1) a) EPR paradox and the Bell inequalities

1) Historical Context
2) Example of entangled state
3) EPR paradox (Albert EINSTEIN, Boris PODOLSKY
Nathan ROSEN)
4) Local Variables
5) Bell inequality
6) Entanglement and special relativity

1) Not so long after the full development of QM (~1920) the Bohr - Einstein debates started.
Einstein was convinced that QM was incomplete, he believed that one could find a more fundamental theory, he wrote in a 1926 he was convinced that God «is not playing dice». While Bohr defended the fundamental probabilistic character of quantum measurement.

Einstein and his collaborators Boris PODOLSKY and Nathan Rosen published an argument in 1935 to defend their thesis. In this paper they proposed a thought experiment later termed EPR paradox.

① b) I will now present you the EPR paradox.
↳ I will use spin insted of position/momentum in the original paper.

First Einstein was claiming that QM was incomplete but Einstein was not questioning the prediction of QM.
These predictions has already been verified experimentally
This will be an absurdity
⓪ QM predictions are correct

Now we can study a particular state allowed by QM

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\downarrow\rangle_1 |\uparrow\rangle_2 - |\uparrow\rangle_1 |\downarrow\rangle_2\right)$$

Here we have two elections in a superposed state of spin 0.

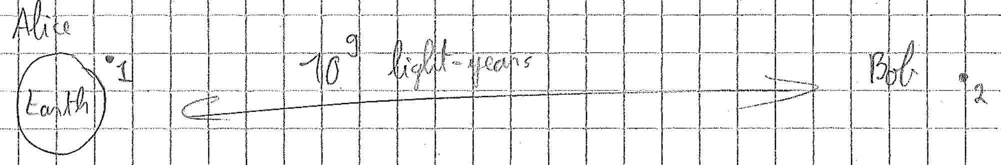
$$\vec{S}^2 |\Psi\rangle = 0$$

This implies that the two elections must always have opposite sign when measured along the same axis.
Let's take an example:
election 1 is measured along z and we get $+1$
↳ new state is: (after the collapse of the wave function)

$$\left(|\uparrow\rangle_1\langle\uparrow|_1 \otimes \mathbb{1}\right)|\Psi\rangle = -\frac{1}{\sqrt{2}}|\uparrow\rangle_1 |\downarrow\rangle_2$$

the new state after measurement is $-|\uparrow\rangle_1|\downarrow\rangle_2$
the spin of the second election along z is now -1
This is true for any other axis.

② a) Let's start with the thought experiment.
To create the state $|\psi\rangle$ we need the electrons to interact with each other.
The two electrons start at the same place
$\cdot_1 \cdot_2$ At time T the electrons are separated
The electron 1 is given to Alice which remains on Earth while the second is given to Bob who travels billions of light-years away from Earth



Alice and Bob know special relativity so they can coordinate their clock to measure their electron almost at the same time.
There is a spacelike interval $(\Delta\tau < 0)$ between the two measurement so that the electrons can not communicate in any way.

$$\Delta \tau = \Delta t^2 - \Delta x^2 + \Delta y^2 + \Delta z^2 < 0$$

As we have already seen if alice measures $+1$ along $\hat{z}$, the wavefunction callapse instantly and Bob will measure -1 along $\hat{z}$.
Two possible interpretations:

- You can admit that Alice measurement changes the physical state of Bob's electron via the collapse of the wave function

① There is a spooky action at distance (non local)

② b) • Or one can argue that there no spooky action at distance at all. The physical state of Bob's election remains unchanged after Alice measurement.
↳ the state of Bob's election has not changed since $t = T$

(The collapse of the wave function is just a mathematical tool but has no physical interpretation.)

According to QM:

if Alice measures $+1 / \hat{y}$ → Bob will measures $-1 / \hat{y}$

In the absence of interactions between the two elections we must admid that has always been $+1$ along $\hat{y}$

We can use the same reasoning for each axis such that we conclude Bob's election has definite spin along each axis.

This is in contradiction with QM since $[S_i, S_j] \neq 0$

So QM is not complete.

Conclusion

1) Spoophy action at distance → non local

2) Definite values for non commuting observable

→ QM not complete

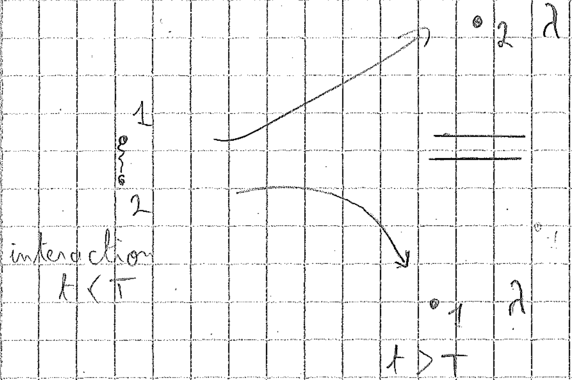
③ a)

Now I will talk a bit more about this second case,
↳ assuming everything is local.
I will talk about local variable.

If we refuse any non locality we have seen that a particle can have perfectly definite spin along several axis simultaneously.

We can wonder how would look like such a theory which reproduces the predictions of QM

Let's come back to the thought experiment
initialy the electrons are interacting before to be separated separated at time T.



to preserve locality the spin projections along all axis must have been determined at $t = T$.

We can imagine that the component of a particle's spin in any direction $\vec{n}$ is given by a function $S(\vec{n}, \lambda) = \pm 1$
λ : a set of hidden variables

In the thought experiment λ is fixed à $t = T$ such that all spin components have a determined value. $S(\vec{n}, \lambda) = S_1(\vec{n}, \lambda) = -S_2(\vec{n}, \lambda)$

In this case QM is incomplete because it does not describe the evolution of λ

What we observe is that $\{\lambda\}$ obeys some probability density function $\rho(\lambda)$.

③ b) λ takes differents value at each iteration of the experiment, making the result of measurement probabilistic.

for example $\langle S_x \rangle_\psi = \int d\lambda \; S(\vec{x}, \lambda) \, \rho(\lambda)$

of course the PDF $\rho(\lambda)$ depends on $|\psi\rangle$.

At this point we have to interpretation of QT

- One defended by Bohr, which is non local
  fundamentally probabilistic

- One defended by Einstein, local
  include hidden variable
  ↳ QT is not complete.

<u>This look like a metaphysical question</u>

But in 1964 John Bell publiched a paper where he described an experiment for which the predictions of QT and the local hidden variable theory

④ a) Bell inequality

I will derive the original Bell inequality published in 1964. This inequality is satisfied in the local hidden variable theory but not in QM.

We are still working with a state of two electrons of spin 0 : $S_1(\vec{a},\lambda) = -S_2(\vec{a},\lambda) = S(\vec{a},\lambda) = \pm 1$

We can write the correlation between the spins of the two electrons

$$\langle (\vec{S}_1 \cdot \vec{a})(\vec{S}_2 \cdot \vec{b}) \rangle = -\int d\lambda \, \rho(\lambda) \, S(\vec{a},\lambda) S(\vec{b},\lambda)$$

Now

$$\langle (\vec{S}_1 \cdot \vec{a})(\vec{S}_2 \cdot \vec{b}) \rangle - \langle (\vec{S}_1 \cdot \vec{a})(\vec{S}_2 \cdot \vec{c}) \rangle =$$

$$-\int d\lambda \, \rho(\lambda) \left[S(\vec{a},\lambda) S(b,\lambda) - S(\vec{a},\lambda) S(\vec{c},\lambda) \right] \quad \text{use } S^2(b,\lambda) = 1$$

$$= -\int d\lambda \, \rho(\lambda) \, S(\vec{a},\lambda) S(b,\lambda) \left[1 - S(b,\lambda) S(c,\lambda) \right]$$

take the absolute value

$$\left| \langle (\vec{S}_1 \cdot \vec{a})(\vec{S}_2 \cdot \vec{b}) \rangle - \langle (\vec{S}_1 \cdot \vec{a})(\vec{S}_2 \cdot \vec{c}) \rangle \right| \leqslant \int d\lambda \, \rho(\lambda) \, \underbrace{|S(\vec{a},\lambda) S(b,\lambda)|}_{1} \left(1 - S(b,\lambda) S(c,\lambda) \right)$$

$$\leqslant \int d\lambda \, \rho(\lambda) \left[1 - S(b,\lambda) S(c,\lambda) \right]$$

$$\boxed{\left| \langle (\vec{S}_1 \cdot \vec{a})(\vec{S}_2 \cdot \vec{b}) \rangle - \langle (\vec{S}_1 \cdot \vec{a})(\vec{S}_2 \cdot \vec{c}) \rangle \right| \leqslant 1 + \langle (\vec{S}_1 \cdot \vec{b})(\vec{S}_2 \cdot \vec{c}) \rangle}$$

original Bell inequality

This inequality is not satisfied by QM correlation function.

④ b) Indeed, in QM

$\vec{S} = \vec{\sigma}$ we have $(\vec{\sigma}.\vec{a})(\vec{\sigma}.\vec{b}) = \vec{a}.\vec{b} + i(\vec{a}\times\vec{b}).\vec{\sigma}$

for the state $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

we have $\langle \vec{S}_1 \rangle = \langle \vec{S}_2 \rangle = 0$

$\langle (\vec{S}_1.\vec{a})(\vec{S}_2.\vec{b}) \rangle_{QM} = -\vec{a}.\vec{b}$

We can chose $\vec{a}.\vec{b} = 0$ and $\vec{c} = \frac{\vec{a}+\vec{b}}{\sqrt{2}}$

then

$$|\langle (\vec{S}_1.\vec{a})(\vec{S}_2.\vec{b}) \rangle_{QM} - \langle (\vec{S}_1.\vec{a})(\vec{S}_2.\vec{c}) \rangle_{QM}| = \frac{1}{\sqrt{2}}$$

while $1 + \langle (\vec{S}_1.\vec{b})(\vec{S}_2.\vec{c}) \rangle_{QM} = 1 - \frac{1}{\sqrt{2}} < \frac{1}{\sqrt{2}}$

The violation of Bell inequalities has been observed experimentally several times since 1964

→ Alain Aspect confirmed the predictions of QM in a paper written in 1982

It refutes the existence of local hidden variables but one can still propose non local hidden variable

(5) a)

I would like to talk a bit about this "spooky action at distance".

According to QM, Alice measurement instantly influences Bob's electron state.
We can wonder this allows to send messages faster than light.

Let $|\Psi_i\rangle$ be a basis of the Hilbert space for electron 1
" " $|\phi_j\rangle$ " " 2

The more general state we can write is

$$\Psi = \sum_{i,j} C_{i,j} |\Psi_i\rangle |\phi_j\rangle \quad , \quad \sum_{ij} |C_{i,j}|^2 = 1$$

Alice measures Ψ_I with probability $P^A(I) = \sum_j |C_{I,j}|^2$
the wave function :

$$\Psi \longrightarrow \Psi' = \sum_j \frac{C_{I,j}}{\sqrt{\sum_k |C_{I,k}|^2}} |\Psi_I\rangle |\phi_j\rangle$$

$$P^A(I) = |\langle \Psi | \Psi' \rangle|^2$$

After Alice measurement Bob measure his electron and find ϕ_J with probability

$$P^B_{(A\to I)}(J) = \frac{|C_{I,J}|^2}{\sum_k |C_{I,k}|^2} = |(\langle \Psi_I | \langle \phi_J |) | \Psi' \rangle|^2$$

If the interval between both measurement is spacelike, then Bob does not know which value has obtained Alice

5 e)

From Bob point of view

$$P^B(J) = \sum_i P^B_{(A \to i)}(J)\, P^A(i)$$

$$= \sum_i \frac{|C_{i,J}|^2}{\sum_k |C_{i,k}|^2} \cdot \sum_k |C_{i,k}|^2$$

$$P^B(J) = \sum_i |C_{i,J}|^2$$

So Bob probability density function does not differ from alice's one.

Bob has no way if Alice have already achieved her measure or what she is measuring.

This is in agreement with special relativity, indeed if the interval between the measurement is spacelike then two observers can disagree on who from Alice and Bob was the first to achieve the measurement.