Variance Reduction

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Outline

What is Variance Reduction?

Path integral formulation of QFT (finite, discrete, Euclidean space-time lattice): QFT vacuum expectation values $\leftrightarrow$ Ensemble Averages

$$\langle 0|\hat{\mathcal{O}}_1(x_1)...\hat{\mathcal{O}}_N(x_N)|0\rangle_{con} = \frac{1}{Z}\int \mathcal{D}\phi\ \mathcal{O}_1(x_1)...\mathcal{O}_N(x_N)\ \mathrm{e}^{-S[\phi]}$$

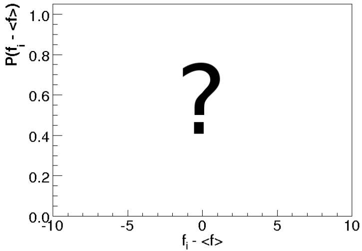
Un-biased estimator of $\langle 0|\hat{\mathcal{O}}_1(x_1)...\hat{\mathcal{O}}_N(x_N)|0\rangle_{con}$:

$$\langle \mathcal{O}_1(x_1)...\mathcal{O}_N(x_N)\rangle_{N_{conf}} \equiv \frac{1}{N_{conf}}\sum_{i=1}^{N_{conf}} \mathcal{O}_1(x_1)...\mathcal{O}_N(x_N)[\phi_i]$$

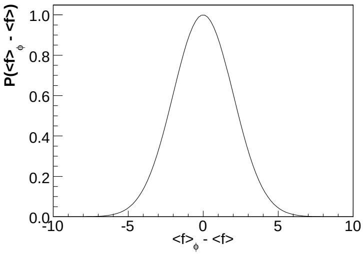
where the ϕ_i are distributed according to $e^{-S[\phi]}/Z$. Treat $f_i = \mathcal{O}_1(x_1)...\mathcal{O}_N(x_N)[\phi_i]$ as a random variable. Variance:

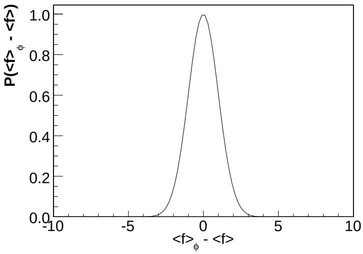
$$\begin{aligned} Var[f] &\equiv \lim_{N_{conf}\to\infty} \langle f^2\rangle_{N_{conf}} - \langle f\rangle^2_{N_{conf}} \\ &= \langle 0|\hat{f}^2|0\rangle_{con} - \langle 0|\hat{f}|0\rangle^2_{con} \end{aligned}$$

Central Limit Theorem: As $N_{conf} \to \infty$, distribution of $\langle f \rangle_{N_{conf}}$ approaches normal with mean $\langle f \rangle = \langle 0 | \hat{f} | 0 \rangle_{con}$ and variance $2\tau_f^{int} Var[f]/N_{conf}$.



As N_{conf} is increased, the distribution of $\langle f \rangle_{N_{conf}}$ narrows, with width $\sqrt{2\tau_f^{int} Var[f]/N_{conf}}$.





Two observables $\hat{f}$ and $\hat{g}$, such that $\langle 0|\hat{f}|0\rangle = \langle 0|\hat{g}|0\rangle$ but $Var[f] > Var[g]$. We should measure g, as it has a smaller error.

Simple example (Lüscher '10, arXiv:1002.4232 [hep-lat]):
Translation Invariant observable $\hat{\mathcal{O}}(x)$.

$$\mathcal{E} = \langle 0|\hat{\mathcal{O}}(0)|0\rangle = \frac{1}{V}\sum_x \langle 0|\hat{\mathcal{O}}(x)|0\rangle$$

Variance of the unsummed observable:

$$v_1 = \langle 0|\hat{\mathcal{O}}(0)^2|0\rangle - \langle 0|\hat{\mathcal{O}}(0)|0\rangle^2$$

Variance of the summed observable:

$$v_2 = \frac{1}{V^2}\sum_{x,y}\langle 0|\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)|0\rangle - \frac{1}{V^2}\Big(\sum_{x}\langle 0|\hat{\mathcal{O}}(x)|0\rangle\Big)^2$$

$$= \frac{1}{V^2}\Bigg[\sum_{\substack{x,y\\ ||x-y||<R}}\langle 0|\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)|0\rangle + \sum_{\substack{x,y\\ ||x-y||>R}}\langle 0|\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)|0\rangle\Bigg] - \mathcal{E}^2$$

$$= \frac{1}{V^2}\Bigg[\sum_{\substack{x,y\\ ||x-y||<R}}\langle 0|\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)|0\rangle + \sum_{\substack{x,y\\ ||x-y||>R}}\langle 0|\hat{\mathcal{O}}(x)|0\rangle\langle 0|\hat{\mathcal{O}}(y)|0\rangle\Bigg] - \mathcal{E}^2$$

$$= \frac{1}{V^2}\sum_{\substack{x,y\\ ||x-y||<R}}\langle 0|\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)|0\rangle + \mathcal{O}\Big(\frac{R^2}{V^2}\Big)$$

$$\sim \frac{1}{V}$$

‘Variance Reduction’: The clever construction of observables $\hat{g}$ which attempt to minimize $Var[g]$.

Variance and the Signal-to-noise problem

Observables are typically extracted from the temporal fall-off of correlation functions between hadron interpolating fields.

$$C^{2pt}(t) = \langle \mathcal{O}(t)\bar{\mathcal{O}}(0)\rangle = \sum_n |A_n|^2 e^{-E_n t}$$

$$C^{3pt}(t_1, t_2) = \langle \mathcal{O}(t_1 + t_2)J(t_2)\bar{\mathcal{O}}(0)\rangle = \sum_{mn} A_m A_n^* \mathcal{M}_{mn} e^{-E_m t_1} e^{-E_n t_2}$$

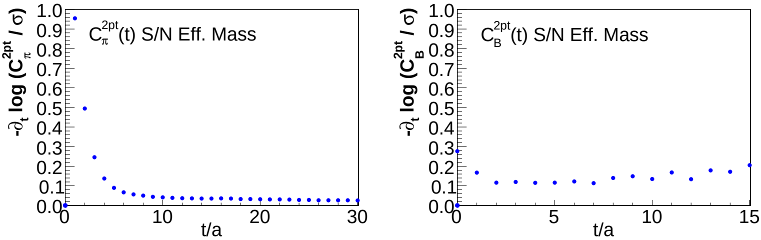
$$\mathcal{M}_{mn} = \langle m|\hat{J}|n\rangle, \quad A_n = \langle 0|\hat{\mathcal{O}}|n\rangle$$

In order to get the desired asymptotic behaviour (i.e. the 'ground state') all t's must be large.

Nearly all correlation functions have exponential decay of signal-to-noise with t.

Treat the signal-to-noise ratio as a correlation function $C(t) = S(t)/N(t)$. Examine the effective 'mass' $m_{eff} = \ln(C(t)/C(t+1))$

Examine two representative cases: Pion and pseudoscalar static-light meson (CLS ensemble, $a = 0.07\mathrm{fm}$, $m_\pi = 230, 400\mathrm{MeV}$).



How can I understand this? Lüscher '10 (arXiv:1002.4232 [hep-lat]), Lepage '89

Examine the variance of a pion correlation function:

$$\mathcal{O}_\pi(t) = \sum_{\mathbf{x}} \bar{u}\gamma_5 d(\mathbf{x}, t)$$

$$C_\pi^{2pt}(t) = \langle \mathcal{O}_\pi(t)\bar{\mathcal{O}}_\pi(0)\rangle$$

$$Var[C_\pi^{2pt}(t)] \sim \langle \mathcal{O}_{\pi\pi}(t)\bar{\mathcal{O}}_{\pi\pi}\rangle - \left[C_\pi^{2pt}(t)\right]^2$$

$$\frac{C_\pi^{2pt}(t)}{\sqrt{Var[C_\pi^{2pt}(t)]}} \sim e^{-(m_\pi - E_{\pi\pi}/2)t} \sim 1$$

And a pseudoscalar static-light meson correlation function:

$$C_B^{2pt}(t) = \langle \mathcal{O}_B(t)\bar{\mathcal{O}}_B(0)\rangle$$

$$Var[C_B^{2pt}(t)] \sim \langle \mathcal{O}_\pi(t)\bar{\mathcal{O}}_\pi\rangle$$

$$\frac{C_B^{2pt}(t)}{\sqrt{Var[C_B^{2pt}(t)]}} \sim e^{-(m_B - m_\pi/2)t}$$

Things I'm not going to talk about

- Autocorrelation:
  - Reduction in autocorrelation $\Rightarrow$ reduction in 'effective' variance.
  - Autocorrelation Refs: Sommer, Schaefer, Virotta '10 (arXiv:1009.5228 [hep-lat]); Wolff '04 (hep-lat/0306017)
- 'Multi-level'-type algorithms
  - In bosonic theories, an observable is factored into sub-lattice expectation values.
  - This achieves an exponential error reduction compared to conventional simulations.
  - Refs: Lüscher, Weisz '01 (hep-lat/0108014); Della Morte, Giusti '10 (0806.2601 [hep-lat]).
- Interpolator Improvement
  - Hasten onset of asymptotic behavior, Improve signal to noise
  - Quark/Link Smearing: Basak, et al. '06 (hep-lat/0509179)
  - Solutions of a GEVP: Blossier, et al. '09 (arXiv:0902.1265 [hep-lat])
  - Clever operator construction/choice: Basak, et al. '05 (hep-lat/0506029)

What I *will* talk about

Correlators with fermion fields must be wick-contracted down fermion two-point functions (quark propagators)

$$\langle \mathcal{O}_\pi(t)\mathcal{O}_\pi(0)\rangle = \sum_{\mathbf{x},\mathbf{y}} \mathrm{Tr}\left[Q(\mathbf{x},t|\mathbf{y},0)^2\right]$$

$$Q(\mathbf{x},t|\mathbf{y},0) = \langle q(\mathbf{x},t)\bar{q}(\mathbf{y},0)\rangle$$

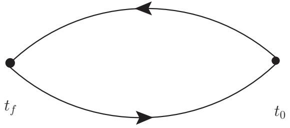
Quark propagators are the inverse of the ($V^3 \times T \times N_{spin} \times N_{color}$ dimensional) fermionic Dirac matrix, and cannot be inverted directly.

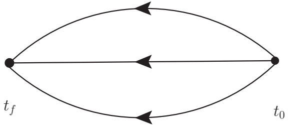
(Costly) Algorithms (which iteratively apply the matrix) can obtain the inverse ($Q = M^{-1}$) acting on something.

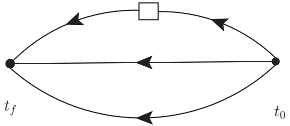
$$M\psi = \eta \Rightarrow \psi = M^{-1}\eta$$

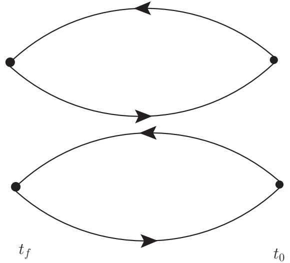
Some correlation functions only require 'point-to-all' propagators, i.e $Q(\mathbf{x}, t|\mathbf{x}_0, t_0) = M^{-1}\delta_{\mathbf{x}\mathbf{x}_0}\delta_{tt_0}$.

- 'connected' correlation functions (without definite $\mathbf{p}_0$)
- Mom. proj. at 'sink' will suffice
- Some over source sites will increase statistics. (Beware!)



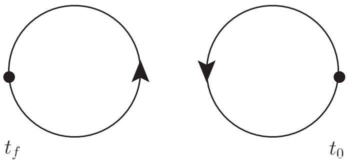


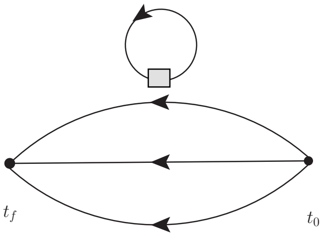


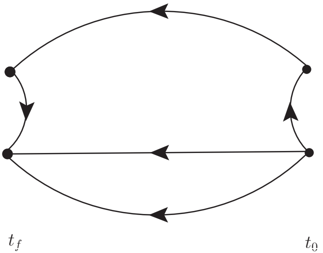


Others require ‘all-to-all’ propagators

- Hadrons with definite $\mathbf{p}_0$
- Flavor singlet quantites
- Multi-hadron states







All-to-all propagators cannot be done trivially. Clever estimators can be constructed.

The topic of these lectures will be the construction and improvement (i.e. variance reduction) of estimators for all-to-all quark propagators.

Outline

Stochastic estimates of inverse matricies

Simple Idea: Choose random numbers η_i from some probability distribution p with the properties

- $\langle \eta \rangle \equiv \int d\eta \, \eta p(\eta) = 0$
- $\langle \eta \eta^* \rangle = \langle |\eta|^2 \rangle = 1$

Then for N independent random numbers we have: $(i, j = 1...N)$

$$\langle \eta_i \eta_j^* \rangle = \begin{cases} \langle |\eta|^2 \rangle = 1, & i = j \\ |\langle \eta \rangle|^2 = 0, & i \neq j \end{cases}$$

$$Var[\eta_i \eta_j^*] = \begin{cases} Var[|\eta|^2] \,, & i = j \\ Var[\eta] Var[\eta^*] \,, & i \neq j \end{cases}$$

So that by the Central Limit Thm.

$$\langle \eta_i \eta_j^* \rangle_R \equiv \frac{1}{R} \sum_{r=1}^{R} \eta_i^{(r)} \eta_j^{(r)*}$$

$$\lim_{R \to \infty} \langle \eta_i \eta_j^* \rangle_R = \langle \eta_i \eta_j^* \rangle + \mathcal{O}(\sqrt{Var[\eta_i \eta_j^*]/R}) \to \delta_{ij}$$

We can estimate a (N-dimensional) matrix inverse:

$$M\psi^{(r)} = \eta^{(r)} \to \psi^{(r)} = M^{-1}\eta^{(r)}$$
$$\lim_{R\to\infty} \langle \psi\eta^* \rangle_R = M^{-1} + \mathcal{O}(\sqrt{Var[\psi\eta^*]/R})$$

What about the variance?

$$\begin{aligned}
Var[\psi_i\eta_j^*] &= \langle |\psi_i\eta_j^*|^2 \rangle - |\langle \psi_i\eta_j^* \rangle|^2 \\
&= \sum_{k,l} M_{ik}^{-1} \left[M_{il}^{-1}\right]^* \langle \eta_k\eta_l^*\eta_j\eta_j^* \rangle - |M_{ij}^{-1}|^2 \\
&= \sum_{k\neq j} |M_{ik}^{-1}|^2 \langle |\eta|^2 \rangle^2 + |M_{ij}^{-1}|^2 \left(\langle |\eta|^4 \rangle - 1\right) \\
&= \sum_{k\neq j} |M_{ik}^{-1}|^2 + |M_{ij}^{-1}|^2 \left(\langle |\eta|^4 \rangle - 1\right)
\end{aligned}$$

What type of noise should I use?

Real Gaussian: η real, $p(\eta) = e^{-\eta^2/2}/\sqrt{2\pi}$

$$\langle \eta \rangle = \int_{-\infty}^{\infty} d\eta \; \eta p(\eta) = 0$$

$$\langle \eta^2 \rangle = \int_{-\infty}^{\infty} d\eta \; \eta^2 p(\eta) = 1$$

$$\langle \eta^4 \rangle = \int_{-\infty}^{\infty} d\eta \; \eta^4 p(\eta) = 3$$

$$\langle \eta_i \eta_j \rangle = \delta_{ij}, \quad Var[\eta_i \eta_j] = 1 + \delta_{ij}$$

So the variance is:

$$Var[\psi_i \eta_j] = \sum_{k \neq j} |M^{-1}_{ik}|^2 + |M^{-1}_{ij}|^2 (3 - 1)$$

$$= \sum_{k} |M^{-1}_{ik}|^2 + |M^{-1}_{ij}|^2$$

Complex Gaussian: η Complex, $p(\eta) = \mathrm{e}^{-\eta^*\eta}/\pi$

$$\langle|\eta|^4\rangle = \int_{\mathcal{C}} d\eta \; (\eta^*\eta)^2 p(\eta) = 2$$

$$Var[|\eta|^2] = \int_{\mathcal{C}} d\eta \; (|\eta|^2 - 1)^2 p(\eta) = 1$$

So the variance is:

$$Var[\psi_i \eta_j] = \sum_{k \neq j} \left|M^{-1}_{ik}\right|^2 + |M^{-1}_{ij}|^2$$

Unimodular noise

Z_2: $p(1) = p(-1) = 1/2$

$$\langle \eta^4 \rangle = 1$$
$$Var[\eta^2] = \langle (\eta^2 - \langle \eta^2 \rangle)^2 \rangle = 0$$

Z_4 : $p(1) = p(-1) = p(i) = p(-i) = 1/4$

(or $Z_2 \times Z_2$: $p(1+i) = p(1-i) = p(-1+i) = p(-1-i) = 1/4$)

$$Var[|\eta|^2] = 0$$
$$\langle |\eta|^4 \rangle = 1$$

$U(1)$: $\eta = \mathrm{e}^{i\theta}, \quad p(\eta) = 1/(2\pi)$

$$\langle |\eta|^4 \rangle = 1$$
$$Var[|\eta|^2] = 0$$

Stochastic Estimate of products of M^{-1}

For an unbiased estimator, need independent stochastic estimates for each factor of M^{-1}.

Caveat (the 'sourceless' method): I can obtain $\sum_k M^{-1}_{ik}[M^{-1}_{jk}]^*$ with a single stochastic estimate

$$\langle \psi_i \psi_j^* \rangle = \sum_{k,l} M^{-1}_{ik}[M^{-1}_{jl}]^* \langle \eta_k \eta_l^* \rangle$$
$$= \sum_k M^{-1}_{ik}[M^{-1}_{jk}]^*$$

The variance of products of M^{-1} can be easily calculated: *EXERCISE*!!!

Dilution

Introduce a set of projectors $P_{ij}^{[d]},\ d = 1 \ldots N_d$

- $\sum_d P_{ij}^{[d]} = \delta_{ij}$
- $\sum_k P_{ik}^{[d]} P_{kj}^{[d']} = \delta_{dd'} P_{ij}^{[d]}$

Now the matrix inverse can be estimated by

$$M_{ij}^{-1} = \sum_d \langle \psi_i^{[d]} \eta_j^{[d]*} \rangle$$

$$\eta_j^{[d]} = P_{jk}^{[d]} \eta_k, \quad \psi_i^{[d]} = M_{ij}^{-1} \eta_j^{[d]}$$

Variance Revisited

$$Var[\sum_d \psi_i^{[d]} \eta_j^{[d]*}] = \sum_d Var[\psi_i^{[d]} \eta_j^{[d]*}]$$

$$Var[\psi_i^{[d]} \eta_j^{[d]*}] = \sum_{k,l} M_{ik}^{-1} [M_{il}^{-1}]^* \langle \eta_j^{[d]} \eta_j^{[d]*} \eta_k^{[d]} \eta_l^{[d]*} \rangle - |\langle \psi_i^{[d]} \eta_j^{[d]*} \rangle|^2$$

If $j \notin d$, $\langle \psi_i^{[d]} \eta_j^{[d]} \rangle = 0$ and $Var[\psi_i^{[d]} \eta_j^{[d]}] = 0$. So

$$Var[\sum_d \psi_i^{[d]} \eta_j^{[d]}] = Var[\psi_i^{[d_j]} \eta_j^{[d_j]}]$$

$$= \sum_{k \in d_j} |M_{ik}^{-1}|^2 + |M_{ij}^{-1}|^2 \left(\langle |\eta|^4 \rangle - 1 \right)$$

There are now *significantly* fewer terms in the sum over k.

The Homeopathic Limit (Maximal Dilution)

Take $N_d = N$. Examine the Variance again:

$$Var[\sum_d \psi_i^{[d]} \eta_j^{[d]}] = \sum_{k \in d_j} |M_{ik}^{-1}|^2 + |M_{ij}^{-1}|^2 (\langle |\eta|^4 \rangle - 1)$$
$$= |M_{ij}^{-1}|^2 (\langle |\eta|^4 \rangle - 1)$$

This estimate has zero variance for unimodular noise. The Homeopathic limit can be reached with a single noise vector.

Start with a single (unimodular) noise vector. I can reduce the error by

- Adding additional noise sources: decreases like $\sim \sqrt{Var/R}$
- Adding additional dilution projectors: reaches the exact answer with $N_d = N$

Fermionic Fields

All the above considerations apply to the Dirac matrix. Compound indicies on η_i: $i = \{a, \alpha, \mathbf{x}, t\}$

Dilution Schemes: (Possible choices of dilution projectors)

- Time Dilution : $N_d = N_t$, $P^{[d]} = \delta_{td}(\otimes 1_{spin} \otimes 1_{color} \otimes 1_{space})$
- Spin Dilution : $N_d = N_s$, $P^{[d]} = \delta_{d\alpha}$
- Color Dilution : $N_d = N_c$, $P^{[d]} = \delta_{da}$
- Space Even-Odd Dilution : $N_d = 2$. Each projector covers the even or odd sites of the lattice.

Time dilution is a good idea

For connected correlation functions, full time dilution is a good idea. Look at the variance:

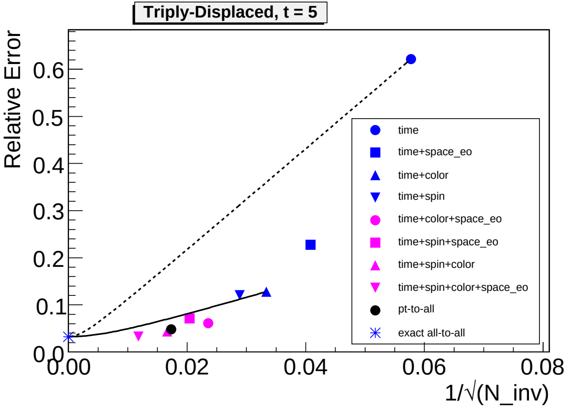
$$Var[\sum_d \psi_{a\alpha}(\mathbf{x},t)^{[d]}\eta_{a_0\alpha_0}(\mathbf{x}_0,t_0)^{[d]*}] = \sum_{(a',\alpha',\mathbf{x}',t')\in d} |M_{(a\alpha|a'\alpha')}(\mathbf{x},t|\mathbf{x}',t')|^2$$

If time dilution is not used, $Var[\psi(t)\eta^*(t_0)]$ is independent of $t - t_0$ resulting in a signal-to-noise problem.

For fully disconnected correlators, variance on the exact all-to-all is independent of time, so full time dilution may be overkill.

Dilution tests

Examine a single observable (Nucleon correlator at a fixed time seperation). Bulava, et al '08 (arXiv:0810.1469 [hep-lat])



Solid/Dashed lines: $\sigma_s/\sqrt{N_{inv}} + \sigma_g$

The rel. error falls off faster than $1/\sqrt{N_d}$

Correlator Construction

With full time dilution:

$$Q^{-1}(t|t_0) = \sum_d \langle \psi^{[d]}(t)\eta^{[d]*}(t_0)\rangle$$
$$= \sum_{d\in t_0} \langle \psi^{[d,t_0]}(t)\eta^{[d,t_0]*}(t_0)\rangle$$

Meson 2pt correlation function:

$$C^{2pt}_{mn}(t-t_0) = \sum_{d,d'} \rho_m^{[d,d';t_0]}(t)\omega_n^{[d,d',t_0]*}(t_0)$$
$$\rho_m^{[d,d';t_0]}(t) = \psi^{[d,t_0]\dagger}(t)\Gamma_m\psi^{[d',t_0]}(t)$$
$$\omega_n^{[d,d';t_0]}(t) = \eta^{[d,t_0]\dagger}(t_0)\Gamma_n\eta^{[d',t_0]}(t_0)$$

Baryon 2pt correlation function:

$$C^{2pt}_{mn}(t-t_0) = \sum_{d_1,d_2,d_3} \Delta_m^{[d_1,d_2,d_3;t_0]}(t)\Omega_n^{[d_1,d_2,d_3,t_0]*}(t_0)$$

Outline

Quark Smearing

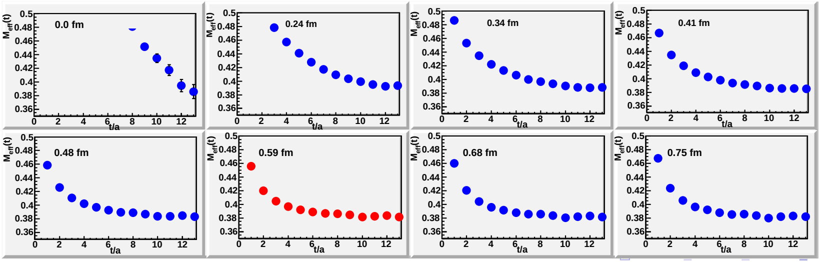
Quark Smearing reduces the level of excited state contamination.

$$Q^{-1} \to \tilde{Q}^{-1} = SQ^{-1}S^{\dagger}$$

S is diagonal in time, independent of spin, gauge covariant, and isotropic.

Typical choice: $S = \mathrm{e}^{-\frac{\sigma}{2}\Delta}$, σ is a tunable parameter controling the 'width' of the 'wavefunction'.

To tune, examine the effective mass:
$am_{eff}(t) = \ln(C(t)/C(t+1))$.



A new quark smearing algorithm

Examine the smearing operator in its eigenbasis:

$$S = \mathrm{e}^{-\frac{\sigma}{2}\Delta} = \sum_n \mathrm{e}^{-\frac{\sigma}{2}\lambda_n} v_n v_n^\dagger$$

Gaussian decreases the weight of high-lying laplacian eigenmodes.
$\Rightarrow$ a new kind of smearing: truncation in laplacian eigenmodes.
(Peardon, et al. '10 arXiv:0905.2160 [hep-lat])

$$S = \Theta(\sigma_s^2 + \Delta) = \sum_{n,\lambda_n<\sigma_s^2} v_n v_n^\dagger \approx \sum_{n=1}^{N_v} v_n v_n^\dagger \tag{1}$$

Exact Smeared all-to-all

This 'distillation' allows the exact smeared quark propagator (or the smeared-unsmeared quark propagator) to be calculated with $N_{inv} = N_v \times N_t \times N_{spin}$ inversions.

$$\eta^{[n,t_0,\alpha_0]} = v_n[t_0] \otimes \delta_{\alpha\alpha_0} \otimes \delta_{tt_0}$$

$$\psi^{[n,t_0,\alpha_0]} = Q^{-1}\eta^{[n,t_0,\alpha_0]}$$

$$Q^{-1}_{\alpha\alpha_0}(t|t_0)S = \sum_{n=1}^{N_v} \psi_\alpha^{[n,t_0,\alpha_0]}(t)\eta^{[n,t_0,\alpha_0]*}$$

Smeared-Smeared Correlator Construction

Correlator construction is particularly simplified with smeared-smeared propagators.

$$\tilde{Q}^{-1}(t|t_0) = \sum_{m,n=1}^{N_v} v_m[t] v_m^\dagger[t] Q^{-1}(t|t_0) v_n[t_0] v_n^\dagger[t_0]$$

$$= \sum_{m,n=1}^{N_v} v_m[t] K_{mn}(t|t_0) v_n^\dagger[t_0]$$

Meson correlation functions:

$$C_{mn}^{2pt}(t - t_0) = \mathrm{Tr}\left[M^m[t] K(t|t_0) M^{n\dagger}[t_0] K(t_0|t) \right]$$

$$M_{ij}^m[t] = \sum_{\mathbf{x},\mathbf{x}'} v_i^\dagger[t](\mathbf{x}) D_m(\mathbf{x}|\mathbf{x}') v_j[t](\mathbf{x}')$$

Baryon correlation functions:

$$C^{2pt}_{mn}(t-t_0) = \sum_{\substack{ijk \\ i'j'k'}} \Omega^{m}_{ijk}[t] K_{ii'}(t|t_0) K_{jj'}(t|t_0) K_{kk'}(t|t_0) \Omega^{n\dagger}_{i'j'k'}[t_0]$$

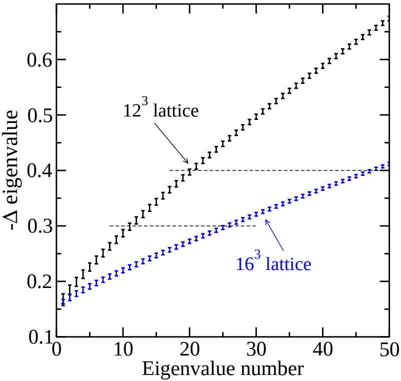
Current Insertions:

$$K_{ij}(t_f|t_0) \to G^{J}_{ij}(t_f|t|t_0)$$
$$G^{J}_{ij}(t_f|t|t_0) = \psi^{[i,t_f,\alpha_f]\dagger}(t) J \psi^{[j,t_0,\alpha_0]}(t)$$

Multi-hadron correlation functions are more complicated products of M, K, Ω, G

The V^2 Problem

To maintain fixed σ_s, N_v (and N_{inv}) increases like $\sim V$.



Each inversion cost increases like $V \rightarrow V^2$ total cost increase.

Error only goes down like $\sim V$.

Preliminary Distillation Results (Small Volumes)

- Nucleon, Δ, and Ω baryon excitation spectra: Bulava, et al. '10 (arXiv:1004.5072 [hep-lat])
- Isovector meson excitation spectra: Morningstar, et al. '10 (arXiv:1011.6573 [hep-lat])
- Isoscalar meson excitation spectra: Dudek et al. '11 (arXiv:1102.4299 [hep-lat])
- $I = 2\ \pi - \pi$ scattering phase shifts: Dudek et al. '10 (arXiv:1011.6352 [hep-lat])

Outline

A new way to introduce noise

Ordinary Lattice Noise : $\eta^{(r)}_{a\alpha}(\mathbf{x}, t) \in Z_4$

LapH(Laplacian-Heaviside) Noise :

$\eta^{(r)}_{a\alpha}(\mathbf{x}, t) = \sum_n d^{(r)}_{n\alpha t} v_{na}[t](\mathbf{x}), \quad d^{(r)}_{n\alpha t} \in Z_4$

Two distinct features of LapH Noise :

- *Dramatically* fewer random numbers than standard noise
- η is now gauge-covariant

Correlator construction is the same as discussed previously, with source smearing.

Types of LapH dilution schemes

Dilution can now occur in *time* $\otimes$ *spin* $\otimes$ *eigenmode* space.

For each type of dilution, one has three simple options:

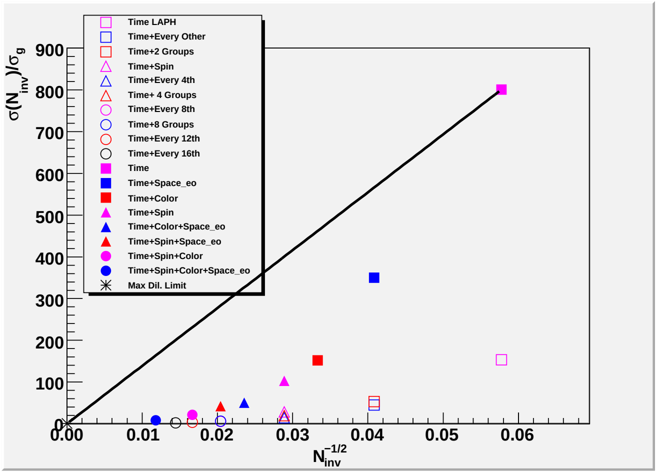
- Full (F): full dilution
- Interlace-N (IN): Each projector has support seperated by N
- Block-N (BN): Each projector has support on N adjacent

Examples:

- $[F, F, F]$: The maximal dilution limit.
- $[F, F, I8]$: Full time and spin dilution, support on every $8th$ eigenvector
- $[I16, F, I8]$: Support on every $16th$ timeslice, full spin dilution, support on every $8th$ eigenvector.

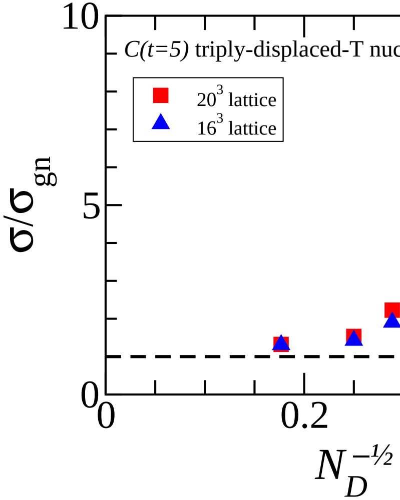
Numerical Tests/Demonstrations

Laph Noise vs. Lattice Noise (J. Bulava '09, P.H.D Thesis)
$20^3 \times 128$, $m_\pi \sim 400\mathrm{MeV}$, $a_s = .12\mathrm{fm} = 3a_t$.



Volume dependence of higher LapH dilution schemes (Morningstar, et al. *to appear*)

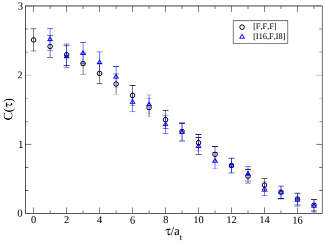
Dilution schemes, from left to right:
$[F, F, I8], [F, N, I16], [F, N, I12], [F, N, I8]$.

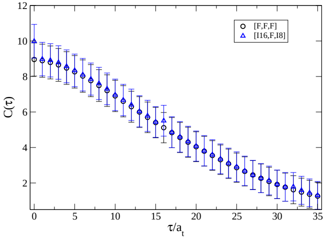


For $[F, F, I8]$, $\sigma/\sigma_g = 1.3$, does not grow with the volume.

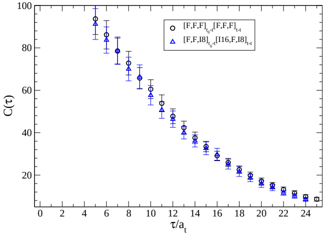
Cost increases like $\sim V$, error decreases like $\sim V$.

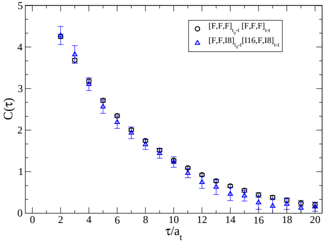
- Isoscalar(σ and η') mesons: Foley, et al. '10 (arXiv:1011.6573 [hep-lat])
- Examine disconnected contributions in Distillation vs. Stochastic LapH.





- Two-pion systems: Foley, et al. '10 (arXiv:1011.6573 [hep-lat])
- Disconnected diagrams from $\pi\pi - \pi\pi$ (box diagram) and $\rho - \pi\pi$(triangle diagram).





Conclusions

- All-to-all propagators are required to evaluate many correlation functions, and (probably) have to be estimated stochastically.
- If you don't have to do all-to-all, a careful cost-benefit analysis should be performed, analyzing the variance, etc.
- Introducing additional dilution projectors reduces the variance faster than $N_{inv}^{-1/2}$, to a point.
- LapH noise seems to perform better (factor of ~ 8) than lattice noise, although the construction of the eigenvectors is a non-trivial (but relatively small) cost.
- Additionally, highly diluted Laph noise has a volume independent effectiveness. The $[F, F, I8]$ scheme has an error only 30% larger than the exact answer, for a factor $\sim 4 - 32$ less inversions (on a $2\mathrm{fm}$) lattice.

Additional all-to-all techniques

- Low-mode preconditioning: Neff, et al. '01 (hep-lat/0106016)
- Even-odd preconditioning: Blossier, et al. '10 (arXiv:1004.2661 [hep-lat])
- Eigenspectrum noise subtraction: Guerrero, et al. '09 (arXiv:1001.4366 [hep-lat])
- Hopping parameter acceleration: Bali et al. '10 (arXiv:0910.3970 [hep-lat])
- Truncated solver method: Bali et al. '10 (arXiv:0910.3970 [hep-lat])
- Domain decomposition improvement: Burch and Hagen '06 (arXiv:hep-lat/0609011)