INSTANTONS AND RESURGENCE IN TOPOLOGICAL STRING THEORY

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+ work in progress with J. Gu, A. Kashani-Poor and A. Klemm

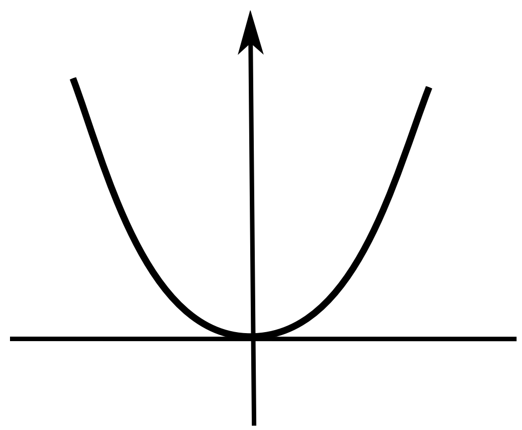
The search for a non-perturbative understanding of topological string theory has been going on for many years.

In 2006-2008 it was proposed in [M.M., M.M.-Schiappa-Weiss] to look at this problem by using the theory of resurgence, i.e. by using traditional tools which had been very successful in QM and QFT.

An key aspect of this approach:
"make use as much as possible of the important pieces of information contained in the coupling constant expansion"
(G. 't Hooft, 1979)

The quartic oscillator

The "role model" for this approach is the Bender-Wu analysis of perturbative series in the quantum anharmonic oscillator



$$\mathsf{H} = \frac{\mathsf{p}^2}{2} + \frac{\mathsf{x}^2}{2} + g\mathsf{x}^4$$

ground state

$$E_0(g) \sim \sum_{n\geq 0} a_n g^n = \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \cdots$$

Sharp question: what is the asymptotic behavior of the coefficients a_n at large *n*?

Large-Order Behavior of Perturbation Theory

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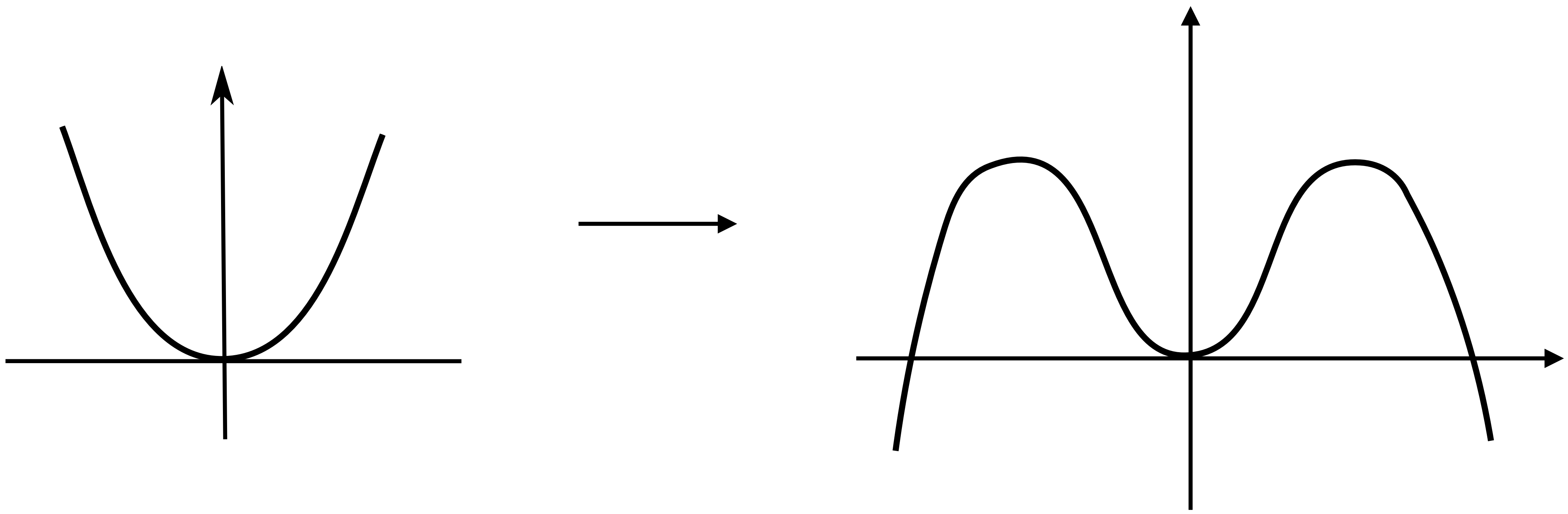
We examine the large-order behavior of perturbation theory for the anharmonic oscillator, a simple quantum-field-theory model. New analytical techniques are exhibited and used to derive formulas giving the precise rate of divergence of perturbation theory for all energy levels of the x^{2N} oscillator. We compute higher-order corrections to these formulas for the x^4 oscillator with and without Wick ordering.

In a remarkable series of papers in 1969-73, Bender and Wu found the answer to this question:

$$a_n \sim \frac{\sqrt{6}}{\pi^{3/2}} 3^n (-1)^{n+1} \Gamma\left(n + \frac{1}{2}\right)$$

Perhaps the most interesting aspect of this formula is the connection to non-perturbative effects (which makes concrete an insight by Dyson in 1952)

The large order behavior can be deduced from an instanton effect in the theory with **negative** coupling!



In the unstable potential, the ground state becomes metastable. Its energy picks an imaginary piece which is exponentially small and can be computed with instanton methods

$$\operatorname{Im} E_0(g) \approx \frac{4\sqrt{2}}{\pi^{1/2}(-g)^{1/2}} \mathrm{e}^{1/(3g)}$$

This determines the asymptotics of perturbation theory. For example, the action of the instanton, which is *A=1/3*, leads to the subleading factorial growth in the Bender-Wu formula

From non-perturbative effects to asymptotics

More generally, if we have a non-perturbative effect of the form

$$\varphi(g) = g^{-b} \mathrm{e}^{-A/g} \sum_{k \geq 0} c_k g^k$$

one obtains the asymptotics

$$\begin{aligned} a_n &\sim \frac{1}{2\pi} \sum_{k \geq 0} c_k \Gamma(n + b - k) A^{-n-b+k} \\ &= \frac{c_0}{2\pi} A^{-n-b} \Gamma(n+b) \left(1 + \mathcal{O}(n^{-1})\right) \end{aligned}$$

The take home message is that **perturbation theory encodes information about non-perturbative effects**. Conversely, one can use non-perturbative corrections to derive asymptotic results

In particular, proposals about putative non-perturbative effects in a theory can (and should) be tested against the large order behavior of the perturbative series

This is particularly relevant for string theory, where in many cases perturbation theory is all we have. Can we answer similar questions in e.g. topological string theory?

Topological string theory

Let *X* be a Calabi-Yau (CY) threefold. At each genus *g* we can compute the topological string free energy $F_g(t)$, which depends on the Kahler modulus *t* (for simplicity I will only consider one-modulus CYs)

At large *t* this has an expansion encoding Gromov-Witten invariants of *X*, which "count" holomorphic curves of genus *g* in *X*:

$$F_g(t) = \sum_{d \geq 1} N_{g,d}\, \mathrm{e}^{-dt}$$

I will use mirror symmetry throughout, so for example the Kahler modulus will be given by a mirror map

$$t = t(z)$$

relating it to the complex structure modulus z of the mirror CY

String perturbation theory tells us that the **total free energy** is given by a genus expansion in the string coupling constant

$$F(t, g_s) = \sum_{g \geq 0} F_g(t) g_s^{2g-2}$$

How much do we know about this series? How easy it is to effectively compute the coefficients?

This is important if we want e.g. to check explicit statements about asymptotics. In the Bender-Wu case, they tested their formula with ~150 terms of the perturbative series.

The holomorphic anomaly equations

One of the most successful methods to calculate the topological string perturbation series is the holomorphic anomaly equations (HAE) of BCOV.

For simplicity, I will first consider the so-called **local** or toric, non-compact CY case. The total free energy of the topological string satisfies a partial differential equation involving a **propagator** *S* and the complex modulus *z* of the CY

$$\frac{\partial F}{\partial S} = \frac{g_s^2}{2} D_z^2 F + \frac{1}{2}(D_z F)^2$$

This equation can be solved recursively, genus by genus. One obtains expressions for the genus g free energies which are polynomials in the propagator, and involve known functions of the complex modulus z [BCOV, Yamaguchi-Yau, Grimm-Klemm-M.M.-Weiss, Alim-Lange, Klemm et al., …]

$$F_2(S, z) = \frac{5Y(z)^2 S^3}{24} + \cdots$$

$Y(z)$: Yukawa coupling (third derivative of F_0)

In this formulation, *S* is essentially an arbitrary variable. The conventional topological string free energies are recovered in the so-called **holomorphic limit**, where *S* becomes a (known) function of *z*.

By using this method, one can calculate the genus expansion efficiently, up to an integration constant called the **holomorphic ambiguity**, which is a function of z.

In the local case, the ambiguity can be determined at all genera by using the behavior at the conifold point , where the CY becomes singular [Haghighat-Klemm-Rauch].
In practice, one can calculate ~150 terms of the series

One can set up more complicated HAE in the compact case, but it is more difficult to fix the ambiguity. For some compact CYs, like the famous quintic, it can be fixed up to genus ~50 by using additional geometric constraints [Huang-Klemm-Quackenbush].

Sharp question: what is the asymptotic behavior of the $F_g(t)$ at large genus?

General string theory arguments [Shenker] predict that they grow double-factorially,

$$F_g(t) \sim (2g)!, \qquad g \gg 1$$

but we want to be much more precise than that.

Note that this asymptotic problem is more difficult than in Bender-Wu, since the coefficients of the perturbative series are themselves functions of the modulus *t* (or *z*)!

Their general lesson should still apply, though, and the physics approach to the problem would be to calculate a (spacetime) **instanton amplitude in topological string theory**, exponentially small in the string coupling constant.

However, it is not even clear what is the framework to do instanton calculus!

In the case of toric CYs there are explicit, non-perturbative matrix model duals [M.M.-Zakany], but the calculation of large *N* instantons in these models seems difficult.

The CESV framework

In two remarkable papers in 2013-4, CESV [Couso-Edelstein-Schiappa-Vonk] proposed to use the holomorphic anomaly equations of BCOV to calculate instanton amplitudes.

This was inspired by **trans-series solutions** in ODEs, which involve exponentially small terms [Ecalle, Costin, …]

Euler equation: $$x^2 y'(x) - y(x) = -x$$

perturbative solution: $$y_{\mathrm{p}}(x) = \sum_{n \geq 0} n! x^{n+1}$$

trans-series solution: $$y(x) = y_{\mathrm{p}}(x) + \boxed{C\mathrm{e}^{-1/x}}$$

We then solve $$\frac{\partial F}{\partial S} = \frac{g_s^2}{2} D_z^2 F + \frac{1}{2}(D_z F)^2$$

with a **trans-series ansatz**

$$F = \sum_{g \geq 0} F_g(S,z) g_s^{2g-2} + g_s^{-b} \mathrm{e}^{-\mathcal{A}/g_s} \sum_{k \geq 0} F_k^{(1)}(S,z) g_s^k$$

perturbative series

instanton correction

One additional assumption of the ansatz is that the instanton action $\mathcal{A}$ is a **CY period** [Drukker-M.M.-Putrov]

In the local case, this means that

$$\mathcal{A} = \alpha \frac{\partial F_0}{\partial t} + \beta t + \gamma$$

This conjecture is motivated by results in quantum mechanics and matrix models. It can be verified empirically by looking at the asymptotic behavior of the perturbative series.

I will assume that $\alpha \neq 0$, otherwise the one-instanton amplitude is very simple: it is two-loop exact and given by [Pasquetti-Schiappa] (see also [Alim et al.,Grassi et al.])

$$(\mathcal{A} + g_s)\mathrm{e}^{-\mathcal{A}/g_s}$$

In the trans-series ansatz one can include subleading instanton corrections. For example, we can have multi-instanton amplitudes, involving an action $m\,\mathcal{A}$, where m is a positive integer

CESV solved for the very first $F_k^{(1)}(S,z)$ in the toric case. One finds e.g.

$$F_0^{(1)}(S,z) = \mathcal{A} \exp\left(\frac{1}{2}(\partial_z \mathcal{A})^2 S\right)$$

which gives the prefactor in the Bender-Wu-like asymptotic formula. They verified in detail these predictions against perturbative data

Exact solutions

The CESV results for instanton corrections get increasingly complicated for higher order terms, and their physical interpretation is not clear.

In a recent paper with Jie Gu, we found the **exact** solution for all multi-instanton trans-series.

In the holomorphic limit, it can be shown that the instanton corrections only involve the perturbative topological string free energies and their derivatives.

[Technical point: we will redefine the genus zero free energy in such a way that $\mathcal{A} = \alpha \partial_t F_0$]

In the one-instanton case, one finds the all-orders solution

$$F^{(1)} = \left(g_s + g_s^2 \alpha \partial_t F(t - \alpha g_s)\right) \mathrm{e}^{F(t-\alpha g_s) - F(t)}$$

In the case of *m*-instantons there are different solutions, but they all involve the exponent

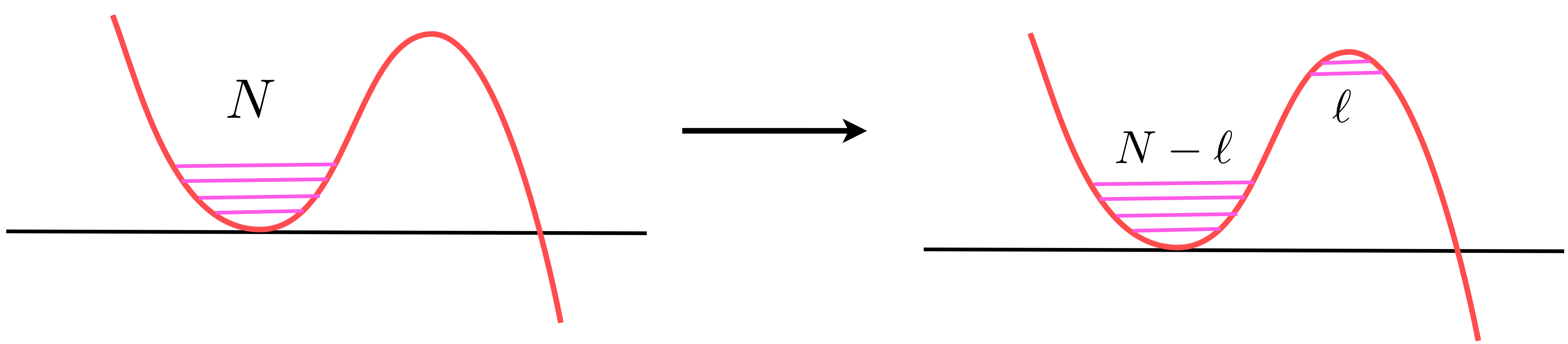
$$\mathrm{e}^{F(t - m\alpha g_s) - F(t)}$$

This has an easy interpretation: the *m*-th instanton sector corresponds to a background where the Kahler parameter *t* is shifted

$$t \to t - m\alpha g_s$$

This suggests that t is **quantized**, as postulated in large N dualities. However, in reaching this conclusion we have only used the holomorphic anomaly equations.

The resulting amplitude is also similar to large N instanton effects in matrix models, which are obtained by "eigenvalue tunneling"



The compact case

So far I have only considered toric CYs where life is comparatively simple. However, in ongoing work with Gu, Kashani-Poor and Klemm, we have generalized some of these results to compact, one-modulus CYs, like the famous quintic.

Now the BCOV equations involve three propagators instead of one, but we can still compute instanton amplitudes.

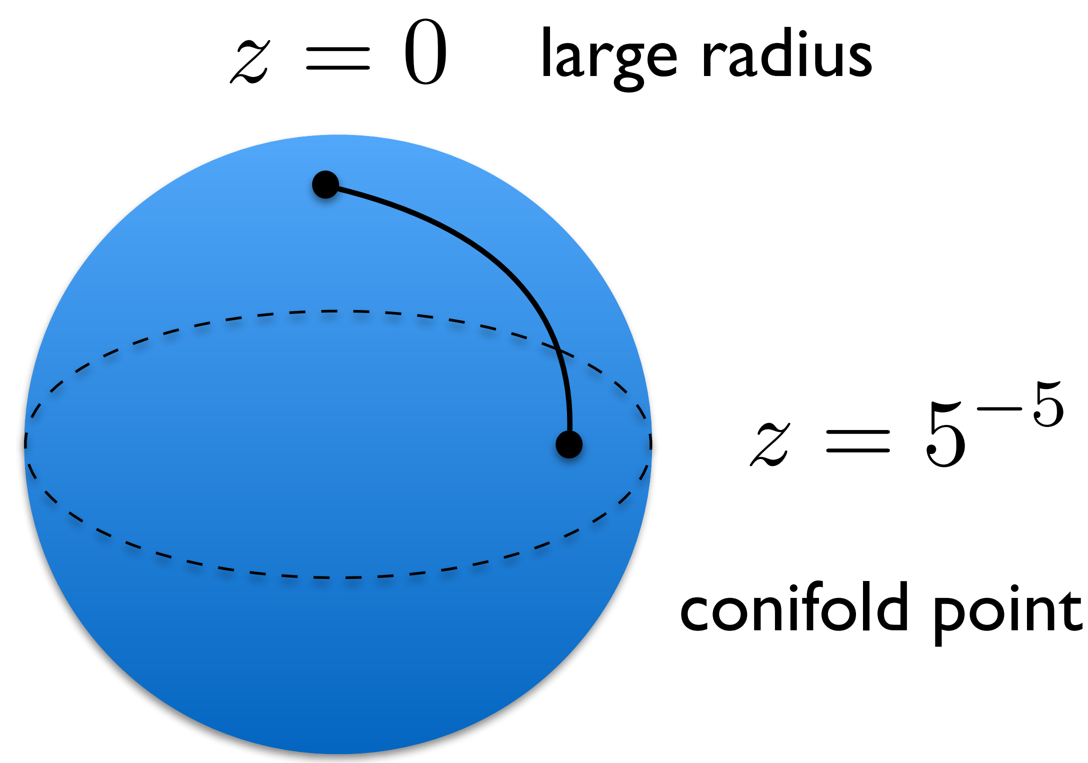
Let me present a concrete asymptotic result, which is is the analogue of the Bender-Wu formula for the topological string on the quintic CY.

For real values of the modulus *z* in between the LR point and the conifold, we find the asymptotics

$$F_g(t) \sim \frac{\exp\left(\frac{\tau_{00}}{8\pi^2}\right)}{2\pi^2} \mathcal{A}^{-2g+2}\Gamma(2g-1)$$

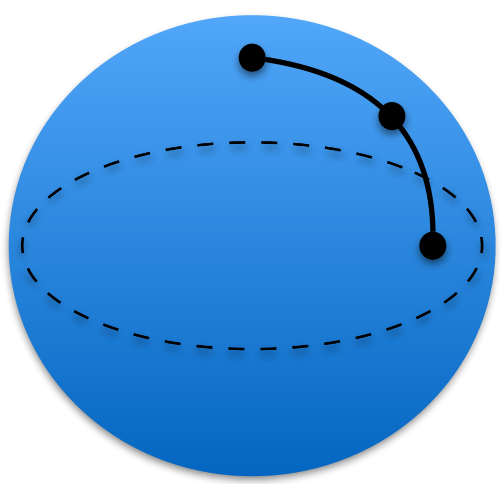
$$\mathcal{A} = 2\pi(F_0 - t\partial_t F_0)$$

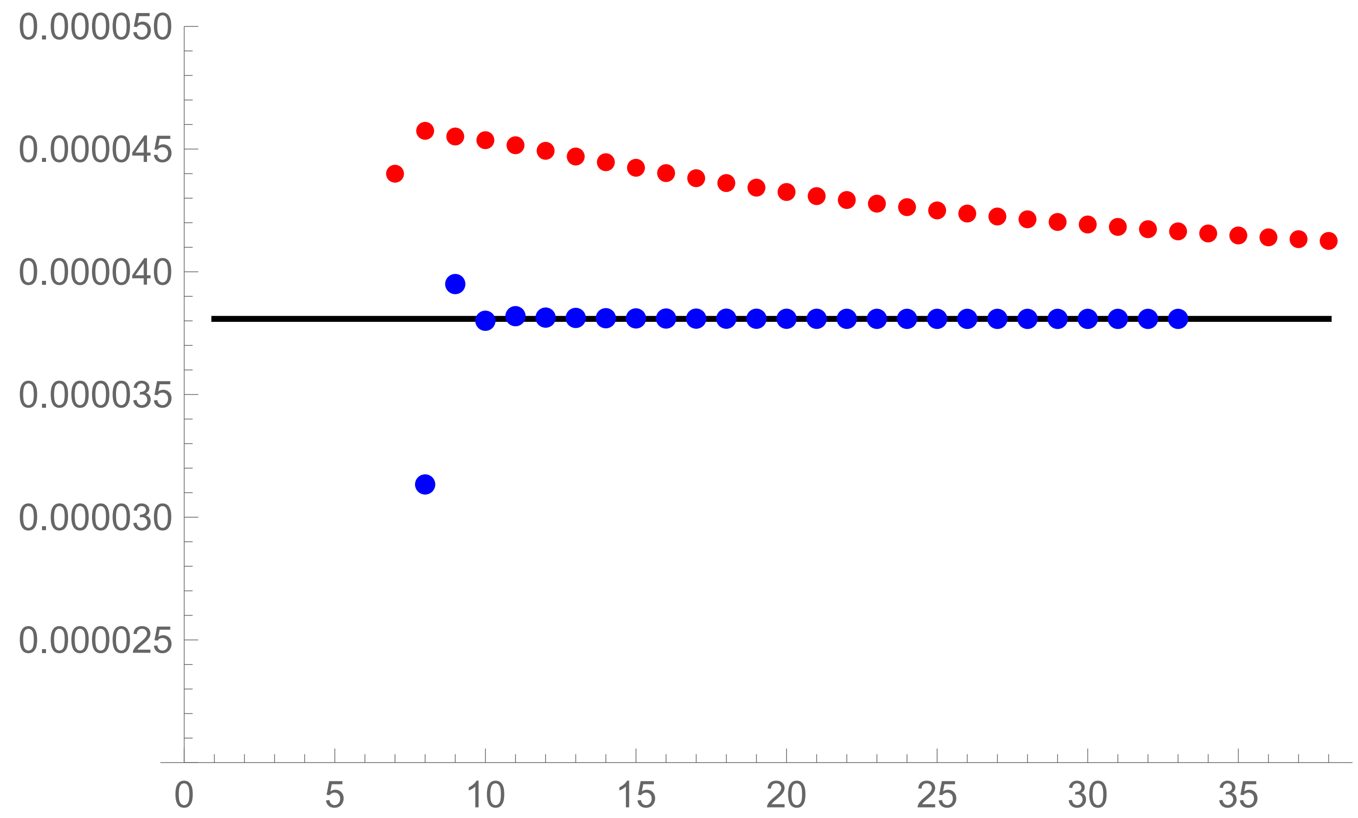
$$\tau_{00} = 2F_0 - 2t\partial_t F_0 + t^2\partial_t^2 F_0$$



Experimental evidence

$$z = \frac{5^{-5}}{2}$$





red dots: auxiliary sequence from the Fg's to extract the prefactor

blue dots: Richardson transform to accelerate the convergence

black line: prediction $$\frac{\mathcal{A}}{2\pi^2} \exp\left(\frac{\tau_{00}}{8\pi^2}\right)$$

What are the possible instanton actions?

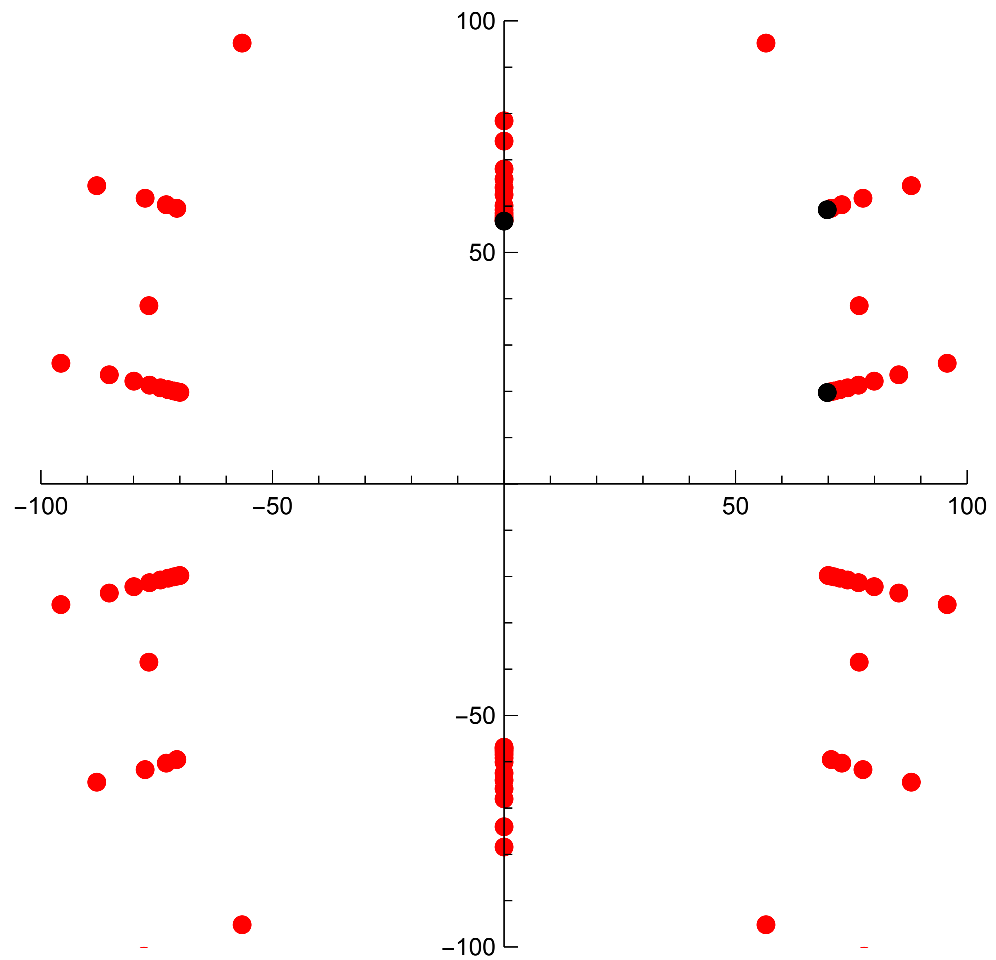
The procedure explained above tells us what are the instanton amplitudes, up to an overall multiplicative constant, once the instanton action is known.

In general, it does not tell us what are the possible instantons, nor their actual multiplicative constant. This information is part of what I call the **resurgent structure** of the theory, and has to be determined independently.

The possible instanton actions turn out to be the **singularities of the Borel transform**

$$\widehat{F}(t,\zeta) = \sum_{g\geq 0} \frac{F_g(t)}{(2g)!}\zeta^{2g}$$

An example of singularities in the Borel place of local P2, for a point in moduli space near large radius



More examples and plots in 2211.0143

The local behavior of the Borel transform near these singularities is determined by the instanton amplitudes we have computed, plus a numerical constant called **Stokes constant**

Recent work indicates that Stokes constants have interesting enumerative information on the CY. Finding efficient ways to extract this information is one of the main challenges of the theory.

Conclusions and outlook

We have developed the framework of CESV to calculate instanton amplitudes in topological string theory, by using a trans-series ansatz in the holomorphic anomaly equations of BCOV. We are doing instanton calculus in Kodaira-Spencer theory!

We managed to find **exact** answers for these amplitudes, in the toric case, and show that they involve only perturbative information. The underlying physics involves the quantization of flat coordinates, as in large N dualities.

We are now extending these results to the **compact** case, and much of the underlying physics seems to be the same. What is the meaning of the quantization there ?

Is it possible to realize these instantons in terms of extended objects? They have the form of D-brane amplitudes, as in the non-critical string.

The full resurgent structure requires determining the possible instanton actions and the Stokes constants, and we expect very rich mathematics and physics in this resurgent structure

Thank you for your attention!