Integration by parts

Andrey Grozin

Institut für Theoretische Teilchenphysik
Karlsruher Institut für Technologie

Steps to solve a problem

- Diagrams generation, classification into topologies, routing momenta
- Tensor and Dirac algebra in numerators, reduction to scalar Feynman integrals
- Reduction of scalar Feynman integrals to master integrals
- Calculation of master integrals

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Expansion in small ratios of momenta and masses (the method of regions)

Feynman graphs and Feynman integrals

Loop momenta $k_1, \ldots, k_L$
External momenta $p_1, \ldots, p_E$
$q_i = k_1, \ldots, k_L, p_1, \ldots, p_E$
Line momenta $l_1, \ldots, l_I$ — linear combinations of q_i

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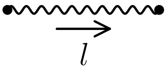
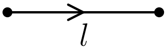
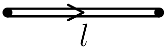


$$= \frac{1}{-l^2 - i0}$$



$$= \frac{1}{m^2 - l^2 - i0}$$



$$= \frac{1}{-2l \cdot v - i0}$$

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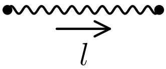

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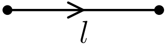

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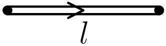
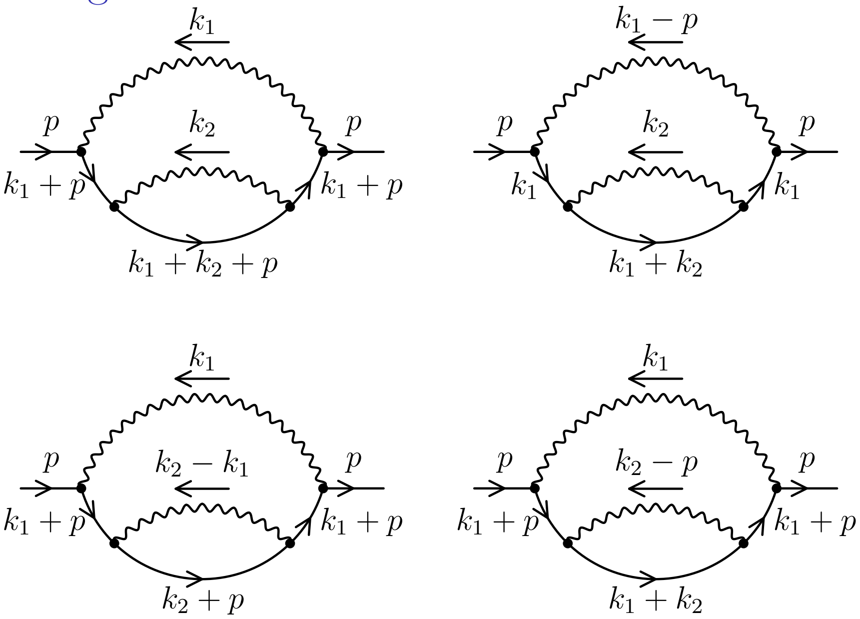


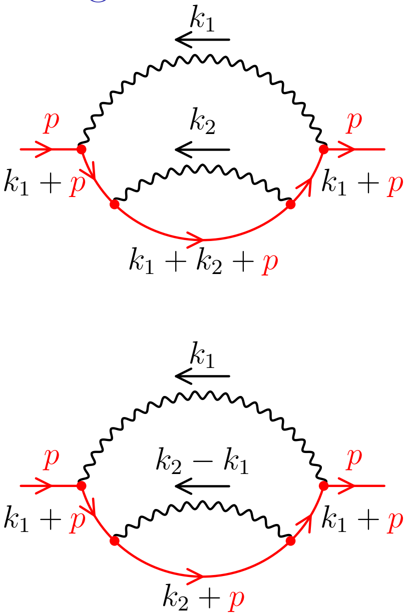
$$= \frac{1}{-2l \cdot v - i0}$$

Denominators D_i are linear in $s_{ij} = k_i \cdot q_j$

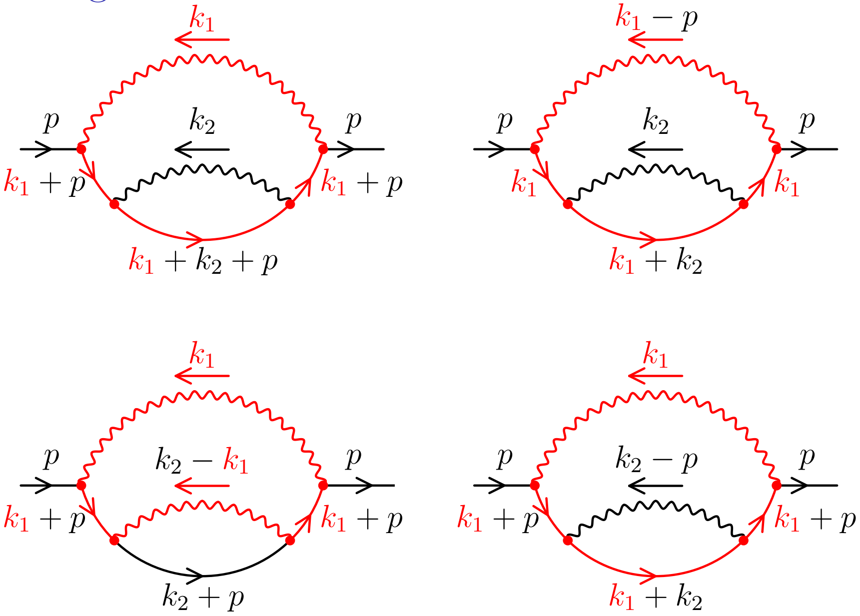
Routing momenta



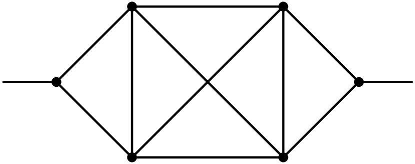
Routing momenta



Routing momenta



Symmetries



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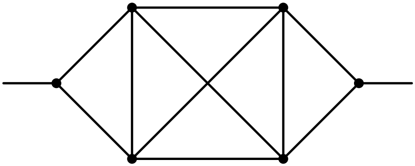

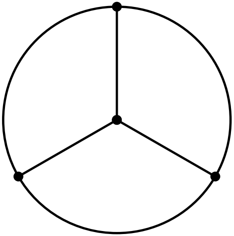

Irreducible numerators

There are

$$N = \frac{L(L+1)}{2} + LE$$

scalar products $s_{ij} = k_i \cdot q_j$

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$(E+1)$-legged L-loop diagrams:
the maximum number of denominators

$$M = 3L + E - 2 \qquad N - M = \frac{(L-1)(L+2E-4)}{2}$$

Vacuum diagrams

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Vacuum diagrams

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Irreducible numerators $D_{M+1}, \ldots, D_N$

Scalar Feynman integral

$$I(n_1, \ldots, n_N) = \int d^d k_1 \cdots d^d k_L \, f(k_1, \ldots, k_L)$$

$$f(k_1, \ldots, k_L) = \frac{1}{D_1^{n_1} \cdots D_N^{n_N}}$$

D_i are linear functions of $s_{ij} = k_i \cdot q_j$

Point in L-dimensional integer space

For irreducible numerators, $n_i \leq 0$

Dimensionality: 0 scales

$$[\text{mass}]^{Ld-n}$$

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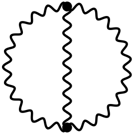
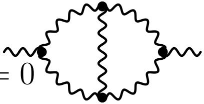


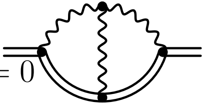
$= 0$

$p^2 = 0$



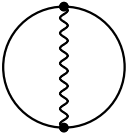
$= 0$

$p \cdot v = 0$

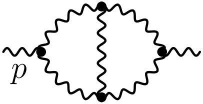


$= 0$

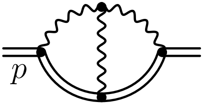
Dimensionality: 1 scale



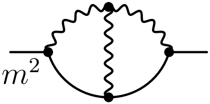
$$= m^{Ln-n} I(d)$$



$$= (-p^2)^{Ld/2-n} I(d)$$



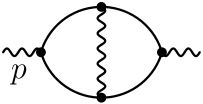
$$= (-2p \cdot v)^{Ld-n} I(d)$$



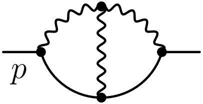
$p^2 = m^2$

$$= m^{Ld-n} I(d)$$

Dimensionality: 2 scales



$$= m^{Ld-n} I\left(d; \frac{-p^2}{m^2}\right)$$



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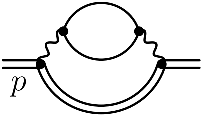
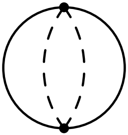


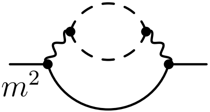
$$= m^{Ld-n} I\left(d; \frac{-2p \cdot v}{m^2}\right)$$

Dimensionality: 2 scales



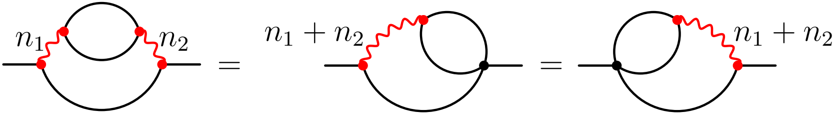
$$= m^{Ld-n} I\left(d; \frac{m'}{m}\right)$$

$p^2 = m^2$

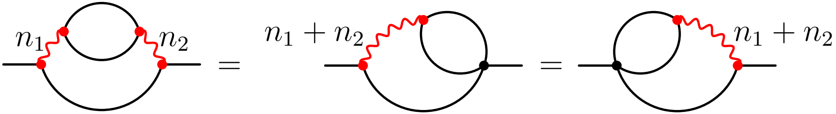


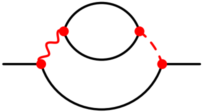
$$= m^{Ld-n} I\left(d; \frac{m'}{m}\right)$$

Self-energy insertions



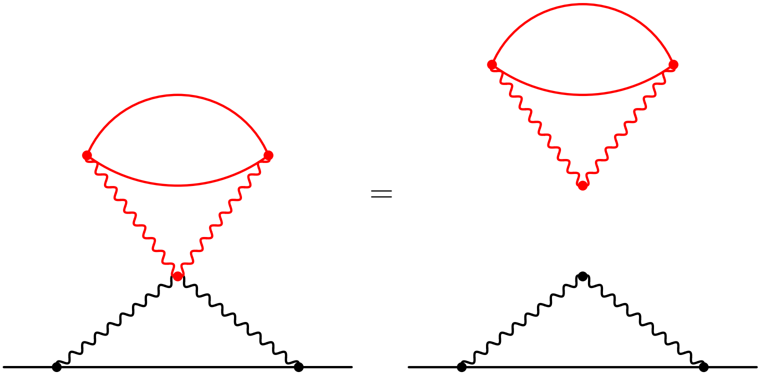
Self-energy insertions





Linearly-dependent denominators

Subdiagrams connected at 1 vertex



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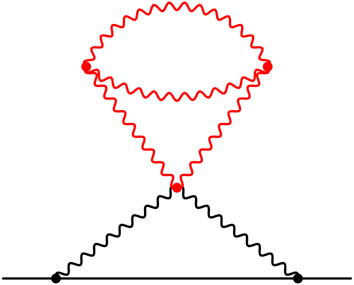
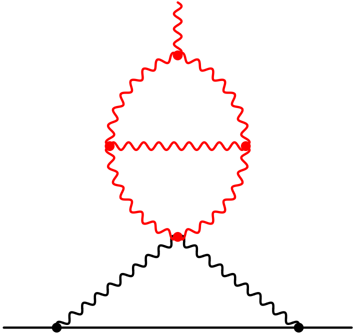


$= 0$

Subdiagrams connected at 1 vertex



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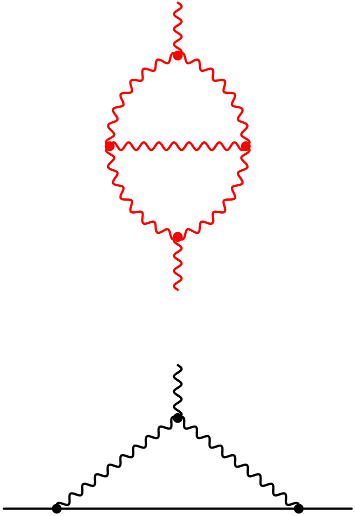

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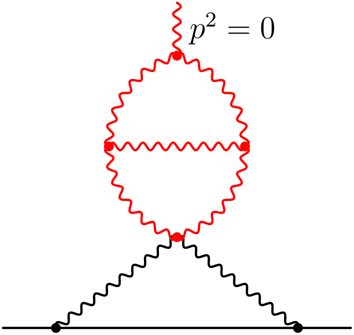
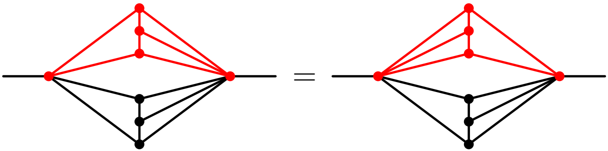
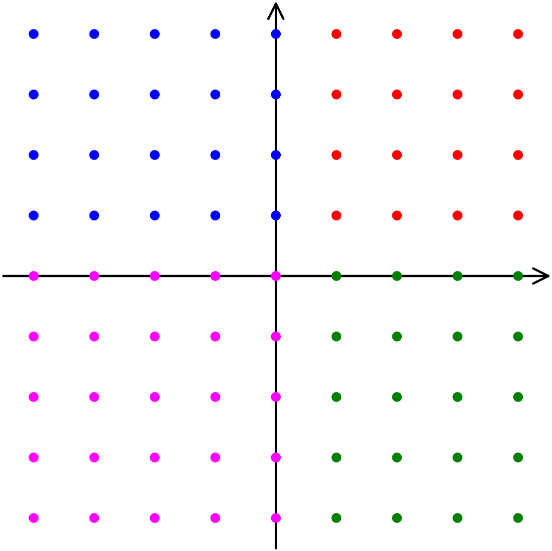


$= 0$

Subdiagrams connected at 2 vertices



Sectors

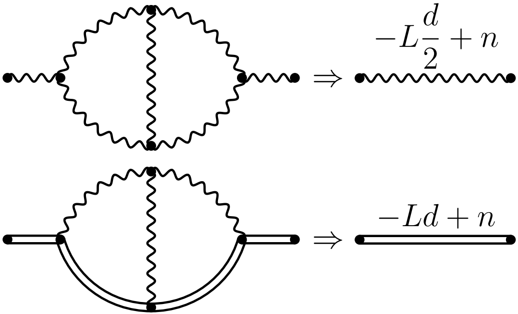


Partial ordering

Sectors

- For irreducible numerators, sectors $n_i > 0$ don't exist
- Trivial sectors: $I = 0$
  (at least, the sector with all $n_i \leq 0$ is trivial)
- Sectors just above trivial ones:
  often an explicit formula via Γ functions
- Some sectors are related by symmetries

Non-integer indices



d cannot be compared with integers
$\Rightarrow$ No sectors along this index

Integration momenta substitutions

Lie group

$$k_i \to M_{ij}q_j = A_{ij}k_j + B_{ij}p_j$$

$$M = \begin{pmatrix} A_{11} & \cdots & A_{1L} & B_{11} & \cdots & B_{1E} \\ \vdots & \ddots & \vdots & B_{11} & \cdots & B_{1E} \\ A_{L1} & \cdots & A_{LL} & B_{L1} & \cdots & B_{LE} \end{pmatrix}$$

$$\det A \neq 0$$

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Infinitesimal transformations $k_i \to k_i + \alpha q_j$

$$f \to f + \alpha q_j \cdot \partial_i f$$

If $j = i$

$$d^d k_i \to (1+\alpha)^d d^d k_i = (1 + \alpha d) d^d k_i$$

Lie algebra

$$\int d^d k_1 \cdots d^d k_L \, O_{ij} f = 0$$

$$O_{ij} = \partial_i \cdot q_j \qquad \partial_i = \frac{\partial}{\partial k_i}$$

R. Lee (2008)

$$[O_{ij}, O_{i'j'}] = \delta_{ij'} O_{i'j} - \delta_{i'j} O_{ij'}$$

Lie algebra

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$$O_{ij} = d\delta_{ij} + q_j \cdot \sum_n \frac{\partial D_n}{\partial k_i} \frac{\partial}{\partial D_n}$$

Operator notation

$$(\mathbf{n}_i F)(n_1, \ldots, n_i, \ldots, n_N) = n_i F(n_1, \ldots, n_i, \ldots, n_N)$$
$$(\mathbf{i}^+ F)(n_1, \ldots, n_i, \ldots, n_N) = F(n_1, \ldots, n_i + 1, \ldots, n_N)$$
$$(\mathbf{i}^- F)(n_1, \ldots, n_i, \ldots, n_N) = F(n_1, \ldots, n_i - 1, \ldots, n_N)$$
$$\mathbf{i}^+\mathbf{i}^- = \mathbf{i}^-\mathbf{i}^+ = 1 \qquad [\mathbf{i}^\pm, \mathbf{n}_j] = \pm\delta_{ij}\mathbf{i}^\pm$$

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$$\hat{\mathbf{i}}^+ = \mathbf{n}_i \mathbf{i}^+$$
$$[\hat{\mathbf{i}}^+, \mathbf{j}^-] = \delta_{ij}$$

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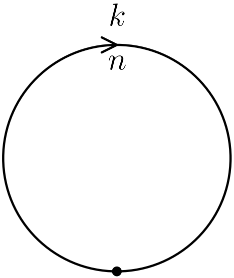
$$\hat{\mathbf{i}}^+ = \mathbf{n}_i \mathbf{i}^+$$
$$[\hat{\mathbf{i}}^+, \mathbf{j}^-] = \delta_{ij}$$

$$\int d^d k_1 \cdots d^d k_L \, O_{ij} f = 0 = P_{ij}(\hat{\mathbf{i}}^+, \mathbf{i}^-) I(n_1, \ldots, n_N)$$
$$[P_{ij}, P_{i'j'}] = \delta_{ij'} P_{i'j} - \delta_{i'j} P_{ij'}$$

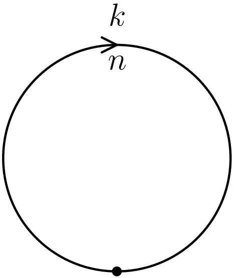
1-loop vacuum diagram



$$\frac{1}{i\pi^{d/2}} \int \frac{d^d k}{D^n} = m^{d-2n} V(n) \qquad D = m^2 - k^2 - i0$$

by dimensionality

1-loop vacuum diagram



$$\frac{1}{i\pi^{d/2}} \int \frac{d^d k}{D^n} = m^{d-2n} V(n) \qquad D = m^2 - k^2 - i0$$

by dimensionality

$$V(n) = \frac{1}{i\pi^{d/2}} \int \frac{d^d k}{D^n} \qquad D = 1 - k^2 - i0$$

1-loop vacuum diagram

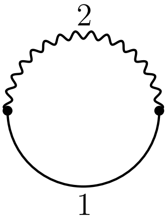
$$(d - 2n + 2n\mathbf{1}^{+})V(n) = 0$$

1-loop vacuum diagram

$$(d - 2n + 2n\mathbf{1}^{+})V(n) = 0$$

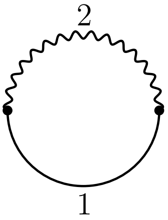
$$V(n) = \frac{1}{\Gamma(n)} \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma\left(1 - \frac{d}{2}\right)} V(1)$$

Vacuum diagram with masses m and 0



$$D_1 = 1 - k^2 \qquad D_2 = -k^2$$

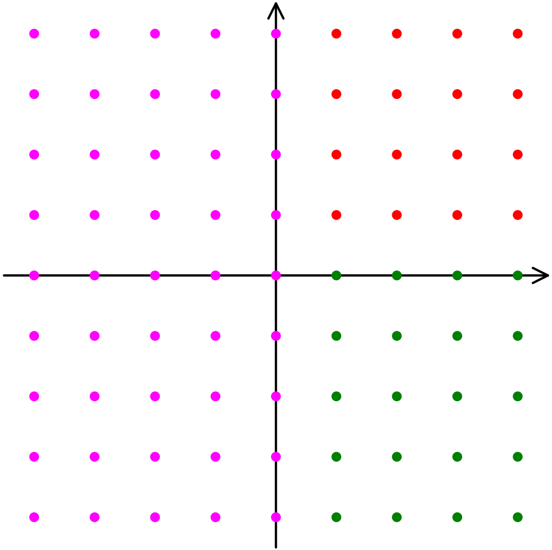
Vacuum diagram with masses m and 0



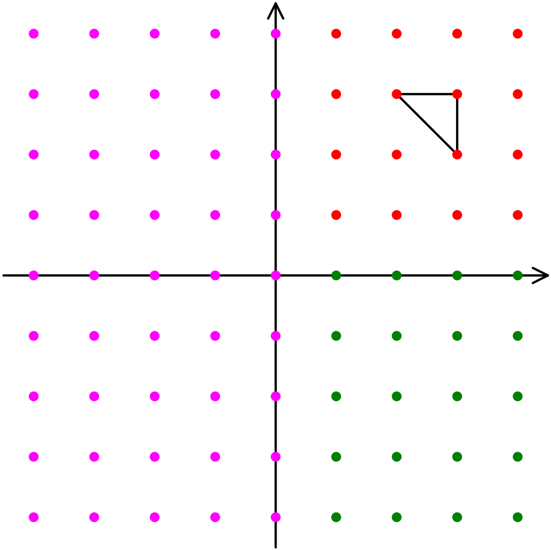
$$D_1 = 1 - k^2 \qquad D_2 = -k^2$$

$$D_1 - D_2 = 1$$

Vacuum diagram with masses m and 0

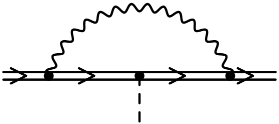


Vacuum diagram with masses m and 0



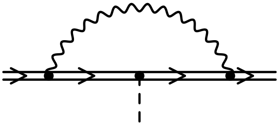
$$(1 - \mathbf{1}^{-} + \mathbf{2}^{-})I = 0$$

Linear dependent HQET denominators

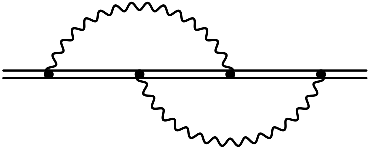


$$D_1 = -2(k + p_1) \cdot v = -2(k \cdot v + \omega_1)$$
$$D_2 = -2(k + p_2) \cdot v = -2(k \cdot v + \omega_2)$$
$$D_1 - D_2 + 2(\omega_1 - \omega_2) = 0$$
$$\mathbf{1}^- - \mathbf{2}^- + 2(\omega_1 - \omega_2) = 0$$

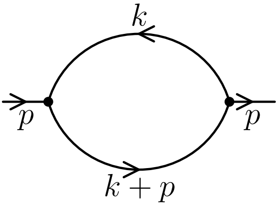
Linear dependent HQET denominators



$$D_1 = -2(k + p_1) \cdot v = -2(k \cdot v + \omega_1)$$
$$D_2 = -2(k + p_2) \cdot v = -2(k \cdot v + \omega_2)$$
$$D_1 - D_2 + 2(\omega_1 - \omega_2) = 0$$
$$\mathbf{1}^- - \mathbf{2}^- + 2(\omega_1 - \omega_2) = 0$$



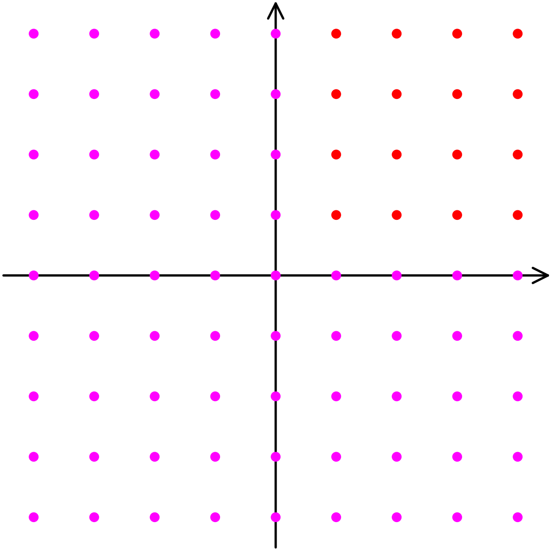
1-loop massless self-energy



$p^2 = -1$ (restore by dimensionality)

$$D_1 = -(k+p)^2 \qquad D_2 = -k^2$$
$$p^2 = -1 \qquad k^2 = -D_2 \qquad 2p \cdot k = 1 - D_1 + D_2$$
$$\partial \cdot k = d - 2k \cdot (k+p) \frac{\partial}{\partial D_1} - 2k \cdot k \frac{\partial}{\partial D_2}$$

1-loop massless self-energy



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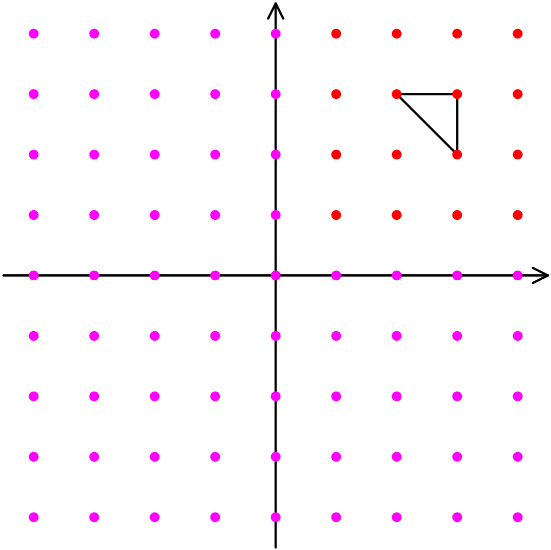

$$\left[d - n_1 - 2n_2 + n_1 \mathbf{1}^+ (1 - \mathbf{2}^-)\right] I = 0$$

Some codes

- `Mincer` (`Form`) — 3-loop massless self-energies
- `Recursor` (`Reduce`) — 2-loop massive on-shell self-energies, 3-loop massive vacuum diagrams
- `SHELL2` (`Form`) — 2-loop massive on-shell self-energies
- `Matad` (`Form`) — 3-loop massive vacuum diagrams
- `Slicer` (`Reduce`) — 3-loop massless self-energies
- `Grinder` (`Reduce`) — 3-loop HQET self-energies
- `SHELL3` (`Form`) — 3-loop massive on-shell self-energies

Homogeneity relations

$$\left(\sum_i p_i \cdot \frac{\partial}{\partial p_i} + \sum_i m_i \frac{\partial}{\partial m_i} \right) I = \left(Ld - 2 \sum_i n_i \right) I$$

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$$\left(\sum_i p_i \cdot \frac{\partial}{\partial p_i} + \sum_i m_i \frac{\partial}{\partial m_i} - Ld + 2\sum_i n_i\right) f$$
$$= \left(\sum_i q_i \cdot \frac{\partial}{\partial q_i} + \sum_i m_i \frac{\partial}{\partial m_i} - \sum_i k_i \cdot \frac{\partial}{\partial k_i} - Ld + 2\sum_i n_i\right) f$$
$$= \left(-\sum_i k_i \cdot \frac{\partial}{\partial k_i} - Ld\right) f = -\left(\sum_i \frac{\partial}{\partial k_i} \cdot k_i\right) f$$

Lorentz-invariance relations

$E \geq 2$ $(i \neq j)$

$$p_i^\mu p_j^\nu \left(\sum_n p_n^{[\mu} \frac{\partial}{\partial p_n^{\nu]}} \right) I = 0$$

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They are linear combinations of IBP relations

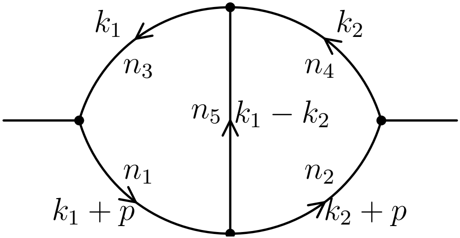
$$p_i^\mu p_j^\nu \left(\sum_n p_n^{[\mu} \frac{\partial}{\partial p_n^{\nu]}} \right) f = p_i^\mu p_j^\nu \left(\sum_n q_n^{[\mu} \frac{\partial}{\partial q_n^{\nu]}} - \sum_n k_n^{[\mu} \frac{\partial}{\partial k_n^{\nu]}} \right) f$$

$$= -p_i^\mu p_j^\nu \left(\sum_n k_n^{[\mu} \frac{\partial}{\partial k_n^{\nu]}} \right) f$$

$$= \sum_n \left(p_j \cdot k_n \, p_i \cdot \frac{\partial}{\partial k_n} - p_i \cdot k_n \, p_j \cdot \frac{\partial}{\partial k_n} \right) f$$

$$= \sum_n \frac{\partial}{\partial k_n} \cdot (p_i \, p_j \cdot k_n - p_j \, p_i \cdot k_n) \, f$$

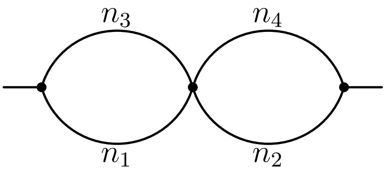
2-loop massless self-energy

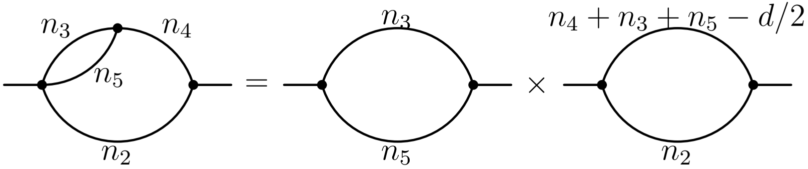


$$D_1 = -(k_1 + p)^2 \qquad D_2 = -(k_2 + p)^2$$

$$D_3 = -k_1^2 \qquad D_4 = -k_2^2 \qquad D_5 = -(k_1 - k_2)^2$$

Trivial cases

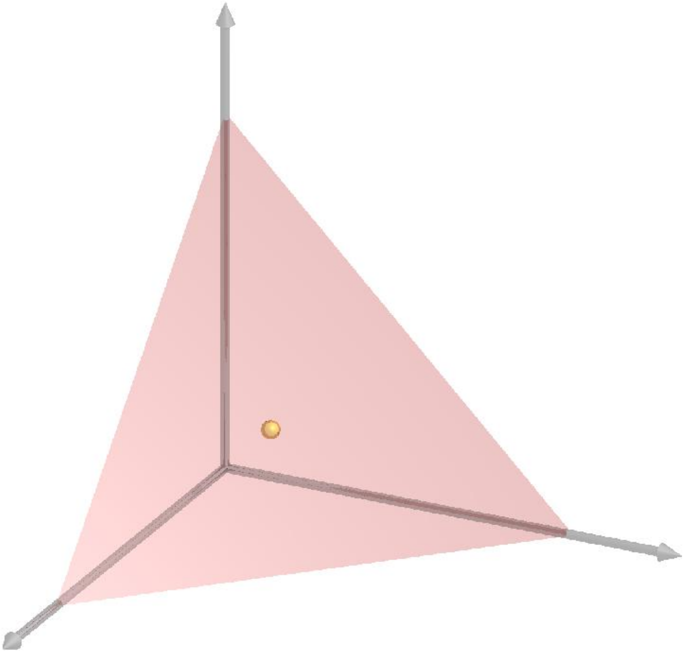


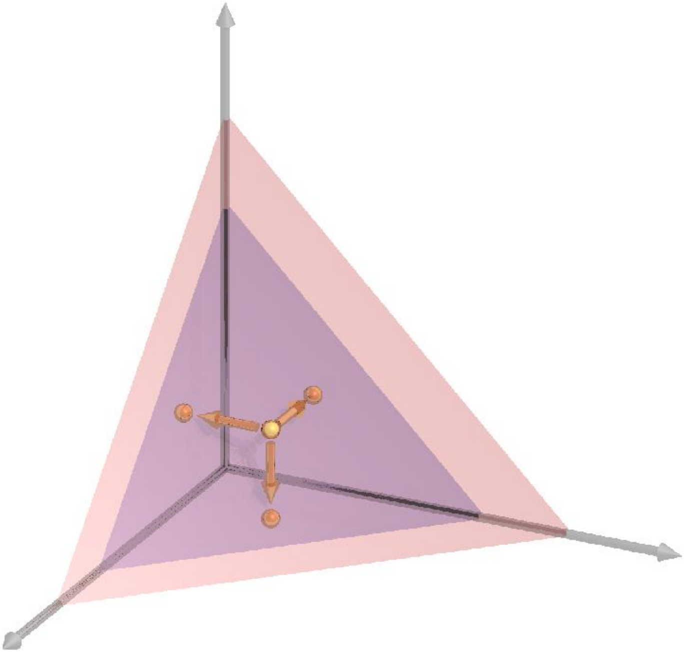


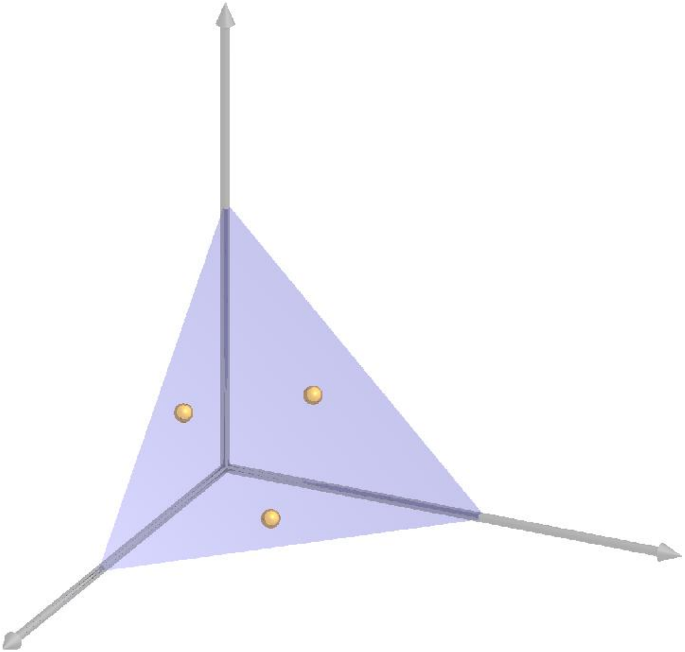
IBP

$$\partial_2\cdot(k_2-k_1) = d-n_2-n_4-2n_5-(D_1-D_5)\frac{\partial}{\partial D_2}-(D_3-D_5)\frac{\partial}{\partial D_4}$$

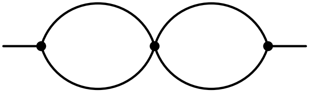
$$[d - n_2 - n_4 - 2n_5 + n_2\mathbf{2}^+(\mathbf{1}^- - \mathbf{5}^-) + n_4\mathbf{4}^+(\mathbf{3}^- - \mathbf{5}^-)]G = 0$$



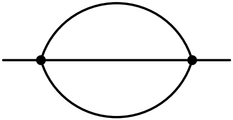




Master integrals



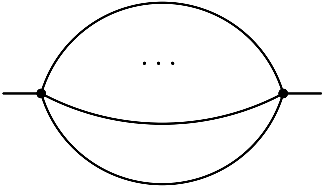
$= G_1^2$



$= G_2$

$$G_n = \frac{g_n}{\left(n+1-n\frac{d}{2}\right)_n \left((n+1)\frac{d}{2}-2n-1\right)_n}$$

$$g_n = \frac{\Gamma(1+n\varepsilon)\Gamma^{n+1}(1-\varepsilon)}{\Gamma(1-(n+1)\varepsilon)}$$



Homogeneity relation

$\partial_1 \cdot k_1$

$$[d - n_2 - n_5 - 2n_4 + n_2 \mathbf{2}^+ (1 - \mathbf{4}^-) + n_5 \mathbf{5}^+ (\mathbf{3}^- - \mathbf{4}^-)] G = 0$$

$\partial_1 \cdot k_1$ mirrir-symmetric

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$\partial_1 \cdot k_1$ mirrir-symmetric

$p \cdot (\partial/\partial p)G$

$$[2(d-n_3-n_4-n_5)-n_1-n_2+n_1\mathbf{1}^+(1-\mathbf{3}^-)+n_2\mathbf{2}^+(1-\mathbf{4}^-)]G = 0$$

Larin relation

Insert $(k_1 + p)^\mu$. The vector integral $\sim p^\mu$:

$$k_1 + p \to \frac{(k_1 + p) \cdot p}{p^2} p = \left(1 + \frac{D_1 - D_3}{-p^2}\right) \frac{p}{2}$$

$\partial / \partial p^\mu$

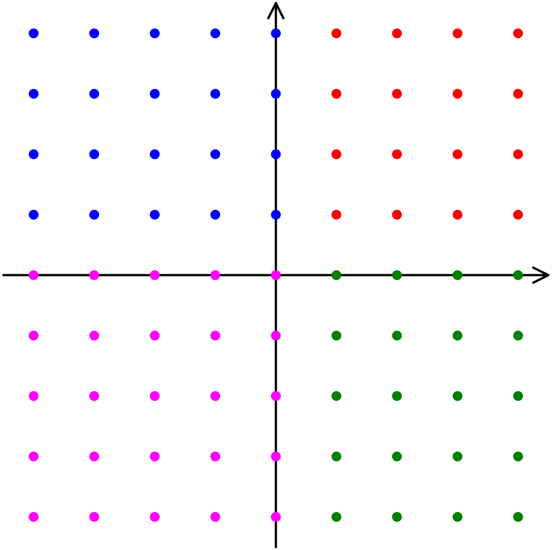
$$\left(\tfrac{3}{2}d - \sum n_i\right) \left(1 + \frac{D_1 - D_3}{-p^2}\right)$$

Explicit differentiation

$$d + \frac{n_1}{D_1} 2(k_1 + p)^2 + \frac{n_2}{D_2} 2(k_2 + p) \cdot (k_1 + p)$$

$$\Big[\tfrac{1}{2}d + n_1 - n_3 - n_4 - n_5 + \left(\tfrac{3}{2}d - \sum n_i\right)(\mathbf{1}^- - \mathbf{3}^-)$$
$$+ n_2 \mathbf{2}^+ (\mathbf{1}^- - \mathbf{5}^-)\Big] G = 0$$

Ordering



Statement of the problem

Suppose we have n variables x_1, ..., x_n. They are not independent, but satisfy some polynomial equations $p_1 = 0$, ..., $p_m = 0$ (p_j are polynomials of x_i). Let's consider a polynomial q. Is it equal to 0 due to the constraints on our variables? If there is another polynomial q_2, there is the question of their equality.

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These questions would become very easy if we had an algorithm reducing polynomials of dependent variables to a canonical form. Two equal polynomials reduce to the same canonical form; a polynomial equal to 0 reduces to the canonical form 0.

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We can try to use the equations $p_j = 0$ for simplifying the polynomial q, i.e. for replacing its more complicated terms by combinations of simpler ones. But to do so we first have to accept some convention which terms are more complicated and which are more simple.

Monomial orders

We need a total order of monomials (i.e. products of powers of the variables $x_1^{n_1} \cdots x_n^{n_n}$). An order is total if for any monomials s and t either $s < t$, or $s > t$, or $s = t$ is true. An order is admissible if two properties are satisfied:

- $1 \leq s$ for any monomial s,
- if $s < t$ then $su < tu$ for any monomial u.

Monomial orders

Lexicographic order

We are comparing two monomials: $s = x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}$ and $t = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$

- $n_1 > m_1 \Rightarrow s > t$
- $n_1 < m_1 \Rightarrow s < t$
- $n_1 = m_1$
    - $n_2 > m_2 \Rightarrow s > t$
    - $n_2 < m_2 \Rightarrow s < t$
    - $n_2 = m_2$

      $\ldots$

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  - $n_2 < m_2 \Rightarrow s < t$
  - $n_2 = m_2$

    $\ldots$

By total degree than lexicographic

First we compare the total degree $n = n_1 + n_2 + \cdots + n_n$ of the monomial s and the total degree $m = m_1 + m_2 + \cdots + m_n$ of the monomial t.

- $n > m \Rightarrow s > t$
- $n < m \Rightarrow s < t$
- $n = m$ — compare lexicographically

Reduction of polynomials

Let's fix some admissible monomial order. We'll write polynomials in descending order: the leading term first, followed by the rest ones. We'll normalize all polynomials p_j in such a way that the coefficient of the leading term is 1. Now they can be used as substitutions which replace the leading term by minus sum of the remaining ones.

Example

Lexicographic order with $x > y$

$$p_1 = x^2 + y^2 - 1 \qquad p_2 = xy - \frac{1}{4}$$

$$q = x^2 y$$

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$$q_1 = -y^3 + y$$

p_1 :
$x^2 \to -y^2 + 1$

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$$q = x^2 y$$

$p_2 :$
$xy \to \frac{1}{4}$

$$q_2 = \frac{1}{4}x$$

Definition

Every time when more than one substitution can be applied to a term of a polynomial q, a fork appears; maybe, its branches join later, but maybe, they don't.

A set of polynomials $p_1, \ldots, p_n$ is called a *Gröbner basis* (for a given monomial order) if reduction of any polynomial q with respect to this set is unique.

(This definition is not constructive.)

S-polynomials

The constraints $p_1 = 0$ and $p_2 = 0$ allow us to simplify the monomials x^2 and xy. Do these constraints contain an extra information usable for simplification but not obvious? Yes, they do!

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$$p_1 = x^2 + y^2 - 1 = 0$$

$$p_2 = xy - \frac{1}{4} = 0$$

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$$p_1 = x^2 + y^2 - 1 = 0 \qquad \times\, y \qquad x^2y + y^3 - y = 0$$

$$p_2 = xy - \tfrac{1}{4} = 0 \qquad \times\, x \qquad x^2y - \tfrac{1}{4}x = 0$$

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\begin{array}{lcl}
p_1 = x^2 + y^2 - 1 = 0 & \times\, y & x^2y + y^3 - y = 0 \\
p_2 = xy - \frac{1}{4} = 0 & \times\, x & x^2y - \frac{1}{4}x = 0 \\
 & & \frac{1}{4}x + y^3 - y = 0
\end{array}
$$

S-polynomials

The constraints $p_1 = 0$ and $p_2 = 0$ allow us to simplify the monomials x^2 and xy. Do these constraints contain an extra information usable for simplification but not obvious? Yes, they do!

$p_1 = x^2 + y^2 - 1 = 0$	$\times\, y$	$x^2 y + y^3 - y = 0$
$p_2 = xy - \frac{1}{4} = 0$	$\times\, x$	$x^2 y - \frac{1}{4} x = 0$
		$\frac{1}{4} x + y^3 - y = 0$

S-polynomial $S(p_1, p_2)$

$$p_3 = x + 4y^3 - 4y$$

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Lexicographic order with $x > y$

$$p_1 = x^2 + y^2 - 1 \qquad p_2 = xy - \frac{1}{4}$$

$$q_1 = -y^3 + y$$

$p_1 :$

$x^2 \to -y^2 + 1$

$$q = x^2 y$$

$p_2 :$

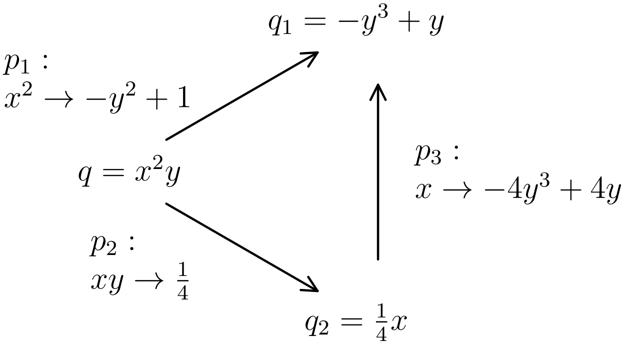
$xy \to \frac{1}{4}$

$$q_2 = \frac{1}{4}x$$

Example

Lexicographic order with $x > y$

$$p_1 = x^2 + y^2 - 1 \qquad p_2 = xy - \frac{1}{4}$$



Reduced Gröbner basis

p_1, p_2, p_3 form a Gröbner basis (though we have not proven this). We can reduce them with respect to each other (omitting vanishing polynomials). The reduced Gröbner basis is

$$p_1 = y^4 - y^2 + \tfrac{1}{16}$$
$$p_2 = x + 4y^3 - 4y$$

It has triangular structure.

Buchberger algorithm

Given a set of polynomials $P = \{p_j\}$

- $S =$ set of all pairs (i, j) of integer numbers from 1 to n with $i < j$
- **while** S is not empty
- Choose and remove some pair (i, j) from S
- Calculate S-polynomial $S(p_i, p_j)$
- Reduce it with respect to P
- if the result is not 0, add this polynomial to P and the corresponding pairs to S

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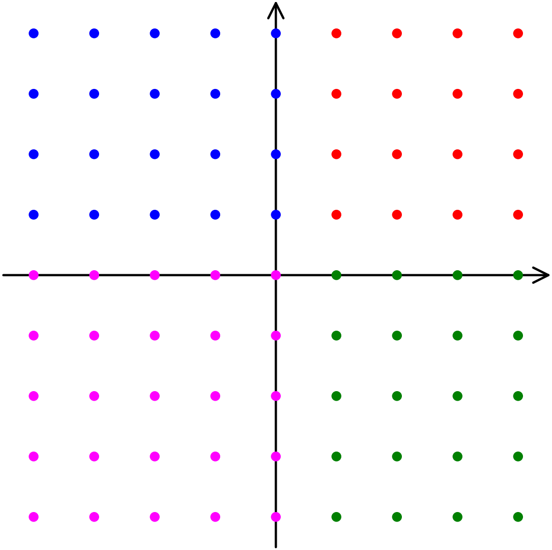
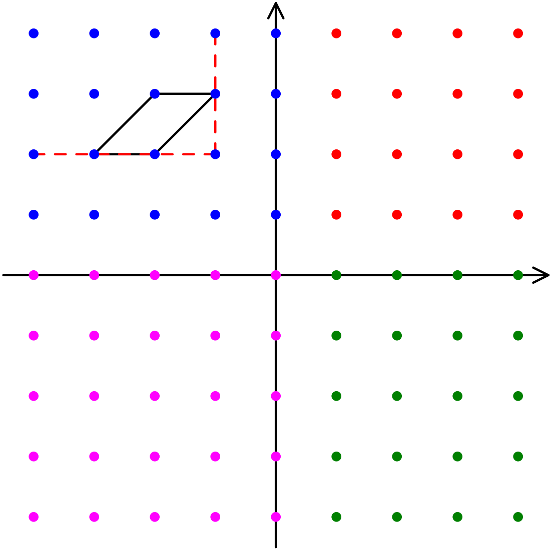
The set of pairs S alternatingly shrinks and grows. But it can be proved that this process terminates after a finite number of steps, and produces a Gröbner basis P.

Sectors and corners



$$I(n_1, n_2) = \left(\mathbf{1}^-\right)^{-n_1} \left(\mathbf{2}^+\right)^{n_2-1} I(0,1)$$

Normal form of IBP relations in a sector



$$\sum_{j_1 j_2} C_{j_1 j_2}(n_i) \left(\mathbf{1}^-\right)^{j_1} \left(\mathbf{2}^+\right)^{j_1} \sim 0$$

S-bases

Find a Gröbner-like basis, reduce $(\mathbf{1}^-)^{-n_1} (\mathbf{2}^+)^{n_2-1}$ and apply to $I(0,1)$

Gröbner bases for PDE

Each line has a separate mass m_i

$$\frac{\partial}{\partial m_i^2} \Rightarrow -\mathbf{n}_i \mathbf{i}^+$$

IBP

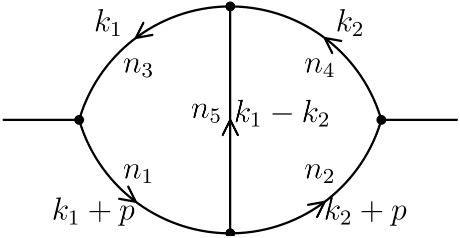
$$\sum C_{j_1 \ldots j_N}(m_1^2, \ldots, m_N^2) \left(\frac{\partial}{\partial m_1^2} \right)^{j_1} \cdots \left(\frac{\partial}{\partial m_N^2} \right)^{j_N} \sim 0$$

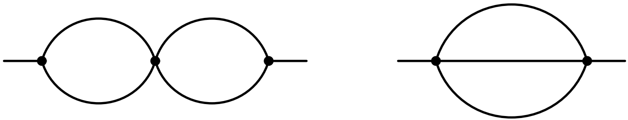
2-loop self-energy diagrams with all different masses

Approaches

- Generic n_i
  - Construct an algorithm and implement by hand: `Mincer`, ...
  - More automated approaches
    - Gröbner-based approaches
    - Lie-algebra based approaches
    - Baikov's methods
- Specific numeric n_i: Laporta algorithm (`Air`, `FIRE`, `reduze`...)

2-loop self-energy diagram





3-loop vacuum diagram