}\gamma_\mu P_L b)(\bar{\mu}\gamma^\mu\mu)$
electromagnetic dipole:	$\mathcal{O}_7^\gamma$	$=$	$\frac{e}{16\pi^2} m_b(\bar{s}_L\sigma_{\mu\nu}b_R)F^{\mu\nu}$

The $B_s \to \mu^+\mu^-$ decay amplitude

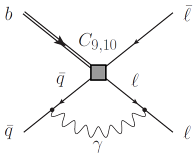
Without QED corrections:

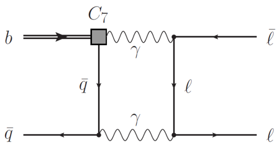
- only $\mathcal{O}_{10}$ contributes
- only hadronic uncertainty from decay constant f_{B_s}
- helicity suppression with the muon mass



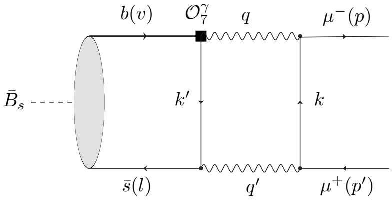
With QED corrections: [Beneke, Bobeth, Szafron, 2017]

- also $\mathcal{O}_9$ and $\mathcal{O}_7^\gamma$ contribute
- resolves momentum distribution of the strange quark
- helicity suppression lifted by strange-quark propagator
- single-log enhancement for $\mathcal{O}_9$ contribution
- double-log enhancement for $\mathcal{O}_7^\gamma$ contribution from endpoint-divergent convolution integrals





O_7^γ contribution to the exclusive decay amplitude



$$\propto \; -\frac{\alpha^2}{2\pi} C_7^{\text{eff}} \, m \, f_{B_s} \left[\bar{u}(p)\,(1+\gamma_5)\,v(p')\right] \mathcal{F}^{\text{LO}}(E,m)$$

Hadronic information contained in form factor:

$$\mathcal{F}^{\text{LO}}(E,m) = \int_0^\infty \frac{d\omega}{\omega}\,\phi_+(\omega)\left[\frac{1}{2}\ln^2\frac{m^2}{2E\omega} + \ln\frac{m^2}{2E\omega} + \frac{\pi^2}{3}\right]$$

- $\phi_+(\omega)$: light-cone distribution amplitude (LCDA) for B_s meson
- ω: light-cone projection of strange-quark momentum $(\langle\omega\rangle \sim \mathcal{O}(\Lambda_{\text{had}}))$
- logarithmic enhancement in small expansion parameter $\lambda^2 = \frac{m^2}{2E\omega} \sim \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{m_b}\right) \sim \mathcal{O}\left(\frac{m}{m_b}\right)$

Momentum regions in QED box diagram

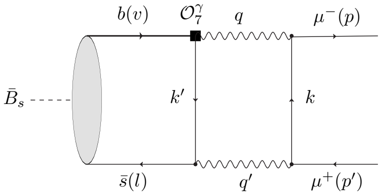
- introduce light-cone coordinates for muon momentum,

$$k^\mu = (\overline{n}\cdot k)\frac{n^\mu}{2} + (n\cdot k)\frac{\overline{n}^\mu}{2} + k_\perp^\mu$$

with $n^2 = \bar{n}^2 = 0$ and $n\cdot\bar{n} = 2$, and $(\bar{n}\cdot p) = (n\cdot p') \simeq 2E$, and $\omega = (\bar{n}\cdot \ell)$

- relevant momentum regions depend on regulator for endpoint divergences !
we considered two alternatives

$$\mathcal{R}_a(k) = \left(\frac{\nu^2}{-(n\cdot k)(\bar{n}\cdot l) + i0}\right)^\delta, \qquad \mathcal{R}_b(k) = \left(\frac{\nu^2}{(\bar{n}\cdot k)(n\cdot p') - (n\cdot k)(\bar{n}\cdot l) + i0}\right)^\delta$$



region	$\{(\bar{n}\cdot k),\ k_\perp, (n\cdot k)\}/m_b$	regulator
$\overline{hc}$	$(\lambda^2, \lambda, 1)$	$\mathcal{R}_{a,b}$
$\overline{c}$	$(\lambda^4, \lambda^2, 1)$	$\mathcal{R}_{a,b}$
s	$(\lambda^2, \lambda^2, \lambda^2)$	$\mathcal{R}_{a,b}$
$\overline{sc}$	$(\lambda^3, \lambda^2, \lambda)$	$\mathcal{R}_b$

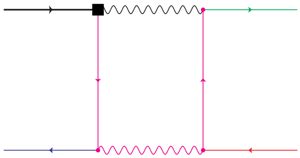
The bare QCD factorization theorem

- Considering version (a) for the analytic regulator,
  the **bare** QCD factorization theorem for the $\mathcal{O}_7^\gamma$ contribution takes the form:

$$
\begin{aligned}
\mathcal{F}(E,m) \;=\; & \int_0^\infty \frac{d\omega}{\omega}\,\phi_+(\omega)\Bigg\{\int_0^1 \frac{du}{u}\,H_1(u)\,\bar{J}_1(u;\omega) \\
& + \bar{J}_2(1,\omega)\int_0^1 \frac{du}{u}\,H_1(u)\,\bar{C}(u;\omega) \\
& + H_1(0)\int_0^\infty \frac{du}{u}\int_0^\infty \frac{d\rho}{\rho}\,S(u,\rho;\omega)\,\bar{J}_2(1+\rho,\omega)\Bigg\}_{\text{bare}}
\end{aligned}
$$

- Each term contains an endpoint-divergent convolution integral, that is regularised for finite δ and ϵ.
- Individual functions in $\{\ldots\}$ follow from momentum regions for given loop diagram ...

- Feynman integral takes the form:



$$I_{\overline{hc}}(\omega) = \int_0^1 \frac{du}{u} H_1^{(0)}(u)\, \bar{J}_1^{(1)}(u;\omega)$$

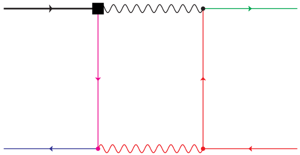
with

$$H_1^{(0)}(u) = 1\,, \qquad \bar{J}_1^{(1)}(u;\omega) = -\Gamma(\epsilon)\left(\frac{\mu^2 e^{\gamma_E}}{2E\omega\, u(1-u)}\right)^{\epsilon}(1-u)$$

- endpoint divergence from $u \to 0$ regularized for $\epsilon < 0$ [δ has been set to zero]
- expanding in ϵ yields:

$$I_{\overline{hc}}(\omega) = \frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2}{2E\omega} + \frac{1}{2}\ln^2\frac{\mu^2}{2E\omega} - \frac{\pi^2}{12} + \frac{1}{\epsilon} + \ln\frac{\mu^2}{2E\omega} + 2$$

Anti-collinear region: with $k \sim (\lambda^4, \lambda^2, 1)$



- Feynman integral takes the form:

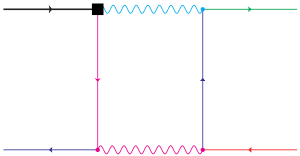
$$I_{\bar{c}}(\omega) \quad = \quad \bar{J}_2^{(0)}(1,\omega) \int_0^1 \frac{du}{u} \, H_1^{(0)}(u) \, \bar{C}^{(1)}(u;\omega)$$

with

$$\bar{J}_2^{(0)}(z,\omega) \quad = \quad \frac{1}{z}\,, \qquad \bar{C}^{(1)}(u;\omega) = \left(\frac{\mu^2 e^{\gamma_E}}{m^2}\right)^{\epsilon} \left(\frac{\nu^2}{2E\omega}\right)^{\delta} (1-u)^{1-2\epsilon} \; u^{-\delta}$$

- endpoint divergence from $u \to 0$ requires analytic regulator $\delta \neq 0$
- expanding in δ and subsequently in ϵ yields

$$I_{\bar{c}}(\omega) \quad = \quad \left(-\frac{1}{\delta} - \ln\frac{\nu^2}{2E\omega}\right)\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{m^2}\right) + \frac{\pi^2}{3} - \frac{1}{\epsilon} - \ln\frac{\mu^2}{m^2} - 2$$



- Feynman integral takes the form:

$$\tilde{I}_s(\omega) = H_1^{(0)}(0) \int_0^\infty \frac{du}{u} \int_0^\infty \frac{d\rho}{\rho} S^{(1)}(u,\rho;\omega)\, \bar{J}_2^{(0)}(1+\rho,\omega)$$

with

$$S^{(1)}(u,\rho;\omega) = \theta(u\rho - \lambda^2) \left(\frac{\mu^2 e^{\gamma_E}}{2E\omega}\right)^\epsilon \left(\frac{\nu^2}{2uE\omega}\right)^\delta \frac{(u\rho-\lambda^2)^{-\epsilon}}{\Gamma(1-\epsilon)}$$

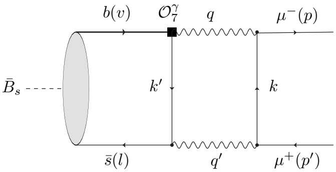
- endpoint divergence requires analytic regulator $\delta \neq 0$
- expanding in δ and subsequently in ϵ yields

$$I_s(\omega) = \left(\frac{1}{\delta} + \ln\frac{\nu^2}{m^2}\right)\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{m^2}\right) - \frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2}{m^2} - \frac{1}{2}\ln^2\frac{\mu^2}{m^2} + \frac{\pi^2}{12}$$

Summing up

$$I(\omega) \;=\; I_{\overline{hc}}(\omega) + I_{\bar{c}}(\omega) + I_s(\omega) = \frac{1}{2}\ln^2\frac{m^2}{2E\omega} + \ln\frac{m^2}{2E\omega} + \frac{\pi^2}{3}$$

- all poles cancel ✓
- double-logarithmic enhancement ✓



$$\mathcal{F}^{\rm LO} = \int_0^\infty \frac{d\omega}{\omega}\, I(\omega) \qquad \checkmark$$

- but how to renormalize functions *before* the limit $\delta \to 0$???

Construction of the renormalized QCD factorization theorem

Crucial ingredients:

[Liu/Neubert, Bell/Böer/Feldmann ...]

- use refactorization conditions,

$$[[\bar{C}(u;\omega)]] \equiv \bar{C}(u;\omega)\big|_{u\to 0} = \int_0^\infty \frac{d\rho}{\rho}\, S(u,\rho;\omega) + \mathcal{O}(\alpha)$$

  to redistribute endpoint-divergent contributions within factorization theorem
- take into account that scaleless integrals vanish for analytic regulators,

$$\int_0^\infty \frac{du}{u} \int_0^\infty \frac{d\rho}{\rho}\, S(u,\rho;\omega)\big|_{\lambda^2\to 0} = 0\,,$$

Construction of the renormalized QCD factorization theorem

$$\mathcal{F}(E,m) = \int_0^\infty \frac{d\omega}{\omega}\,\phi_+(\omega)\Bigg\{\int_0^\infty \frac{du}{u}\Big[H_1(u)\,\bar{J}_1(u;\omega)\,\theta(1-u) - H_1(0)\,\bar{J}_2(1,\omega)\,\theta(u-1)\int_1^\infty \frac{d\rho}{\rho}\,S(u\rho;\omega)\Big]_{\lambda^2\to 0}$$

$$+\,\bar{J}_2(1,\omega)\int_0^1 \frac{du}{u}\Big[H_1(u)\,\bar{C}(u) - H_1(0)\,\big[\big[\bar{C}(u)\big]\big]\Big]$$

$$+\,H_1(0)\int_0^\infty \frac{du}{u}\int_0^\infty \frac{d\rho}{\rho}\,\Big[\bar{J}_2(1+\rho,\omega) - \theta(1-\rho)\,\bar{J}_2(1,\omega)\Big]\,S(u\rho;\omega)$$

$$+\,\bar{J}_2(1,\omega)\,H_1(0)\int_0^1 \frac{du}{u}\int_0^1 \frac{d\rho}{\rho}\,S(u\rho;\omega)\Big|_{\lambda^2\to 0}\Bigg\}$$

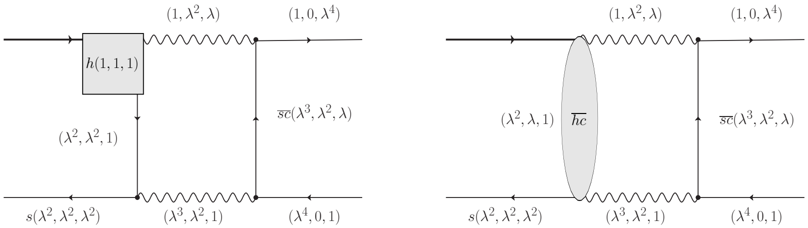
- Each individual line in the factorization theorem is now endpoint finite!
- The last line contains the large double-logarithmic enhancement, and corresponds to *anti-soft-collinear* region with cut-off longitudinal integrals

[see also: Liu/Penin, [1809.04950].]

$$\propto \int_0^1 \frac{du}{u}\int_0^1 \frac{d\rho}{\rho}\,\theta(u\rho-\lambda^2) = \frac{1}{2}\ln^2\lambda^2$$

- In the following we concentrate on this term of the factorization theorem . . .

Leading-Logarithmic QCD-Corrections



- 1-loop RGE is multiplicative in so-called dual space, where

$$\phi_+(\omega) = \int \frac{d\omega'}{\omega'} \sqrt{\frac{\omega}{\omega'}}\, J_1\left(2\sqrt{\frac{\omega}{\omega'}}\right) \rho_+(\omega')$$

- leading-logarithmic QCD corrections to double-logarithmic term in form factor take the form

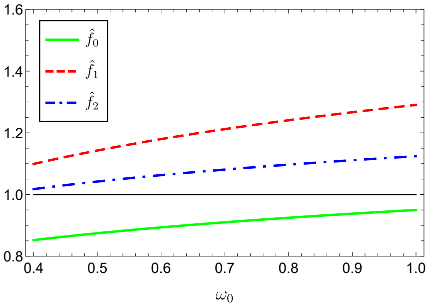
$$\mathcal{F}(E,m)\Big|_{\rm LL} = \frac{1}{2}\, e^{V(\mu_{hc},\mu_h)}\, e^{V(\mu_{hc},\mu_0)} \int \frac{d\omega'}{\omega'} \ln^2 \frac{m^2}{2E\hat{\omega}'} \left(\frac{\hat{\mu}_0\, e^{2\gamma_E}}{\omega'}\right)^{-g(\mu_{hc},\mu_0)} \rho_+(\omega',\mu_0)\,,$$

Leading-Logarithmic QCD-Corrections

- numerical estimate requires systematic expansion of LCDA
- we employ the parametrization as in [Feldmann/Lüghausen/van Dyk 2022]

$$\rho_+(\omega', \mu_0) = \frac{e^{-\omega_0/\omega'}}{\omega'} \sum_{k=0}^{K} \frac{(-1)^k \, a_k(\mu_0)}{1+k} L_k^{(1)}(2\omega_0/\omega')$$

- depending on the shape of the LCDA, the RG effects can be of the order $10 - 30\%$ (see figure)
- Note, however, that the overall effect of the electromagnetic dipole operator is already small!



Next steps:

- confirm factorization formula by calculating anti-hard-collinear functions at 2 loops !
- conceptually understand factorization of QED corrections for this case ?

Backup Slides

Soft Region. Details

Taking the analytic regulator

$$\mathcal{R}_a(k) \to \left(\frac{\nu^2}{-\omega\,(n\cdot k)+i0} \right)^{\delta} ,$$

we can first perform the $k_\perp$ integration, to end up with

$$I_s(\omega) = \frac{\Gamma(\epsilon)}{2\pi i} \int \frac{d(n\cdot k)}{-n\cdot k + i0} \int \frac{d(\bar{n}\cdot k)}{\bar{n}\cdot k + i0} \, \frac{\omega}{\bar{n}\cdot k - \omega + i0}$$
$$\times \left(\frac{\mu^2 e^{\gamma_E}}{-(n\cdot k)(\bar{n}\cdot k) + m^2 - i0} \right)^{\epsilon} \mathcal{R}_a(k) \,.$$

Now consider the analytic structure in $(\bar{n}\cdot k)$-plane:

- There are two poles at

$$\mathrm{Re}(\bar{n}\cdot k) = 0,\, \omega \quad \textit{below} \text{ the real axis.}$$

- There is a branch cut from the dimensional regulator,

$$\begin{array}{lcll} \text{for } (n\cdot k) > 0 & : & m^2/(n\cdot k) \leq \mathrm{Re}(\bar{n}\cdot k), & \textit{below} \text{ the real axis,} \\ \text{for } (n\cdot k) < 0 & : & \mathrm{Re}(\bar{n}\cdot k) \leq m^2/(n\cdot k), & \textit{above} \text{ the real axis.} \end{array}$$

Soft Region. Details 2

- For $(n \cdot k) > 0$, the integrand is analytic in the upper half plane, and therefore the integral vanishes.
- For $(n \cdot k) < 0$, the branch cut from the dimensional regulator is in the upper half plane.

$$\mathrm{Disc}\left(\frac{\mu^2}{-(n\cdot k)(\bar{n}\cdot k)+m^2-i0}\right)^{\epsilon} = \frac{2\pi i\,\theta((n\cdot k)(\bar{n}\cdot k)-m^2)}{\Gamma(\epsilon)\Gamma(1-\epsilon)}\left(\frac{\mu^2}{(n\cdot k)(\bar{n}\cdot k)-m^2}\right)^{\epsilon}$$

- we arrive at the representation

$$I_s(\omega) = \frac{1}{\Gamma(1-\epsilon)}\int\limits_{-\infty}^{0}\frac{d(n\cdot k)}{n\cdot k}\int\limits_{-\infty}^{m^2/n\cdot k}\frac{d(\bar{n}\cdot k)}{\bar{n}\cdot k}\,\frac{\omega}{\omega-\bar{n}\cdot k}\left(\frac{\mu^2 e^{\gamma_E}}{(n\cdot k)(\bar{n}\cdot k)-m^2}\right)^{\epsilon}\mathcal{R}_a(k)$$

$$= \frac{1}{\Gamma(1-\epsilon)}\left(\frac{\mu^2 e^{\gamma_E}}{2E\omega}\right)^{\epsilon}\left(\frac{\nu^2}{2E\omega}\right)^{\delta}\int_0^{\infty}\frac{du}{u}\int_{\lambda^2/u}^{\infty}\frac{d\rho}{\rho}\,\frac{1}{1+\rho}\left(u\rho-\lambda^2\right)^{-\epsilon}u^{-\delta}$$

QCD corrections

- Concentrating on the anti-soft-collinear region which generates the leading double-logarithmic term
- Include the leading logarithmic radiative QCD corrections the QCD effects in the factorization formula

$$\mathcal{F}(E,m)\Big|_{\mathrm{LL}} = \frac{1}{2} H_1(0;\mu) \int \frac{d\omega'}{\omega'} \rho_+(\omega';\mu) \ln^2 \frac{m^2 e^{2\gamma_E}}{2E\omega'} \times \bar{\mathcal{J}}_2(1,\omega';\mu)$$

- Hard QCD corrections at the $b \to s\gamma^*$ vertex $(H_1(0,\mu))$
- Hard-collinear corrections to the strange-quark jet function $\bar{\mathcal{J}}_2(1,\omega',\mu)$
- Soft corrections to the B-meson LCDA

QCD corrections 2

- Inserting the net leading-logarithmic RG factor into the form factor for the $\mathcal{O}_7$ contribution, we end up with

$$\mathcal{F}(E,m)\Big|_{\rm LL} = \frac{1}{2}\, e^{V(\mu_{hc},\mu_h)}\, e^{V(\mu_{hc},\mu_0)} \int \frac{d\omega'}{\omega'}\, \ln^2 \frac{m^2}{2E\hat{\omega}'}\, \left(\frac{\hat{\mu}_0}{\omega'}\right)^{-g(\mu_{hc},\mu_0)} \rho_+(\omega',\mu_0)\,,$$

  with $\hat{\omega}' = \omega'\, e^{-2\gamma_E}$.

- If we define the generating function for logarithmic moments of $\rho_+(\omega',\mu_0)$ as

$$F_{[\rho_+]}(t;\mu_0,\mu_m) \quad = \quad \int_0^\infty \frac{d\omega'}{\omega'} \left(\frac{\hat{\mu}_m}{\omega'}\right)^{-t} \rho_+(\omega',\mu_0)\,,$$

- This can also be written as ($\hat{\mu}_m = \mu_m e^{2\gamma_E}$)

$$\mathcal{F}(E,m)\Big|_{\rm LL} = \frac{1}{2} e^{V(\mu_{hc},\mu_h)} e^{V(\mu_{hc},\mu_0)} \left(\frac{2E\mu_0}{m^2}\right)^{-g(\mu_{hc},\mu_0)} \frac{d^2}{dt^2} F_{[\rho_+]}(t+g(\mu_{hc},\mu_0);\mu_0,\frac{m^2}{2E})\Bigg|_{t=0}\,.$$

QCD corrections 3

- Using the explicit parametrization as in [Feldmann/Lüghausen/van Dyk 2022]

$$\rho_+(\omega', \mu_0) \quad = \quad \frac{e^{-\omega_0/\omega'}}{\omega'} \sum_{k=0}^{K} \frac{(-1)^k \, a_k(\mu_0)}{1+k} \, L_k^{(1)}(2\omega_0/\omega')$$

- Truncating the parametrization for $\rho_+(\omega', \mu_0)$ at $K = 2$, we obtain

$$\begin{aligned}
\mathcal{F}(E,m)\Big|_{\mathrm{LL}} \quad &\simeq \quad \frac{\Gamma(1-g)}{2\omega_0} \, e^{V(\mu_{hc},\mu_h)+V(\mu_{hc},\mu_0)} \left(\frac{\hat{\mu}_0}{\omega_0}\right)^{-g} \\
&\times \left\{ \left(a_0 - g\,a_1 + \frac{1+2g^2}{3}\,a_2\right) \left[\left(\ln\hat{\lambda}_0^2 + \psi(1-g)\right)^2 + \psi'(1-g)\right] \right. \\
&\quad \left. + \left(2a_1 - \frac{8}{3}\,g\,a_2\right)\left(\ln\hat{\lambda}_0^2 + \psi(1-g)\right) + \frac{4a_2}{3} \right\} \\
\quad &\equiv \quad \sum_{k=0}^{2} a_k \, f_k(\omega_0)\,,
\end{aligned}$$