Notes on quantum mechanics

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November 29, 2022

Abstract

Lecture notes on quantum mechanics: precision tests of Bell's inequalities.

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Chapter 1

Precision tests of Bell's inequalities

1.1 Overview

Bell's inequalities test possibility of a replacement of quantum mechanics (QM) by classical theories where the probabilistic nature of the QM is reproduced by statistical average over certain hidden classical variables λ. This was inspired by the famous Einstein, Podolsky, Rosen (EPR) paper [1] where they argued that either

1. the QM description of reality given by the wave function is not complete, or
2. when operators corresponding to two physical quantities that do not commute, the two properties described by them cannot have simultaneous reality.

While the second does hold true in QM, EPR argued that if local realism was to be taken seriously, independent measurements on entangled particles at space-like separations can indeed imply a simultaneous reality of two non-commuting observables. The only way out of this paradox was to treat even space-like separated particles (or the wavefunction thereof) as one entity, such that a measurement of properties of one immediately affects that of the other, such that simultaneous measurement of two noncommuting observables on either of the particles is no longer permitted.

John S. Bell in 1964 pointed out [2] that all attempts to construct local, realist model of quantum phenomena must lead to statistical correlations that are distinctly different from those predicted by quantum mechanics. Such (hidden variable)theories were shown by Bell to satisfy inequalities constructed out of specific measurements, that QM necessarily will violate in certain situations. Bell considered the gedankenexperiment of Bohm [3] where a pair of entangled spins measured in different directions, each of which take discrete values ± 1 (such as electron spin in unites of $\hbar/2$)

$$\left|\left\langle(\mathbf{s}_1\cdot\hat{a})(\mathbf{s}_2\cdot\hat{b})\right\rangle-\left\langle(\mathbf{s}_1\cdot\hat{a})(\mathbf{s}_2\cdot\hat{c})\right\rangle\right|\leq\frac{\hbar^2}{2}+\left\langle(\mathbf{s}_1\cdot\hat{b})(\mathbf{s}_2\cdot\hat{c})\right\rangle. \tag{1.1}$$

This inequality is violated by QM for certain directions and hence provides a pathway for definitively testing validity of QM.

1.2 Clauser's proposal

In practice, however, it is challenging to consider correlated spins. Instead a more reasonable option is to consider correlated photon emissions. The first definitive step in this direction was taken by Clauser, Horne, Shimony and Holt (CHSH) [4]. The considered correlations between the polarizations of the photons in four directions $\hat{a},\hat{a}',\hat{b},\hat{b}'$ and considered the quantity:

$$\left\langle S_1(\hat{a})S_2(\hat{b})\right\rangle-\left\langle S_1(\hat{a})S_2(\hat{b}')\right\rangle+\left\langle S_1(\hat{a}')S_2(\hat{b})\right\rangle+\left\langle S_1(\hat{a})S_2(\hat{b}')\right\rangle. \tag{1.2}$$

Here $S_1(\hat{a})$ corresponds to measurement of (linear) polarization of photon-1 in the direction $\hat{a}$. If the photon is found polarized in the direction (orthogonal to) $\hat{a}$ then $S_1(\hat{a})=+1(-1)$. Likewise

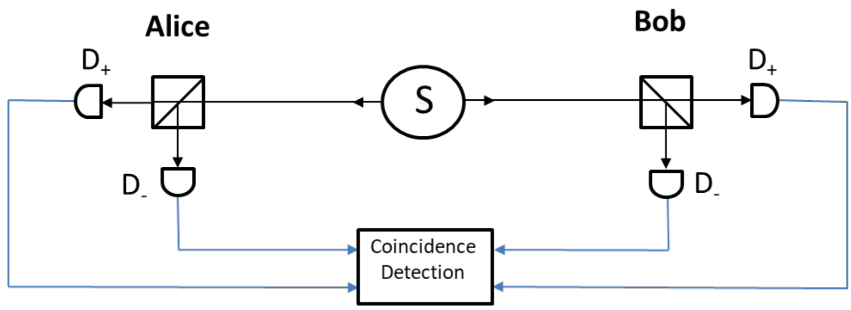

Figure 1.1 Schematic of the setup proposed by Clauser *et al.* (involving measurement only in single channel) and later on improved by Aspect *et al.*. Taken from [5].

for photon-2. The directions $a, \hat{a}$ $(b, \hat{b})$ correspond to two different choices for polarizer at the first (second) detector. See Fig. 1.1.

In a hidden variable theory, we assume the photons to carry their spin all-along with them during the flight, which is determined at their production at the source. Assuming such a production involves certain "hidden-variables" λ with a probability distribution $\rho(\lambda)$, the value of the correlation above will be given by

$$\int \mathrm{d}\lambda\, \rho(\lambda) \Big[S_1(\hat{a},\lambda)S_2(\hat{b},\lambda) - S_1(\hat{a},\lambda)S_2(\hat{b}',\lambda) + S_1(\hat{a}',\lambda)S_2(\hat{b},\lambda) + S_1(\hat{a},\lambda)S_2(\hat{b}',\lambda) \Big] \,. \tag{1.3}$$

Since each particle in this theory carries a definite value of polarizations in any given direction, we find that for any given λ the magnitude of the quantity in the brackets is at most $+2$, such that the absolute value of the correlation in Eq. (1.2) is constrained to be less than 2. This is because demanding the first three terms to be $+1$ constrains the last term to be -1. Hence, we have

$$\left| \left\langle S_1(\hat{a})S_2(\hat{b}) \right\rangle - \left\langle S_1(\hat{a})S_2(\hat{b}') \right\rangle + \left\langle S_1(\hat{a}')S_2(\hat{b}) \right\rangle + \left\langle S_1(\hat{a}')S_2(\hat{b}') \right\rangle \right|_{\rm cl} \leq 2\,. \tag{1.4}$$

Let us now derive the expectation in QM. In their experiment, CHSH considered double photon emission between energy levels 6^1S_0 and 4^1S_0 of Calcium ions, such that the probability is given by square of the amplitude

$$\langle \gamma\gamma(4^1S_0)|(6^1S_0)\rangle \tag{1.5}$$

Both the states have $j = 0$, such that the two photon state must be a scalar function of the polarizations. The two possibilities are $\hat{k} \cdot (\mathbf{e}_1 \times \mathbf{e}_2)$ or $\mathbf{e}_1 \cdot \mathbf{e}_2$, where $\hat{k}$ is the direction of the photon. The matrix element must be even in parity due to the even parity states, such that $\mathbf{e}_1 \cdot \mathbf{e}_2$ is the only allowed possibility. The probability will involve squaring this amplitude, such that probability for photon 1 polarized in direction $\hat{a}$ and the other in direction $\hat{b}$ is given by

$$P(++) \propto (\hat{a} \cdot \hat{b})^2 = \cos^2\theta_{ab} \tag{1.6}$$

The other possibilities with one of them minus correspond to photon polarized in a direction orthogonal to $\hat{a}$ (and also orthogonal to direction of the photon $\hat{k}$ itself), such that

$$P(-+) \propto \sin^2\theta_{ab}\,, \tag{1.7}$$

Demanding that the probability for the four cases add up to one fixes the coefficient to be 1/2. Thus, the QM expectation is

$$\left\langle S_1(\hat{a})S_2(\hat{b})\right\rangle_{\text{QM}} = P(++) - P(-+) - P(+-) + P(--) = \cos 2\theta_{ab} \tag{1.8}$$

This can be plugged into the formula above in Eq. (1.4). One finds that this is maximum when $\theta_{ab} = \theta_{a'b} = \theta_{a'b'} = 22.5$ and the fourth $\theta_{ab'} = 67.5$, in which case the expectation value is $2\sqrt{2}$. Note that at any given point we only have two directions, one from $\{\hat{a}, \hat{a}'\}$ and the other from $\{\hat{b}, \hat{b}'\}$.

1.3 Aspect's group

Subsequent to Bell's significant theoretical discovery, several experimental tests followed. Each experimental test was required to satisfy the following requirements as best as possible:

1. Observations on the entangled particles must be made at space-like distances.
2. Observations must involve two non-commuting observables
3. The directions $\hat{a}$, $\hat{a}'$ be chosen independently of $\hat{b}$, $\hat{b}'$, such that any possible hidden correlations between the two detectors are ruled out.
4. The directions $\hat{a}$, etc. be randomly chosen while the particles are in flight, such that any possible correlations between the directions and the original event leading to production of entangled particles are ruled out.
5. Observations be made at high efficiency, so that violations of the inequalities due to incorrect observations may be ruled out.

The first two points test the weirdest property of QM, "spooky action at a distance". This must necessarily involve non-commuting observables which in the words of EPR paper cannot have a "simultaneous reality". In the tests conducted by Clauser *et al.* the first two assumptions were definitely incorporated. They, however, used single channel polarizers which meant that their detectors could only detect the photons if they had certain polarization, whereas the opposite polarization went undetected. This however, is not ideal as the non-appearance of the opposite polarization can also result from simply having missed the photon. The third criteria was satisfied to certain degree: their setup involved static polarizers that could not be rotated during the flight of the particles. Thus, in demonstrating violation of the Bell's inequality, they had to make a crucial assumption that the rates of photons impinging on the detectors with any given polarization are independent of the directions of the two polarizers. However, the static nature of the experiment left the fourth point as a loop hole.

This was overcome to certain extent by later experiments by Aspect, Dalibard and Roger [6]. Their setup involved using ultrasonic standing waves in the water to enable fast switching between two polarizer directions during the flight of the photons. The detectors were positioned 12m apart such that $L/c = 40$ ns. Their setup, shown in Fig. 1.2, involved double channel polarizers, and hence they were able to tell apart between $\pm$ polarizations of the impinging photons. The acoustic switching was achieved at 10ns, and the lifetime of the intermediate cascade as 5 ns. Hence their setup enabled randomly choosing direction of either detector while photons were *en-route*. They found the Bell inequality violated by 5 standard deviations. However, it was noted that the polarizers were switched in a quasiperiodic fashion, and the ideal scheme wasn't fully completed. One could argue that the sinusoidal switching using ultrasonic waves can be

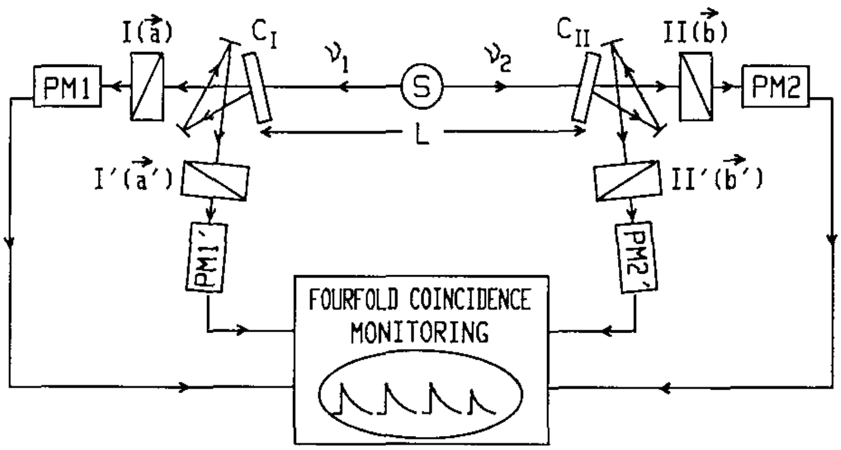

Figure 1.2 Schematic of setup by Aspect *et al.*. Taken from [6].

predictable into the future, and one instead requires a truly random switching of the directions while the photons are in the flight. They proposed that "a more ideal experiment with random and complete switching would be necessary for a fully conclusive argument against the whole class of supplementary-parameter (hidden-parameter) theories obeying Einstein's causality" [6].

1.4 Zeilinger's group

1.4.1 Bell's theorem with inequalities

The final milestone of random switching of detectors during photon-flight was achieved in a remarkable experiment by Weihs, Jennewein, Simon, Weinfurter and Zeilinger [7] in 1998. Their experiment was conducted using optical fibers stretched 400 meters apart across the Innsbruck university science campus. This gave them 1.3μs to perform individual measurements. They used a *physical* random number generator, a light-emitting diode, for fast switching of polarizer directions. Their random number generator did not have a perfectly even distribution though they argued that they normalized all the correlation functions to total number of events for a certain combination of the analyzers' settings. They managed to keep the distribution within 2%. With their electronics under control they ensured that their analyzer setting wouldn't have been influenced by any event more than 100 ns earlier, clearly much shorter than 1.3 μs.

This set up succeeded in achieving completely the locality criterion of the gedankenexperiment. One could argue if an unfair sampling of all the photon pairs that were created was responsible for the violation of the ineqaulity. This was overcome in the Orsay experiments where two-channel polarizers were used. Here the orthogonal polarization was deflected and detected as $-$. Lastly, the efficiency of the detectors in the last experiment was about 5%. Their final results were violation of the Bell's inequality by 30 standard deviations.

1.4.2 Bell's theorem without inequalities

Next we discuss another set-up where Bell's theorem can be recast without inequalities and without statistical terms. For two particle state, local realism can be only tested using statistical predictions of the theory. We will now see that for three particles, we see a conflict even for definite predictions. The statistics now is limited to the inevitable limitations of the experiments that are also present in classical physics.

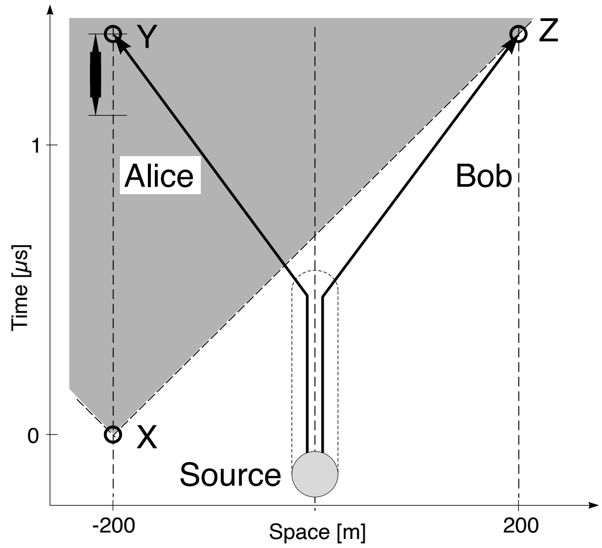

Figure 1.3 Setup of experiment by Zeilinger's group. Here the black vertical bar in the light-cone shows the amount of time they needed to implement random switching of the polarizer direction, which was about one tenth of the total flight time of the entangled photons. Taken from [7].

Let us first consider three spin-1/2 particles a, b, c and a set of observables,

$$\mathcal{O}_1 = \sigma_x^a \sigma_y^b \sigma_y^c \,, \qquad \mathcal{O}_2 = \sigma_y^a \sigma_x^b \sigma_y^c \,, \qquad \mathcal{O}_3 = \sigma_y^a \sigma_y^b \sigma_x^c \,. \tag{1.9}$$

It can be checked that the three observables commute and hence we can decompose any arbitrary state as simultaneous eignevectors of these observables. When applied on the state

$$|\psi\rangle = \frac{|+++\rangle - |---\rangle}{\sqrt{2}} \,, \tag{1.10}$$

using

$$\sigma_x|\pm\rangle = |\mp\rangle \,, \qquad \mathrm{i}\sigma_y|\pm\rangle = \mp|\mp\rangle \,, \tag{1.11}$$

we find

$$\sigma_x^a \sigma_y^c \sigma_y^c |\psi\rangle = |\psi\rangle \,, \tag{1.12}$$

and likewise $+1$ eigenvalue for the other two observables. Thus we have

$$\mathcal{O}_{1,2,3}|\psi\rangle = |\psi\rangle \,, \tag{1.13}$$

If we instead consider the state

$$|\phi\rangle = \frac{|+++\rangle + |---\rangle}{\sqrt{2}} \,, \tag{1.14}$$

we find

$$\mathcal{O}_{1,2,3}|\phi\rangle = -|\phi\rangle \,. \tag{1.15}$$

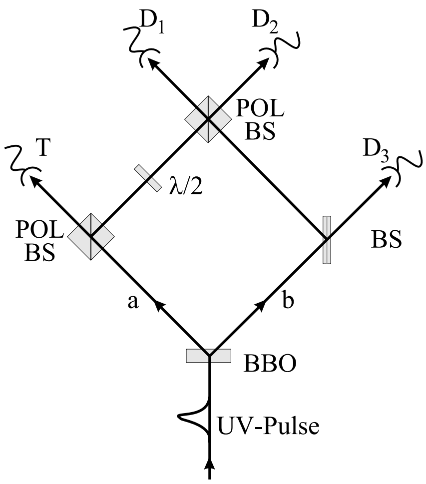

Figure 1.4 Setup to produce Greenberger-Horne-Zeilinger entangled state. Taken from [9].

The state $|\phi\rangle$ is termed as the Greenberger-Horne-Zeilinger state [8]. To understand the significance of these eigenstates, consider applying the operator

$$\mathcal{O}_x \equiv \sigma_x^a \sigma_x^b \sigma_x^c = -\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \,. \tag{1.16}$$

This must result -1 when applied on $|\psi\rangle$ (or +1 for $|\phi\rangle$). However, it turns out that if these three particles in this state were detected at space-like separations by three observers as in the experiments described above, with each observer making a random choice between x and y directions, the last result will be in contradiction with local realism where each of the three particles carry information about x and y spin components from the point they are created. In other words, local realism implies that measurement of $\mathcal{O}_x$ must result in +1 if the three measurements $\mathcal{O}_{1,2,3}$ also result in +1, in direct contradiction with QM!

To see how this works, let us consider the case when $\mathcal{O}_i|\psi\rangle = +|\psi\rangle$. In a hidden-variable theory, the three measurements using $\mathcal{O}_i$ will result in +1 outcome only for certain specific combinations. We can check explicitly the outcome of measuring $\mathcal{O}_x$ for all these configurations, the product of the spins in x directions must be positive, unlike the quantum mechanical result above. Suppose we consider first operating with $\mathcal{O}_1$ that results in +1 times the state. Thus, the particles can be assumed to carry spins, for example

$$|\psi\rangle_{\rm cl} = \begin{pmatrix} + \\ \ \end{pmatrix} \begin{pmatrix} \ \\ - \end{pmatrix} \begin{pmatrix} \ \\ - \end{pmatrix} \tag{1.17}$$

Here the first row represents outcome of the x-component spin and the second y. The three matrices represent three particles (not to be confused with the column vector labeling S_z components!). The empty slots are not constrained by $\mathcal{O}_1$ measurement. We can now consider application of $\mathcal{O}_2$ and again demand a +1 eigenvalue. Note that $\mathcal{O}_1$ has already fixed the σ_y^c eigenvalue. This is because once the particles are created, in the local realism explanation, they must carry these values to the detector where any of the two directions can be measured. Thus, for example, a viable configuration is

$$|\psi\rangle_{\rm cl} = \begin{pmatrix} + \\ - \end{pmatrix} \begin{pmatrix} + \\ - \end{pmatrix} \begin{pmatrix} \ \\ - \end{pmatrix} \tag{1.18}$$

Finally, application of $\mathcal{O}_3$ on this state must now fully constrain all the entries. Since both $\sigma_y^{a,b}$ are -1, the σ_x^c ought to be $+1$, such that

$$|\psi\rangle_{\text{cl}} = \begin{pmatrix} + \\ - \end{pmatrix} \begin{pmatrix} + \\ - \end{pmatrix} \begin{pmatrix} + \\ - \end{pmatrix} \tag{1.19}$$

However, now the application of $\sigma_x^a \sigma_x^b \sigma_x^c$ results in positive eigenvalue. It can be checked that the remaining cases also result in a plus sign, in direct contraction with quantum mechanics.

This state was prepared by Zeilinger's group in 1999 [9] and used for testing Bell's theorem in 2000 [10]. In their setup, shown in Fig. 1.4, they employed a β-barium Borate source which almost always emits a pair of entangled photons, each pair with zero total angular momentum. These photons are directed towards a setup consisting of polarizing and normal beam splitters and four detectors T, D_1, D_2 and D_3. In the event when all the four detectors detect photons, with the one in T being always horizontally polarized, the three photons in measured in $D_{1,2,3}$ correspond to a measurement on the GHZ state. This can be seen through a series of checks. For definiteness, let us stick to the terminology of [9] and refer to + as horizontal polarization (H), and $-$ as the vertical (V). Now, let us consider the event where all the four detectors are triggered:

1. The detector T must have H-polarized photon, so let's call it H_1, and it's companion V_1
2. The companion V_1 must go through arm **b**. It can either be reflected at the BS or transmitted. Let us say it was simply transmitted, then it will be detected at detector D_3.
3. Now let us consider the other pair. Since we have found a photon in the trigger T, one of the photons from the other pair traveling along the arm **a** must have had polarization V so as to be reflected by the PBS. Let's call it V_2, but leave this here for a moment.
4. The other photon from the second pair thus carries horizontal polarization, and let's call it H_2. From point 2 above, we've already assigned D_3 to V_1, so H_2 must be reflected at BS. Eventually it will encounter the PBS on top, and having horizontal polarization, it will be transmitted and registered at D_1.
5. Let us now return to the V_2. If upon passing through $\lambda/2$ plate its polarization does not rotate, it will be detected at D_1, which we have already assigned to H_2. Thus, the only possibility that remains is that it does get rotated $V_2 \to H_2'$, and goes right through PBS into D_2.

Hence, the outcome of this is

$$|T\rangle \otimes |D_1 D_2 D_3\rangle = |H_1\rangle \otimes |H_2 H_2' V_1\rangle \to |H\rangle \otimes |HHV\rangle\,, \tag{1.20}$$

Similarly, the other outcome when photon V_1 gets reflected at BS, is given by

$$|T\rangle \otimes |D_1 D_2 D_3\rangle = |H_1\rangle \otimes |V_2 V_1 H_2\rangle \to |H\rangle \otimes |VVH\rangle\,. \tag{1.21}$$

In the second outcome we see that the photon that was initially V_2 does not get rotated into horizontal polarization. Thus we see that the two outcomes occuring with equal probability lead to the state

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}\big(|HHV\rangle + |VVH\rangle\big)\,. \tag{1.22}$$

This may not look quite like $|\phi\rangle$ in Eq. (1.14), but it's just a matter of redefining the relative orientation of the third detector D_3 so as to call $V = +$ and $H = -$. Using this state, Zeilinger's group confirmed the validity of QM to 8 standard deviations [10].

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